PDE Approach to Valuation and Hedging of Credit Derivatives

Tomasz R. Bielecki*
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA

Monique Jeanblanc†
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France

Marek Rutkowski
School of Mathematics
University of New South Wales
Sydney, NSW 2052, Australia
and
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland

February 23, 2005

*The research of the first author was supported in part by NSF Grant 0202851 and by Moody's Corporation grant 5-55411.

†The research of the second author was supported in part by Zéliade and Itô33

1 Introduction

Our aim is to examine the PDE approach to valuation and hedging of a defaultable claim in various settings; this allows us to emphasize the importance of the choice of the traded assets. We start with a general model for the dynamics of the traded primary assets. Subsequently, we specify some particular models and we deal with particular defaultable claims such as, for instance, survival claims. For the sake of notational simplicity, we deal throughout with a model with only three primary traded assets. A generalization to the case of k primary assets is rather straightforward, though notationally more cumbersome.

The paper is organized as follows. In Section 2, we examine the no-arbitrage property of a model in terms of a martingale measure. The next section is devoted to the study of the PDE approach to valuation of defaultable claims and we give the hedging strategies of a contingent claim under the assumption that prices of primary assets are strictly positive. Section 4 shows how to adapt the valuation PDE and replicating strategies if one of the primary assets is a defaultable security with zero recovery, so that its price vanishes after default. In Section 5, we modify the original market model by replacing the fixed horizon date for trading activities by a random time horizon determined by the default time. Finally, in Section 6, we examine briefly the possible extensions to the case of several default times. We refer to the companion works by Bielecki et al. (2004a)-(2004d) for notation and related results.

2 Martingale Measures

In this section, we start, as in standard modelling of prices, with historical dynamics of the traded assets. For ease of computation, we restrict our attention to the case of three traded assets and two sources of noise: a Brownian motion and a random default time. Our goal is to derive the PIDE satisfies by the price of a defaultable claim in a general model under market completeness. Subsequently, we examine some examples corresponding to specific choices of the underlying assets.

2.1 Default Time

Let τ be a strictly positive random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, referred to as a *default time*. Note that $\mathbb{Q}$ is the real-life (or statistical) probability measure. In order to exclude trivial cases, we assume that $\mathbb{Q}\{\tau > 0\} = 1$ and $\mathbb{Q}\{\tau \leq T\} > 0$. Let us introduce the jump process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and denote by $\mathbb{H}$ the filtration generated by this process.

Assume that we are given, in addition, a *reference filtration* $\mathbb{F}$ such that $\mathcal{F}_t \subseteq \mathcal{G}$ for every $t \in [0, T]$ and the σ-field $\mathcal{F}_0$ is trivial. We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as to the *full filtration*; it includes the observations of default event. We assume that any $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale. Such an assumption is sometimes called *Hypothesis H*. For more details on this assumption, we refer to Bielecki et al. (2004a).

We denote $F_t = \mathbb{Q}\{\tau \leq t \,|\, \mathcal{F}_t\}$, so that $G_t = 1 - F_t = \mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}$ is the *survival process* with respect to $\mathbb{F}$. It is easily seen that F is a bounded, non-negative, $\mathbb{F}$-submartingale. It is well known that, under Hypothesis H, we have $F_t = \mathbb{Q}\{\tau \leq t \,|\, \mathcal{F}_\infty\}$, and thus the process F is increasing. Assume, in addition, that $F_t < 1$ for every $t \in \mathbb{R}_+$. The $\mathbb{F}$-*hazard process* Γ of a random time τ with respect to a filtration $\mathbb{F}$ is defined through the equality $1 - F_t = e^{-\Gamma_t}$, that is, $\Gamma_t = -\ln G_t$.

It is well known that if the $\mathbb{F}$-hazard process Γ of τ is a continuous, increasing process then the process $M_t = H_t - \Gamma_{t\wedge\tau}$, $t \in \mathbb{R}_+$, is a $\mathbb{G}$-martingale. The process M is referred to as the compensated martingale of the default process H. If the hazard process is absolutely continuous with respect to the Lebesgue measure, so that $\Gamma_t = \int_0^t \gamma_u \, du$ for some $\mathbb{F}$-progressively measurable process γ, then γ is called the $\mathbb{F}$-*intensity* of τ. In what follows, we are working mainly with a constant or deterministic intensity γ, in order to give the main ideas, which, somewhat surprisingly, do not seem to be commonly known.

We assume throughout that the $\mathbb{F}$-hazard process is absolutely continuous. Hence, the process

$$M_t = H_t - \int_0^{t\wedge\tau} \gamma_u \, du = H_t - \int_0^t \gamma_u (1 - H_u) \, du = H_t - \int_0^t \xi_u \, du, \tag{1}$$

where $\xi_t = \gamma \mathbb{1}_{\{t\leq\tau\}}$, is a $\mathbb{G}$-martingale under $\mathbb{Q}$. Also, let us recall that if the representation theorem holds for the filtration $\mathbb{F}$ and a finite family $Z^i, i \leq n$ of $\mathbb{F}$-martingales then, under Hypothesis H, the representation theorem holds also for the filtration $\mathbb{G}$ and with respect to the $\mathbb{G}$-martingales $Z^i, i \leq n$ and M .

2.2 Primary Traded Assets

We assume that we are given a family Y^1, Y^2, Y^3 of semimartingales defined on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$. We interpret Y^i_t as the cash price at time t of the ith primary traded asset, and we examine a market model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$, where Φ is the class of all self-financing trading strategies. In order to get more explicit valuation formulae, we postulate that the process Y^i is governed by the SDE

$$dY^i_t = Y^i_{t-}\big(\mu_i \, dt + \sigma_i \, dW_t + \kappa_i \, dM_t\big), \quad i = 1, 2, 3, \tag{2}$$

with the initial condition $Y^i_0 > 0$. Here W is a standard one-dimensional Brownian motion and the process M, given by (1), is the compensated martingale of the default process H. Note that in view of Hypothesis H, the process W follows a Brownian motion not only with respect to its natural filtration $\mathbb{F}^W$, but also with respect to the enlarged filtration $\mathbb{G}$. In Bielecki et al. (2005), we extend this model to more general dynamics, involving also a Poisson process.

In the first step, we restrict our attention to the case where the coefficients $\mu_i, \sigma_i > 0, \kappa_i$ and the default intensity $\gamma > 0$ are constant. We assume that $\kappa_i \geq -1$ for $i = 1, 2, 3$ in order to ensure that the price processes Y^i, $i = 1, 2, 3$ are non-negative. Note that the equality $\kappa_i = 0$ corresponds to the case where the ith asset is *default-free*, while the inequality $\kappa_i \neq 0$ means that the ith asset is formally classified as a *defaultable* security. In particular, the equality $\kappa_i = -1$ corresponds to the case where the price Y^i vanishes after default, i.e., the ith asset is defaultable and is submitted to the *zero recovery* (or *total default*) scheme. It should be stressed that most of our results can be easily extended to the case of coefficients with Markovian-type dependence on the underlying stochastic processes, as made precise by the following assumption.

Assumption A. The coefficients $\mu_i, \sigma_i, \kappa_i$ in dynamics (2) and the intensity γ are given by some functions on $\mathbb{R}_+ \times \mathbb{R}^3 \times \{0, 1\}$, so that $\mu_i = \mu_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, H_{t-})$, $\sigma_i = \sigma_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, H_{t-})$, $\kappa_i = \kappa_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-})$ and $\gamma = \gamma(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-})$. Moreover, the coefficients are regular so that the SDE (2) admits a unique strong solution for $i = 1, 2, 3$.

The crucial observation is that, under Assumption A, the process (Y^1, Y^2, Y^3, H), taking values in $\mathbb{R}^3 \times \{0, 1\}$, possesses the Markov property under the statistical probability $\mathbb{Q}$. It is worth noting that the triple (Y^1, Y^2, Y^3) does not follow a Markov process under $\mathbb{Q}$.

2.2.1 Recovery Schemes

Let us observe that case where the ith asset pays a pre-determined recovery at default is covered by the present set-up. For instance, the case of a constant recovery payoff $\delta \geq 0$ at default time τ corresponds to the coefficient $\kappa_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}) = \delta(Y^i_{t-})^{-1} - 1$. Hence, the dynamics of the ith asset under $\mathbb{Q}$ are

$$dY^i_t = Y^i_{t-}\big(\mu_i \, dt + \sigma_i \, dW_t + (\delta(Y^i_{t-})^{-1} - 1) \, dM_t\big).$$

If the recovery is proportional to the pre-default asset's value and is paid at default time τ (i.e., under the *fractional recovery of market value*), we deal with the constant coefficient $\kappa_i = \delta - 1$, and thus the dynamics of Y^i become

$$dY^i_t = Y^i_{t-}\big(\mu_i \, dt + \sigma_i \, dW_t + (\delta - 1) \, dM_t\big).$$

If the ith asset is no longer traded after default time, we may assume that the price process is stopped at time τ and thus the coefficients in the dynamics of the ith asset vanish after time τ.

2.3 Change of a Numeraire

We assume throughout that Y^i, $i = 1, 2, 3$ are governed by (2) and that $\kappa_1 > -1$ so that $Y^1_t > 0$ for every $t \in \mathbb{R}_+$. This assumptions allows us to take the first asset as a numeraire. Let us recall that the constant coefficient $\kappa_1 > -1$ in dynamics (2) corresponds to a fractional recovery of market value for the first asset.

In general, we do not refer to the theory where under the risk-neutral probability associated with the choice of a risk-free asset (a savings account) as a numeraire. In fact, we do not make the assumption that a risk-free security exists. We shall instead use an equivalent martingale measure $\mathbb{Q}^1$ such that, under $\mathbb{Q}^1$ the asset prices expressed in units of the numeraire Y^1 are martingales. In other words, the martingale measure $\mathbb{Q}^1$ is characterized by the property that the relative prices $Y^i(Y^1)^{-1}$, $i = 1, 2, 3$ are $\mathbb{Q}^1$-martingales.

