Dynamic Modeling of Portfolio Credit Risk with Common Shocks

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Abstract

We consider a bottom-up Markovian model of portfolio credit risk where dependence among credit names stems from the possibility of simultaneous defaults. A common shocks interpretation of the model is possible so that efficient convolution recursion procedures are available for pricing and hedging CDO tranches, conditionally on any given state of the Markov model. Calibration of marginals and dependence parameters can be performed separately using a two-steps procedure, much like in a standard static copula set-up. As a result this model allows us to hedge CDO tranches using single-name CDS-s in a theoretically sound and practically convenient way. To illustrate this we calibrate the model against market data on CDO tranches and the underlying single-name CDS-s. We then study the loss distributions as well as the min-variance hedging strategies in the calibrated portfolios.
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1 Introduction

The CDO market have been deeply and adversely impacted by the credit crisis of 2008-2009. In particular, CDO issuances have become quite rare. Nevertheless, there are huge notionals of CDO contracts outstanding and market participants continue to be confronted with the task to hedge their positions in these contracts up to maturity date. Moreover, according to the recent revision of Basel II regulation on trading book capital requirement [3], tranches on standard indices and their associated liquid hedging positions will continue to be charged as hedge-sets under internal VaR-based method, which makes the issue of hedging even more important for standardized CDO tranches. For previous studies of this issue we refer the reader to, among others, Laurent, Cousin and Fermanian [28], Frey and Backhaus [21], Cont and Kan [13] or Cousin, Crépey and Kan [15]. In particular it has been established empirically in [13] and [15] that a single-instrument hedge of a CDO tranche by the corresponding credit index is often not good enough. In this paper we deal with a bottom-up Markovian model, in which hedging loss derivatives by single-name instruments can be performed in a theoretically sound and practical way.

The main theoretical contribution of the paper is the common shocks interpretation of a (specific) Markovian model. From a practical point of view this interpretation underlies semi-explicit, convolution based pricing and Greeking schemes for basket credit derivatives. Such numerical schemes play a crucial role when calibrating credit portfolio models and in related applications such as counterparty risk valuation for portfolios (see [2, 6]). This allows one to address in a dynamic and theoretically consistent way the issues of pricing and hedging basket credit derivatives, whilst preserving the static common factor tractability.

The framework presented in this paper is related to the work by Elouerkhaoui [19] (see also Brigo et al. [11, 12]). The approach in [19] and our set-up are both aiming at deriving a dynamic credit portfolio model which can be jointly calibrated to individual and basket credit derivatives market-data. Hence, both approaches are syntheses of the so-called bottom-up and top-down approaches. However, while Elouerkhaoui [19] works in a point-process set-up, we use a Markovian model. The practical interest of our framework is an increased model tractability.

The paper is organized as follows. In Section 2 we formulate a bottom-up Markovian model, in which individual default processes for various credit names are coupled together by means of simultaneous defaults. We then prove that conditionally on the full information in this model, the dependence structure of surviving names is equivalent to a Marshall-Olkin copula. In Section 3 we use this equivalence with the Marshall-Olkin framework in order to derive a common shocks model interpretation of our Markovian setting. This enables us to derive fast deterministic computational tractable algorithms for pricing and Greeking schemes in our heterogeneous model. In Section 4 we present numerical results of calibration against market data from CDO tranches as well as individual CDS spreads. We also discuss hedging sensitivities computed in the models thus calibrated. Technical proofs are deferred to Appendix A.

In the rest of the paper we consider a risk neutral pricing model $(\Omega, \mathcal{F}, \mathbb{P})$, for a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ which will be specified below and where $T \geq 0$ is a fixed time horizon. We denote $\mathbb{N}_n = \{1, \dots, n\}$ and let $\mathcal{N}_n$ denote the set of all subsets of $\mathbb{N}_n$ where n represents the number of obligors in the underlying credit portfolio. Further, we set $\max \emptyset = -\infty$.

2 Model of Default Times

In this section we construct a bottom-up Markovian model consisting of a multivariate factor process $\mathbf{X}$ and a vector $\mathbf{H}$ representing the default indicator processes in a pool of n different credit names. More specifically, $\mathbf{H}_t$ is a vector in $\{0,1\}^n$ where the i-th entry of $\mathbf{H}_t$ is the indicator function for the event of a default of obligor i up to time t. The purpose of the factor process $\mathbf{X}$ is to more realistically model diffusive randomness of credit spreads.

In our model defaults are the consequence of some "triggering-events" associated with groups of obligors. We then define the following pre-specified set of groups

$$\mathcal{Y} = \{\{1\}, \dots, \{n\}, I_1, \dots, I_m\},$$

where $I_1, \dots, I_m$ are m subsets of $\mathbb{N}_n$ (elements of the set $\mathcal{N}_n$ of all parts of $\mathbb{N}_n$), and each group I_l contains at least two obligors or more. The triggering events are divided in two categories: the ones associated with singletons $\{1\}, \dots, \{n\}$ can only trigger the default of name $1, \dots, n$ individually, while the others associated with multi-name groups $I_1, \dots, I_m$ may simultaneously trigger the default of all names in these groups. Note that several triggering events may affect the same particular name, so that only the one occurring first effectively triggers the default of that name. As a result, when a triggering-event associated with a specific group occurs at time t, it only triggers the default of names that are still alive in that group at time t. In the following, the elements Y of $\mathcal{Y}$ will be used to designate triggering events and we let $\mathcal{I} = (I_l)_{1\le l\le m}$ denote the pre-specified set of multi-name groups of obligors.

Let $\nu = |\mathcal{Y}| = n + m$ denote the cardinality of $\mathcal{Y}$. Given a multivariate Brownian motion $\mathbf{W} = (W^Y)_{Y\in\mathcal{Y}}$ with independent components, we assume that the factor process $\mathbf{X} = (X^Y)_{Y\in\mathcal{Y}}$ is a strong solution to

$$dX_t^Y = b_Y(t, X_t^Y)\, dt + \sigma_Y(t, X_t^Y)\, dW_t^Y, \tag{1}$$

for suitable drift and diffusion functions $b_Y = b_Y(t,x)$ and $\sigma_Y = \sigma_Y(t,x)$. By application of Theorem 32 page 100 of Protter [31], this makes $\mathbf{X}$ an $\mathcal{F}^{\mathbf{W}}$-Markov process admitting the following generator acting on functions $v = v(t,\mathbf{x})$ with $\mathbf{x} = (x_Y)_{Y\in\mathcal{Y}}$

$$A_t v(t,\mathbf{x}) = \sum_{Y\in\mathcal{Y}} \Big(b_Y(t,x_Y)\partial_{x_Y} v(t,\mathbf{x}) + \tfrac{1}{2}\sigma_Y^2(t,x_Y)\partial^2_{x_Y^2} v(t,\mathbf{x}) \Big). \tag{2}$$

Let $\mathcal{F} := \mathcal{F}^{(\mathbf{W},\mathbf{H})}$ be the filtration generated by the Brownian motion $\mathbf{W}$ and the point process $\mathbf{H}$. Given the "intensity functions"[1] of triggering-events, say $\lambda_Y = \lambda_Y(t, x_Y)$ for every triggering-event $Y \in \mathcal{Y}$, we would like to construct a model in which the $\mathcal{F}$-predictable intensity of a jump of $\mathbf{H} = (H^i)_{1\le i\le n}$ from $\mathbf{H}_{t-} = \mathbf{k}$ to $\mathbf{H}_t = \mathbf{l}$, with $\mathbf{l} \neq \mathbf{k}$ in $\{0,1\}^n$, is given by

$$\lambda(t, \mathbf{X}_t, \mathbf{k}, \mathbf{l}) := \sum_{\{Y\in\mathcal{Y};\, \mathbf{k}^Y = \mathbf{l}\}} \lambda_Y(t, X_t^Y), \tag{3}$$

where, for any $Z \in \mathcal{N}_n$, the expression $\mathbf{k}^Z$ denotes the vector obtained from $\mathbf{k} = (k_1, \dots, k_n)$ by replacing the components k_i, $i \in Z$, by numbers one (whenever k_i is not equal to one already). The intensity of a jump of $\mathbf{H}$ from $\mathbf{k}$ to $\mathbf{l}$ at time t is thus equal to the sum of the

[1] These functions will indeed be interpreted as shock intensity functions in subsection 2.3.

intensities of the triggering-events $Y \in \mathcal{Y}$ such that, if the joint default of the survivors in group Y occurred at time t, then the state of $\mathbf{H}$ would move from $\mathbf{k}$ to $\mathbf{l}$.

To achieve (3) we follow the classical methodology: we construct $\mathbf{H}$ by an $\mathbf{X}$-related change of probability measure, starting from a continuous-time Markov chain with intensity one. This construction is detailed in Appendix A.1.

2.1 Itô formula

In this subsection we state the Itô formula for functions of the Markov process $(\mathbf{X}, \mathbf{H})$.

For any set $Z \in \mathcal{N}_n$, let the set-event indicator process H^Z denote the indicator process of a joint default of the names in Z and only in Z. For $\mathbf{k} = (k_1, \ldots, k_n) \in \{0,1\}^n$, we introduce $\text{supp}(\mathbf{k}) = \{i \in \mathbb{N}_n;\, k_i = 1\}$ and $\text{supp}^c(\mathbf{k}) = \{i \in \mathbb{N}_n;\, k_i = 0\}$. Hence, $\text{supp}(\mathbf{k})$ denotes the obligors who have defaulted in state $\mathbf{k}$ and similarly $\text{supp}^c(\mathbf{k})$ are the survived names in the portfolio-state $\mathbf{k}$.

The following lemma provides the structure of the so called compensated set-event martingales M^Z-s, which we will use later as fundamental martingales to represent the pure jump martingale components of the various price processes involved.

Lemma 2.1 *For every set $Z \in \mathcal{N}_n$ the intensity of H^Z is given by $\ell_Z(t, \mathbf{X}_t, \mathbf{H}_t)$, so*

$$dM_t^Z = dH_t^Z - \ell_Z(t, \mathbf{X}_t, \mathbf{H}_t)dt$$

is a martingale, and the set-event intensity function $\ell_Z(t, \mathbf{x}, \mathbf{k})$ is defined as

$$\ell_Z(t, \mathbf{x}, \mathbf{k}) = \sum_{Y \in \mathcal{Y};\, Y \cap supp^c(\mathbf{k}) = Z} \lambda_Y(t, x_Y). \tag{4}$$

Proof. See Appendix A.1.1.

So $\ell_Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) = \sum_{Y \in \mathcal{Y};\, Y_t = Z} \lambda_Y(t, X_t^Y)$, where for every Y in $\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$ we define

$$Y_t = Y \cap \text{supp}^c(\mathbf{H}_{t-}), \tag{5}$$

the set-valued process representing the survived obligors in Y right before time t. Let also $\mathcal{Z}_t = \{Z \in \mathbb{N}_n; Z = Y_t \text{ for at least one } Y \in \mathcal{Y}\} \setminus \emptyset$ denote the set of all non-empty sets of survivors of sets Y in $\mathcal{Y}$ right before time t.

We now derive a version of the Itô formula, which is relevant for our model. It will be used below for establishing the Markov properties of our set-up, as well as for deriving price dynamics. Let $\sigma(t, \mathbf{x})$ denote the diagonal matrix with diagonal $(\sigma_Y(t, x_Y))_{Y \in \mathcal{Y}}$. Given a function $u = u(t, \mathbf{x}, \mathbf{k})$ with $\mathbf{x} = (x_Y)_{Y \in \mathcal{Y}}$ and $\mathbf{k} = (k_i)_{1 \le i \le n}$ in $\{0,1\}^n$, we denote

$$\nabla u(t, \mathbf{x}, \mathbf{k}) = (\partial_{x_1} u(t, \mathbf{x}, \mathbf{k}), \ldots, \partial_{x_\nu} u(t, \mathbf{x}, \mathbf{k})).$$

Let also δu^Z represent the sensitivity of u to the event $Z \in \mathcal{N}_n$, so

$$\delta u^Z(t, \mathbf{x}, \mathbf{k}) = u(t, \mathbf{x}, \mathbf{k}^Z) - u(t, \mathbf{x}, \mathbf{k}).$$

Proposition 2.2 *Given a regular enough function $u = u(t, \mathbf{x}, \mathbf{k})$, one has*

$$\begin{aligned} du(t, \mathbf{X}_t, \mathbf{H}_t) = \Big(\partial_t + \mathcal{A}_t\Big) u(t, \mathbf{X}_t, \mathbf{H}_t)dt \,+\, \nabla u(t, \mathbf{X}_t, \mathbf{H}_t)\sigma(t, \mathbf{X}_t)d\mathbf{W}_t \\ +\, \sum_{Z \in \mathcal{Z}_t} \delta u^Z(t, \mathbf{X}_t, \mathbf{H}_{t-})dM_t^Z, \end{aligned} \tag{6}$$

where

$$\begin{aligned}\mathcal{A}_t u(t,\mathbf{x},\mathbf{k}) = &\sum_{Y\in\mathcal{Y}} \Big(b_Y(t,x_Y)\partial_{x_Y} u(t,\mathbf{x},\mathbf{k}) + \frac{1}{2}\sigma_Y^2(t,x_Y)\partial^2_{x_Y^2} u(t,\mathbf{x},\mathbf{k})\Big) \\ &+ \sum_{Y\in\mathcal{Y}} \lambda_Y(t,x_Y)\delta u^Y(t,\mathbf{x},\mathbf{k}).\end{aligned} \tag{7}$$

Proof. See Appendix A.1.2.

In the Itô formula (6), the jump term may involve any of the 2^n set-events martingales M^Z for $Z \in \mathcal{N}_n$. This suggests that the martingale dimension[2] of the model is $\nu + 2^n$, where $\nu = n + m$ corresponds to the dimension of the Brownian motion $\mathbf{W}$ driving the factor process $\mathbf{X}$ and 2^n corresponds to the jump component $\mathbf{H}$. Yet by a reduction which is due to specific structure of the intensities in our set-up, the jump term of $\mathcal{A}_t$ in (7) is a sum over the set of triggering-events $\mathcal{Y}$, which has cardinality ν.

Note that our model excludes direct contagion effects in which intensities of surviving names would be affected by past defaults, for example as in the bottom-up models treated by e.g. [14, 24, 25, 28]. To provide some understanding in this regard, we give a simple illustrative example.

Example 2.3 Take $\mathbb{N}_n = \{1,2,3\}$, so that the state space of $\mathbf{H}$ contains 8 elements:

$$\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}.$$

Now, let $\mathcal{Y}$ be given as $\mathcal{Y} = \{\{1\},\{2\},\{3\},\{1,2\},\{1,2,3\}\}$. This is an example of the nested structure of $\mathcal{I}$ with $I_1 = \{1,2\} \subset I_2 = \{1,2,3\}$. Suppose for simplicity that λ_Y does not depend either on t or on $\mathbf{x}$ (dependence in $t, \mathbf{x}$ will be dealt with in Subsection 2.2). Then, the generator $\mathcal{A}$ of the chain $\mathbf{H}$ is given by

$$\mathcal{A} \equiv \begin{bmatrix} \cdot & \lambda_{\{1\}} & \lambda_{\{2\}} & \lambda_{\{3\}} & \lambda_{\{1,2\}} & 0 & 0 & \lambda_{\{1,2,3\}} \\ 0 & \cdot & 0 & 0 & \lambda_{\{2\}}+\lambda_{\{1,2\}} & \lambda_{\{3\}} & 0 & \lambda_{\{1,2,3\}} \\ 0 & 0 & \cdot & 0 & \lambda_{\{1\}}+\lambda_{\{1,2\}} & 0 & \lambda_{\{3\}} & \lambda_{\{1,2,3\}} \\ 0 & 0 & 0 & \cdot & 0 & \lambda_{\{1\}} & \lambda_{\{2\}} & \lambda_{\{1,2,3\}}+\lambda_{\{1,2\}} \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \lambda_{\{3\}}+\lambda_{\{1,2,3\}} \\ 0 & 0 & 0 & 0 & 0 & \cdot & 0 & \lambda_{\{2\}}+\lambda_{\{1,2,3\}}+\lambda_{\{1,2\}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \lambda_{\{1\}}+\lambda_{\{1,2,3\}}+\lambda_{\{1,2\}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

where '$\cdot$' represents the sum of all other elements in the row multiplied with -1. Now, consider group $\{1,2,3\}$. Suppose, that at some point of time obligor 2 is defaulted, but obligors 1 and 3 are still alive, so that process $\mathbf{H}$ is in state $(0,1,0)$. In this case the two survivors in the group $\{1,2,3\}$ may default simultaneously with intensity $\lambda_{\{1,2,3\}}$. Of course, here $\lambda_{\{1,2,3\}}$ cannot be interpreted as intensity of all three defaulting simultaneously,

[2] Minimal number of fundamental martingales which can be used as integrators to represent all the martingales in the model, see Appendix A.1.

as obligor 2 has already defaulted. In fact, the only state of the model in which $\lambda_{\{1,2,3\}}$ can be interpreted as the intensity of all three defaulting, is state $(0,0,0)$. Note that obligor 1 defaults with intensity $\lambda_{\{1\}} + \lambda_{\{1,2,3\}} + \lambda_{\{1,2\}}$ regardless of the state of the pool, as long company 1 is alive. Similarly, obligor 2 will default with intensity $\lambda_{\{2\}} + \lambda_{\{1,2,3\}} + \lambda_{\{1,2\}}$ regardless of the state of the pool, as long company 1 is alive. Also, obligors 1 and 2 will default together with intensity $\lambda_{\{1,2,3\}} + \lambda_{\{1,2\}}$ regardless of the state of the pool, as long as company 1 and 2 still are alive.

