Optimal Investment Decisions for a Portfolio with a Rolling Horizon Bond and a Discount Bond

Tomasz R. Bielecki[1] , Stanley Pliska[2], and Jiongmin Yong[3]

Abstract. An optimal investment problem is considered for a continuous-time market consisting of the usual bank account, a rolling horizon bond, and a discount bond whose maturity coincides with the planning horizon. Two economic factors, namely, the short rate and the risk-free yield of some fixed maturity, are modeled as Gaussian processes. For the problem of maximizing expected HARA utility of terminal wealth, the optimal portfolio is obtained through a Bellman equation. The results are noteworthy because the discount bond, which is the riskless asset for the investor, causes a degeneracy due to its zero volatility at the planning horizon. Indeed, this delicate matter is treated rigorously for what seems to be the first time, and it is shown that there exists an optimal, admissible (but unbounded) trading strategy.

Keywords. Rolling horizon bond, discount bond, Bellman equation, Riccati equation, stochastic interest rates, optimal portfolio

AMS Subject Classifications. 90A09, 93E20.

§1. Introduction

We suppose there is a security market with two *economic factors* given by the following:

$$\begin{cases} r(t) = \text{the (risk-free) short interest rate at time } t, \\ \rho(t) = \text{risk-free yield for some fixed maturity } \widehat{T} \text{ (say, 10 years),} \\ \qquad\quad \text{i.e., the yield over } [t, t+\widehat{T}] \text{ of a zero-coupon bond maturing at } t+\widehat{T}. \end{cases}$$

We also suppose this security market has three interest rate based assets:

[1] Applied Mathematics Department, Illinois Institute of Technology, Chicago IL 60616, USA.
Partially supported by NSF Grant DMS-0202851.
[2] Department of Finance, University of Illinois at Chicago, Chicago IL 60607-7124 USA.
[3] Laboratory of Mathematics for Nonlinear Sciences, Department of Mathematics, and Institute of Mathematical Finance, Fudan University, Shanghai 200433, China. This work was supported in part by the NSFC, under grant 10131030, the Chinese Education Ministry Science Foundation under grant 2000024605, and the Cheung Kong Scholars Programme. Current mail address: Department of Mathematics, University of Central Florida, Orlando, FL 32816 USA.

- A *bank account* whose price process is denoted by $S_0(\cdot)$.
- A *rolling horizon bond* whose price process is denoted by $S_1(\cdot)$; this is a certain type of fixed income asset having an infinite life.
- A zero-coupon *discount bond* whose price process is denoted by $S_2(\cdot)$; this is risk-free and maturing at some fixed time $T < \infty$.

Note carefully that the maturity time $T < \infty$ for the discount bond coincides precisely with the planning horizon associated with the investor's optimal portfolio problem. Moreover, T can be either small than, equal to, or bigger than $\widehat{T}$.

In this paper we describe the two interest rates by a Gaussian process and model the three fixed income assets in terms of these underlying factors. We then address and solve the optimal portfolio problem of maximizing expected HARA utility of wealth at the fixed, finite planning horizon T. The most important feature of our model is the fact that one of the assets is a zero-coupon, discount bond whose maturity coincides with the investor's planning horizon. This kind of asset is significant for at least two reasons. First, since interest rates are stochastic, it is this asset, not the usual bank account, which serves as the riskless asset for the investor. Having a riskless asset is essential for many kinds of portfolio management problems. Second, because the volatility of this asset goes to zero as the time to maturity for this asset goes to zero, the usual Bellman equation has a degeneracy when time coincides with the maturity date, that is, with the planning horizon. In other words, the volatility matrix corresponding to the assets is singular at the planning horizon, and by usual methods this implies any optimal strategy will entail unbounded positions in at least two of the assets. The main contribution of this paper is that it provides what seems to be the first mathematically rigorous treatment of an optimal portfolio problem for this kind of market, that is, a market where interest rates are stochastic and one asset is a zero-coupon, discount bond that matures at or before the planning horizon.

Only a few papers in the literature have studied portfolio optimization problems that include a zero coupon, discount bond as one of the assets. Apparently working independently, Bajeux-Besnainou, Jordan, and Portait [1], Deelstra, Grasselli, and Koehl [5], Liu [9], and Sorensen [15] assumed one factor is the short rate having either Vasicek [16] or Cox, Ingersoll, and Ross [4] dynamics. In some cases there is also a second factor and/or a third asset that is taken to be a stock. All four studies focused on the problem of maximizing expected utility of wealth at a finite planning horizon T, where the utility function is of the form $u(v) = v^\gamma/\gamma$ and $\gamma < 1$ is a risk aversion parameter. For $\gamma = 0$ one actually has as a special limiting case $u(v) = \ln(v)$, giving rise to what is sometimes called the growth optimal or numeraire portfolio (a term that we use below).