We first derive the dynamics of the process $Y^{i,1} = Y^i(Y^1)^{-1}$ for $i = 1, 2, 3$. From Itô's formula, we obtain the dynamics of the process $(Y^1)^{-1}$:

$$d\left(\frac{1}{Y^1_t}\right) = \frac{1}{Y^1_{t-}}\left\{\left(-\mu_1 + \sigma_1^2 + \xi_t\left(\frac{1}{1+\kappa_1} - 1 + \kappa_1\right)\right)dt - \sigma_1 dW_t - \frac{\kappa_1}{1+\kappa_1}\,dM_t\right\}.$$

Consequently, the integration by parts formula yields the following dynamics for the processes $Y^{i,1}$:

$$\begin{aligned} dY^{i,1}_t &= Y^{i,1}_{t-}\left\{\left(\mu_i - \mu_1 - \sigma_1(\sigma_i - \sigma_1) - \xi_t(\kappa_i - \kappa_1)\frac{\kappa_1}{1+\kappa_1}\right)dt \right. \\ &\qquad \left. +(\sigma_i - \sigma_1)\,dW_t + \frac{\kappa_i - \kappa_1}{1+\kappa_1}\,dM_t\right\}. \end{aligned}$$

As a partial check, we can verify that the jump of $Y^{i,1}_t$ equals

$$\Delta Y^{i,1}_t = Y^{i,1}_t - Y^{i,1}_{t-} = Y^{i,1}_{t-}\left(\frac{1+\kappa_i}{1+\kappa_1} - 1\right)\Delta H_t = Y^{i,1}_{t-}\frac{\kappa_i - \kappa_1}{1+\kappa_1}\,\Delta H_t.$$

2.4 Equivalent Martingale Measure

We now search for a probability measure $\mathbb{Q}^1$, equivalent to the real-life probability $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$, and such that the processes $Y^{i,1}$, $i = 2, 3$ follow martingales under $\mathbb{Q}^1$. From Kusuoka (1999), we know that any probability equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ is defined by means of its Radon-Nikodým density process η satisfying the SDE

$$d\eta_t = \eta_{t-}\big(\theta_t\,dW_t + \zeta_t\,dM_t\big), \quad \eta_0 = 1, \tag{3}$$

where θ and ζ are $\mathbb{G}$-predictable processes satisfying mild technical condition. Since the martingale M is stopped at τ, we may and do assume in what follows that the process ζ is stopped at τ. Moreover, the processes $\widehat{W}$ and $\widehat{M}$, given as, for $t \in [0, T]$,

$$\begin{aligned} \widehat{W}_t &= W_t - \int_0^t \theta_u\,du, \\ \widehat{M}_t &= M_t - \int_0^t \xi_u\zeta_u\,du = H_t - \int_0^t \xi_u(1+\zeta_u)\,du = H_t - \int_0^t \widehat{\xi}_u\,du, \end{aligned}$$

where $\widehat{\xi}_u = \xi_u(1+\zeta_u)$, are $\mathbb{G}$-martingales under $\mathbb{Q}^1$. The relative prices $Y^{i,1}$, $i = 2, 3$ follow $\mathbb{Q}^1$-martingales if and only if the drift term in their dynamics expressed in terms of $\widehat{W}$ and $\widehat{M}$ vanishes.

This in turn means that the following equality holds, for $i = 2, 3$ and every $t \in [0, T]$,

$$Y_{t-}^{i,1} \left\{ \mu_i - \mu_1 + (\sigma_1 - \sigma_i)(\theta_t - \sigma_1) + \xi_t(\kappa_1 - \kappa_i) \frac{\zeta_t - \kappa_1}{1 + \kappa_1} \right\} = 0. \tag{4}$$

Equivalently, we have, on the set $Y_{t-}^{i,1} \neq 0$,

$$\mu_1 - \mu_i + (\sigma_1 - \sigma_i)(\theta_t - \sigma_1) + \xi_t(\kappa_1 - \kappa_i) \frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0, \quad i = 2, 3. \tag{5}$$

Remarks. In the case $\kappa_1 = \sigma_1 = 0$ and $\mu_1 = r$, the dynamics of Y^1 are $dY_t^1 = rY_t^1 \, dt$, where r is the short-term interest rate. Of course, in this case the process Y^1 represents the savings account and the martingale measure $\mathbb{Q}^1$ is the usual risk-neutral probability.

2.4.1 Case of Strictly Positive Primary Assets

We work under the standing assumption that $\kappa_1 > -1$ so that $Y_t^1 > 0$ for every t. We assume, in addition, that $\kappa_i > -1$ for $i = 2, 3$, so that the price processes Y^2 and Y^3 are strictly positive as well. From the general theory of arbitrage pricing, it follows that the market model $\mathcal{M}$ is complete and arbitrage-free provided that there exists a unique solution (θ, ζ) of (4) such that the process ζ is strictly greater than -1. Since $Y^{i,1} > 0$, we seek a pair of processes (θ, ζ) for which we have:

$$\theta_t(\sigma_1 - \sigma_i) + \zeta_t \xi_t \frac{\kappa_1 - \kappa_i}{1 + \kappa_1} = \mu_i - \mu_1 + \sigma_1(\sigma_1 - \sigma_i) + \xi_t(\kappa_1 - \kappa_i) \frac{\kappa_1}{1 + \kappa_1}, \quad i = 2, 3. \tag{6}$$

Recall that $\xi_t = \gamma \mathbb{1}_{\{t \le \tau\}}$, so that we deal here with four linear equations, specifically,

$$\begin{aligned}
\theta_t(\sigma_1 - \sigma_2) + \zeta_t \gamma \frac{\kappa_1 - \kappa_2}{1 + \kappa_1} &= \mu_2 - \mu_1 + \sigma_1(\sigma_1 - \sigma_2) + \gamma(\kappa_1 - \kappa_2) \frac{\kappa_1}{1 + \kappa_1}, \text{ for } t \le \tau, && (7) \\
\theta_t(\sigma_1 - \sigma_3) + \zeta_t \gamma \frac{\kappa_1 - \kappa_3}{1 + \kappa_1} &= \mu_3 - \mu_1 + \sigma_1(\sigma_1 - \sigma_3) + \gamma(\kappa_1 - \kappa_3) \frac{\kappa_1}{1 + \kappa_1}, \text{ for } t \le \tau, && (8) \\
\theta_t(\sigma_1 - \sigma_2) &= \mu_2 - \mu_1 + \sigma_1(\sigma_1 - \sigma_2), \text{ for } t > \tau, && (9) \\
\theta_t(\sigma_1 - \sigma_3) &= \mu_3 - \mu_1 + \sigma_1(\sigma_1 - \sigma_3), \text{ for } t > \tau. && (10)
\end{aligned}$$

Equations (7)-(8) (equations (9)-(10) respectively) are referred to as the *pre-default* (*post-default* respectively) no-arbitrage restrictions. To solve explicitly these equations, we find it convenient to write $a = \det A$, $b = \det B$, and $c = \det C$, where A, B and C are the following matrices:

$$A = \begin{bmatrix} \sigma_1 - \sigma_2 & \kappa_1 - \kappa_2 \\ \sigma_1 - \sigma_3 & \kappa_1 - \kappa_3 \end{bmatrix}, \quad B = \begin{bmatrix} \sigma_1 - \sigma_2 & \mu_1 - \mu_2 \\ \sigma_1 - \sigma_3 & \mu_1 - \mu_3 \end{bmatrix}, \quad C = \begin{bmatrix} \kappa_1 - \kappa_2 & \mu_1 - \mu_2 \\ \kappa_1 - \kappa_3 & \mu_1 - \mu_3 \end{bmatrix}.$$

The following lemma follows from (6) by simple algebra.

Lemma 2.1 *The pair (θ, ζ) satisfies the following equations*

$$\begin{aligned}
\theta_t a &= \sigma_1 a + c, \\
\zeta_t \xi_t a &= \kappa_1 \xi_t a - (1 + \kappa_1) b.
\end{aligned}$$

To ensure the validity of the second equation in Lemma 2.1 not only prior to, but also after the default time τ (i.e., on the set $\{\xi_t = 0\}$), we need to impose an additional condition $b = 0$, or more explicitly,

$$(\sigma_1 - \sigma_2)(\mu_1 - \mu_3) - (\sigma_1 - \sigma_3)(\mu_1 - \mu_2) = 0. \tag{11}$$

If (11) holds, we arrive at the following equations:

$$\begin{aligned}
\theta_t a &= \sigma_1 a + c, \\
\zeta_t \xi_t a &= \kappa_1 \xi_t a.
\end{aligned}$$

We are thus in the position to formulate an auxiliary result.

Proposition 2.1 *Assume that the processes Y^1, Y^2, Y^3 satisfy* (2) *with $\kappa_i > -1$ for $i = 1,2,3$.*
(i) *If $a \neq 0$ and $b = 0$ then the unique martingale measure $\mathbb{Q}^1$ has the Radon-Nikodým density of the form*

$$\frac{d\mathbb{Q}^1}{d\mathbb{Q}} = \mathcal{E}_T(\theta W)\mathcal{E}_T(\zeta M), \tag{12}$$

where the constants θ and ζ are given by

$$\theta = \sigma_1 + \frac{c}{a}, \quad \zeta = \kappa_1 > -1, \tag{13}$$

and where we write, for $t \in [0,T]$,

$$\mathcal{E}_t(\theta W) = \exp\Big(\theta W_t - \frac{1}{2}\theta^2 t\Big) \tag{14}$$

and

$$\mathcal{E}_t(\zeta M) = \big(1 + \mathbb{1}_{\{\tau \le t\}}\zeta\big)\exp\Big(-\zeta\gamma(t\wedge\tau)\Big). \tag{15}$$

The model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is arbitrage-free and complete. Moreover, the process (Y^1, Y^2, Y^3, H) has the Markov property under $\mathbb{Q}^1$.
(ii) *If $b = 0$ and $a = 0$ then a solution (θ, ζ) exists provided that $c = 0$ and the uniqueness of a martingale measure $\mathbb{Q}^1$ fails to hold. In this case, the model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is arbitrage-free, but it is not complete.*
(iii) *If $b \neq 0$ then a martingale measure does not exist and the model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is not arbitrage-free.*

Proof. All statements in part (i) are rather obvious, except for the last one. The Markov property of the process (Y^1, Y^2, Y^3, H) under $\mathbb{Q}^1$ can be easily deduced by observing that the dynamics of $Y^{i,1}$, $i = 2,3$ under $\mathbb{Q}^1$ are

$$dY^{i,1}_t = Y^{i,1}_{t-}\left((\sigma_i - \sigma_1)\, d\widehat{W}_t + \frac{\kappa_i - \kappa_1}{1+\kappa_1}\, d\widehat{M}_t\right), \tag{16}$$

and by combining this observation with the fact that the default intensity $\widehat{\gamma}$ under $\mathbb{Q}^1$ is deterministic, specifically, $\widehat{\gamma} = \gamma(1+\zeta) = \gamma(1+\kappa_1)$. It is interesting to note that the default intensity under $\mathbb{Q}^1$ coincides with the default intensity under the real-life probability $\mathbb{Q}$ if and only if the process Y^1 is continuous.