2.2 Markov Properties

Below, for every obligor i, a real-valued marginal factor process X^i will be defined as a suitable function of the above multivariate factor process $\mathbf{X} = (X^Y)_{Y\in\mathcal{Y}}$. We shall then state conditions on the default intensities which enables us to prove that the marginal pair (X^i, H^i) is a Markov process. Markovianity of the model marginals (X^i, H^i)-s is crucial at the stage of calibration of the model, so that these marginals can be calibrated independently.

Observe that in view of (3), the intensity of a jump of H^i from $H^i_{t-} = 0$ to 1 is given by, for $t \in [0,T]$,

$$\sum_{\{Y\in\mathcal{Y};\, i\in Y\}} \lambda_Y(t, X^Y_t), \tag{9}$$

where the sum in this expression is taken over all pre-specified groups that contain name i. We define the marginal factor X^i as a linear functional φ_i of the multivariate factor process $\mathbf{X} = (X^Y)_{Y\in\mathcal{Y}}$ so that $X^i_t := \varphi_i(\mathbf{X}_t)$. In general the transition intensity (9) implies non-Markovianity of the marginal (X^i, H^i). Hence, in order to make the process (X^i, H^i) to be Markov, one needs to impose a more specified parametrization of (9) as well as conditions on the mapping φ_i. To be more specific:

Assumption 2.4 For every obligor i, there exists a linear form $\varphi_i(\mathbf{x})$ and a real-valued function $\lambda_i(t,x)$ such that for every $(t,\mathbf{x})$ with $\mathbf{x} = (x_Y)_{Y\in\mathcal{Y}}$

$$\sum_{\{Y\in\mathcal{Y};\, i\in Y\}} \lambda_Y(t, x_Y) = \lambda_i(t, \varphi_i(\mathbf{x})), \tag{10}$$

where, in addition, $X^i_t := \varphi_i(\mathbf{X}_t)$ is a Markov-process with respect to the filtration $\mathcal{F} = \mathcal{F}^{(\mathbf{W},\mathbf{H})}$, with the following generator acting on functions $v_i = v_i(t,x)$ with $x \in \mathbb{R}$

$$A^i_t v_i(t,x) \;=\; b_i(t,x)\partial_x v_i(t,x) + \frac{1}{2}\sigma_i^2(t,x)\partial^2_{x^2} v_i(t,x) \tag{11}$$

for suitable drift and diffusion coefficients $b_i(t,x)$ and $\sigma_i(t,x)$.

Under such a specification of the intensities, it is clear that dependence between defaults in the model will only stem from the possibility of common jumps, even with a factor process $\mathbf{X}$ (see Example 2.3 and the comment preceding it).

In the above assumption we require that $X^i_t = \varphi_i(\mathbf{X}_t)$ is a Markov process. This assumption is a non-trivial in general, as a process which is a (measurable) functional of a Markov process does not have to be a Markov process itself. We refer to Pitman and Rogers [32] for some discussion of this issue. In our model set-up one, one can show that under appropriate regularity conditions, if for every $(t,\mathbf{x},x)$ with $\mathbf{x} = (x_Y)_{Y\in\mathcal{Y}}$ and $x = \varphi_i(\mathbf{x})$, one has

$$\begin{aligned}\sum_{\{Y\in\mathcal{Y}\}} b_Y(t,\mathbf{x})\partial_{x_Y}\varphi_i(\mathbf{x}) &= b_i(t,x)\\ \sum_{\{Y\in\mathcal{Y}\}} \sigma_Y^2(t,\mathbf{x})(\partial_{x_Y}\varphi_i(\mathbf{x}))^2 &= \sigma_i^2(t,x)\end{aligned} \tag{12}$$

then $X_t^i = \varphi_i(\mathbf{X}_t)$ is an $\mathcal{F}$-Markov process with generator A^i in (10). The proof follows from Lemma A.2 (up to the reservation which is made right after the lemma regarding technicalities about the domain of the generators) since for every regular test-function $v_i = v_i(t,x)$, letting $u(t,\mathbf{x}) := v_i(t,\varphi^i(\mathbf{x}))$, one has

$$\begin{aligned}& v_i(t,X_t^i) - \int_0^t \big(\partial_s + A_s^i\big)v_i(s,X_s^i)ds \\ &\qquad = u(t,\mathbf{X}_t) - \int_0^t \big(\partial_s + A_s\big)u(s,\mathbf{X}_s)ds.\end{aligned}$$

In the two examples given below, the $\mathcal{F}$-Markov property of $X_t^i = \varphi_i(\mathbf{X}_t)$ also rigorously follows, in case of Example 2.5 where φ_i is a coordinate projection operator, from the Markov consistency results of [8], or, in case of Example 2.6, from the semimartingale representation of X^i provided by the SDE (14). The $\mathcal{F}$-Markov property of X^i in Example 2.6 thus follows from the fact that a strong solution to the Markovian SDE (14) driven by the $\mathcal{F}$-Brownian motion W^i, is an $\mathcal{F}$-Markov process, by application of Theorem 32 page 100 of Protter [31]. Example 2.6 is important, as it goes beyond the case of Example 2.5 where the λ_I-s are deterministic functions of time, and it goes towards the fully stochastic specifications of the λ_Y-s (including the λ_I-s).

Example 2.5 (Deterministic Group Intensities) For every group $I \in \mathcal{I}$, the intensity $\lambda_I(t,\mathbf{x})$ does not depend on $\mathbf{x}$.

Letting $\varphi_i(\mathbf{x}) = x_{\{i\}}$, then (10) and (12) hold with

$$\begin{aligned}\lambda_i(t,x) &:= \lambda_{\{i\}}(t,x) + \sum_{\{I\in\mathcal{I};\, i\in I\}} \lambda_I(t)\\ b_i(t,x) &:= b_{\{i\}}(t,x)\\ \sigma_i(t,x) &:= \sigma_{\{i\}}(t,x).\end{aligned}$$

So, $X^i = X^{\{i\}}$ is $\mathcal{F}$-Markov with drift and diffusion coefficients $b_i(t,x)$ and generator $\sigma_i(t,x)$ thus specified.

Example 2.6 (Extended CIR Intensities) For every $Y \in \mathcal{Y}$, the pre-specified group intensities are given by $\lambda_Y(t,X_t^Y) = X_t^Y$, where the factor X^Y is an extended CIR process

$$dX_t^Y = a(b_Y(t) - X_t^Y)dt + c\sqrt{X_t^Y}\,dW_t^Y \tag{13}$$

for non-negative constants a, c and non-negative functions $b_Y(t)$. The SDE-s for the factors X^Y-s have thus the same coefficients except for the $b_Y(t)$-s.

Letting $\varphi_i(\mathbf{x}) = \sum_{\{Y\in\mathcal{Y};\, i\in Y\}} x_Y = x_{\{i\}} + \sum_{\{I\in\mathcal{I};\, i\in I\}} x_I$, and denoting likewise $b_i(t) = \sum_{\{Y\in\mathcal{Y};\, i\in Y\}} b_Y(t) = b_{\{i\}}(t) + \sum_{\{I\in\mathcal{I};\, i\in I\}} b_I(t)$, then (10) and (12) hold with

$$\begin{aligned} \lambda_i(t,x) &:= x \\ b_i(t,x) &:= a(b_i(t) - x) \\ \sigma_i(t,x) &:= c\sqrt{x}. \end{aligned}$$

So, $X^i = \sum_{\{Y\in\mathcal{Y};\, i\in Y\}} X^Y$ is an $\mathcal{F}$-Markov process with drift and diffusion coefficients $b_i(t,x)$ and generator $\sigma_i(t,x)$ thus specified.

Note that X^i satisfies the following extended CIR SDE with parameters a, $b_i(t)$ and c as

$$dX^i_t = a(b_i(t) - X^i_t)dt + c\sqrt{X^i_t}dW^i_t \tag{14}$$

for the $\mathcal{F}$-Brownian motion W^i such that

$$\sqrt{X^i_t}dW^i_t = \sum_{i\in Y}\sqrt{X^Y_t}dW^Y_t,\;\; dW^i_t = \sum_{i\in Y}\frac{\sqrt{X^Y_t}}{\sqrt{\sum_{i\in Y}X^Y_t}}dW^Y_t.$$

Remark 2.7 Both the time-deterministic group intensities specification of Example 2.5 and the affine intensities specification of Example 2.6 have already been fruitfully used in the context of various credit and counterparty credit risk applications (anticipating the theoretical aspects of the model which are dealt with in the present paper), see [7, 2, 6].

For every $Y \in \mathcal{Y}$ and every set of non-negative constants t_i-s, we define the quantities $\Lambda^Y_{s,t}, \Lambda^Y_t$ and θ^Y_t as

$$\Lambda^Y_{s,t} = \int_s^t \lambda_Y(s, X^Y_s)ds\,,\;\; \Lambda^Y_t = \Lambda^Y_{0,t} = \int_0^t \lambda_Y(s, X^Y_s)ds \quad \text{and} \quad \theta^Y_t = \max_{i\in Y\cap \mathrm{supp}^c(\mathbf{H}_t)} t_i$$

where $Y \cap \mathrm{supp}^c(\mathbf{H}_t)$ in θ^Y_t is the set of survivors in Y at time t (and we use in θ^Y_t our convention that $\max\emptyset = -\infty$). Note that Λ^I is a deterministic function of time for every group $I \in \mathcal{I}$. Let τ_i denote the default time for obligor i. Since H^i is the default indicator of name i, we have

$$\tau_i = \inf\{t > 0\,; H^i_t = 1\},\;\; H^i_t = \mathbb{1}_{\{\tau_i \le t\}}.$$

The following Proposition gathers the Markov properties of the model.

Proposition 2.8 **(i)** $(\mathbf{X}, \mathbf{H})$ *is an* $\mathcal{F}$*-Markov process with infinitesimal generator given by* $\mathcal{A}$.
(ii) *For every obligor* i, (X^i, H^i) *is an* $\mathcal{F}$*-Markov process*[3] *admitting the following generator acting on functions* $u_i = u_i(t, x_i, k_i)$ *with* $(x_i, k_i) \in \mathbb{R}\times\{0,1\}$

$$\begin{aligned} \mathcal{A}^i_t u_i(t,x_i,k_i) &= b_i(t,x_i)\partial_{x_i}u_i(t,x_i,k_i) + \frac{1}{2}\sigma^2_i(t,x_i)\partial^2_{x_i^2}u_i(t,x_i,k_i) \\ &\quad + \lambda_i(t,x_i)\big(u_i(t,x_i,1) - u_i(t,x_i,k_i)\big). \end{aligned} \tag{15}$$

[3]And hence an $\mathcal{F}^{(X^i,H^i)}$-Markov process.

Moreover, the $\mathcal{F}$-intensity process[4] *of H^i is given by $\mathbb{1}_{\{\tau_i>t\}}\lambda_i(t,X^i_t)$. In other words, the process M^i defined by*

$$M^i_t = \mathbb{1}_{\{\tau_i \le t\}} - \int_0^t \mathbb{1}_{\{\tau_i>s\}}\lambda_i(s,X^i_s)ds, \tag{16}$$

is an $\mathcal{F}$-martingale.[5]

(iii) *For any fixed non-negative constants $t, t_1, \ldots, t_n$, one has*

$$\begin{aligned}&\mathbb{P}\left(\tau_1 > t_1, \ldots, \tau_n > t_n \mid \mathcal{F}_t\right) \\ &\quad = \mathbb{1}_{\{t_i<\tau_i\,,\ i\in supp(\mathbf{H}_t)\}}\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}}\Lambda^Y_{t,\theta^Y_t}\big)\,\Big|\,\mathbf{X}_t\right\}.\end{aligned} \tag{17}$$

The conditional survival probability function of every obligor i is given by, for every $t_i \ge t$,

$$\begin{aligned}\mathbb{P}(\tau_i > t_i \mid \mathcal{F}_t) &= \mathbb{1}_{\{\tau_i>t\}}\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y},\,i\in Y}\Lambda^Y_{t,t_i}\big)\,\Big|\,\mathbf{X}_t\right\} \\ &= \mathbb{1}_{\{\tau_i>t\}}\mathbb{E}\left\{\exp\big(-\int_t^{t_i}\lambda_i(s,X^i_s)ds\big)\mid X^i_t\right\} \\ &= \mathbb{1}_{\{\tau_i>t\}}G^i_t(t_i),\end{aligned} \tag{18}$$

with

$$G^i_t(t_i) = \mathbb{E}\left\{\exp\big(-\int_t^{t_i}\lambda_i(s,X^i_s)ds\big)\mid X^i_t\right\}.$$

In particular

$$\mathbb{E}\left\{\exp\big(-\Lambda^{\{i\}}_t\big)\right\} = \exp\left\{-\left(\Gamma_i(t) - \sum_{i\in I}\Lambda^I_t\right)\right\}, \tag{19}$$

where $\Gamma_i(t) = -\ln G^i_0(t) = -\ln(\mathbb{P}(\tau_i > t))$ is the hazard function of name i.

Proof. See Appendix A.2.1.

We shall illustrate part (iii) of the above proposition using the following example.

Example 2.9 In case of two obligors and $\mathcal{Y} = \{\{1\},\{2\},\{1,2\}\}$, one can easily check that (17) boils down to

$$\begin{aligned}\mathbb{P}\left(\tau_1 > t_1, \tau_2 > t_2 \mid \mathcal{F}_t\right) = \mathbb{1}_{\{\tau_1>t\}}\mathbb{1}_{\{\tau_2>t\}}\mathbb{E}&\left\{\exp\big(-\sum_{Y\in\mathcal{Y}}\int_t^{t_1\vee t_2}\lambda_Y(s,\mathbf{X}_s)\big)\,\Big|\,\mathbf{X}_t\right\} \\ +\mathbb{1}_{\{t_2<\tau_2\le t\}}\mathbb{1}_{\{\tau_1>t\}}\mathbb{E}&\left\{\exp\big(-\int_t^{t_1}\lambda_1(s,X^1_s)\,ds\big)\,\Big|\,X^1_t\right\} \\ +\mathbb{1}_{\{t_1<\tau_1\le t\}}\mathbb{1}_{\{\tau_2>t\}}\mathbb{E}&\left\{\exp\big(-\int_t^{t_2}\lambda_2(s,X^2_s)\,ds\big)\,\Big|\,X^2_t\right\} \\ &+\mathbb{1}_{\{t_1<\tau_1\le t\}}\mathbb{1}_{\{t_2<\tau_2\le t\}}.\end{aligned}$$

[4] And hence, $\mathcal{F}^{(X^i,H^i)}$-intensity process.

[5] And hence, an $\mathcal{F}^{(X^i,H^i)}$-martingale.