The Bajeux-Besnainou, Jordan, and Portait [1], Deelstra, Grasselli, and Koehl [5], and Sorensen [15] studies are especially pertinent because, just like we do in this paper, they specifically fixed the maturity of the zero coupon bond equal to the planning horizon T. Using the risk neutral computational approach introduced by Pliska [13], all three studies derived the same general form for the optimal trading strategy: at every point in time hold a fixed fraction of one's wealth in the growth optimal portfolio and invest the rest in the zero coupon bond. But their growth optimal portfolios call for proportions of wealth in the discount bond that are unbounded in every neighborhood of the planning horizon T. This unboundedness is the result of the degeneracy issue that was raised above. Unfortunately, however, and with one exception, they ignored the unboundedness of their trading strategies. The exception is the paper by Deelstra, Grasselli, and Koehl [5]. They did recognize the degeneracy problem and attempted to overcome it by making a judicious choice of the class of admissible trading strategies. But it appears their specification of this class is flawed for it involves the solution to the associated SDE which is supposed to be satisfied by the wealth process, resulting in what seems to be a circular argument.

In summary, there are only a few studies of continuous time, portfolio optimization problems where interest rates are stochastic and where one of the assets is a zero coupon, discount bond. But none of these papers satisfactorily addressed the degeneracy problem. They left unanswered some troublesome questions about whether their optimal strategies are meaningful, whether corresponding portfolio value processes are well-defined stochastic processes, whether optimal objective values are finite, and so forth. The main contribution of this paper is to carefully answer questions like these.

We now shift our discussion to another asset in our model, the rolling horizon bond. While the bank account and discount bond are well-known securities, the rolling horizon bond is a new concept that was recently developed in a rigorous manner by Rutkowski [14]. Such financial instruments are theoretical constructs which resemble the so-called *Constant Maturity Treasuries* (CMT's). They can be thought of as mutual funds where discount bonds having a fixed maturity (say ten years) are continuously rolled over in a self-financing manner. The rolling horizon bond plays a secondary role in our model. We include it in order to develop a more interesting and richer model. In particular, by including this asset we can include two underlying factors, namely, the two exogenous interest rates, and yet have a model that is complete.

The rest of the paper is organized as follows. In Section 2 we formulate our securities market and specify our optimal portfolio problem. There considerable emphasis is placed on the dynamics of the assets, making sure these assets are modeled in a logical, rigorous, and consistent manner in terms of the underlying interest rate processes. Attention is also given to conditions that guarantee the market is complete and free of arbitrage opportunities. Section 3 is devoted to the well-posedness of the state equations, for which some estimates on the economic factor processes have to be established. Here we precisely specify the admissible trading strategies and show that for each the corresponding value of the portfolio is a well-defined stochastic process. In Section 4 we study the feasibility and accessibility of our optimal portfolio problem. In particular, here we develop conditions under which the corresponding market is arbitrage-free and, moreover, the optimal objective value is finite. Finally, in Section 5 we look at the corresponding Bellman equation for our problem and construct optimal portfolios. Under still another condition we show that there exists a unique solution to our optimal portfolio problem, and we specify the optimal trading strategy. An appendix has some technical results pertaining to Section 2.

From the standpoint of financial economics it is interesting to note that our optimal trading strategies call for unbounded positions in the discount bond in every neighborhood of the planning horizon, just like in some of the studies cited above. Since our derivation of this result is mathematically rigorous, one well might wonder about the economic implications. In particular, one might wonder whether our optimal strategy is an arbitrage opportunity, perhaps in some kind of asymptotic sense. But the answer is clear. While the position in the discount bond is unbounded, so is the position in the bank account, with the sum of these two proportions having a finite limit as time approaches maturity. Since the rates of return for the two assets converge to the same quantity, namely, the short rate, as time approaches maturity, there is no arbitrage opportunity.

§2. Formulation of the Problem

We denote the factor process by $X(t) \stackrel{\Delta}{=} (r(t), \rho(t))^T$ and suppose that it is governed by the following SDE:

$$\begin{cases} dX(t) = [AX(t) + a]dt + DdW(t), & t \geq 0, \\ X(0) = x. \end{cases} \tag{2.1}$$

where $A, D \in \mathbb{R}^{2\times 2}$, $a \in \mathbb{R}^2$, and $W(\cdot)$ is a two-dimensional standard Brownian motion defined on some complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ satisfying the *usual* *conditions* (see Yong and Zhou [19]) such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $W(\cdot)$ augmented by all the **P**-null sets. We assume that the three assets $S_0(\cdot)$, $S_1(\cdot)$ and $S_2(\cdot)$ satisfy the following:

$$\begin{cases} \dfrac{dS_0(t)}{S_0(t)} = r(t)dt, \\ \dfrac{dS_1(t)}{S_1(t)} = \big[\langle c, X(t)\rangle + c_0\big]dt + \langle \sigma, dW(t)\rangle, \\ \dfrac{dS_2(t)}{S_2(t)} = \big[\langle \mu(t), X(t)\rangle + \mu_0(t)\big]dt + \langle \nu(t), dW(t)\rangle, \end{cases} \tag{2.2}$$

where $c_0 \in \mathbb{R}$, $c \stackrel{\Delta}{=} (c_1, c_2)^T \in \mathbb{R}^2$, and $\sigma = (\sigma_1, \sigma_2)^T \in \mathbb{R}^2$ are some suitable parameters and where $\mu_0(\cdot)$, $\mu(\cdot) \stackrel{\Delta}{=} (\mu_1(\cdot), \mu_2(\cdot))^T$ and $\nu(\cdot) \stackrel{\Delta}{=} (\nu_1(\cdot), \nu_2(\cdot))^T$, with $\mu_i, \nu_j : [0, \infty) \to \mathbb{R}$, $i = 0, 1, 2$, $j = 1, 2$. For a justification of the dynamics for $S_1(\cdot)$ see Section 5 in Bielecki and Pliska [2]. A discussion of the dynamics for $S_2(\cdot)$ is provided in Section 2.2 below.

Note that (2.1) has ten scalar parameters and (2.2) has an additional five scalar parameters plus five deterministic functions. In order for the resulting market to be arbitrage-free and complete, these twenty objects must be properly inter-related. In the following few sub-sections we will develop these relationships. For expositional purposes, it is advisable (for the time being, at least) to suppose the ten interest rate parameters in (2.1) and the three rolling horizon bond appreciation rate parameters c and c_0 are exogenously chosen (suppose, for instance, they are calibrated from market data) and then focus on the implications for the remaining asset parameters and functions in (2.2).

§2.1. The market price of risk and no-arbitrage

Our first requirement is that there cannot exist any arbitrage opportunities in our securities market. It is well known (e.g., see Harrison and Pliska [7]) that arbitrage opportunities will not exist for the market (2.1)–(2.2) if there exists at least one equivalent martingale measure for this market. We shall now provide a sufficient condition for the existence of such a measure. It follows from Theorem 4.2 in Karatzas and Shreve [8] that an equivalent martingale measure will exist for the market (2.1)–(2.2) (considered over the time interval $[0, T]$) if there exists a progressively measurable, two-dimensional process, say $\theta(\cdot)$, that satisfies the following two conditions:

$$\begin{pmatrix} \langle c, X(t)\rangle \\ \langle \mu(t), X(t)\rangle \end{pmatrix} + \begin{pmatrix} c_0 \\ \mu_0(t) \end{pmatrix} - \begin{pmatrix} r(t) \\ r(t) \end{pmatrix} = \begin{pmatrix} \sigma^T \\ \nu(t)^T \end{pmatrix} \theta(t), \qquad \mathbf{P}\text{–a.s.} \tag{2.3}$$

for almost all t in the interval $[0, T]$, and

$$E\Big[e^{-\int_0^T \langle\, \theta(t), dW(t)\,\rangle \,-\frac{1}{2}\int_0^T |\theta(t)|^2 dt}\Big] = 1. \tag{2.4}$$

Such a process $\theta(\cdot)$ is commonly called a *market price of risk* process. In order to be sure that there are no arbitrage opportunities, we therefore will require the existence of a market price of risk process, that is, a process $\theta(\cdot)$ satisfying (2.3) and (2.4).

Now knowing, at least implicitly, the market price of risk process, we know the dynamics of the factor process $X(\cdot)$ under the equivalent martingale measure, and from this we know, at least in principle (see, e.g., Musiela and Rutkowski [11]), considerably about the real world dynamics (2.2) of the securities. However, for tractability it is useful to make a simplifying assumption: we assume that the market price of risk is an affine function of the factors. In other words, we assume $\theta(\cdot)$ has the form:

$$\theta(t) = GX(t) + g, \quad t \geq 0, \tag{2.5}$$

where $g \in \mathbb{R}^2$ is a constant vector and $G \in \mathbb{R}^{2\times 2}$ is a constant matrix. Note that with this assumption the factor process under the equivalent martingale measure will have exactly the same form as in (2.1), that is, it will be Gaussian with a drift coefficient that is an affine function of the level of the factors.