Parts (ii) and (iii) are also easy to check . Let us only observe that under the assumptions of part (ii), the price processes Y^2 and Y^3 are proportional. □

From now on, we work under assumptions of part (i) in the lemma. Recall that the processes $\mathcal{E}(\theta W)$ and $\mathcal{E}(\zeta M)$ given by (14) and (15) are unique solutions to SDEs

$$d\mathcal{E}_t(\theta W) = \theta\mathcal{E}_t(\theta W)\, dW_t, \quad d\mathcal{E}_t(\zeta M) = \zeta\mathcal{E}_{t-}(\zeta M)\, dM_t,$$

with the initial conditions: $\mathcal{E}_0(\theta W) = \mathcal{E}_0(\zeta M) = 1$. Hence, the product $\eta = \mathcal{E}(\theta W)\mathcal{E}(\zeta M)$ satisfies, as expected, the SDE (3) with constant processes θ and ζ, specifically,

$$d\eta_t = \eta_{t-}\left\{\Big(\sigma_1 + \frac{c}{a}\Big)\, dW_t + \kappa_1\, dM_t\right\}.$$

Example 2.1 Assume that the asset Y^1 is risk-free, the asset $Y^2 \neq Y^1$ is default-free, and Y^3 is a defaultable asset with non-zero recovery, so that

$$\begin{aligned} dY^1_t &= rY^1_t\, dt, \\ dY^2_t &= Y^2_t\big(\mu_2\, dt + \sigma_2\, dW_t\big), \\ dY^3_t &= Y^3_{t-}\big(\mu_3\, dt + \sigma_3\, dW_t + \kappa_3\, dM_t\big). \end{aligned} \tag{17}$$

We thus have $\sigma_1 = \kappa_1 = 0$, $\mu_1 = r$, $\sigma_2 \neq 0$, $\kappa_2 = 0$, and $\kappa_3 \neq 0$, $\kappa_3 > -1$. Therefore,

$$a = \sigma_2 \kappa_3 \neq 0, \quad c = \kappa_3(r - \mu_2),$$

and the equality $b = 0$ holds if and only if

$$\sigma_2(r - \mu_3) = \sigma_3(r - \mu_2).$$

It is easy to check that

$$\theta = \frac{r - \mu_2}{\sigma_2}, \quad \zeta = 0, \tag{18}$$

and thus under the martingale measure $\mathbb{Q}^1$ we have (notwithstanding whether $\sigma_3 > 0$ or $\sigma_3 = 0$)

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(r\,dt + \sigma_2\,d\widehat{W}_t\big), \\ dY_t^3 &= Y_{t-}^3\big(r\,dt + \sigma_3\,d\widehat{W}_t + \kappa_3\,dM_t\big). \end{aligned}$$

Note the risk-neutral default intensity $\widehat{\gamma}$ coincides here with the real-life intensity γ.

2.4.2 Case of a Defaultable Asset with Zero Recovery

In this section, we postulate that $\kappa_i > -1$ for $i = 1, 2$ and $\kappa_3 = -1$. This implies that the price of a defaultable asset Y^3 vanishes after τ, and thus the findings of the previous section are no longer valid. Indeed, since the process Y^3 jumps to zero after τ, the first equality in (5), that is,

$$\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) + \xi_t(\kappa_2 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0,$$

should still be satisfied for every $t \in [0, T]$, but the second equality in (5), namely,

$$\mu_3 - \mu_1 + (\sigma_3 - \sigma_1)(\theta_t - \sigma_1) + \xi_t(\kappa_3 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0,$$

is required to hold on the set $\{\tau > t\}$ only (i.e., when $\xi_t = \gamma$). Thus, the unknown processes θ and ζ satisfy the following equations:

$$\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) = 0, \text{ for } t > \tau, \tag{19}$$

$$\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) + \gamma(\kappa_2 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0, \text{ for } t \leq \tau, \tag{20}$$

$$\mu_3 - \mu_1 + (\sigma_3 - \sigma_1)(\theta_t - \sigma_1) + \gamma(-1 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0, \text{ for } t \leq \tau. \tag{21}$$

This leads to the following lemma.

Lemma 2.2 *The pair (θ, ζ) satisfies the following equations, for $t \leq \tau$,*

$$\begin{aligned} \theta_t a &= \sigma_1 a + c, \\ \zeta_t \gamma a &= \kappa_1 \gamma a - (1 + \kappa_1) b. \end{aligned}$$

Moreover, for $t > \tau$,

$$\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) = 0.$$

Assume that $a \neq 0$ and $\sigma_1 \neq \sigma_2$. Then the unique solution (θ, ζ) is

$$\theta_t = \mathbb{1}_{\{t \leq \tau\}}\left(\sigma_1 + \frac{c}{a}\right) + \mathbb{1}_{\{t > \tau\}}\left(\sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}\right), \quad \zeta_t = \kappa_1 - \frac{(1 + \kappa_1) b}{\gamma a}, \tag{22}$$

and the unique martingale measure $\mathbb{Q}^1$ *is given by the formula*

$$\frac{d\mathbb{Q}^1}{d\mathbb{Q}} = \mathcal{E}_T(\theta W)\mathcal{E}_T(\zeta M).$$

The model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ *is arbitrage-free, complete, and has the Markov property under* $\mathbb{Q}^1$.

Example 2.2 Assume that the asset Y^1 is risk-free, the asset $Y^2 \neq Y^1$ is default-free, and Y^3 is a defaultable asset with zero recovery (see (17)). This corresponds to the following set of conditions: $\sigma_1 = \kappa_1 = 0$, $\mu_1 = r$, $\sigma_2 \neq 0$, $\kappa_2 = 0$, and $\kappa_3 = -1$. Under these assumptions, we obtain $a = -\sigma_2 \neq 0$ and

$$\theta = \frac{r - \mu_2}{\sigma_2}, \quad \zeta = \frac{1}{\gamma}\left(r - \mu_3 - \sigma_3 \frac{r - \mu_2}{\sigma_2}\right). \tag{23}$$

Consequently, we have, under the martingale measure $\mathbb{Q}^1$,

$$\begin{aligned} dY_t^1 &= rY_t^1\, dt, \\ dY_t^2 &= Y_t^2\big(r\, dt + \sigma_2\, d\widehat{W}_t\big), \\ dY_t^3 &= Y_{t-}^3\big(r\, dt + \sigma_3\, d\widehat{W}_t - d\widehat{M}_t\big). \end{aligned}$$

We do not assume here that the equality $b = 0$ holds; when it does then $\zeta = 0$, as in Example 2.1. In general, the risk-neutral default intensity $\widehat{\gamma}$ and the real-life intensity γ are different.

Remarks. Assume that $\kappa_2 = \kappa_3 = -1$. Then the pair (θ, ζ) satisfies (20)-(21), so that we have, for $t \leq \tau$ (see (25))

$$\theta_t = \sigma_1 + \frac{c}{a}, \quad \zeta_t = \kappa_1 - \frac{(1+\kappa_1)b}{\gamma a}, \tag{24}$$

provided that $a \neq 0$. Solution of (20)-(21) is not uniquely determined for $t > \tau$.

2.4.3 Case of a Stopped Trading

In some circumstances, the recovery payoff at the time of default is exogenously specified in terms of some economic factors related to prices of traded assets (e.g., credit spreads). In such a case, the valuation problem for a defaultable claim is reduced to finding of its pre-default value, and it is natural to seek a replicating strategy up to the default time (default time included), but not after this random time. Consequently, it suffices to focus on the properties of the *stopped* model, that is, a model in which asset prices, as well as all trading activities, are assumed to be freezed at time τ (for a formal definition, see Section 5). In this case, we search for a pair (θ, ζ) of real numbers satisfying (7)-(8) (or (20)-(21)). Equivalently,

$$\begin{aligned} \theta a &= \sigma_1 a + c, \\ \zeta \gamma a &= \kappa_1 \gamma a - (1+\kappa_1)b. \end{aligned}$$

We no longer postulate that condition (11) is satisfied. It is clear that if $a \neq 0$ then the unique solution (θ, ζ) to the above pair of equations equals

$$\theta = \sigma_1 + \frac{c}{a}, \quad \zeta = \kappa_1 - \frac{(1+\kappa_1)b}{\gamma a}. \tag{25}$$

As expected, in a stopped model, we obtain the same representation (25) of the unique martingale measure $\mathbb{Q}^1$ for any choice of κ_2 and κ_3. Let us repeat that condition (11) (or equality (19)) is needed only if we wish to use the same model (2) to value claims on defaultable and non-defaultable assets. Indeed, according to (2), after time τ the processes Y^1, Y^2 and Y^3 represent the prices of three assets in a model driven by a single source of randomness (a Brownian motion W), and thus condition (11) (or equality (19)) is necessary to exclude arbitrage opportunities when trading is continued up to time T. These conditions are spurious when trading is stopped at default, so that the effective horizon date becomes $\tau \wedge T$.

3 PDE Approach for Strictly Positive Traded Assets

We shall first examine the PDE approach in a model in which the prices of all three primary assets are non-vanishing. In this case, it is natural to focus on the case when the market model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is complete and arbitrage-free. To this end, we shall work under the assumptions of part (i) in Proposition 2.1.

3.1 Valuation PDE

We are interested in the valuation and hedging of a generic contingent claim with maturity T and the terminal payoff $Y = G(Y_T^1, Y_T^2, Y_T^3, H_T)$. As we shall see in what follows, the technique derived for this case can be easily applied to a defaultable claim that is subject to a fairly general recovery scheme (including, of course, the zero recovery scheme).