2.3 Common Shocks Model Interpretation

In this subsection we establish a connection between the dynamic Markovian model $(\mathbf{X}, \mathbf{H})$, and a common shock model with a Marshall-Olkin common factor structure of default times as in Lindskog and McNeil [29], Elouerkhaoui [19] or Brigo et al. [11, 12].

In rough terms, conditionally on any given state $(\mathbf{x}, \mathbf{k})$ of $(\mathbf{X}, \mathbf{H})$ at time t, it is possible to define a common shock model of default times of the surviving names at time t, such that the conditional law of the default times in the common shock model is the same as the corresponding conditional distribution in the original Markov model. This connection between the Markovian model and the common shock framework is the main theoretical contribution of this paper.

We thus introduce a family of common shocks copula models, parameterized by the current time t. For every $Y \in \mathcal{Y}$, we define

$$\tau_Y(t) = \inf\{\theta > t;\, \Lambda_\theta^Y > \Lambda_t^Y + E_Y\},$$

where the random variables E_Y-s are i.i.d. and exponentially distributed with parameter 1. For every obligor i we let

$$\tau_i(t) = \min_{\{Y \in \mathcal{Y};\, i \in Y\}} \tau_Y(t)\ , \tag{20}$$

which defines the default time of obligor i in the common shocks copula model starting at time t. We also introduce the common shock model indicator processes $H_\theta^Y(t) = \mathbb{1}_{\{\tau_Y(t) \le \theta\}}$ and $H_\theta^i(t) = \mathbb{1}_{\{\tau_i(t) \le \theta\}}$, for every triggering-event Y, obligor i and time horizon $\theta \ge t$. Let $Z \in \mathcal{N}_n$ denote a set of obligors, meant in the probabilistic interpretation to represent the set $\mathrm{supp}^c(\mathbf{H}_t)$ of survived obligors in the Markov model at time t. We now prove that on $\{\mathrm{supp}^c(\mathbf{H}_t) = Z\}$, the conditional law of $(\tau_i)_{i \in \mathrm{supp}^c(\mathbf{H}_t)}$ given $\mathcal{F}_t$ in the Markov model, is equal to the conditional law of $(\tau_i(t))_{i \in Z}$ given $\mathbf{X}_t$ in the common shocks framework. Let also $N_\theta = \sum_{1 \le i \le n} H_\theta^i$ denote the cumulative number of defaulted obligors in the Markov model up to time θ. Let $N_\theta(t, Z) = n - |Z| + \sum_{i \in Z} H_\theta^i(t)$, denote the cumulative number of defaulted obligors in the common shocks framework up to time θ where $|Z|$ is the cardinality of the set Z.

Proposition 2.10 *Let $Z \in \mathcal{N}_n$ denote an arbitrary subset of obligors and let $t \ge 0$. Then,* **(i)** *for every $t_1, \ldots, t_n \ge t$, one has*

$$\mathbb{1}_{\{supp^c(\mathbf{H}_t) = Z\}} \mathbb{P}\left(\tau_i > t_i,\, i \in supp^c(\mathbf{H}_t) \,\middle|\, \mathcal{F}_t\right) = \mathbb{1}_{\{supp^c(\mathbf{H}_t) = Z\}} \mathbb{P}\left(\tau_i(t) > t_i,\, i \in Z \,\middle|\, \mathbf{X}_t\right). \tag{21}$$

(ii) *for every $\theta \ge t$, one has that for every $k = n - |Z|, \ldots, n$,*

$$\mathbb{1}_{\{supp^c(\mathbf{H}_t) = Z\}} \mathbb{P}\left(N_\theta = k \,\middle|\, \mathcal{F}_t\right) = \mathbb{1}_{\{supp^c(\mathbf{H}_t) = Z\}} \mathbb{P}\left(N_\theta(t, Z) = k \,\middle|\, \mathbf{X}_t\right).$$

Proof. Part (ii) readily follows from part (i), that we now show. Let, for every obligor i, $\tilde{t}_i = \mathbb{1}_{i \in \mathrm{supp}^c(\mathbf{H}_t)} t_i$. Note that one has, for $Y \in \mathcal{Y}$

$$\max_{i \in Y \cap \mathrm{supp}^c(\mathbf{H}_t)} \tilde{t}_i = \max_{i \in Y \cap \mathrm{supp}^c(\mathbf{H}_t)} t_i = \theta_t^Y.$$

Thus, by application of identity (17) in Proposition 2.8 to the sequence of times $(\tilde{t}_i)_{1\le i\le n}$, it comes,

$$
\begin{aligned}
&\mathbb{1}_{\{\mathrm{supp}^c(\mathbf{H}_t)=Z\}}\mathbb{P}\left(\tau_i > t_i,\, i\in \mathrm{supp}^c(\mathbf{H}_t)\,\middle|\, \mathcal{F}_t\right)\\
&= \mathbb{1}_{\{\mathrm{supp}^c(\mathbf{H}_t)=Z\}}\mathbb{P}\left(\left(\tau_i > t_i,\, i\in Z\right),\, \left(\tau_i > 0,\, i\in Z^c\right)\,\middle|\, \mathcal{F}_t\right)\\
&= \mathbb{1}_{\{\mathrm{supp}^c(\mathbf{H}_t)=Z\}}\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}}\Lambda^Y_{t,\theta^Y_t}\big)\,\middle|\,\mathbf{X}_t\right\}
\end{aligned}
$$

which on $\{\mathrm{supp}^c(\mathbf{H}_t)=Z\}$ coincides with the expression

$$
\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}}\Lambda^Y_{t,\max_{i\in Y\cap Z} t_i}\big)\,\middle|\,\mathbf{X}_t\right\}
$$

derived for $\mathbb{P}(\tau_i(t) > t_i,\, i\in Z \,\big|\, \mathbf{X}_t)$ in the common shocks model of Elouerkhaoui [19]. □

For instance, in the situation of Example 2.3, the shock interpretation at time $t = 0$ is clear: there are five different shocks, corresponding to the elements of $\mathcal{Y}$. In particular, obligors 1 and 2 can default simultaneously if either the shock corresponding to $\{1, 2\}$ arrives, or the shock corresponding to $\{1, 2, 3\}$ arrives.

This interpretation will be used in the next section for deriving fast exact convolution recursion procedures for pricing portfolio loss derivatives.

The common shocks interpretation can also be used for simulation purposes. In view of Proposition 2.10(i) and given $\mathcal{F}_t$, the simulation of the random times $(\tau_i)_{i\in\mathrm{supp}^c(\mathbf{H}_t)}$, or equivalently on $\{\mathrm{supp}^c(\mathbf{H}_t) = Z\}$, $(\tau_i(t))_{i\in Z}$, is fast. One essentially needs to simulate IID exponential random variables E_Y-s.

3 Pricing, Calibration and Hedging Issues

This section treats the pricing, calibration and hedging issues in the Markov copula model of Section 2. First, in Subsection 3.1 we derive the price dynamics for CDS contracts and for CDO tranches in this model. In Subsection 3.2 we use dynamics of Subsection 3.1 to derive min-variance hedging strategies in the Markov copula model. In the case of CDO-s these formulas lead to a very large PDE-system which in practice is difficult to solve. So, in Subsection 3.3 we instead exploit the relationship between our Markov model and the common shock model, which enables us to derive fast, deterministic, computationally tractable algorithms for derivation of the prices and sensitivities.

For notational convenience, we assume zero interest rates. The extension of all theoretical results to time dependent, deterministic interest rates is straightforward but more cumbersome notationally, especially regarding hedging. Time-dependent deterministic interest rates will be used in the numerical part.

3.1 Pricing Equations

In this subsection we derive price dynamics formulas for CDS contracts and CDO tranches in the Markov model; all prices are considered from perspective of the protection buyers. These dynamics will be useful when deriving the min-variance hedging strategies in Subsection 3.2.

In a zero interest-rates environment, the (ex-dividend) price process of an asset is simply given by the risk neutral conditional expectation of future cash flows associated with the asset; the cumulative value process is the sum of the price process and of the cumulative cash-flows process. The cumulative value process is a martingale, as opposed to the price process. When it comes to hedging, the cumulative value process is the main quantity of interest (see for instance Frey and Backhaus [21]).

For a fixed maturity T, we let S_i denote the T-year CDS spread for obligor i, with recovery rate R_i. Similarly, we let S denote the T-year model CDO tranche spread for the tranche $[a,b]$, with payoff process

$$L_t^{a,b} = L_{a,b}(\mathbf{H}_t) = (L_t - a)^+ - (L_t - b)^+ \,, \tag{22}$$

where $L_t = \frac{1}{n}\sum_{i=1}^n (1-R_i)H_t^i$ is the credit loss process for the underlying portfolio. The premium legs in these products are payed at $t_1 < t_2 < \ldots < t_p = T$ where $t_j - t_{j-1} = \mathrm{h}$ and h is typically a quarter. Below, the notation is the same as in the Itô formula (6).

Proposition 3.1 **(i)** *The price P^i and the cumulative value $\widehat{P}^i$ at time $t \in [0,T]$ of the single-name CDS on obligor i with contractual spread S_i are given by*

$$\begin{aligned} P_t^i &= \mathbb{1}_{\{\tau_i > t\}} v_i(t, X_t^i) \\ d\widehat{P}_t^i &= \mathbb{1}_{\{\tau_i > t\}} \partial_{x_i} v_i(t, X_t^i)\sigma_i(t, X_t^i) dW_t^i \;+ \sum_{Z \in \mathcal{Z}_t} \mathbf{1}_{i \in Z} \left(1 - R_i - v_i(t, X_t^i)\right) dM_t^Z \end{aligned} \tag{23}$$

for a pre-default pricing function $v_i(t,x_i)$ such that

$$\mathbb{1}_{\{\tau_i > t\}} v_i(t, X_t^i) = \mathbb{E}[-S_i \mathrm{h} \sum_{t < t_j \le T} \mathbb{1}_{\{\tau_i > t_j\}} \;+\; (1 - R_i)\mathbb{1}_{\{t < \tau_i \le T\}} | \mathcal{F}_t].$$

(ii) *The price process Π and cumulative value $\widehat{\Pi}$ at time $t \in [0,T]$ of a CDO tranche $[a,b]$ with contractual spread S are given by*

$$\begin{aligned} \Pi_t &= u(t, \mathbf{X}_t, \mathbf{H}_t) \\ d\widehat{\Pi}_t &= \nabla u(t, \mathbf{X}_t, \mathbf{H}_t)\sigma(t, \mathbf{X}_t) d\mathbf{W}_t \\ &\qquad + \sum_{Z \in \mathcal{Z}_t} \Big(L_{a,b}(\mathbf{H}_{t-}^Z) - L_{a,b}(\mathbf{H}_{t-}) + \delta u^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) \Big) dM_t^Z \end{aligned} \tag{24}$$

for a pricing function $u(t,\mathbf{x},\mathbf{k})$ such that

$$u(t, \mathbf{X}_t, \mathbf{H}_t) = \mathbb{E}\Big[-S\,\mathrm{h} \sum_{t < t_j \le T} \Big(b - a - L_{t_j}^{a,b} \Big) \;+\; L_T^{a,b} - L_t^{a,b} \Big| \mathcal{F}_t \Big].$$

Proof. See Appendix A.2.2.

Note that in view of the marginal Markov property of the model, the martingale representation (23) of $\widehat{P}^i$ can be reduced to a "univariate" martingale representation based on the compensated martingale M^i of H^i in (16). However, as will be clear from Subsection 3.2, it is more useful to state martingale representations of $\widehat{\Pi}$ and $\widehat{P}^i$ relatively to a common set of fundamental martingales in order to handle the hedging issue.

The pricing functions v_i and u can be characterized as the unique solutions to the related Kolmogorov equations (54) and (56) in Appendix A.2.2. If the pricing functions are known, the prices at a given time are recovered by plugging the corresponding state of the model into the right-hand-side of the first lines of (23) or (24). The pricing equation (56) for a CDO tranche leads to a large system of PDEs which in practice is impossible to handle numerically as soon as n is larger than a few units. As a remedy for this we will in Subsection 3.3 instead use the translation to a Marshall-Olkin framework which allows us to derive practical recursive pricing schemes for CDO tranche price processes.

3.2 Min-Variance Hedging

In this subsection we use the price dynamics from Subsection 3.1 to derive min-variance hedging strategies in the Markov copula model. By min-variance hedging strategies we mean strategies that minimize the variance of the hedging error. Note that one could also try to minimize the variance relatively to the historical probability measure, however in this paper we minimize the risk-neutral variance for simplicity: see Schweizer [33] for a survey about various quadratic hedging approaches. The hedging strategies are theoretically sound due to our bottom-up Markovian framework and they will be shown in Subsection 3.3 to be computationally tractable thanks to the Marshall-Olkin copula interpretation of Subsection 2.3.

Consider a CDO tranche $[a, b]$ with pricing function u specified in Proposition 3.1. Our aim is to find explicit min-variance hedging formulas when hedging this CDO tranche by using the savings account and d single-name CDS-s with pricing functions v_i given by Proposition 3.1. First we introduce the CDS cumulative value vector-function

$$\mathbf{v}(t, \mathbf{x}, \mathbf{k}) = (\mathbb{1}_{k_1=0} v_1(t, x_1) + \mathbb{1}_{k_1=1}(1 - R_1), \ldots, \mathbb{1}_{k_d=0} v_d(t, x_d) + \mathbb{1}_{k_d=1}(1 - R_d))^{\mathsf{T}} .$$

Let $\nabla \mathbf{v}$ denote the Jacobian matrix of $\mathbf{v}$ with respect to $\mathbf{x}$ in the sense of the $d \times \nu$-matrix such that $\nabla \mathbf{v}(t, \mathbf{x}, \mathbf{k})_i^j = \mathbb{1}_{k_j=0} \partial_{x_j} v_i(t, x_i)$, for every $1 \le i \le d$ and $1 \le j \le \nu$. Let $\Delta \mathbf{v}^Z$ represent the vector-function of the sensitivities of v with respect to the event $Z \in \mathcal{N}_n$, so

$$\Delta \mathbf{v}^Z(t, \mathbf{x}, \mathbf{k}) = (\mathbb{1}_{1 \in Z,\, k_1=0} \left((1 - R_1) - v_1(t, x_1) \right), \ldots, \mathbb{1}_{d \in Z,\, k_d=0} \left((1 - R_d) - v_d(t, x_d) \right))^{\mathsf{T}}.$$

By using the vector notation $\widehat{\mathbf{P}} = (\widehat{P}^i)_{1 \le i \le d}$, one has in view of Proposition 3.1(i)

$$d\widehat{\mathbf{P}}_t = \nabla \mathbf{v}(t, \mathbf{X}_t, \mathbf{H}_t) \sigma(t, \mathbf{X}_t) d\mathbf{W}_t + \sum_{Z \in \mathcal{Z}_t} \Delta \mathbf{v}^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) dM_t^Z. \tag{25}$$

Let

$$\Delta u^Z(t, \mathbf{x}, \mathbf{k}) = \delta^Z u(t, \mathbf{x}, \mathbf{k}) + L_{a,b}(\mathbf{k}^Z) - L_{a,b}(\mathbf{k}).$$

represent the function of sensitivity of the CDO tranche $[a, b]$ cumulative value process with respect to the event $Z \in \mathcal{N}_n$. Let ζ be an d-dimensional row-vector process, representing the number of units held in the first d CDS-s which are used in a self-financing[6] hedging strategy for the CDO tranche $[a, b]$. Given (24) and (25), the tracking error (e_t) of the hedged portfolio satisfies $e_0 = 0$ and, for $t \in [0, T]$

$$\begin{aligned} de_t &= d\widehat{\Pi}_t - \zeta_t d\widehat{\mathbf{P}}_t \\ &= \Big(\nabla u(t, \mathbf{X}_t, \mathbf{H}_t) - \zeta_t \nabla \mathbf{v}(t, \mathbf{X}_t, \mathbf{H}_t) \Big) \sigma(t, \mathbf{X}_t) d\mathbf{W}_t \\ &\quad + \sum_{Z \in \mathcal{Z}_t} \Big(\Delta u^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) - \zeta_t \Delta \mathbf{v}^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) \Big) dM_t^Z. \end{aligned} \tag{26}$$

[6] Using also the savings account (constant asset).

Since the martingale dimension of the model is $\nu + 2^n$, replication is typically out-of-reach[7] in the Markov model. However, in view of (26), we still can find min-variance hedging formulas.

Proposition 3.2 *The min-variance hedging strategy ζ is*

$$\zeta_t = \frac{d\langle \widehat{\Pi}, \widehat{\mathbf{P}} \rangle_t}{dt} \left(\frac{d\langle \widehat{\mathbf{P}} \rangle_t}{dt} \right)^{-1} = \zeta(t, \mathbf{X}_t, \mathbf{H}_{t-}) \tag{27}$$

where $\zeta = (u, \mathbf{v})(\mathbf{v}, \mathbf{v})^{-1}$, with

$$\begin{aligned} (u, \mathbf{v}) &= (\nabla u)\sigma^2(\nabla \mathbf{v})^\mathsf{T} + \sum_{Y \in \mathcal{Y}} \lambda_Y \Delta u^Y (\Delta \mathbf{v}^Y)^\mathsf{T} \\ (\mathbf{v}, \mathbf{v}) &= (\nabla \mathbf{v})\sigma^2(\nabla \mathbf{v})^\mathsf{T} + \sum_{Y \in \mathcal{Y}} \lambda_Y \Delta \mathbf{v}^Y (\Delta \mathbf{v}^Y)^\mathsf{T}. \end{aligned} \tag{28}$$

Proof. The first identity in (27) is a classical risk neutral min-variance hedging[8] formula, derived for instance in Section 3.5 of Crépey [16]. Moreover, one has by computation of the oblique brackets based on the second lines in (23) and (24):

$$\begin{aligned} \frac{d\langle \widehat{\Pi}, \widehat{\mathbf{P}} \rangle_t}{dt} &= \left((\nabla u)\sigma^2(\nabla \mathbf{v})^\mathsf{T} + \sum_{Z \in \mathcal{Z}_t} \lambda_Z \Delta u^Z (\Delta \mathbf{v}^Z)^\mathsf{T} \right)(t, \mathbf{X}_t, \mathbf{H}_{t-}) = (u, \mathbf{v})(t, \mathbf{X}_t, \mathbf{H}_{t-}) \\ \frac{d\langle \widehat{\mathbf{P}} \rangle_t}{dt} &= \left((\nabla \mathbf{v})\sigma^2(\nabla \mathbf{v})^\mathsf{T} + \sum_{Z \in \mathcal{Z}_t} \lambda_Z \Delta \mathbf{v}^Z (\Delta \mathbf{v}^Z)^\mathsf{T} \right)(t, \mathbf{X}_t, \mathbf{H}_{t-}) = (\mathbf{v}, \mathbf{v})(t, \mathbf{X}_t, \mathbf{H}_{t-}) \end{aligned} \tag{29}$$

where the second identities in both lines of (29) use simplifications similar to those used in the proof of the Itô formula (6) in Appendix A.1.2. □

In (28), the u-related terms can be computed by using the conditional convolution-recursion procedures presented in Subsection 3.3; the v_i-related terms can be computed very quickly (actually semi-explicitly in either of the specifications of examples 2.5 and 2.6). We will illustrate in Subsection 4.4 the tractability of this approach for computing min-variance hedging deltas.

We refer the reader to Elouerkhaoui [19] for analogous results in the context of the common shock model presented in Subsection 2.3. A nice feature of our set-up however is that due to the specific structure of the intensities, the sums in (28) are over the set $\mathcal{Y}$ of triggering-events $\mathcal{Y}$ which is of cardinality $\nu = n + m$ rather than over the set $\mathcal{N}_n$ of all set-events Z, which would be of cardinality 2^n.

We also refer the reader to Frey and Backhaus [21] for other related min-variance hedging formulas.

3.3 Convolution Recursion Pricing Schemes

In this subsection we use the common shock model interpretation to derive fast convolution recursion algorithms for computing the portfolio loss distribution. In the case where the recovery rate is the same for all names, i.e., $R_i = R$, $i = 1, \ldots, n$, the aggregate loss L_t at

[7]See the comments following Proposition 2.2.
[8]See Schweizer [33]

time t is equal to $(1-R)N_t$, where we recall N_t is the total number of defaults that have occurred in the Markov model up to time t. It is well known, see, e.g., [13, 21, 23, 28], and Proposition 3.1(ii), that the price process for a CDO tranche $[a, b]$ is determined by the probabilities $\mathbb{P}[N_\theta = k \,|\, \mathcal{F}_t]$ for $k = |\mathbf{H}_t|, \ldots, n$ and $\theta \geq t \geq 0$. Thanks to the common shock model interpretation of Subsection 2.3, one has from Proposition 2.10(ii) that

$$\mathbb{P}[N_\theta = k \,|\, \mathcal{F}_t] = \mathbb{P}[N_\theta(t, Z) = k \,|\, \mathbf{X}_t]$$

on the event $\{\mathrm{supp}^c(\mathbf{H}_t) = Z\}$, so we will focus on computation of the latter probabilities, which are derived in formula (31) below. Furthermore, recall that $\mathrm{supp}^c(\mathbf{H}_t)$ denotes the obligors who have survived in state $\mathbf{H}_t$ at time t.

We henceforth assume a nested structure of the sets I_j-s given by

$$I_1 \subset \ldots \subset I_m. \tag{30}$$

This structure implies that if all obligors in group I_k have defaulted, then all obligors in group $I_1, \ldots, I_{k-1}$ have also defaulted. The nested structure (30) is important since it implies a particularly tractable expression for the portfolio loss distribution. Denoting conventionally $I_0 = \emptyset$ and $H_\theta^{I_0}(t) = 1$, then in view of (30), the events

$$\Omega_\theta^j(t) := \{H_\theta^{I_j}(t) = 1, H_\theta^{I_{j+1}}(t) = 0, \ldots, H_\theta^{I_m}(t) = 0\}, \ \ 0 \leq j \leq m$$

form a partition of Ω. Hence, we have

$$\mathbb{P}(N_\theta(t, Z) = k \,|\, \mathbf{X}_t) = \sum_{0 \leq j \leq m} \mathbb{P}\big(N_\theta(t, Z) = k \,|\, \Omega_\theta^j(t), \mathbf{X}_t\big)\mathbb{P}\big(\Omega_\theta^j(t) \,|\, \mathbf{X}_t\big) \tag{31}$$

where, by construction of the $H_\theta^I(t)$-s and independence of the $\lambda_I(t, X_t^I)$-s we have

$$\mathbb{P}\big(\Omega_\theta^j(t) \,|\, \mathbf{X}_t\big) = \Big(1 - \mathbb{E}\big(e^{-\Lambda_{t,\theta}^{I_j}} \,|\, X_t^{I_j}\big)\Big) \prod_{j+1 \leq l \leq m} \mathbb{E}\big(e^{-\Lambda_{t,\theta}^{I_l}} \,|\, X_t^{I_l}\big) \tag{32}$$

which in our Markov setup can be computed very quickly (actually, semi-explicitly in either of the specifications of examples 2.5 and 2.6). We now turn to the computation of the term

$$\mathbb{P}\big(N_\theta(t, Z) = k \,|\, \Omega_\theta^j(t), \mathbf{X}_t\big) \tag{33}$$

appearing in (31). Recall first that $N_\theta(t, Z) = n - |Z| + \sum_{i \in Z} H_\theta^i(t)$ with $|Z|$ denoting the cardinality of Z. We know that for every group $j = 1, \ldots, m$, given $\Omega_\theta^j(t)$, the marginal default indicators $H_\theta^i(t)$ for $i \in Z$ are such that:

$$H_\theta^i(t) = \begin{cases} 1, & i \in I_j, \\ H_\theta^{\{i\}}(t), & \text{else.} \end{cases} \tag{34}$$

Hence, the $H_\theta^i(t)$-s are conditionally independent given $\Omega_\theta^j(t)$. Finally, conditionally on $(\Omega_\theta^j(t), \mathbf{X}_t)$ the random vector $\mathbf{H}_\theta(t) = (H_\theta^i(t))_{i \in \mathbb{N}_n}$ is a vector of independent Bernoulli random variables with parameter $p = (p_\theta^{i,j}(t))_{i \in \mathbb{N}_n}$, where

$$p_\theta^{i,j}(t) = \begin{cases} 1, & i \in I_j, \\ 1 - \mathbb{E}\Big\{\exp\big(-\Lambda_{t,\theta}^{\{i\}}\big) \,|\, X_t^{\{i\}}\Big\} = 1 - \dfrac{G_t^i(\theta)}{\prod_{i \in I} \mathbb{E}\Big(e^{-\sum_{i \in I} \Lambda_{t,\theta}^I} \,\Big|\, X_t^I\Big)}, & \text{else} \end{cases} \tag{35}$$

in which $G_t^i(\theta)$ is given by (18). The conditional probability (33) can therefore be computed by a standard convolution recursive procedure (see, for instance, Andersen and Sidenius [1]).

Remark 3.3 The linear number of terms in the sum of (31) is due to the nested structure of the groups I_j-s in (31). Note that the convolution recursion procedure works for arbitrary structuring of the groups I_j-s. However, a general structuring of the m groups I_j-s would imply 2^m terms instead of m in the sum of (31), which in practice would only work for very few groups m. The nested structure (31) of the I_j-s, or equivalently, the tranched structure of the $I_j \setminus I_{j-1}$-s, is also quite natural from the point of view of application to CDO tranches.

4 Numerical Results

In this section we briefly discuss the calibration of the model and some few numerical results connected to the loss-distributions and the min-variance hedging. Subsection 4.1 outlines the calibration methodology while Subsection 4.2 presents the numerical calibration of the Markov copula model against market data. Furthermore, in Subsection 4.3 we study the implied loss-distributions in our fitted model. Finally, Subsection 4.4 discusses the min-variance hedging strategies in the calibrated models using constant recoveries. A more extensive numerical study of the model can be found in the paper [5]. In the present paper we do not use a factor process $\mathbf{X}$ in the individual or joint default intensities. For examples of uses of $\mathbf{X}$ we refer to [5, 7]. Throughout this paper we consider constant recoveries. The case with random recoveries is treated separately in [5] since including stochastic recoveries in the model requires a substantial theoretical treatment as well as extended studies of the numerical implementation issues.

4.1 Calibration methodology

In this subsection we discuss the calibration methodology that will be used when fitting the Markov copula model against CDO tranches on the iTraxx Europe and CDX.NA.IG series in Subsection 4.2.

The calibration of the Markov copula model is in general a three-stages process when including the factor process $\mathbf{X}$. In this paper we do not use a factor process $\mathbf{X}$ so the calibration methodology will then reduce to a two-step procedure (see [5] for the general three-stage calibration with a factor process $\mathbf{X}$ included).

The first step is to calibrate the single-name CDS for every obligor. Given the T-year market CDS spread S_i^* for obligor we want to find the individual default parameters for obligor i so that $P_0^i(S_i^*) = 0$, or in view of Proposition 3.1(i),

$$S_i^* = \frac{(1 - R_i)\mathbb{P}\left(\tau_i < T\right)}{\mathrm{h}\sum_{0<t_j\leq T}\mathbb{P}\left(\tau_i > t_j\right)} \tag{36}$$

where we used the facts that interest rate is zero and that the recovery R_i is constant. Hence, the first step is to extract the implied hazard function $\Gamma_i^*(t)$ in (19) from the CDS curve of every obligor i by using a standard bootstrapping procedure based on (36).

Given the marginal hazard functions, the law of the total number of defaults N at a fixed time t is a function of the joint default intensity functions $\lambda_I(t)$, as described by the recursive algorithm of Subsection 3.3. The second step is therefore to calibrate the common-shock intensities $\{\lambda_I(t)\}$ so that the model CDO tranche spreads coincide with the corresponding market spreads. This is done by using the recursive algorithm of Subsection 3.3, for $\lambda_I(t)$-s parameterized as non-negative and piecewise constant functions of time.

Moreover, in view of (19), for every obligor i and at each time t we impose the constraint

$$\sum_{I\in\mathcal{I};\, i\in I} \lambda_I(t) \leq \lambda_i^*(t) \tag{37}$$

where $\lambda_i^* := \frac{d\Gamma_i^*}{dt}$ denotes the hazard rate (or hazard intensity) of name i. For constant joint default intensities $\lambda_I(t) = \lambda_I$ the constraints (37) reduce to

$$\sum_{I\ni i} \lambda_I \leq \underline{\lambda}_i := \inf_{t\in[0,T]} \lambda_i^*(t) \quad \text{for every obligor } i.$$

Given the nested structure of the groups I_j-s specified in (30), this is equivalent to

$$\sum_{j=l}^{m} \lambda_{I_j} \leq \underline{\lambda}_{I_l} := \min_{i\in I_l\setminus I_{l-1}} \underline{\lambda}_i \text{ for every group } l. \tag{38}$$

Furthermore, for piecewise constant common shock intensities on a time grid (T_k), the condition (38) extends to the following constraint

$$\sum_{j=l}^{m} \lambda_{I_j}^k \leq \underline{\lambda}_{I_l}^k := \min_{i\in I_l\setminus I_{l-1}} \underline{\lambda}_i^k \quad \text{for every } l,k \quad \text{where } \underline{\lambda}_i^k := \inf_{t\in[T_{k-1},T_k]} \lambda_i^*(t). \tag{39}$$

We remark that insisting on calibrating all CDS names in the portfolio, including the safest ones, implies via (38) or (39) a very constrained region for the common shock parameters. This region can be expanded by relaxing the system of constraints for the joint default intensities, by excluding the safest CDS-s from the calibration.

In this paper we will use a time grid consisting of two maturities T_1 and T_2. Hence, the single-name CDS-s constituting the entities in the credit portfolio are bootstrapped from their market spreads for $T = T_1$ and $T = T_2$. This is done by using piecewise constant individual default intensity λ_i-s on the time intervals $[0, T_1]$ and $[T_1, T_2]$.

Before we leave this subsection, we give some few more details on the calibration of the common shock intensities for the m groups in the second calibration step. From now on we assume that the joint default intensities $\{\lambda_{I_j}(t)\}_{j=1}^m$ are piecewise constant functions of time, so that $\lambda_{I_j}(t) = \lambda_{I_j}^{(1)}$ for $t \in [0, T_1]$ and $\lambda_{I_j}(t) = \lambda_{I_j}^{(2)}$ for $t \in [T_1, T_2]$ and for every group j. Next, the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k} = \{\lambda_{I_j}^{(k)} : j = 1, \dots, m \text{ and } k = 1, 2\}$ are then calibrated so that the five-year model spread $S_{a_l,b_l}(\boldsymbol{\lambda}) =: S_l(\boldsymbol{\lambda})$ will coincide with the corresponding market spread S_l^* for each tranche l. To be more specific, the parameters $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ are obtained according to

$$\boldsymbol{\lambda} = \operatorname*{argmin}_{\widehat{\boldsymbol{\lambda}}} \sum_l \left(\frac{S_l(\widehat{\boldsymbol{\lambda}}) - S_l^*}{S_l^*}\right)^2$$

under the constraints that all elements in $\boldsymbol{\lambda}$ are nonnegative and that $\boldsymbol{\lambda}$ satisfies the inequalities (39) for every group I_l and in each time interval $[T_{k-1}, T_k]$ where $T_0 = 0$. In $S_l(\widehat{\boldsymbol{\lambda}})$ we have emphasized that the model spread for tranche l is a function of $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ but we suppressed the dependence in other parameters like interest rate, payment frequency or λ_i, $i = 1, \dots, n$. In the calibration we used an interest rate of 3%, the payments in the premium leg were quarterly and the integral in the default leg was discretized on a quarterly mesh. For each data-set we use a constant recovery of 40%.

4.2 Calibration Results

In this subsection we calibrate our model against CDO tranches on the iTraxx Europe and CDX.NA.IG series with maturity of five years. We use the calibration methodology described in the previous subsection.