We assume that $a \neq 0$ and $b = 0$, and we work under the unique martingale measure $\mathbb{Q}^1$ corresponding to the choice of Y^1 as a numeraire. Recall that we have

$$\frac{d\mathbb{Q}^1}{d\mathbb{Q}} = \mathcal{E}_T(\theta W)\mathcal{E}_T(\zeta M),$$

where the pair (θ, ζ) is given by (13). If the random variable $Y(Y_T^1)^{-1}$ is $\mathbb{Q}^1$-integrable then the arbitrage price of a claim Y can be represented as follows, for every $t \in [0, T]$,

$$\pi_t(Y) = Y_t^1\, \mathbb{E}_{\mathbb{Q}^1}\big((Y_T^1)^{-1} Y \,\big|\, \mathcal{G}_t\big) = Y_t^1\, \mathbb{E}_{\mathbb{Q}^1}\left((Y_T^1)^{-1} G(Y_T^1, Y_T^2, Y_T^3, H_T) \,\middle|\, Y_t^1, Y_t^2, Y_t^3, H_t\right),$$

where the second equality is a consequence of the Markov property of (Y^1, Y^2, Y^3, H) under $\mathbb{Q}^1$. Let $C : [0,T] \times \mathbb{R}_+^3 \times \{0,1\} \to \mathbb{R}$ be a function such that $\pi_t(Y) = C(t, Y_t^1, Y_t^2, Y_t^3, H_t)$ for every $t \in [0,T]$. It is clear that we have, for $h = 0$ and $h = 1$,

$$C(T, y_1, y_2, y_3, h) = G(y_1, y_2, y_3, h), \quad \forall\, (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Moreover, the process $\widetilde{C}_t$, $t \in [0,T]$, given by the formula

$$\widetilde{C}_t = (Y_t^1)^{-1} C(t, Y_t^1, Y_t^2, Y_t^3, H_t), \quad \forall\, t \in [0,T],$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}^1$. As expected, our next goal is to use this property in order to derive the equation satisfied by the *valuation function* C. To this end, we shall apply Itô's formula to the process $\widetilde{C}$. For brevity, we write $\partial_i C = \partial_{y_i} C, \partial_{ij} C = \partial_{y_i}\partial_{y_j} C$. Also, we denote (it is easy to check that if $b = 0$ then the right-hand side of the formula below does not depend on i)

$$\alpha = \mu_i + \sigma_i \frac{c}{a}.$$

Proposition 3.1 *Let the price processes Y^i, $i = 1,2,3$ satisfy*

$$dY_t^i = Y_{t-}^i \big(\mu_i dt + \sigma_i\, dW_t + \kappa_i\, dM_t\big)$$

with $\kappa_i > -1$ for $i = 1,2,3$. Assume that $a \neq 0$ and $b = 0$. Then the arbitrage price of a contingent claim Y with the terminal payoff $G(Y_T^1, Y_T^2, Y_T^3, H_T)$ equals

$$\pi_t(Y) = C(t, Y_t^1, Y_t^2, Y_t^3, H_t) = \mathbb{1}_{\{t<\tau\}} C(t, Y_t^1, Y_t^2, Y_t^3, 0) + \mathbb{1}_{\{t \geq \tau\}} C(t, Y_t^1, Y_t^2, Y_t^3, 1)$$

for some function $C : [0,T] \times \mathbb{R}_+^3 \times \{0,1\} \to \mathbb{R}$. Assume that for $h = 0$ and $h = 1$ the auxiliary function $C(\cdot, h) : [0,T] \times \mathbb{R}_+^3 \to \mathbb{R}$ belongs to the class $\mathrm{C}^{1,2}([0,T] \times \mathbb{R}_+^3, \mathbb{R})$. Then the functions $C(\cdot, 0)$ and $C(\cdot, 1)$ solve the following PDEs

$$\begin{aligned} &\partial_t C(\cdot, 0) + \sum_{i=1}^{3} (\alpha - \gamma\kappa_i) y_i \partial_i C(\cdot, 0) + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot, 0) - \alpha C(\cdot, 0) \\ &\quad + \gamma\big[C(t, y_1(1+\kappa_1), y_2(1+\kappa_2), y_3(1+\kappa_3), 1) - C(t, y_1, y_2, y_3, 0)\big] = 0 \end{aligned}$$

and

$$\partial_t C(\cdot,1) + \alpha \sum_{i=1}^{3} y_i \partial_i C(\cdot,1) + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C(\cdot,1) - \alpha C(\cdot,1) = 0$$

subject to the terminal conditions

$$C(T,y_1,y_2,y_3,0) = G(y_1,y_2,y_3,0), \quad C(T,y_1,y_2,y_3,1) = G(y_1,y_2,y_3,1).$$

Proof. The first statement is an immediate consequence of the Markov property of the process (Y^1,Y^2,Y^3,H) under $\mathbb{Q}^1$. Let us denote

$$\Delta C(t,Y^1_{t-},Y^2_{t-},Y^3_{t-}) = C(t,Y^1_{t-}(1+\kappa_1),Y^2_{t-}(1+\kappa_2),Y^3_{t-}(1+\kappa_3),1) - C(t,Y^1_{t-},Y^2_{t-},Y^3_{t-},0).$$

We write $C_t = C(t,Y^1_t,Y^2_t,Y^3_t,H_t)$, and we typically omit the variables $(t,Y^1_{t-},Y^2_{t-},Y^3_{t-},H_{t-})$ in expressions $\partial_t C$, $\partial_i C$, ΔC, etc. An application of Itô's formula yields

$$\begin{aligned}
dC_t &= \partial_t C\,dt + \sum_{i=1}^{3}\partial_i C\,dY^i_t + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C\,dt + \Big(\Delta C - \sum_{i=1}^{3}\kappa_i Y^i_{t-}\partial_i C\Big)\,dH_t \\
&= \partial_t C\,dt + \sum_{i=1}^{3}\partial_i C\,dY^i_t + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C\,dt + \Big(\Delta C - \sum_{i=1}^{3}\kappa_i Y^i_{t-}\partial_i C\Big)\big(dM_t + \xi_t\,dt\big) \\
&= \partial_t C\,dt + \sum_{i=1}^{3} Y^i_{t-}\partial_i C\big(\mu_i\,dt + \sigma_i\,dW_t\big) + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C\,dt \\
&\quad + (\Delta C)\,dM_t + \Big(\Delta C - \sum_{i=1}^{3}\kappa_i Y^i_{t-}\partial_i C\Big)\xi_t\,dt \\
&= \left\{\partial_t C + \sum_{i=1}^{3}\mu_i Y^i_{t-}\partial_i C + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C + \Big(\Delta C - \sum_{i=1}^{3}\kappa_i Y^i_{t-}\partial_i C\Big)\xi_t\right\}dt \\
&\qquad + \Big(\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i C\Big)\,dW_t + \Delta C\,dM_t.
\end{aligned}$$

We now write integration by parts formula for $\widetilde{C}$. Since $d[M]_t = dH_t = dM_t + \xi_t\,dt$, we obtain

$$\begin{aligned}
d\widetilde{C}_t &= \widetilde{C}_{t-}\left\{\left[-\mu_1 + \sigma_1^2 + \xi_t\left(\frac{1}{1+\kappa_1} - 1 + \kappa_1\right)\right]dt - \sigma_1\,dW_t - \frac{\kappa_1}{1+\kappa_1}\,dM_t\right\} \\
&\quad + (Y^1_{t-})^{-1}\left\{\partial_t C + \sum_{i=1}^{3}\mu_i Y^i_{t-}\partial_i C + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C + \Big(\Delta C - \sum_{i=1}^{3}\kappa_i Y^i_{t-}\partial_i C\Big)\xi_t\right\}dt \\
&\quad + (Y^1_{t-})^{-1}\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i C\,dW_t + (Y^1_{t-})^{-1}\Delta C\,dM_t \\
&\quad - (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^{3}\sigma_i Y^i_{t-}\partial_i C\,dt - (Y^1_{t-})^{-1}\frac{\kappa_1}{1+\kappa_1}\,\Delta C\big(dM_t + \xi_t\,dt\big) \\
&= \widetilde{C}_{t-}\left\{-\mu_1 + \sigma_1^2 + \xi_t\left(\frac{1}{1+\kappa_1} - 1 + \kappa_1\right)\right\}dt \\
&\quad + \widetilde{C}_{t-}\left\{-\sigma_1\,d\widehat{W}_t - \sigma_1\theta\,dt - \frac{\kappa_1}{1+\kappa_1}\,d\widehat{M}_t - \frac{\zeta\xi_t\kappa_1}{1+\kappa_1}\,dt\right\} \\
&\quad + (Y^1_{t-})^{-1}\left\{\partial_t C + \sum_{i=1}^{3}\mu_i Y^i_{t-}\partial_i C + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C + \Big(\Delta C - \sum_{i=1}^{3}\kappa_i Y^i_{t-}\partial_i C\Big)\xi_t\right\}dt
\end{aligned}$$

$$
\begin{aligned}
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i C\, d\widehat{W}_t + (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i C\, dt\\
&+ (Y^1_{t-})^{-1}\Delta C\, d\widehat{M}_t + (Y^1_{t-})^{-1}\zeta\xi_t\Delta C\, dt\\
&- (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma^i Y^i_{t-}\partial_i C\, dt - (Y^1_{t-})^{-1}\frac{\kappa_1}{1+\kappa_1}\Delta C\big(d\widehat{M}_t + \xi_t(1+\zeta)\, dt\big)\\
= \;& \widetilde{C}_{t-}\left\{-\mu_1+\sigma_1^2+\xi_t\left(\frac{1}{1+\kappa_1}-1+\kappa_1\right)\right\}dt + \widetilde{C}_{t-}\left\{-\sigma_1\theta - \frac{\zeta\xi_t\kappa_1}{1+\kappa_1}\right\}dt\\
&+ (Y^1_{t-})^{-1}\left\{\partial_t C + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i C + \frac12\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C + \Big(\Delta C - \sum_{i=1}^3 \kappa_i Y^i_{t-}\partial_i C\Big)\xi_t\right\}dt\\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i C\, dt + (Y^1_{t-})^{-1}\zeta\xi_t\Delta C\, dt\\
&- (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i C\, dt - (Y^1_{t-})^{-1}\frac{\kappa_1}{1+\kappa_1}\,\xi_t(1+\zeta)\Delta C\, dt\\
&+ \text{ a martingale under } \mathbb{Q}.
\end{aligned}
$$