Hence, the 125 single-name CDS-s constituting the entities in these series are bootstrapped from their market spreads for $T_1 = 3$ and $T_2 = 5$ using piecewise constant individual default intensities on the time intervals $[0, 3]$ and $[3, 5]$. Figure 1 displays the 3 and 5-year market CDS spreads for the 125 obligors used in the single-name bootstrapping, for the two portfolios CDX.NA.IG sampled on December 17, 2007 and the iTraxx Europe series sampled on March 31, 2008. The CDS spreads are sorted in decreasing order.

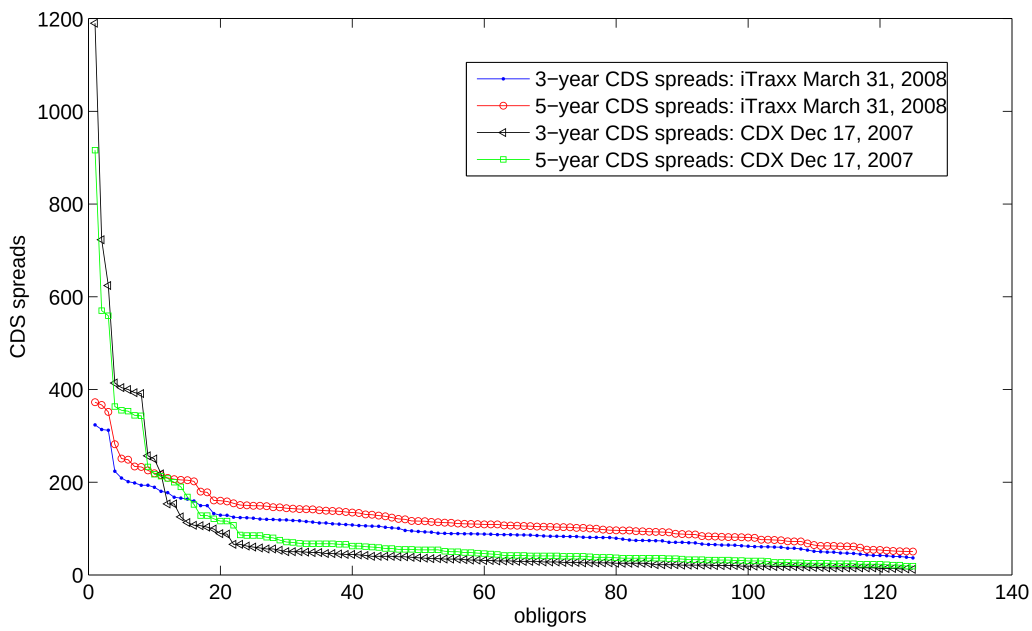


Figure 1: The 3 and 5-year market CDS spreads for the 125 obligors used in the single-name bootstrapping, for the two portfolios CDX.NA.IG sampled on December 17, 2007 and the iTraxx Europe series sampled on March 31, 2008. The CDS spreads are sorted in decreasing order.

When calibrating the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ for the CDX.NA.IG Series 9, December 17, 2007 we used 5 groups $I_1, I_2, \ldots, I_5$ where $I_j = \{1, \ldots, i_j\}$ for $i_j = 6, 19, 25, 61, 125$. Recall that we label the obligors by decreasing level of riskiness. We use the average over 3-year and 5-year CDS spreads as a measure of riskiness. Consequently, obligor 1 has the highest average CDS spread while company 125 has the lowest average CDS spread. Moreover, the obligors in the set $I_5 \setminus I_4$ consisting of the 64 safest companies are assumed to never default individually, and the corresponding CDS-s are excluded from the calibration, which in turn relaxes the constraints for $\boldsymbol{\lambda}$ in (39). Hence, the obligors in $I_5 \setminus I_4$ can only bankrupt due to a simultaneous default of the companies in the group $I_5 = \{1, \ldots, 125\}$, i.e., in an Armageddon event. With this structure the calibration against the December 17, 2007 data-set is very good as can be seen in Table 1. By using stochastic recoveries we can get a perfect fit of the same data-set and we refer the reader to [5] for more on this issue.

Table 1: CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0, 3]$ spread is quoted in %. All maturities are for five years.

CDX 2007-12-17

Tranche	$[0,3]$	$[3,7]$	$[7,10]$	$[10,15]$	$[15,30]$
Market spread	48.07	254.0	124.0	61.00	41.00
Model spread	48.07	254.0	124.0	61.00	38.94
Absolute error in bp	0.010	0.000	0.000	0.000	2.061
Relative error in %	0.0001	0.000	0.000	0.000	5.027

iTraxx Europe 2008-03-31

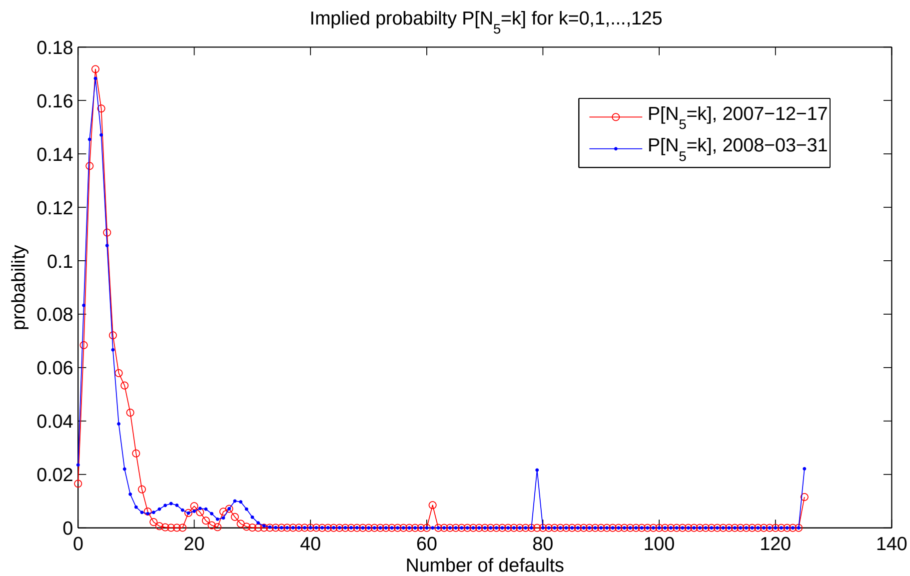
Tranche	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
Market spread	40.15	479.5	309.5	215.1	109.4
Model spread	41.67	429.8	309.4	215.1	103.9
Absolute error in bp	152.5	49.69	0.0359	0.0348	5.592
Relative error in %	3.798	10.36	0.0116	0.0162	5.109

The calibration of the joint default intensities $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ for the data sampled at March 31, 2008 is more demanding. This time we use 19 groups $I_1, I_2, \ldots, I_{19}$ where $I_j = \{1, \ldots, i_j\}$ for $i_j = 1, 2, \ldots, 15, 19, 25, 79, 125$. In order to improve the fit, as in the 2007-case, we relax the constraints for $\boldsymbol{\lambda}$ in (39) by excluding from the calibration the CDS-s corresponding to the obligors in $I_{19} \setminus I_{18}$. Hence, we assume that the obligors in $I_{19} \setminus I_{18}$ never default individually, but can only bankrupt due to an simultaneous default of all companies in the group $I_{19} = \{1, \ldots, 125\}$. In this setting, the calibration of the 2008 data-set with constant recoveries yields an acceptable fit except for the $[3, 6]$ tranche, as can be seen in Table 1. However, by including stochastic recoveries this fit can be substantially improved and we refer the reader to [5] about this.

4.3 The implied loss distribution

After the fit of the model against market spreads we can use the calibrated portfolio parameters $\boldsymbol{\lambda} = (\lambda_{I_j}^{(k)})_{j,k}$ together with the calibrated individual default intensities, to study the credit-loss distribution in the portfolio. In this paper we only focus on some few examples derived from the loss distribution with constant recoveries evaluated at $T = 5$ years.

The allowance of joint defaults of the obligors in the groups I_j-s together with the restriction of the most safest obligors not being able to default individually, will lead to some interesting effects of the loss distribution, as can be seen in Figures 2 and 3. For example, we clearly see that the support of the loss-distributions will in practice be limited to a rather compact set. To be more specific, the upper and lower graphs in Figure 2 indicate that $\mathbb{P}[N_5 = k]$ roughly has support on the set $\{1, \ldots, 35\} \cup \{61\} \cup \{125\}$ for the 2007 case and on $\{1, \ldots, 40\} \cup \{79\} \cup \{125\}$ for the 2008 data-set. This becomes even more clear in a log-loss distribution, as is seen in the upper and lower graphs in Figure 3.



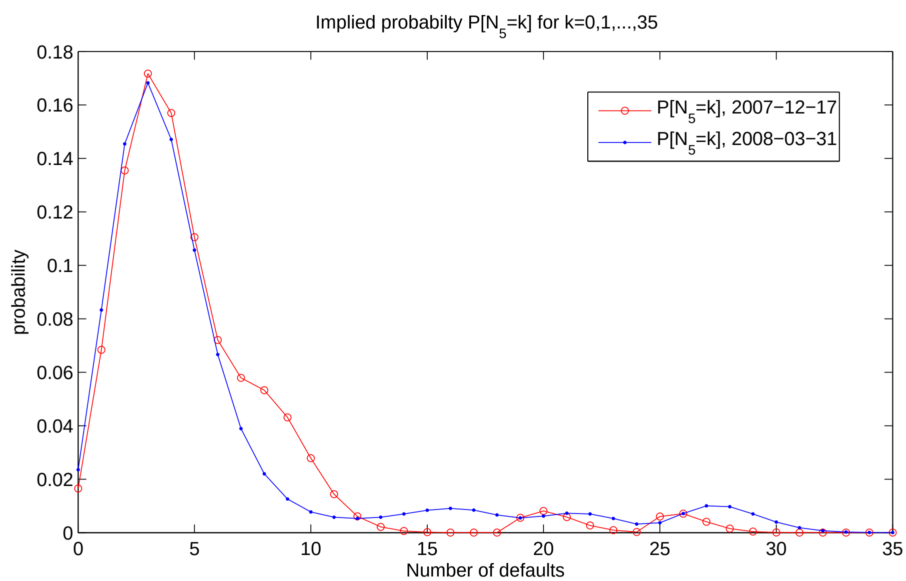


Figure 2: The implied distribution $\mathbb{P}[N_5 = k]$ on $\{0, 1, \ldots, \ell\}$ where $\ell = 125$ (top) and $\ell = 35$ (bottom) when the model is calibrated against CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008.

From the upper graph in Figure 3 we see that the default-distribution is nonzero on $\{36, \ldots, 61\}$ in the 2007-case and nonzero on $\{41, \ldots, 79\}$ for the 2008-sample, but the actual size of the loss-probabilities are in the range 10^{-10} to 10^{-70}. Such low values will obviously be treated as zero in any practically relevant computation. Furthermore, the reasons for the empty gap in the upper graph in Figure 3 on the interval $\{62, \ldots, 124\}$ for the 2007-case is due to the fact that we forced the obligors in the set $I_5 \setminus I_4$ to never default individually, but only due to an simultaneous common shock default of the companies in the group $I_5 = \{1, \ldots, 125\}$. This Armageddon event is displayed as an isolated nonzero 'dot' at default nr 125 in the upper graph of Figure 3. The gap on $\{80, \ldots, 124\}$ in the 2008 case is explained similarly due to our assumption on the companies in the set $I_{19} \setminus I_{18}$. Also note that the two 'dots' at default nr 125 in the top plot of Figure 3 are manifested as spikes in the upper graph displayed in Figure 2. The shape of the multimodal loss distributions presented in Figure 2 and Figure 3 are typical for models allowing simultaneous defaults, see for example Figure 2, page 59 in [12] and Figure 2, page 710 in [19].

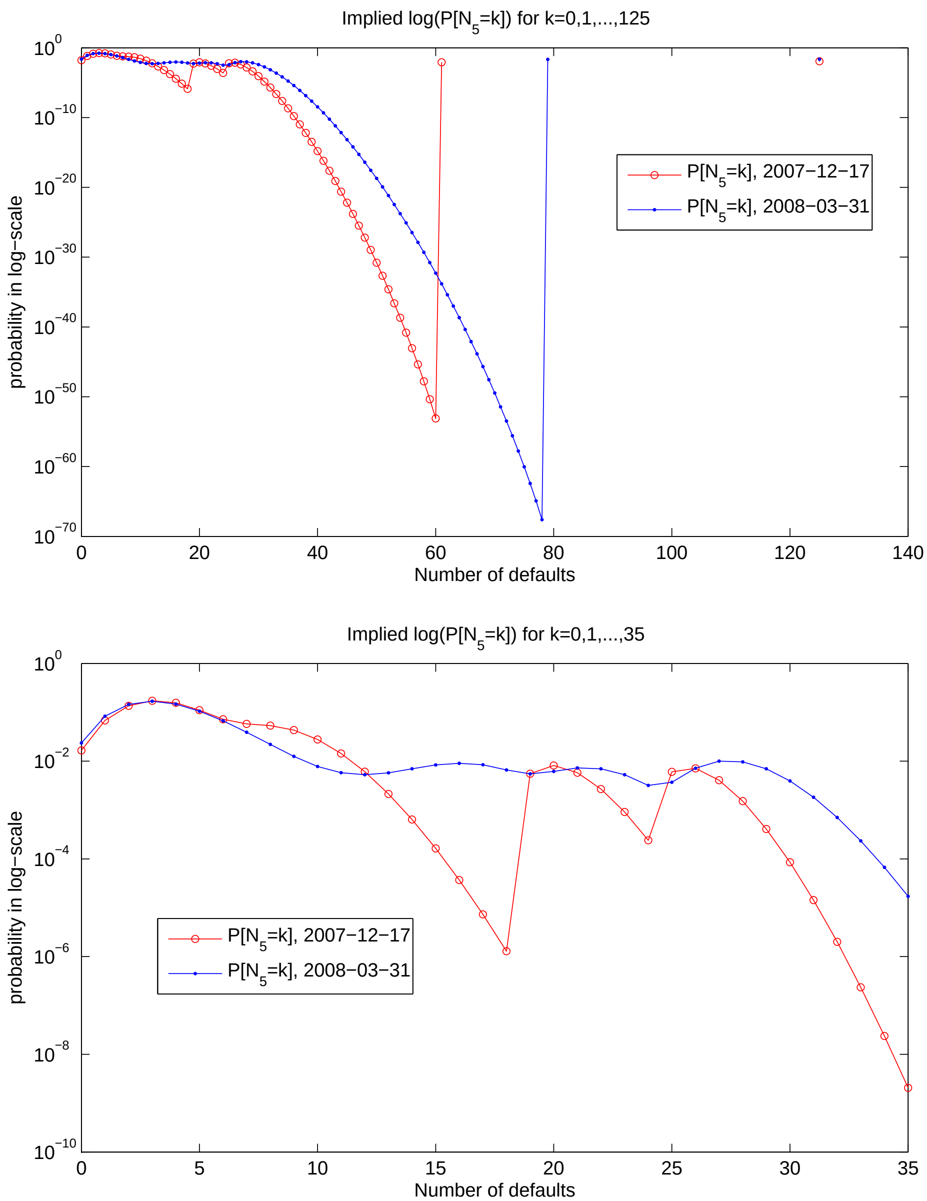


Figure 3: The implied log distribution $\ln(\mathbb{P}[N_5 = k])$ on $\{0, 1, \ldots, \ell\}$ where $\ell = 125$ (top) and $\ell = 35$ (bottom) when the model is calibrated against CDX.NA.IG Series 9, December 17, 2007 and iTraxx Europe Series 9, March 31, 2008.

4.4 Min-variance hedging strategies

In this subsection we present some numerical results illustrating performance of the min-variance hedging strategies given in Proposition 3.2. We here only focus on hedging strategies for the data of CDX.NA.IG Series 9 on December 17, 2007 calibrated in the constant recovery model presented in Subsection 4.1. A more extensive numerical study of the min-variance hedging strategies and related issues can be found in the paper [5].