Since the process $\widetilde{C}$ follows a martingale under $\mathbb{Q}^1$, the finite variation part in its canonical decomposition necessarily vanishes, that is,

$$
\begin{aligned}
0 \;=\;& C_{t-}(Y^1_{t-})^{-1}\left\{-\mu_1+\sigma_1^2+\xi_t\left(\frac{1}{1+\kappa_1}-1+\kappa_1\right)-\sigma_1\theta-\frac{\zeta\xi_t\kappa_1}{1+\kappa_1}\right\}\\
&+ (Y^1_{t-})^{-1}\left\{\partial_t C + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i C + \frac12\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C + \left(\Delta C - \sum_{i=1}^3 \kappa_i Y^i_{t-}\partial_i C\right)\xi_t\right\}\\
&+ (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i C + (Y^1_{t-})^{-1}\zeta\xi_t\Delta C\\
&- (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i C - (Y^1_{t-})^{-1}\frac{\kappa_1}{1+\kappa_1}\,\xi_t(1+\zeta)\Delta C.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
0 \;=\;& C_{t-}\left\{-\mu_1+\sigma_1^2-\sigma_1\theta+\xi_t\kappa_1-\xi_t(1+\zeta)\frac{\kappa_1}{1+\kappa_1}\right\}\\
&+ \partial_t C + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i C + \frac12\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C + \left(\Delta C - \sum_{i=1}^3 \kappa_i Y^i_{t-}\partial_i C\right)\xi_t\\
&+ \sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i C + \zeta\xi_t\Delta C - \sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i C - \xi_t(1+\zeta)\Delta C\frac{\kappa_1}{1+\kappa_1}.
\end{aligned}
$$

Since, in view of (13), we have

$$
\begin{aligned}
-\mu_1+\sigma_1^2-\sigma_1\theta+\xi_t\kappa_1-\xi_t(1+\zeta)\frac{\kappa_1}{1+\kappa_1} &= -\alpha,\\
\mu_i+\sigma_i(\theta-\sigma_1)-\kappa_i\xi_t &= \alpha-\kappa_i\xi_t,
\end{aligned}
$$

we finally obtain

$$
\partial_t C + \sum_{i=1}^3 (\alpha-\kappa_i\xi_t)\, Y^i_{t-}\partial_i C + \frac12\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}C - \alpha C_{t-} + \xi_t\Delta C = 0.
$$

Recall that $\xi_t = \gamma 1\!\!1_{\{\tau > t\}}$. We conclude that the price of a contingent claim Y with the terminal payoff $G(Y_T^1, Y_T^2, Y_T^3, H_T)$ can be represented as $C(t, Y_t^1, Y_t^2, Y_t^3, H_t)$ where, for $h = 0$ and $h = 1$,

$$C(T, y_1, y_2, y_3, h) = G(y_1, y_2, y_3, h), \quad \forall\, (y_1, y_2, y_3) \in \mathbb{R}_+^3$$

and the auxiliary functions $C(\cdot, 0)$ and $C(\cdot, 1)$ satisfy the PDEs given in the statement of the proposition. □

Remarks. Note that the valuation problem splits in a natural way into two pricing PDEs that can be solved recursively. In the first step, we solve the PDE satisfied by the *post-default pricing function* $C(\cdot, 1)$. Next, we substitute this function into the first PDE, and we solve it for the *pre-default pricing function* $C(\cdot, 0)$. The assumption that we deal with only three primary assets and the coefficients are constant can be easily relaxed, but a general result is too heavy to be stated here. It is also interesting to observe that the real-life default intensity γ, rather than the intensity $\widehat{\gamma}$ under the martingale measure $\mathbb{Q}^1$, enters the valuation PDE. This shows once again that the martingale measure $\mathbb{Q}^1$ is merely a technical tool, and the properties of the default time under the real-life probability are essential for valuation and hedging of a defaultable claim through the PDE approach in a complete market model.

Example 3.1 Black and Scholes PDE. As a rather trivial application of Proposition 3.1, let us observe that if there are only two assets with $Y_t^1 = e^{rt}$ and $Y_t^2 = Y_t$ and if there are no jumps, the valuation PDEs of Proposition 3.1 reduce to

$$\partial_t C + (\mu - \sigma\theta) y C_y + \frac{1}{2}\sigma^2 y^2 C_{yy} - rC = 0$$

with $\theta = -(\mu - r)/\sigma$. After simplifications, we obtain the following equation

$$\partial_t C + ry C_y + \frac{1}{2}\sigma^2 y^2 C_{yy} - rC = 0$$

which is, of course, the classic Black and Scholes PDE.

3.2 Replicating Strategies

Our next goal is to derive a universal representation for a replicating strategy of a generic claim. Recall that $\phi = (\phi^1, \phi^2, \phi^3)$ is a *self-financing strategy* if the processes ϕ^1, ϕ^2, ϕ^3 are $\mathbb{G}$-predictable and the wealth process, given as

$$V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3,$$

satisfies

$$dV_t(\phi) = \phi_t^1\, dY_t^1 + \phi_t^2\, dY_t^2 + \phi_t^3\, dY_t^3.$$

We say that ϕ replicates a contingent claim Y if $V_T(\phi) = Y$. If ϕ is a replicating strategy for a claim Y then we have, for every $t \in [0, T]$,

$$\pi_t(Y) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3.$$

The next result shows that in order to find a replicating strategy it suffices, as in the classical case, to make use of sensitivities of the valuation function C with respect to prices of primary assets, and to take into account the jump ΔC associated with default event. Recall that

$$\Delta C = \Delta C(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3) = C(t, Y_{t-}^1(1+\kappa_1), Y_{t-}^2(1+\kappa_2), Y_{t-}^3(1+\kappa_3), 1) - C(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3, 0).$$

As before, for the sake of better readability, the variables in $C, \partial_i C$ and ΔC are suppressed. Note, however, that we deal here with the two functions $C(\cdot, 0)$ and $C(\cdot, 1)$ depending on whether a replicating portfolio is examined prior to or after default.

Proposition 3.2 *Under the assumptions of Proposition 3.1, the replicating strategy for a claim* $G(Y_T^1, Y_T^2, Y_T^3, H_T)$ *is* $\phi = (\phi^1, \phi^2, \phi^3)$*, where the components* ϕ^i*,* $i = 2, 3$ *are given in terms of the valuation functions* $C(\cdot, 0)$ *and* $C(\cdot, 1)$ *by the following expressions*

$$\phi_t^2 = \frac{1}{aY_{t-}^2}\left((\kappa_3 - \kappa_1)\left(\sum_{i=1}^3 \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\right) - (\sigma_3 - \sigma_1)(\Delta C - \kappa_1 C)\right), \tag{26}$$

$$\phi_t^3 = \frac{-1}{aY_{t-}^3}\left((\kappa_2 - \kappa_1)\left(\sum_{i=1}^3 \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\right) - (\sigma_2 - \sigma_1)(\Delta C - \kappa_1 C)\right). \tag{27}$$

Moreover, the component ϕ^1 *satisfies*

$$\phi_t^1 = (Y_t^1)^{-1}\left(C_t - \sum_{i=2}^3 \phi_t^i Y_t^i\right). \tag{28}$$

Proof. Using the dynamics (see equation (16))

$$dY_t^{i,1} = Y_{t-}^{i,1}\left((\sigma_i - \sigma_1)\, d\widehat{W}_t + \frac{\kappa_i - \kappa_1}{1 + \kappa_1}\, d\widehat{M}_t\right),$$

and setting

$$D = (\sigma_2 - \sigma_1)\frac{\kappa_3 - \kappa_1}{1 + \kappa_1} - (\sigma_3 - \sigma_1)\frac{\kappa_2 - \kappa_1}{1 + \kappa_1} = \frac{a}{1 + \kappa_1},$$

we get (note that obviously $(Y_t^{2,1})^{-1} = Y_t^{1,2}$)

$$\begin{aligned} d\widehat{W}_t &= \frac{1}{D}\left(\frac{\kappa_3 - \kappa_1}{1 + \kappa_1}\, Y_{t-}^{1,2}\, dY_t^{2,1} - \frac{\kappa_2 - \kappa_1}{1 + \kappa_1}\, Y_{t-}^{1,3}\, dY_t^{3,1}\right), \\ d\widehat{M}_t &= -\frac{1}{D}\left((\sigma_3 - \sigma_1)\, Y_{t-}^{1,2}\, dY_t^{2,1} - (\sigma_2 - \sigma_1)\, Y_{t-}^{1,3}\, dY_t^{3,1}\right). \end{aligned}$$

Consequently, we have that

$$\begin{aligned} d\widetilde{C}_t &= (Y_{t-}^1)^{-1}\left(\sum_{i=1}^3 \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\right) d\widehat{W}_t + (Y_{t-}^1)^{-1}\frac{\Delta C - \kappa_1 C}{1 + \kappa_1}\, d\widehat{M}_t \\ &= (Y_{t-}^1)^{-1}\left(\sum_{i=1}^3 \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\right)\frac{1}{D}\left(\frac{\kappa_3 - \kappa_1}{1 + \kappa_1}\, Y_{t-}^{1,2}\, dY_t^{2,1} - \frac{\kappa_2 - \kappa_1}{1 + \kappa_1}\, Y_{t-}^{1,3}\, dY_t^{3,1}\right) \\ &\quad - (Y_{t-}^1)^{-1}\frac{\Delta C - \kappa_1 C}{(1 + \kappa_1)D}\left((\sigma_3 - \sigma_1)\, Y_{t-}^{1,2}\, dY_t^{2,1} - (\sigma_2 - \sigma_1)\, Y_{t-}^{1,3}\, dY_t^{3,1}\right) \\ &= (Y_{t-}^1)^{-1}\frac{1}{a}\left\{(\kappa_3 - \kappa_1)\left(\sum_{i=1}^3 \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\right) - (\sigma_3 - \sigma_1)(\Delta C - \kappa_1 C)\right\} Y_{t-}^{1,2} dY_t^{2,1} \\ &\quad - (Y_{t-}^1)^{-1}\frac{1}{a}\left\{(\kappa_2 - \kappa_1)\left(\sum_{i=1}^3 \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\right) - (\sigma_2 - \sigma_1)(\Delta C - \kappa_1 C)\right\} Y_{t-}^{1,3} dY_t^{3,1}. \end{aligned}$$

This completes the derivation of equalities (26) and (27). Relationship (28) is also clear. □

Assume that Y^1 is the savings account, so that $\mu_1 = r$ and $\sigma_1 = \kappa_1 = 0$. Then, under the assumption that $a = \sigma_2\kappa_3 - \sigma_3\kappa_2 \neq 0$, expressions (26)-(27) simplify as follows:

$$\phi_t^2 = \frac{1}{aY_{t-}^2}\left(\kappa_3 \sum_{i=2}^3 \sigma_i Y_{t-}^i \partial_i C - \sigma_3 \Delta C\right), \tag{29}$$

$$\phi_t^3 = \frac{-1}{aY_{t-}^3}\left(\kappa_2 \sum_{i=2}^3 \sigma_i Y_{t-}^i \partial_i C - \sigma_2 \Delta C\right). \tag{30}$$

3.3 Case: Y^1 Risk-Free, Y^2 Default-Free, Y^3 Defaultable

We now study a particular case, where $Y_t^1 = e^{rt}$ is a risk-free asset, $Y^2 \neq Y^1$ is a default-free asset, i.e., $\sigma_2 \neq 0$, $\kappa_2 = 0$. Finally, we assume that $\kappa_3 \neq 0$ and $\kappa_3 > -1$ (see Example 2.1). Since, by definition, the valuation function C depends on t, we may assume, without loss of generality, that it does not depend explicitly on the variable y_1. The following result combines and specifies Propositions 3.1 and 3.2 to the present situation. Note that we now assume that $a = \sigma_2\kappa_3 \neq 0$.