The aim of this subsection is to analyze the composition of the hedging portfolio at time $t = 0$ (the calibration date) when standardized CDO tranches are hedged with a group of d single-name CDS-s, which are included in the underlying CDS index. Since no spread factor $\mathbf{X}$ is used in the model, Proposition 3.2 then implies that the min-variance hedging ratios at time $t = 0$ is given by $\zeta^{va}(0, \mathbf{H}_0) = (u, \mathbf{v})(\mathbf{v}, \mathbf{v})^{-1}(0, \mathbf{H}_0)$ where

$$(u, \mathbf{v}) = \sum_{Y \in \mathcal{Y}} \lambda_Y(0) \Delta u^Y (\Delta \mathbf{v}^Y)^\mathsf{T} \quad \text{and} \quad (\mathbf{v}, \mathbf{v}) = \sum_{Y \in \mathcal{Y}} \lambda_Y(0) \Delta \mathbf{v}^Y (\Delta \mathbf{v}^Y)^\mathsf{T}.$$

Hence, computing the min-variance hedging ratios involves a summation of the "jump differentials" $\lambda_Y(0)\Delta u^Y(\Delta \mathbf{v}^Y)^\mathsf{T}$ and $\lambda_Y(0)\Delta \mathbf{v}^Y(\Delta \mathbf{v}^Y)^\mathsf{T}$ over all possible triggering events $Y \in \mathcal{Y}$ where $\mathcal{Y} = \{\{1\}, \ldots, \{n\}, I_1, \ldots, I_m\}$.

In the calibration of the CDX.NA.IG Series 9, we used $m = 5$ groups $I_1, I_2, \ldots, I_5$ where $I_j = \{1, \ldots, i_j\}$ for $i_j = 6, 19, 25, 61, 125$ and the obligors have been labeled by decreasing level of riskiness. At the calibration date $t = 0$ associated with December 17, 2007, no name has defaulted in CDX Series 9 so we set $\mathbf{H}_0 = \mathbf{0}$. In our empirical framework, the intensities $\lambda_Y(0)$, $Y \in \mathcal{Y}$ are computed from the constant default intensities λ_i-s that fit market spreads of 3-year maturity CDS-s and from the 3-year horizon joint default intensities λ_{I_j}-s calibrated to CDO tranche quotes. The terms $\Delta u^Y(0, \mathbf{H}_0)$ and $\Delta \mathbf{v}^Y(0, \mathbf{H}_0)$ corresponds to the change in value of the tranche and the single-name CDS-s, at the arrival of the triggering event affecting all names in group Y. Recall that the cumulative change in value of the tranche is equal to

$$\Delta u^Y(0, \mathbf{H}_0) = L_{a,b}(\mathbf{H}_0^Y) - L_{a,b}(\mathbf{H}_0) + u(0, \mathbf{H}_0^Y) - u(0, \mathbf{H}_0)$$

where $\mathbf{H}_0^Y$ is the vector of $\{0,1\}^n$ such that only the components $i \in Y$ are equal to one. Hence, the tranche sensitivity $\Delta u^Y(0, \mathbf{H}_0)$ includes both the protection payment on the tranche associated with the default of group Y and the change in the ex-dividend price u of the tranche. Note that the price sensitivity is obtained by computing the change in the present value of the default leg and the premium leg. The latter quantity involves the contractual spread that defines cash-flows on the premium leg. As for CDX.NA.IG Series 9, the contractual spreads were equal to 500 bps, 130 bps, 45 bps, 25 bps and 15 bps for the tranches [0-3%], [3-7%], [7-10%], [10-15%] and [15-30%]. We use the common-shock interpretation to compute $u(0, \mathbf{H}_0^Y)$ and $u(0, \mathbf{H}_0)$ with the convolution recursion pricing scheme detailed in Subsection 3.3. More precisely, using the same notation as in Subsection 3.3, the CDO tranche price $u(0, \mathbf{H}_0^Y)$ (resp. $u(0, \mathbf{H}_0)$) is computed using the recursion procedure with $Z = \mathbb{N}_n \setminus Y$ (resp. $Z = \mathbb{N}_n$). We let $i_1, \ldots i_d$ be the CDS-s used in the min-variance hedging and assume that they all are initiated at time $t = 0$. Hence, the market value at $t = 0$ for these CDS-s are zero. As a result, when group Y defaults simultaneously, the change in value $\Delta \mathbf{v}^Y(0, \mathbf{H}_0)$ for buy-protection positions on these CDS-s is only due to protection payment associated with names in group Y. Hence, for one unit of nominal exposure on hedging CDS-s, the corresponding vector of sensitivities is equal to $\Delta \mathbf{v}^Y(0, \mathbf{H}_0) = ((1-R)\mathbb{1}_{i_1 \in Y}, \ldots, (1-R)\mathbb{1}_{i_d \in Y})^\mathsf{T}$ where the recovery rate R is assumed to be constant and equal to 40%.

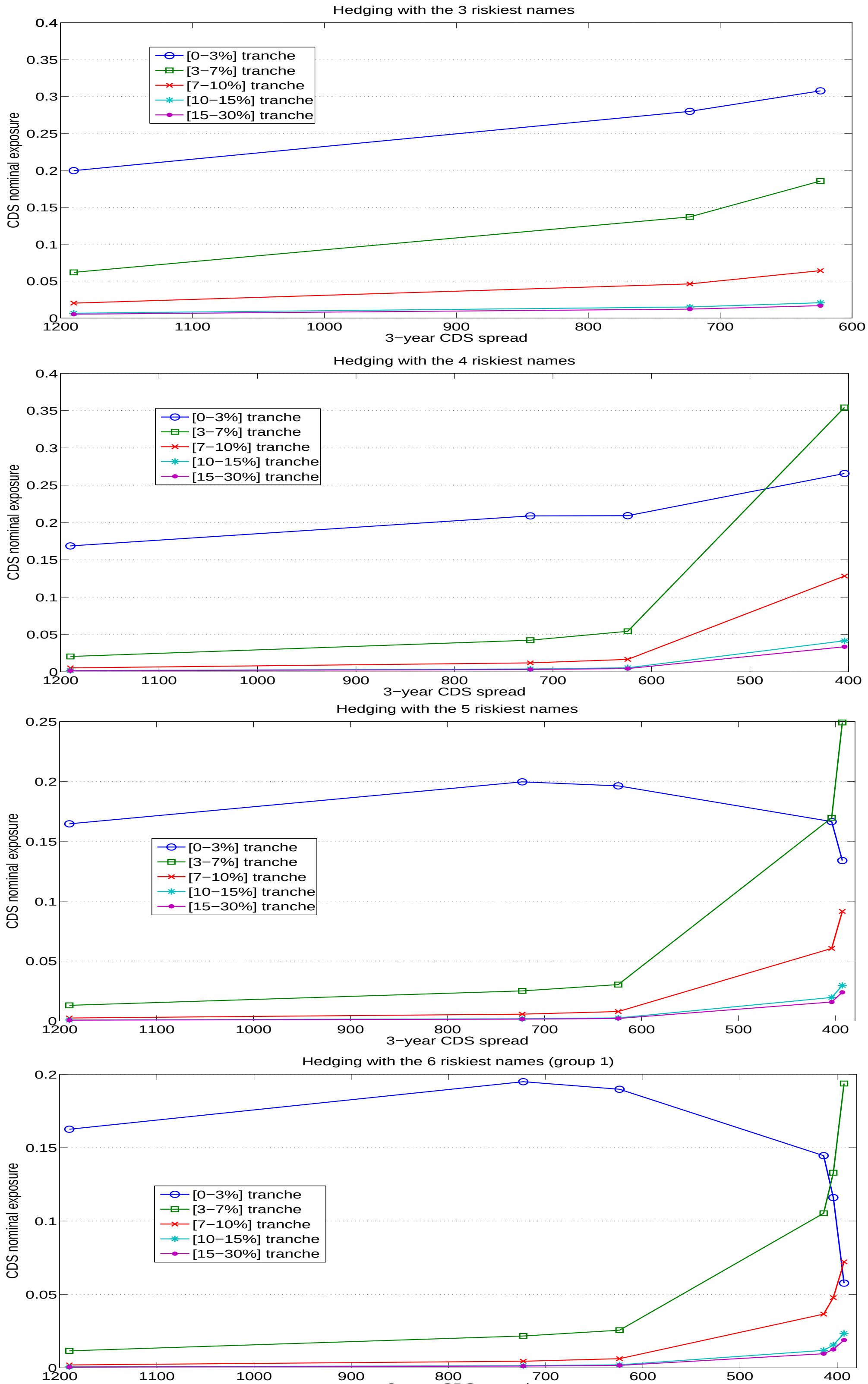


Figure 4: Min-variance hedging strategies associated with the d riskiest CDS-s, $d = 3, 4, 5, 6$ for one unit of nominal exposure of different CDO tranches in a model calibrated to market spreads of CDX.NA.IG Series 9 on December 17, 2007.

Table 2: The names and CDS spreads (in bp) of the six riskiest obligors used in the hedging strategy displayed by Figure 4.

Company (Ticker)	CCR-HomeLoans	RDN	LEN	SFI	PHM	CTX
3-year CDS spread	1190	723	624	414	404	393

Figure 4 displays the nominal exposure for the d most riskiest CDS-s when hedging one unit of nominal exposure in a CDO by using the min-variance hedging strategy in Proposition 3.2. We use $d = 3, 4, 5$ and $d = 6$ in our computations. Furthermore, Table 2 displays the names and sizes of the 3-year CDS spreads used in the hedging strategy. Each plot in Figure 4 should be interpreted as follows: in every pair (x, y) the x-component represents the size of the 3-year CDS spread at the hedging time $t = 0$ while the y-component is the corresponding nominal CDS-exposure computed via Proposition 3.2 using the d riskiest CDS-s. The graphs are ordered from top to bottom, where the top panel corresponds to hedging with the $d = 3$ riskiest CDS and the bottom panel corresponds to hedging with the $d = 6$ riskiest names. Note that the x-axes are displayed from the riskiest obligor to the safest. Thus, hedge-sizes y for riskier CDS-s are aligned to the left in each plot while y-values for safer CDS-s are consequently displayed more to the right. In doing this, going from the top to the bottom panel consists in observing the effect of including new safer names from the right part of the graphs. We have connected the pairs (x, y) with lines forming graphs that visualizes possible trends of the min-variance hedging strategies for the d most riskiest CDS-s.

For example, when the three riskiest names are used for hedging (top panel), we observe that the amount of nominal exposure in hedging instruments decreases with the degree of subordination, i.e., the [0-3%] equity tranche requires more nominal exposure in CDS-s than the upper tranches. Note moreover that the min-variance hedging portfolio contains more CDS-s on names with lower spreads. When lower-spread CDS-s are added in the portfolio, the picture remains almost the same for the 3 riskiest names. For the remaining safer names however, the picture depends on the characteristics of the tranche. For the [0-3%] equity tranche, the quantity of the remaining CDS-s required for hedging sharply decrease as additional safer names are added. One possible explanation is that adding too many names in the hedging strategy will be useless when hedging the equity tranche. This is intuitively clear since one expects that the most riskiest obligors will default first and consequently reduce the equity tranche substantially, explaining the higher hedge-ratios for riskier names, while it is less likely that the more safer names will default first and thus incur losses on the first tranche which explains the lower hedge ratios for the safer names. We observe the opposite trend for the senior (safer) tranches: adding new (safer) names in the hedging portfolio seems to be useful for "non equity" tranches since the nominal exposure required for these names increases when they are successively added.

Figure 5 and 6 display min-variance hedging strategies when hedging a standard tranche with the 61 riskiest names, i.e., all names excepted names in group $I_5 \setminus I_4$. Contrary to Figure 4, these graphs allow to visualize the effect of the "grouping structure" on the composition of the hedging portfolio. In this respect, we use different marker styles in order to distinguish names in the different disjoint groups I_1, $I_2 \setminus I_1$, $I_3 \setminus I_2$, $I_4 \setminus I_3$. As one can see, the min-variance hedging strategies are quite different among tranches. Moreover, whereas nominal exposures required for hedging are clearly monotone for names belonging to the same disjoint group, this tendency is broken when we consider names in different groups. This suggests that the grouping structure has a substantial impact on the distribution of names in the hedging portfolio. For the equity tranche, we observe in Figure 4 that less safer-names are required for hedging. This feature is retained in Figure 5 when we look at names in specific disjoint groups. Indeed, names in a given disjoint group are affected by the same common-shocks which in turn affect the equity tranche with the same severity. The only effect that may explain differences in nominal exposure among names in the same disjoint group is spontaneous defaults: names with wider spreads are more likely to default first, then we need them in greater quantity for hedging than names with tighter spreads.

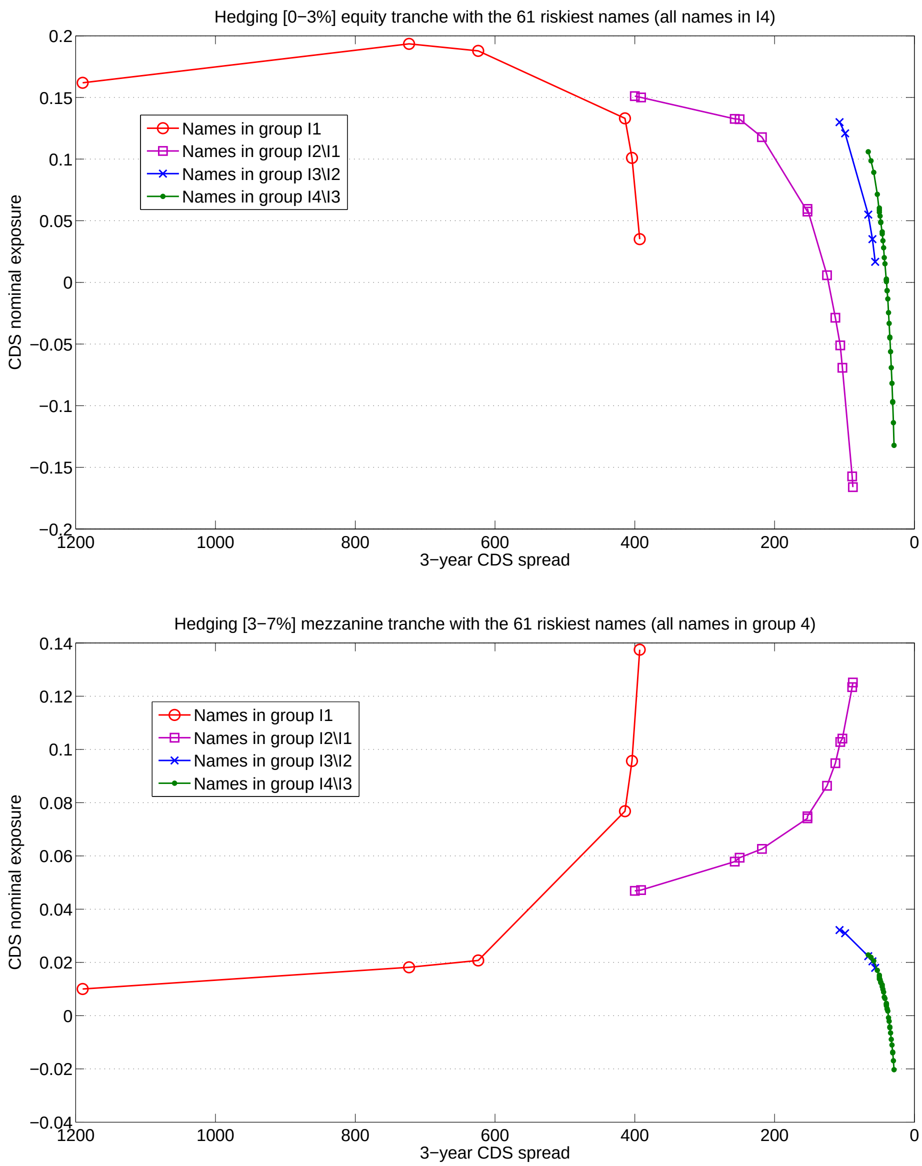


Figure 5: Min-variance hedging strategies when hedging one unit of nominal exposure in the [0-3%] equity tranche (top) and the [3-7%] mezzanine tranche (bottom) using the d riskiest CDS-s, $d = 61$ (all names excepted names in group $I_5 \setminus I_4$) for one unit of nominal exposure.

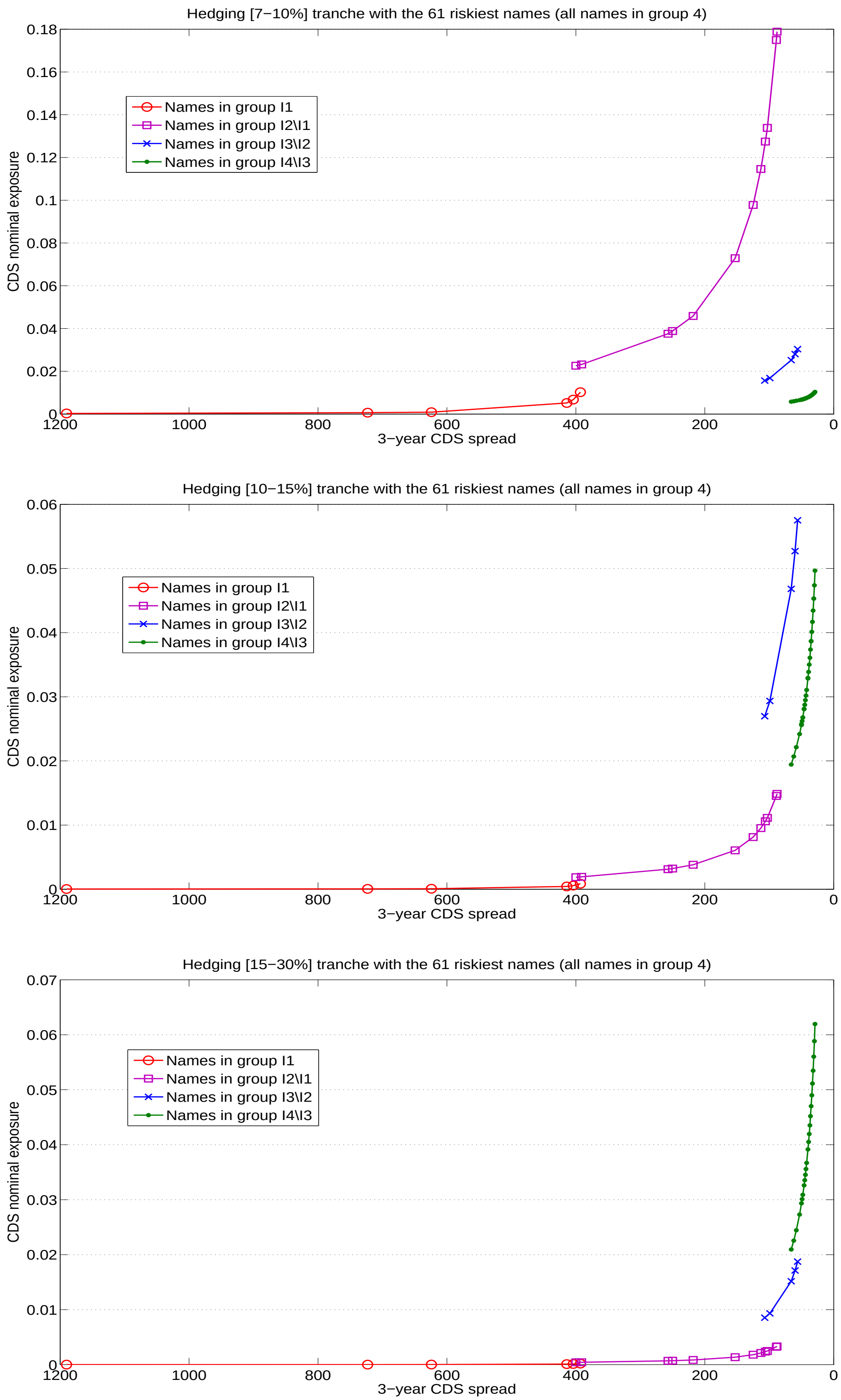


Figure 6: Min-variance hedging strategies when hedging one unit of nominal exposure in the [7-10%] tranche (top), the [10-15%] tranche (middle) and the [15-30%] tranche (bottom) with the d riskiest CDS-s, $d = 61$ (all names excepted names in group $I_5 \setminus I_4$).

Note that nominal exposure in hedging CDS even becomes negative for names within groups $I_2 \setminus I_1$ and $I_4 \setminus I_3$ when spreads are low. However, in Figure 5 we observe that, for the equity tranche, some of the riskiest names in $I_4 \setminus I_3$ are more useful in the hedging than some of the safest names in group I_1, which may sound strange at a first glance, given that the credit spread of the latter is much larger than the credit spread of the former. Recall that the equity tranche triggers protection payments corresponding to the few first defaults, if these occur before maturity. Even if names in group $I_4 \setminus I_3$ have a very low default probability, the fact that they can affect the tranche at the arrival of common-shocks I_4 or I_5 makes these names appealing for hedging because they are less costly (they require less premium payments) than names in I_1.

Figure 5 suggests that names with the lowest spreads should be ineffective to hedge the [0-3%] and the [3-7%] tranches. As can be seen in Figure 6, this is the contrary for the other tranches, i.e., the amount of low-spread names in the hedging portfolio increases as the tranche becomes less and less risky. For the [15-30%] super-senior tranche, we can see on the lowest graph of Figure 6 that the safer a name is, the larger the quantity which is required for hedging. Furthermore, Figure 6 also shows that in a consistent dynamic model of portfolio credit risk calibrated to a real data set, the [15-30%] super-senior tranche has significant (in fact, most of its) sensitivity to very safe names with spreads less than a few dozens of bp-s. For this tranche it is actually likely that one could improve the hedge by inclusion of even safer names to the set of hedging instruments, provided these additional names could also be calibrated to. Recall that on the data of CDX.NA.IG Series 9 on December 17, 2007, we calibrated our model to the 64 safest names in the portfolio.

A Appendix

A.1 Model Construction

The point process $\mathbf{H}$ with intensity depending on the factor process $\mathbf{X}$ in (3), is constructed by an $\mathbf{X}$-related change of probability measure, starting from an independent continuous-time Markov chain. So, given a factor process $\mathbf{X}$ as in (1), let $\mathbf{H}$ denote a continuous-time Markov chain with $\widehat{\mathbb{P}}$-intensity one of transition from $\mathbf{k}$ to $\mathbf{l}$, for every $\mathbf{l} \neq \mathbf{k}$. Let then the $\widehat{\mathbb{P}}$-martingale[9] Γ be defined by $\Gamma_0 = 1$ and, for $t \in [0, T]$,

$$\begin{aligned}\frac{d\Gamma_t}{\Gamma_{t-}} &= \sum_{l \in \{0,1\}^n} (\lambda(t, \mathbf{X}_t, \mathbf{H}_{t-}, \mathbf{l}) - 1) \left(dN_t(\mathbf{H}_{t-}, \mathbf{l}) - \mathbb{1}_{\mathbf{l} \neq \mathbf{H}_{t-}} dt \right) \\ &= \sum_{\mathbf{l} \neq \mathbf{H}_{t-}} (\lambda(t, \mathbf{X}_t, \mathbf{H}_{t-}, \mathbf{l}) - 1) \left(dN_t(\mathbf{H}_{t-}, \mathbf{l}) - dt \right),\end{aligned}$$

where the functions $\lambda(t, \mathbf{x}, \mathbf{k}, \mathbf{l})$-s are those of (3), and where $N_t(\mathbf{k}, \mathbf{l})$ is the point process with $\widehat{\mathbb{P}}$-intensity $\mathbb{1}_{\{\mathbf{k}=\mathbf{H}_{t-}, \mathbf{l} \neq \mathbf{k}\}}$ counting the transitions of $\mathbf{H}$ from $\mathbf{k}$ to $\mathbf{l}$, for every $\mathbf{k}, \mathbf{l} \in \{0,1\}^n$. Defining the measure $\mathbb{P}$ by $\frac{d\mathbb{P}}{d\widehat{\mathbb{P}}} = \Gamma_T$, it is then standard to check[10] that the point process $\mathbf{H}$ has intensity (3) under $\mathbb{P}$. To be precise the intensity of $N_t(\mathbf{k}, \mathbf{l})$ is given by (3), with respect to the model filtration $\mathcal{F} = \mathcal{F}^{(\mathbf{W}, \mathbf{H})}$, and the probability measure $\mathbb{P}$. Moreover, process $\mathbf{W}$

[9] Under suitable regularity and growth assumptions on the model coefficients, see Ethier and Kurtz [20] or Crépey [16].

[10] See for instance the proof of Lemma 8 in Crépey [16], or Lemma 7.5 in the online pre-print version of [16].

remains a Brownian motion under $\mathbb{P}$, the measure-change preserves Markov property of $\mathbf{X}$ with respect to filtration $\mathcal{F}$, and the generator of $\mathbf{X}$ under the new measure is still A_t.

Note that since martingale representation holds under $\widehat{\mathbb{P}}$,[11] martingale representation also holds under the equivalent measure $\mathbb{P}$.

Remark A.1 The prevailing risk neutral probability measure in the paper is $\mathbb{P}$, whereas the auxiliary measure $\widehat{\mathbb{P}}$ is only a mathematical tool used for constructing the model, with no particular financial interpretation.

A.1.1 Proof of Lemma 2.1

By definition of the set-event indicator process H^Z, where $Z \in \mathcal{N}_n$, one has in our model, for $t \in [0, T]$,

$$dH_t^Z = \sum_{\{\mathbf{k},\mathbf{l} \in \{0,1\}^n\,;\, \mathrm{supp}(\mathbf{l}) \setminus \mathrm{supp}(\mathbf{k}) = Z\}} dN_t(\mathbf{k}, \mathbf{l}).$$

So, by (3),

$$\begin{aligned}
\ell_t^Z &= \sum_{\{\mathbf{k},\mathbf{l} \in \{0,1\}^n\,;\, \mathrm{supp}(\mathbf{l}) \setminus \mathrm{supp}(\mathbf{k}) = Z\}} \mathbb{1}_{\{\mathbf{H}_{t-} = \mathbf{k}\}} \sum_{\{Y \in \mathcal{Y};\, \mathbf{k}^Y = \mathbf{l}\}} \lambda_Y(t, X_t^Y) \\
&= \sum_{\{\mathbf{l} \in \{0,1\}^n\,;\, \mathrm{supp}(\mathbf{l}) \setminus \mathrm{supp}(\mathbf{H}_{t-}) = Z\}} \sum_{\{Y \in \mathcal{Y};\, \mathbf{H}_{t-}^Y = \mathbf{l}\}} \lambda_Y(t, X_t^Y) \\
&= \sum_{\{Y \in \mathcal{Y};\, \mathrm{supp}(\mathbf{H}_{t-}^Y) \setminus \mathrm{supp}(\mathbf{H}_{t-}) = Z\}} \lambda_Y(t, X_t^Y) \\
&= \sum_{\{Y \in \mathcal{Y};\, Y_t = Z\}} \lambda_Y(t, X_t^Y).
\end{aligned}$$

A.1.2 Proof of Proposition 2.2

Observe that $[M^Y, M^Z] = 0$ for $Y \neq Z$. One thus has the following Itô formula (see for instance Theorem 3.89 page 109 of Jacod [26] or Crépey [16])

$$\begin{aligned}
du(t, \mathbf{X}_t, \mathbf{H}_t) = \Big(\partial_t + \mathcal{A}_t^c\Big) u(t, \mathbf{X}_t, \mathbf{H}_t) dt \quad + \nabla u(t, \mathbf{X}_t, \mathbf{H}_t) \sigma(t, \mathbf{X}_t) d\mathbf{W}_t \\
+ \sum_{Z \in \mathcal{N}_n} \delta u^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) dH_t^Z
\end{aligned} \tag{40}$$

where

$$\mathcal{A}_t^c u(t, \mathbf{x}, \mathbf{k}) = \sum_{Y \in \mathcal{Y}} \Big(b_Y(t, x_Y) \partial_{x_Y} u(t, \mathbf{x}, \mathbf{k}) + \frac{1}{2} \sigma_Y^2(t, x_Y) \partial_{x_Y^2}^2 u(t, \mathbf{x}, \mathbf{k}) \Big). \tag{41}$$

Moreover, the structure (4) of the set intensities implies that

$$\sum_{Z \in \mathcal{N}_n} \delta u^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) dH_t^Z = \sum_{Z \in \mathcal{Z}_t} \delta u^Z(t, \mathbf{X}_t, \mathbf{H}_{t-}) dH_t^Z,$$

[11] In virtue of standard arguments, see for instance Chapter 10 of [27].

which we may further rewrite as

$$\sum_{Z\in\mathcal{Z}_t} \ell_Z(t,\mathbf{X}_t,\mathbf{H}_{t-})\delta u^Z(t,\mathbf{X}_t,\mathbf{H}_{t-})dt \\ + \sum_{Z\in\mathcal{Z}_t} \Big(\delta u^Z(t,\mathbf{X}_t,\mathbf{H}_{t-})dH_t^Z - \ell_Z(t,\mathbf{X}_t,\mathbf{H}_t)\delta u^Z(t,\mathbf{X}_t,\mathbf{H}_t)dt\Big).$$

Here the second term is $\sum_{Z\in\mathcal{Z}_t}\delta u^Z(t,\mathbf{X}_t,\mathbf{H}_{t-})dM_t^Z$, whereas one has by (4) in the first term:

$$\sum_{Z\in\mathcal{Z}_t} \ell_Z(t,\mathbf{X}_t,\mathbf{H}_{t-})\delta u^Z(t,\mathbf{X}_t,\mathbf{H}_{t-}) \\ = \sum_{Z\in\mathcal{Z}_t}\sum_{Y\in\mathcal{Y};\,Y_t=Z} \lambda_Y(t,X_t^Y)\delta u^Z(t,\mathbf{X}_t,\mathbf{H}_{t-}) \\ = \sum_{Y\in\mathcal{Y}} \lambda_Y(t,X_t^Y)\delta u^Y(t,\mathbf{X}_t,\mathbf{H}_{t-})$$

using in the last identity that

$$\delta u^Z(t,\mathbf{x},\mathbf{k}) = \delta u^Y(t,\mathbf{x},\mathbf{k}),$$

for every $t,\mathbf{x},\mathbf{k},Y$ and Z such that $Y_t = Z$. Thus (40) indeed reduces to (6).

A.2 Markov Properties

Let us first recall the following local martingale characterization of a Markov process with generator $\mathcal{L}$. We work under the standing assumption that uniqueness holds for the solution of the martingale problem defined by $\mathcal{L}$.

Lemma A.2 (See, e.g., Ethier and Kurtz [20]) *Let X be a right-continuous process with Euclidean state space E, adapted to some filtration $\mathcal{F}$. For X to be an $\mathcal{F}$- Markov process with infinitesimal generator $\mathcal{L}$, it is necessary and sufficient that, for every real-valued function φ in the domain of $\mathcal{L}$,*

$$\varphi(t,X_t) - \int_0^t (\partial_s + \mathcal{L}_s)\varphi(s,X_s)ds \tag{42}$$

is an $\mathcal{F}$- local martingale.

We shall use this characterization informally in this paper, ignoring the technicalities related to the notion of domain of an operator. Furthermore, throughout the paper we work under the standing assumption that the valuation equation associated to any infinitesimal generator that we use, is well posed in an appropriate functional space. Finally, we assume that uniqueness holds for the solution of the related martingale problem. The reader is referred to Ethier and Kurtz [20] for more details and for specific conditions which can be postulated in these regards.