Proposition 3.3 *Let the price processes Y^1, Y^2, Y^3 satisfy* (17) *with $\sigma_2 \neq 0$. Assume that the relationship $\sigma_2(r-\mu_3) = \sigma_3(r-\mu_2)$ holds and $\kappa_3 \neq 0, \kappa_3 > -1$. Then the price of a contingent claim $Y = G(Y_T^2, Y_T^3, H_T)$ can be represented as $\pi_t(Y) = C(t, Y_t^2, Y_t^3, H_t)$, where the pricing functions $C(\cdot, 0)$ and $C(\cdot, 1)$ satisfy the following PDEs*

$$\begin{aligned}\partial_t C(t,y_2,y_3,0) &+ ry_2\partial_2 C(t,y_2,y_3,0) + y_3\left(r-\kappa_3\gamma\right)\partial_3 C(t,y_2,y_3,0) - rC(t,y_2,y_3,0)\\ &+ \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} C(t,y_2,y_3,0) + \gamma\big(C(t,y_2,y_3(1+\kappa_3),1) - C(t,y_2,y_3,0)\big) = 0\end{aligned}$$

and

$$\begin{aligned}\partial_t C(t,y_2,y_3,1) &+ ry_2\partial_2 C(t,y_2,y_3,1) + ry_3\partial_3 C(t,y_2,y_3,1) - rC(t,y_2,y_3,1)\\ &+ \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} C(t,y_2,y_3,1) = 0\end{aligned}$$

subject to the terminal conditions

$$C(T,y_2,y_3,0) = G(y_2,y_3,0), \quad C(T,y_2,y_3,1) = G(y_2,y_3,1).$$

The replicating strategy equals $\phi = (\phi^1, \phi^2, \phi^3)$, where ϕ_t^1 is given by (28) *and*

$$\begin{aligned}\phi_t^2 &= \frac{1}{\sigma_2\kappa_3 Y_{t-}^2}\left(\kappa_3\sum_{i=2}^{3}\sigma_i y_i\partial_i C(t,Y_{t-}^2,Y_{t-}^3,H_{t-}) - \sigma_3\big(C(t,Y_{t-}^2,Y_{t-}^3(1+\kappa_3),1) - C(t,Y_{t-}^2,Y_{t-}^3,0)\big)\right),\\ \phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3}\big(C(t,Y_{t-}^2,Y_{t-}^3(1+\kappa_3),1) - C(t,Y_{t-}^2,Y_{t-}^3,0)\big).\end{aligned}$$

3.3.1 Replication of a Survival Claim

By a *survival claim* we mean a contingent claim of the form $Y = \mathbb{1}_{\{\tau>T\}}X$, where a $\mathcal{F}_T$-measurable random variable X represents the *promised payoff.* We assume that the promised payoff has the form $X = G(Y_T^2, Y_T^3)$, where Y_T^i is the (pre-default) value of the ith asset at time T. It is obvious that the pricing function $C(\cdot, 1)$ is now equal to zero, and thus we are only interested in the pre-default pricing function $C(\cdot, 0)$.

Corollary 3.1 *Under the assumptions of Proposition 3.3, the pre-default pricing function $C(\cdot, 0)$ of a survival claim $Y = \mathbb{1}_{\{\tau>T\}}G(Y_T^2, Y_T^3)$ is a solution of the following PDE*

$$\partial_t C(\cdot,0) + ry_2\partial_2 C(\cdot,0) + y_3(r-\kappa_3\gamma)\partial_3 C(\cdot,0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j\partial_{ij} C(\cdot,0) - (r+\gamma)C(\cdot,0) = 0$$

with the terminal condition $C(T,y_2,y_3,0) = G(y_2,y_3)$. The components ϕ^2 and ϕ^3 of a replicating strategy ϕ are given by the following expressions

$$\phi_t^2 = \frac{1}{\kappa_3\sigma_2 Y_{t-}^2}\left(\kappa_3\sum_{i=2}^{3}\sigma_i Y_{t-}^i\partial_i C(\cdot,0) - \sigma_3 C(\cdot,0)\right), \quad \phi_t^3 = -\frac{C(\cdot,0)}{\kappa_3 Y_{t-}^3}.$$

4 PDE Approach: Case of Zero Recovery

In this section, we assume that the prices Y^1 and Y^2 are strictly positive, but $\kappa_3 = -1$ so that Y^3 is a defaultable asset with zero recovery. Of course, the price Y^3_t vanishes after default, that is, on the set $\{t \ge \tau\}$. We assume here that $a \neq 0$ and $\sigma_1 \neq \sigma_2$ (see Lemma 2.2), but we no longer postulate that $b = 0$. Let us denote

$$\alpha_i = \mu_i + \sigma_i \frac{c}{a}, \quad \beta_i = \mu_i - \sigma_i \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}.$$

Proposition 4.1 *Let the price processes Y^i, $i = 1, 2, 3$ satisfy*

$$dY^i_t = Y^i_{t-}\big(\mu_i dt + \sigma_i \, dW_t + \kappa_i \, dM_t\big)$$

with $\kappa_i > -1$ for $i = 1, 2$ and $\kappa_3 = -1$. Assume that $a \neq 0$ and $\sigma_1 \neq \sigma_2$. Consider a contingent claim Y with maturity T and the terminal payoff $G(Y^1_T, Y^2_T, Y^3_T, H_T)$. If the pricing functions $C(\cdot, 0)$ and $C(\cdot, 1)$ belong to the class $\mathrm{C}^{1,2}([0,T] \times \mathbb{R}^3_+, \mathbb{R})$, then the function $C(t, y_1, y_2, y_3, 0)$ satisfies the PDE

$$\begin{aligned} \partial_t C(\cdot,0) + \sum_{i=1}^{3} (\alpha_i - \gamma\kappa_i) y_i \partial_i C(\cdot,0) + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,0) - \left(\alpha_1 + \kappa_1 \frac{b}{a}\right) C(\cdot,0) \\ + \left(\gamma - \frac{b}{a}\right) \big[C(t, y_1(1+\kappa_1), y_2(1+\kappa_2), 0, 1) - C(t, y_1, y_2, y_3, 0)\big] = 0 \end{aligned}$$

and the function $C(t, y_1, y_2, 1)$ solves

$$\partial_t C(\cdot,1) + \sum_{i=1}^{2} \beta_i y_i \partial_i C(\cdot,1) + \frac{1}{2} \sum_{i,j=1}^{2} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,1) - \beta_1 C(\cdot,1) = 0$$

subject to the terminal conditions

$$C(T, y_1, y_2, y_3, 0) = G(y_1, y_2, y_3, 0), \quad C(T, y_1, y_2, 1) = G(y_1, y_2, 0, 1).$$

The replicating strategy ϕ for Y is given by formulae (26)-(28).

Proof. Since the proof is analogous to the proof of Proposition 3.1, we do not give details. We are interested in the martingale property of relative price $\widetilde{C} = C(Y^1)^{-1}$ under the unique martingale measure $\mathbb{Q}^1$ of Lemma 2.2. Using the same computations as in the proof of Proposition 3.1, we arrive at the following condition:

$$\begin{aligned} 0 \;=\; & C_{t-} \left\{ -\mu_1 + \sigma_1^2 - \sigma_1 \theta + \xi_t \kappa_1 - \xi_t (1+\zeta) \frac{\kappa_1}{1+\kappa_1} \right\} \\ & + \partial_t C + \sum_{i=1}^{3} \mu_i Y^i_{t-} \partial_i C + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y^i_{t-} Y^j_{t-} \partial_{ij} C + \left(\Delta C - \sum_{i=1}^{3} \kappa_i Y^i_{t-} \partial_i C\right) \xi_t \\ & + \sum_{i=1}^{3} \sigma_i Y^i_{t-} \theta \partial_i C + \zeta \xi_t \Delta C - \sigma_1 \sum_{i=1}^{3} \sigma_i Y^i_{t-} \partial_i C - \xi_t (1+\zeta) \Delta C \frac{\kappa_1}{1+\kappa_1}. \end{aligned}$$

Using (22), we obtain, for $t \le \tau$,

$$\begin{aligned} -\mu_1 + \sigma_1^2 - \sigma_1 \theta + \xi_t \kappa_1 - \xi_t (1+\zeta) \frac{\kappa_1}{1+\kappa_1} &= -\alpha_1 - \kappa_1 \frac{b}{a}, \\ \mu_i + \sigma_i(\theta - \sigma_1) - \kappa_i \xi_t &= \alpha_i - \gamma \kappa_i. \end{aligned}$$

Hence, for $t \le \tau$,

$$\partial_t C + \sum_{i=1}^{3} (\alpha_i - \gamma\kappa_i) Y^i_{t-} \partial_i C + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y^i_{t-} Y^j_{t-} \partial_{ij} C - \left(\alpha_1 + \kappa_1 \frac{b}{a}\right) C_{t-} + \left(\gamma - \frac{b}{a}\right) \Delta C = 0.$$

Using again (22), we obtain, for $t > \tau$,

$$\begin{aligned} -\mu_1 + \sigma_1^2 - \sigma_1\theta + \xi_t\kappa_1 - \xi_t(1+\zeta)\frac{\kappa_1}{1+\kappa_1} &= -\beta_1, \\ \mu_i + \sigma_i(\theta - \sigma_1) - \kappa_i\xi_t &= \beta_i, \end{aligned}$$

and thus on this set the pricing function satisfies

$$\partial_t C + \sum_{i=1}^{2} \beta_i Y^i_{t-} \partial_i C + \frac{1}{2}\sum_{i,j=1}^{3} \sigma_i\sigma_j Y^i_{t-} Y^j_{t-} \partial_{ij} C - \beta_1 C_{t-} = 0.$$

This completes the proof. □

Remarks. The pre-default valuation PDE of Proposition 4.1 can be seen as an extension of the pre-default valuation PDE established in Proposition 3.1 to the case where $b \neq 0$. In particular, both PDEs are identical if $b = 0$.