A.2.1 Proof of Proposition 2.8

(i) In view of the Itô formula (6), $(\mathbf{X},\mathbf{H})$ solves the martingale problem with generator $\mathcal{A}$ in the filtration $\mathcal{F}$, and is thus an $\mathcal{F}$-Markov process.
(ii) By application of the local martingale characterization of an $\mathcal{F}$-Markov process $(\mathbf{X},\mathbf{H})$ with generator $\mathcal{A}$ to test-functions of the form $u(t,\mathbf{x},\mathbf{k}) = v_i(t,x_i,k_i)$, we get the local martingale characterization of an $\mathcal{F}$- Markov process with generator $\mathcal{A}^i$ for (X^i,H^i). Considering $v_i(t,x_i,k_i) = \mathbb{1}_{k_i=1}$ therein yields that M^i in (16) is an $\mathcal{F}$-local martingale.
(iii) We denote $t_Z = \max_{i\in Z} t_i$, for every $Z \in \mathcal{N}_n$. Formula (17) follows directly from Lemma A.3 below since one has, for every $t, t_1, \dots, t_n \geq 0$,

$$\begin{aligned}\mathbb{P}\left(\tau_1 > t_1, \dots, \tau_n > t_n \mid \mathcal{F}_t\right) &= \sum_{Z\in\mathcal{N}_n} \mathbb{1}_{\{\mathrm{supp}^c(\mathbf{H}_t)=Z\}}\mathbb{P}\left(\tau_1 > t_1, \dots, \tau_n > t_n \mid \mathcal{F}_t\right)\\ &= \sum_{Z\in\mathcal{N}_n}\left(\prod_{i\notin Z}\mathbb{1}_{t_i<\tau_i\le t}\right)\mathbb{E}\left\{\prod_{i\in Z}\mathbb{1}_{\tau_i>t_i\vee t}\,\Big|\,\mathcal{F}_t\right\}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{1}_{\{t_i<\tau_i\,,\,i\in\mathrm{supp}(\mathbf{H}_t)\}}\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}}\Lambda^Y_{t,\theta^Y_t}\big)\,\Big|\,\mathbf{X}_t\right\}\\ &= \sum_{Z\in\mathcal{N}_n}\mathbb{1}_{\{\mathrm{supp}^c(\mathbf{H}_t)=Z\}}\mathbb{1}_{\{t_i<\tau_i\,,\,i\notin Z\}}\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}}\Lambda^Y_{t,\theta^Y_t}\big)\,\Big|\,\mathbf{X}_t\right\}\\ &= \sum_{Z\in\mathcal{N}_n}\left(\prod_{i\notin Z}\mathbb{1}_{t_i<\tau_i\le t}\right)\left(\prod_{i\in Z}\mathbb{1}_{\tau_i>t}\right)\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}}\Lambda^Y_{t,t_{Y\cap Z}}\big)\,\Big|\,\mathbf{X}_t\right\}.\end{aligned}$$

Given (17), the other formulas of part (iii) in Proposition 2.8 are straightforward.

Lemma A.3 *For every $t, t_1, \dots, t_n \geq 0$, and for every $Z \in \mathcal{N}_n$, one has,*

$$\mathbb{E}\left\{\textstyle\prod_{i\in Z}\mathbb{1}_{\tau_i>t_i\vee t}\,\Big|\,\mathcal{F}_t\right\} = \left(\textstyle\prod_{i\in Z}\mathbb{1}_{\tau_i>t}\right)\mathbb{E}\left\{\exp\big(-\textstyle\sum_{Y\in\mathcal{Y}\,;\,Y\cap Z\neq\emptyset}\Lambda^Y_{t,t_Y}\big)\,\Big|\,\mathbf{X}_t\right\}. \tag{43}$$

Proof. It is enough to prove that for t_i-s $\geq t$ one has, for every $Z \in \mathcal{N}_n$,

$$\mathbb{E}\left\{\prod_{i\in Z}\mathbb{1}_{\tau_i>t_i}\,\Big|\,\mathcal{F}_t\right\} = \left(\prod_{i\in Z}\mathbb{1}_{\tau_i>t}\right)\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}}\Lambda^Y_{t,t_{Y\cap Z}}\big)\,\Big|\,\mathbf{X}_t\right\}. \tag{44}$$

Indeed, for general t_i-s, applying (44) to the $t_i \vee t$-s yields

$$\begin{aligned}\mathbb{E}\left\{\prod_{i\in Z}\mathbb{1}_{\tau_i>t_i\vee t}\,\Big|\,\mathcal{F}_t\right\} &= \left(\prod_{i\in Z}\mathbb{1}_{\tau_i>t}\right)\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}}\Lambda^Y_{t,\max_{i\in Y\cap Z}t_i\vee t}\big)\,\Big|\,\mathbf{X}_t\right\}\\ &= \left(\prod_{i\in Z}\mathbb{1}_{\tau_i>t}\right)\mathbb{E}\left\{\exp\left(-\sum_{Y\in\mathcal{Y}\,;\,Y\cap Z\neq\emptyset}\Lambda^Y_{t,\max_{i\in Z}t_i}\right)\,\Big|\,\mathbf{X}_t\right\},\end{aligned}$$

which is (43). Let us thus show (44) for t_i-s $\geq t$, by induction on the cardinality d of Z. For $d = 0$, the result is trivial. Assuming the result at rank $d-1 \geq 0$, let us show the result at rank d. Let us suppose, without loss of generality, that $Z = \mathbb{N}_d$ and $t_1 \geq t_2 \geq \cdots \geq t_d \geq t$. One then needs to prove that, using the notation $J^l = 1 - H^l$ for every $l \in \mathbb{N}_d$,

$$\mathbb{E}(\prod_{l=1}^{d} J^l_{t_l} \,|\, \mathcal{F}_t) \quad = \quad (\prod_{l=1}^{d} J^l_t)\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}} \Lambda^Y_{t,t_{\mathbb{N}_d\cap Y}}\big)\,\Big|\,\mathbf{X}_t\right\}. \tag{45}$$

To establish (45) one first observes that

$$\mathbb{E}(\prod_{l=1}^{d} J^l_{t_l} \,|\, \mathcal{F}_t) = \mathbb{E}\left\{J^d_{t_d}\mathbb{E}\left\{\prod_{l=1}^{d-1} J^l_{t_l}\,\Big|\,\mathcal{F}_{t_d}\right\}\,\Big|\,\mathcal{F}_t\right\}, \tag{46}$$

where by the induction hypothesis at rank $d-1$ the inner conditional expectation can be represented as

$$\big(\prod_{l=1}^{d-1} J^l_{t_d}\big)\mathbb{E}\left\{\exp\big(-\sum_{Y\in\mathcal{Y}} \Lambda^Y_{t_d,t_{\mathbb{N}_{d-1}\cap Y}}\big)\,\Big|\,\mathbf{X}_{t_d}\right\} = \big(\prod_{l=1}^{d-1} J^l_{t_d}\big)v(t_d,\mathbf{X}_{t_d}) \tag{47}$$

for a suitable function $v = v(t,\mathbf{x})$ over $[0,t_{d-1}]\times\mathbb{R}^{\mathcal{Y}}$, by the Markov property of $\mathbf{X}$. Here the upper bound t_{d-1} for the domain of definition of the function v follows from the fact that $t_d \leq t_{d-1} \leq t_{\mathbb{N}_{d-1}\cap Y}$, for every $Y \in \mathcal{Y}$ with $\mathbb{N}_{d-1}\cap Y \neq \emptyset$. Inserting (47) into (46) yields by the Markov property of $(\mathbf{X},\mathbf{H})$ that

$$\mathbb{E}(\prod_{l=1}^{d} J^l_{t_l} \,|\, \mathcal{F}_t) = \mathbb{E}\left\{(\prod_{l=1}^{d} J^l_{t_d})v(t_d,\mathbf{X}_{t_d})\,\Big|\,\mathcal{F}_t\right\} = u(t,\mathbf{X}_t,\mathbf{H}_t),$$

for a function $u = u(t,\mathbf{x},\mathbf{k})$ over $[0,t_d]\times\mathbb{R}^{\mathcal{Y}}\times\{0,1\}^n$ characterized by:

$$\begin{cases} u(t_d,\mathbf{x},\mathbf{k}) = \big(\prod_{l=1}^{d}(1-k_l)\big)v(t_d,\mathbf{x}), \quad \mathbf{x} = (x_Y)_{Y\in\mathcal{Y}},\ \mathbf{k} = (k_1,\ldots,k_n)\in\{0,1\}^n \\ \Big(\partial_t + \mathcal{A}_t\Big)u(t,\mathbf{x},\mathbf{k}) = 0, \quad t < t_d,\ \mathbf{x} = (x_Y)_{Y\in\mathcal{Y}},\ \mathbf{k}\in\{0,1\}^n. \end{cases} \tag{48}$$

One finally shows that the RHS in (45) admits a representation of the form $\big(\prod_{l=1}^{d} J^l_t\big)w(t,\mathbf{X}_t)$, where the function $\tilde{u}(t,\mathbf{x},\mathbf{k}) = \big(\prod_{l=1}^{d}(1-k_l)\big)w(t,\mathbf{x})$ solves (48). By our standing assumption in this paper equation (48) has a unique solution. Thus $\tilde{u} = u$, which proves (45).

Since $\mathbf{X}$ is Markov with generator A (cf. (i)), the conditional expectation in the RHS of (45) can be represented as $w(t,\mathbf{X}_t)$, for a deterministic function $w = w(t,\mathbf{x})$ over the domain $[0,t_d]\times\mathbb{R}^{\mathcal{Y}}$. In order to get the analytic characterization of w, first note that for every $Y\in\mathcal{Y}$, one has:

$$\Lambda^Y_{t_d,t_{\mathbb{N}_{d-1}\cap Y}} = \Lambda^Y_{t_d,t_{\mathbb{N}_d\cap Y}} = \Lambda^Y_{t_d,t_{\mathbb{N}_d\cap Y}}.$$

This yields the terminal condition $w(t_d,\mathbf{x}) = v(t_d,\mathbf{x})$, $\mathbf{x} = (x_Y)_{Y\in\mathcal{Y}}$. One further has by an application of the Feynman-Kac formula that (see, e.g., Jeanblanc et al. [27])

$$\Big(\partial_t + A_t\Big)w(t,\mathbf{x}) = \Big(\sum_{Y\in\mathcal{Y};\,Y\cap Z\neq\emptyset} \lambda_Y(t,x_Y)\Big)w(t,\mathbf{x}), \quad t < t_d,\ \mathbf{x} = (x_Y)_{Y\in\mathcal{Y}}.$$

As a result the function $w = w(t,\mathbf{x})$ is the solution on $[0,t_d]\times\mathbb{R}^{\mathcal{Y}}$ to the following Kolmogorov pricing PDE:

$$\begin{cases} w(t_d,\mathbf{x}) = v(t_d,\mathbf{x}), \quad \mathbf{x} = (x_Y)_{Y\in\mathcal{Y}} \\ \Big(\partial_t + A_t\Big)w(t,\mathbf{x}) = \Big(\sum_{Y\in\mathcal{Y};\,Y\cap Z\neq\emptyset} \lambda_Y(t,x_Y)\Big)w(t,\mathbf{x}), \quad t < t_d,\ \mathbf{x} = (x_Y)_{Y\in\mathcal{Y}}. \end{cases} \tag{49}$$

Denoting $\tilde{u}(t,\mathbf{x},\mathbf{k}) = \big(\prod_{l\in\mathbb{N}_d}(1-k_l)\big)w(t,\mathbf{x})$, an application of the operator $\mathcal{A}_t$ of (7) yields:

$$\begin{aligned}\Big(\partial_t + \mathcal{A}_t\Big)\tilde{u}(t,\mathbf{x},\mathbf{k}) &= \big(\prod_{l\in\mathbb{N}_d}(1-k_l)\big)\Big(\partial_t + A_t\Big)w(t,\mathbf{x}) + w(t,\mathbf{x})\times \\ &\quad \times \sum_{Y\in\mathcal{Y}}\lambda_Y(t,x_Y)\Big(\big(\prod_{l\in\mathbb{N}_d}(1-k_l^Y)\big) - \prod_{l\in\mathbb{N}_d}(1-k_l)\Big), \end{aligned} \tag{50}$$

where we set, for $Y \in \mathcal{Y}$ and $l \in \mathbb{N}_d$,

$$k_l^Y = \begin{cases} 1, & Y \ni l, \\ k_l, & \text{else.}\end{cases}$$

Therefore

$$\begin{aligned}\sum_{Y\in\mathcal{Y}}\lambda_Y(t,x_Y)\Big(\big(\prod_{l\in\mathbb{N}_d}(1-k_l^Y)\big) - \prod_{l\in\mathbb{N}_d}(1-k_l)\Big) \\ = -\prod_{l\in\mathbb{N}_d}(1-k_l)\sum_{Y\in\mathcal{Y}\,;\,Y\cap\mathbb{N}_d\neq\emptyset}\lambda_Y(t,x_Y).\end{aligned} \tag{51}$$

Plugging (49) and (51) in the RHS of (50) yields that $\big(\partial_t + \mathcal{A}_t\big)\tilde{u}(t,\mathbf{x},\mathbf{k}) = 0$. Finally $\tilde{u}$ solves (48), which finishes the demonstration. □

A.2.2 Proof of Proposition 3.1

Given a function $f = f(t,y)$, let $f(t_j-,x)$ be a notation for the formal limit

$$\lim_{(t,y)\to(t_j,x) \text{ with } t<t_j} f(t,y). \tag{52}$$

In view of the Markov properties of the model gathered in Proposition 2.8, the following lemma holds in virtue of the Feynman-Kac formula.[12]

Lemma A.4 **(i)** *Given real numbers ϕ_i and ψ_i, one has $\mathbb{E}[\phi_i\sum_{t<t_j\le T}(1-H^i_{t_j}) + \psi_i H^i_T|\mathcal{F}_t] = (1-H^i_t)v_i(t,X^i_t) + \psi_i H^i_t$, where the pre-default function $v_i(t,x_i)$ solves the following Kolmogorov equation: $v_i(T,x_i) = 0$, $x_i\in\mathbb{R}$, and for j decreasing from p to 1:*
- *At $t = t_j$,*

$$v_i(t_j-,x_i) = v_i(t_j,x_i) + \phi_i,\ x_i\in\mathbb{R}, \tag{53}$$

- *On the time interval $[t_{j-1},t_j)$,*

$$\Big(\partial_t + A^i_t\Big)v_i(t,x_i) + \lambda_i(t,x_i)\psi_i = 0,\ x_i\in\mathbb{R}. \tag{54}$$

(ii) *Given real-valued functions $\phi(\mathbf{k})$ and $\psi(\mathbf{k})$, one has $\mathbb{E}[\sum_{t<t_j\le T}\phi(\mathbf{H}_{t_j}) + \psi(\mathbf{H}_T)|\mathcal{F}_t] = w(t,\mathbf{X}_t,\mathbf{H}_t)$, where the function $w(t,\mathbf{x},\mathbf{k})$ is the solution to the following Kolmogorov pricing PDE system: $w(T,\mathbf{x},\mathbf{k}) = \psi(\mathbf{k})$, $\mathbf{x} = (x_Y)_{Y\in\mathcal{Y}}$, $\mathbf{k}\in\{0,1\}^n$, and for j decreasing from p to 1:*
- *At $t = t_j$,*

$$w(t_j-,\mathbf{x},\mathbf{k}) = w(t_j,\mathbf{x},\mathbf{k}) + \phi(\mathbf{k}),\ \mathbf{x} = (x_Y)_{Y\in\mathcal{Y}},\ \mathbf{k}\in\{0,1\}^n, \tag{55}$$

[12] See, e.g., Jeanblanc et al. [27].

• *On the time interval* $[t_{j-1}, t_j)$,

$$\Big(\partial_t + \mathcal{A}_t\Big) w(t, \mathbf{x}, \mathbf{k}) = 0,\ \mathbf{x} = (x_Y)_{Y \in \mathcal{Y}},\ \mathbf{k} \in \{0,1\}^n. \tag{56}$$

Applying this Lemma with

$$\psi_i = (1 - R_i),\ \ \phi_i = -S_i \mathrm{h}$$

in part (i) and

$$\psi = L_{a,b},\ \ \phi = -S\mathrm{h}\big(b - a - L_{a,b}\big)$$

in part (ii), establishes the first lines in identities (23) and (24). Regarding the latter, note that the ex-dividend pricing function $u(t, \mathbf{k}, \mathbf{x})$ in (24), is provided by $w(t, \mathbf{k}, \mathbf{x}) - L_{a,b}(\mathbf{k})$ here.

Moreover, in the filtration $\mathcal{F} = \mathcal{F}^{\mathbf{W},\mathbf{H}}$, a martingale can only jump at totally unpredictable stopping times. In particular, the cumulative value processes cannot jump at the fixed times t_j-s. Given the first lines in (23) and (24), the second lines then readily follow using the Itô formula (6).

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