4.1 Case: Y^1 Risk-Free, Y^2 Default-Free, Y^3 Defaultable

We assume that the processes Y^1, Y^2, Y^3 satisfy (see Example 2.2)

$$\begin{aligned} dY^1_t &= rY^1_t\, dt, \\ dY^2_t &= Y^2_t\big(\mu_2\, dt + \sigma_2\, dW_t\big), \\ dY^3_t &= Y^3_{t-}\big(\mu_3\, dt + \sigma_3\, dW_t - dM_t\big). \end{aligned}$$

Let us write $\widehat{r} = r + \widehat{\gamma}$, where

$$\widehat{\gamma} = \gamma(1+\zeta) = \gamma - \frac{b}{a} = \gamma + \mu_3 - r + \frac{\sigma_3}{\sigma_2}(r - \mu_2)$$

stands for the default intensity under $\mathbb{Q}^1$. The number $\widehat{r}$ is interpreted as the *credit-risk adjusted short-term rate.* Straightforward calculations show that the following corollary to Proposition 4.1 is valid.

Corollary 4.1 *Assume that $\sigma_1 = \kappa_1 = \kappa_2 = 0$. Then the functions $C(\cdot, 0)$ and $C(\cdot, 1)$ satisfy the following pricing PDEs:*

$$\begin{aligned} &\partial_t C(t, y_2, y_3, 0) + r y_2 \partial_2 C(t, y_2, y_3, 0) + \widehat{r} y_3 \partial_3 C(t, y_2, y_3, 0) - \widehat{r} C(t, y_2, y_3, 0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3, 0) + \widehat{\gamma} C(t, y_2, 1) = 0 \end{aligned}$$

and

$$\partial_t C(t, y_2, 1) + r y_2 \partial_2 C(t, y_2, 1) + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} C(t, y_2, 1) - r C(t, y_2, 1) = 0$$

with the terminal conditions

$$C(T, y_2, y_3, 0) = G(y_2, y_3, 0), \quad C(T, y_2, 1) = G(y_2, 0, 1).$$

4.1.1 Replication of a Survival Claim

In the special case of a survival claim, we have $C(\cdot, 1) = 0$, and thus the following result can be easily established.

Corollary 4.2 *The pre-default pricing function $C(\cdot,0)$ of a survival claim $Y = \mathbb{1}_{\{\tau>T\}}G(Y_T^2, Y_T^3)$ is a solution of the following PDE:*

$$\begin{aligned}\partial_t C(t,y_2,y_3,0) + ry_2\partial_2 C(t,y_2,y_3,0) + \widehat{r}y_3\partial_3 C(t,y_2,y_3,0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} C(t,y_2,y_3,0) \\ - \widehat{r}C(t,y_2,y_3,0) = 0\end{aligned}$$

with the terminal condition $C(T,y_2,y_3,0) = G(y_2,y_3)$. The components ϕ^2 and ϕ^3 of the replicating strategy are

$$\begin{aligned}\phi_t^2 &= \frac{1}{\sigma_2 Y_{t-}^2}\left(\sum_{i=2}^{3}\sigma_i Y_{t-}^i \partial_i C(t,Y_{t-}^2,Y_{t-}^3,0) + \sigma_3 C(t,Y_{t-}^2,Y_{t-}^3,0)\right),\\ \phi_t^3 &= \frac{1}{Y_{t-}^3}\,C(t,Y_{t-}^2,Y_{t-}^3,0).\end{aligned}$$

Note that we have $\phi_t^3 Y_{t-}^3 = C(t,Y_{t-}^2,Y_{t-}^3,0)$ for every $t\in[0,T]$. Hence, the following relationships holds, for every $t<\tau$,

$$\phi_t^3 Y_t^3 = C(t,Y_t^2,Y_t^3,0), \quad \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0.$$

The last equality is a special case of a *balance condition* that was introduced in Bielecki et al. (2004d) in a general semimartingale set-up. It ensures that the wealth of a replicating portfolio falls to 0 at default time.

Example 4.1 Let us first consider a survival claim $Y = \mathbb{1}_{\{\tau>T\}}G(Y_T^2)$, that is, a *vulnerable claim* with default-free underlying asset. Its pre-default pricing function $C(\cdot,0)$ does not depend on y_3, and satisfies the PDE

$$\partial_t C(t,y_2,0) + ry_2\partial_2 C(t,y_2,0) + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} C(t,y_2,0) - \widehat{r}C(t,y_2,0) = 0 \tag{31}$$

with the terminal condition $C(T,y_2,0) = G(y_2)$. The present set-up covers the case of a vulnerable option written on a default-free asset Y^2. For examples of explicit pricing formulae for vulnerable options, see Section 5.1 in Bielecki et al. (2004b).

Example 4.2 Let us now consider a survival claim $Y = \mathbb{1}_{\{\tau>T\}}G(Y_T^3)$, where $G(0)=0$ so that Y can also be represented as $Y = G(Y_T^3)$. Its pre-default price is equal to $C(t,Y_t^3,0)$, where the function $C(t,y_3,0)$ is such that

$$\partial_t C(t,y_3,0) + \widehat{r}y_3\partial_3 C(t,y_3,0) + \frac{1}{2}\sigma_3^2 y_3^2 \partial_{33} C(t,y_3,0) - \widehat{r}C(t,y_3,0) = 0 \tag{32}$$

and $C(T,y_3,0) = G(y_3)$. We conclude that in this case the pre-default value of a survival claim formally coincides with the price of a claim $G(\widehat{Y}_T^3)$ computed in a default-free model

$$\begin{aligned} d\widehat{Y}_t^1 &= \widehat{r}\widehat{Y}_t^1\,dt,\\ d\widehat{Y}_t^3 &= \widehat{Y}_t^3\big(\mu_3\,dt + \sigma_3\,dW_t\big),\end{aligned}$$

with the risk-free interest rate $\widehat{r} = r + \widehat{\gamma} = r + \gamma(1+\zeta)$. This example covers, in particular, the case of a call option written on a defaultable asset with zero recovery. Explicit pricing formulae for such options can be found in Section 5.2 of Bielecki et al. (2004b).

Remarks. It is important to stress that in both particular cases considered in Example 4.1, all three primary assets are needed to perfectly hedge a survival claim. The minor, but important, difference between the PDEs (31) and (32) shows that it is essential to examine in detail all assumptions underpinning a credit risk model used for valuation and hedging of a defaultable claim. Let us finally mention that equation (32) coincides with equation (3.8) in the paper by Ayache et al. (2003) in which the authors examine valuation and hedging of convertible bonds with credit risk.

5 Stopped Trading Strategies

In this section, we adopt a more practical convention regarding the specification of a defaultable claim. Though formally equivalent to the previous one, it is more convenient, since it allows us to directly specify the post-default pricing function $C(\cdot, 1)$ (at least at time τ) in terms of the so-called recovery process. This approach has the following advantages. First, in some circumstances the recovery payoff at default time is exogenously given, and thus the study of a claim after default is unnecessary. Second, since a market model is no longer used after the default time, some technical assumptions regarding the behavior of prices Y^1, Y^2, Y^3 can be relaxed. We can thus cover different cases regarding the behavior after default of primary defaultable assets by a common result.

5.1 Generic Defaultable Claim

According to our convention (see, for instance, Bielecki et al. (2004a)), a generic defaultable claim is determined by a default time τ, a $\mathcal{F}_T$-measurable random variable X, interpreted as the *promised payoff* at maturity T, and a $\mathbb{F}$-predictable process Z interpreted as the *recovery payoff* at the time of default. Formally, a generic defaultable claim can thus be represented as a triple (X, Z, τ). The *dividend process* D of a claim (X, Z, τ), which settles at time T, equals

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{\{t \geq T\}} + \int_{]0,t]} Z_u \, dH_u. = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{\{t \geq T\}} + Z_\tau \mathbb{1}_{\{\tau \leq t\}}.$$

The *ex-dividend price* of a defaultable claim $Y = (X, Z, \tau)$ is given as (all necessary integrability conditions are implicitly assumed to hold with regard to X and Z)

$$\pi_t(Y) = Y_t^1 \, \mathbb{E}_{\mathbb{Q}^1} \Big(\int_{]t,T]} (Y_u^1)^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big). \tag{33}$$

Note that, by definition, the ex-dividend price equals 0 at default time τ and after this date. Hence, we are only interested in the value prior to default, that is, the pre-default value. We denote by $\widetilde{U}(Y)$ the pre-default value Y, so that $\pi_t(Y) = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_t(Y)$. It it rather clear that $\widetilde{U}(Y) = \widetilde{U}(X) + \widetilde{U}(Z)$. For computations of $\widetilde{U}(X)$ and $\widetilde{U}(Z)$ in terms of the intensity of τ, see Bielecki et al. (2004a).

Within the present set-up, it is convenient to assume that X satisfies $X = G(Y_T^1, Y_T^2, Y_T^3)$ and the recovery process Z is given as $Z_t = z(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3)$ for some function $z : [0,T] \times \mathbb{R}_+^3 \to \mathbb{R}$. Under these assumptions, the pre-default value is given by the pre-default pricing function $C(\cdot, 0)$.

The proof of the next result is almost identical to the proof of Proposition 4.1. Note, however, that we now work in a set-up descirbed in Section 2.4.3, so that we do need to assume that $\kappa_2 > -1$. Though we assume here that $\kappa_1 > -1$, it is plausible that this result remains valid also in the case when $\kappa_1 = -1$ (for instance, when Y^1, Y^2, Y^3 are defaultable assets with zero recovery).

Proposition 5.1 *Let the price processes Y^i, $i = 1, 2, 3$ satisfy*

$$dY_t^i = Y_{t-}^i \big(\mu_i \, dt + \sigma_i \, dW_t + \kappa_i \, dM_t \big)$$

with $\kappa_1 > -1$. Assume that $a \neq 0$. If the pre-default pricing function $C(t, y_1, y_2, y_3, 0)$ belongs to the class $\mathrm{C}^{1,2}([0,T] \times \mathbb{R}_+^3, \mathbb{R})$, then it satisfies the PDE

$$\partial_t C(\cdot,0) + \sum_{i=1}^{3} (\alpha_i - \gamma \kappa_i) y_i \partial_i C(\cdot, 0) + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot, 0) - \left(\alpha_1 + \kappa_1 \frac{b}{a} \right) C(\cdot, 0)$$
$$+ \left(\gamma - \frac{b}{a} \right) \big[z(t, y_1(1+\kappa_1), y_2(1+\kappa_2), y_3(1+\kappa_3)) - C(t, y_1, y_2, y_3, 0) \big] = 0$$

subject to the terminal condition $C(T, y_1, y_2, y_3, 0) = G(y_1, y_2, y_3)$.

Example 5.1 An important special case is when a defaultable claim is subject to the fractional recovery of the (pre-default) market value. Under the assumption that the recovery is proportional to the pre-default market value, the value of the claim at the moment of default is equal to δ times the value just before the default. Hence, $z(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}) = \delta C(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, 0)$, and the valuation PDE becomes

$$\partial_t C(\cdot,0) + \sum_{i=1}^{3} (\alpha_i - \gamma\kappa_i) y_i \partial_i C(\cdot,0) + \frac{1}{2}\sum_{i,j=1}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C(\cdot,0)$$
$$+ \left[(\delta - 1)\left(\gamma - \frac{b}{a}\right) - \left(\alpha_1 + \kappa_1 \frac{b}{a}\right)\right] C(\cdot,0) = 0.$$

6 Extension to the Case of Multiple Defaults

We place ourselves within the framework introduced by Kusuoka (1999) (see also Bielecki and Rutkowski (2003). Let τ^1 and τ^2 be strictly positive random variables on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. We introduce the corresponding jump processes $H^i_t = \mathbb{1}_{\{\tau^i \le t\}}$ for $i = 1, 2$, and we denote by $\mathbb{H}^i$ the filtration generated by the process H^i. Finally, we set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$, where $\mathbb{F}$ is generated by a Brownian motion W.

For the sake of simplicity, we assume that $Y^1_t = 1$, so that Y^1 represents the savings account corresponding to the short-term rate $r = 0$. We postulate that the asset price Y^i satisfies, for $i = 2, 3, 4$,

$$dY^i_t = Y^i_{t-}\big(\mu_i\, dt + \sigma_i\, dW_t + \kappa_i\, dM^1_t + \psi_i\, dM^2_t\big), \tag{34}$$

where M^i is the martingale associated with the default process H^i, that is,

$$M^i_t = H^i_t - \int_0^t \gamma^i_u (1 - H^i_u)\, du.$$

In order to ensure the Markov property, we assume that $\gamma^i_u = g_i(u, H^1_u, H^2_u)$. We assume that the coefficients in (34) are such that the market model $\mathcal{M} = (Y^1, Y^2, Y^3, Y^4, \Phi)$ is arbitrage-free and complete. As before, we denote by $\mathbb{Q}^1$ the unique martingale measure for processes $Y^i = Y^i(Y^1)^{-1}$, $i = 2, 3, 4$.

Consider a contingent claim of the form $Y = G(Y^2_T, Y^3_T, Y^4_T, H^1_T, H^2_T)$. Its arbitrage price can be represented as a function $C(t, Y^2_t, Y^3_t, Y^4_t, H^1_t, H^2_t)$, or equivalently, as a quadruplet of functions $C(\cdot,1,1)$ (when t is after the two default times), $C(\cdot,0,1)$, $C(\cdot,1,0)$ and $C(\cdot,0,0)$. The pricing functions satisfy the terminal condition

$$C(T, y_2, y_3, y_4, h_1, h_2) = G(y_2, y_3, y_4, h_1, h_2).$$

The process $C_t = C(t, Y^2_t, Y^3_t, Y^4_t, H^1_t, H^2_t)$ follows a $\mathbb{G}$-martingale under $\mathbb{Q}^1$. The dynamics of Y^i under $\mathbb{Q}^1$ are

$$dY^i_t = Y^i_{t-}\big(\sigma_i\, d\widehat{W}_t + \kappa_i\, d\widehat{M}^1_t + \psi_i\, d\widehat{M}^2_t\big), \quad i = 2, 3, 4,$$

where $\widehat{W}$ is a Brownian motion under $\mathbb{Q}^1$, and the processes

$$\widehat{M}^i_t = H^i_t - \int_0^t \widehat{\xi}^i_u\, du = H^i_t - \int_0^t \widehat{\gamma}^i_u (1 - H^i_u)\, du, \quad i = 1, 2,$$

are $\mathbb{G}$-martingales. An application of Itô's formula yields

$$\begin{aligned} dC_t \;=\; & \partial_t C\, dt + \sum_{i=2}^{4} Y^i_{t-}\partial_i C \left(\sigma_i\, d\widehat{W}_t - \big(\kappa_i \widehat{\xi}^1_t + \psi_i \widehat{\xi}^2_t\big)\, dt\right) + \frac{1}{2}\sum_{i,j=2}^{4} \sigma_i\sigma_j Y^i_{t-} Y^j_{t-} \partial_{ij} C\, dt \\ & + \big(C(\cdot,1,0) - C(\cdot,0,0)\big)(1 - H^2_t)\Delta H^1_t + \big(C(\cdot,0,1) - C(\cdot,0,0)\big)(1 - H^1_t)\Delta H^2_t \\ & + \big(C(\cdot,1,1) - C(\cdot,0,1)\big) H^2_t \Delta H^1_t + \big(C(\cdot,1,1) - C(\cdot,1,0)\big) H^1_t \Delta H^2_t \\ & + \big(C(\cdot,1,1) - C(\cdot,0,0)\big)\Delta H^1_t \Delta H^2_t. \end{aligned}$$

If defaults cannot occur simultaneously (so that $\Delta H_t^1 \Delta H_t^2 = 0$), we obtain the following martingale condition:

$$
\begin{aligned}
0 \;=\; & \partial_t C - \sum_{i=2}^{4} (\kappa_i \widehat{\xi}_t^1 + \psi_i \widehat{\xi}_t^2) Y_{t-}^i \partial_i C + \frac{1}{2} \sum_{i,j=2}^{4} \sigma_i \sigma_j Y_{t-}^i Y_{t-}^j \partial_{ij} C \\
& + \big(C(\cdot,1,0) - C(\cdot,0,0)\big)(1 - H_t^2)\widehat{\xi}_t^1 + \big(C(\cdot,0,1) - C(\cdot,0,0)\big)(1 - H_t^1)\widehat{\xi}_t^2 \\
& + \big(C(\cdot,1,1) - C(\cdot,0,1)\big) H_t^2 \widehat{\xi}_t^1 + \big(C(\cdot,1,1) - C(\cdot,1,0)\big) H_t^1 \widehat{\xi}_t^2 .
\end{aligned}
$$

This condition leads to the four valuation PDEs:

$$
\begin{aligned}
& \partial_t C(\cdot,0,0) - \sum_{i=2}^{4} (\kappa_i \widehat{\gamma}_0^1 + \psi_i \widehat{\gamma}_0^2) y_i \partial_i C(\cdot,0,0) + \frac{1}{2} \sum_{i,j=2}^{4} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,0,0) \\
& \quad + \widehat{\gamma}_0^1 \big(C(\cdot,1,0) - C(\cdot,0,0)\big) + \widehat{\gamma}_0^2 \big(C(\cdot,0,1) - C(\cdot,0,0)\big) = 0,
\end{aligned}
$$

$$
\partial_t C(\cdot,1,0) - \sum_{i=2}^{4} \psi_i \widehat{\gamma}_1^2 y_i \partial_i C(\cdot,1,0) + \frac{1}{2} \sum_{i,j=2}^{4} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,1,0) + \widehat{\gamma}_1^2 \big(C(\cdot,1,1) - C(\cdot,1,0)\big) = 0,
$$

$$
\partial_t C(\cdot,0,1) - \sum_{i=2}^{4} \kappa_i \widehat{\gamma}_2^1 y_i \partial_i C(\cdot,0,1) + \frac{1}{2} \sum_{i,j=2}^{4} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,0,1) + \widehat{\gamma}_2^1 \big(C(\cdot,1,0) - C(\cdot,0,0)\big) = 0,
$$

and

$$
\partial_t C(\cdot,1,1) + \frac{1}{2} \sum_{i,j=2}^{4} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,1,1) = 0.
$$

Here, $\widehat{\gamma}_0^1$ and $\widehat{\gamma}_0^2$ are (possibly time-dependent) intensities of τ^1 and τ^2 prior to the first default, and $\widehat{\gamma}_2^1$ ($\widehat{\gamma}_1^2$ respectively) is the intensity of the default time τ^1 on the set $\tau^2 \le t < \tau^1$ (the intensity of the default time τ^2 on the set $\tau^1 \le t < \tau^2$ respectively).

References

E. Ayache, P.A. Forsyth, K.R. Vetzal (2003) The valuation of convertible bonds with credit risk. Working paper.

T.R. Bielecki, M. Jeanblanc and M. Rutkowski (2004a) Modeling and valuation of credit risk. In: *Stochastic Methods in Finance*, M. Frittelli and W. Runggaldier, eds., Springer-Verlag, Berlin Heidelberg New York, pp. 27-126.

T.R. Bielecki, M. Jeanblanc and M. Rutkowski (2004b) Hedging of defaultable claims. In: *Paris-Princeton Lectures on Mathematical Finance 2003*, R.A. Carmona, E. Cinlar, I. Ekeland, E. Jouini, J.E. Scheinkman, N. Touzi, eds., Springer-Verlag, Berlin Heidelberg New York, pp. 1-132.

T.R. Bielecki, M. Jeanblanc and M. Rutkowski (2004c) Completeness of a general semimartingale market under constrained trading. Working paper.

T.R. Bielecki, M. Jeanblanc and M. Rutkowski (2004d) Completeness of a reduced-form credit risk model with discontinuous asset prices. Working paper.

T.R. Bielecki and M. Rutkowski (2003) Dependent defaults and credit migrations. *Applicationes Mathematicae* 30, 121-145.

J. Jacod and A.N. Shiryaev (1987) *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin Heidelberg New York.

I. Karatzas and S.E. Shreve (1998) *Methods of Mathematical Finance.* Springer-Verlag, Berlin Heidelberg New York.

S. Kusuoka (1999) A remark on default risk models. *Advances in Mathematical Economics* 1, 69-82.

P. Protter (2001) A partial introduction to financial asset pricing theory. *Stochastic Processes and Their Applications* 91, 169-203.

N. Vaillant (2001) A beginner's guide to credit derivatives. Working paper, Nomura International.