Note that by introducing G and g in (2.5) we have introduced six new parameters. However, we can now start establishing some relationships between the various parameters in our model. If we substitute (2.5) into (2.3), then

$$\begin{pmatrix} c^T - e_1^T - \sigma^T G \\ \mu(t)^T - e_1^T - \nu(t)^T G \end{pmatrix} X(t) + \begin{pmatrix} c_0 - \sigma^T g \\ \mu_0(t) - \nu(t)^T g \end{pmatrix} = 0, \tag{2.6}$$

where we introduced the notation c^T for the transpose of c and $e_1 \stackrel{\Delta}{=} (1, 0)^T$. Now (2.6) must be true for all values of the factor process $X(\cdot)$, so (2.6) immediately implies

$$\begin{cases} c = G^T \sigma + e_1, \qquad c_0 = g^T \sigma, \\ \mu(t) = G^T \nu(t) + e_1, \qquad \mu_0(t) = g^T \nu(t). \end{cases} \tag{2.7}$$

In other words, the eleven scalars in G, g, c, c_0, and σ and the five deterministic functions in $\nu(\cdot), \mu(\cdot)$, and $\mu_0(\cdot)$ must satisfy these six scalar-valued equations. Moreover, the expression for the rolling horizon bond's volatility σ actually follows directly and immediately from

Rutkowski [14] (see also Bielecki and Pliska [2]), namely, $\sigma = -\widehat{T}D^T e_2$. In view of (2.7) we thus have

$$\begin{cases} \sigma = -\widehat{T}D^T e_2, \\ c = -\widehat{T}G^T D^T e_2 + e_1, \\ c_0 = -\widehat{T}g^T D^T e_2. \end{cases} \tag{2.8}$$

To proceed with the analysis of the relationships between the parameters of our model, it is necessary to delve deeply into some theory of interest rate models, the subject of the next subsection. But first we shall pause here to comment on condition (2.4). In view of Novikov's criterion, condition (2.4) will be satisfied if the following sufficient condition holds

$$E\Big[e^{1/2\int_0^T |\theta(t)|^2 dt}\Big] < \infty. \tag{2.9}$$

But in view of (2.5), condition (2.9) will be satisfied if

$$E\Big[e^{\beta(G,g)\int_0^T |X(t)|^2 dt}\Big] < \infty, \tag{2.10}$$

where $\beta(G,g)$ is a constant (depending upon G and g). We demonstrate below (c.f. Corollary 3.2) that (2.10) holds provided that $\beta(G,g)$ satisfies (3.14).

§2.2. The Duffie-Kan type model and consistency of the market

Using some ideas from Duffie and Kan [6] and Ma and Yong [10], we can determine the discount bond's volatility $\nu(\cdot)$. A detailed proof will be provided in Appendix A. Here we briefly present the main ideas.

Suppose the risk premium $\theta(\cdot)$ exists in the form given by (2.5) satisfying (2.3)–(2.4). By defining

$$d\widetilde{\mathbf{P}} = e^{-\frac{1}{2}\int_0^T |\theta(s)|^2 ds - \int_0^T \langle\, \theta(s), dW(s)\,\rangle} d\mathbf{P}, \tag{2.11}$$

we know that

$$\widetilde{W}(t) \stackrel{\Delta}{=} W(t) + \int_0^t \theta(s)ds, \qquad t \in [0,T], \tag{2.12}$$

is a standard Brownian motion under $\widetilde{\mathbf{P}}$. This is an equivalent martingale measure for the market, and so the discounted versions of the processes $S_1(\cdot)$ and $S_2(\cdot)$ are $\widetilde{\mathbf{P}}$-martingales. Further, the economic factor process $X(\cdot)$ satisfies the following (note (2.1) and (2.5)):

$$\begin{cases} dX(t) = \big[BX(t) + b\big]dt + Dd\widetilde{W}(t), \\ X(0) = x, \end{cases} \tag{2.13}$$

where we have defined

$$B := A - DG, \qquad b := a - Dg. \tag{2.14}$$

Now consider a zero-coupon, discount bond such as $S_2(\cdot)$ maturing at a fixed time $T > 0$ and whose price process is denoted by $Y(\cdot)$. And suppose the locally riskless short interest rate is $R(X(t))$ for some function R. Then $Y(\cdot)$ satisfies the following backward stochastic differential equation (BSDE, for short):

$$\begin{cases} dY(t) = R(X(t))Y(t)dt + \langle\, Z(t), d\widetilde{W}(t)\,\rangle, \\ Y(T) = 1. \end{cases} \tag{2.15}$$

If $(Y(\cdot), Z(\cdot))$ is the adapted solution of (2.15), then

$$Y(t) = E_{\widetilde{\mathbf{P}}}\Big[e^{-\int_t^T R(X(s))ds} \,\Big|\, \mathcal{F}_t\Big], \qquad t \in [0,T]. \tag{2.16}$$

Using an idea of Ma and Yong [10], one can show that

$$Y(t) = u(t, X(t)), \qquad t \in [0,T], \tag{2.17}$$

with $u(\cdot\,,\cdot)$ being the solution of the following:

$$\begin{cases} u_t(t,x) + \langle\, u_x(t,x), Bx + b\,\rangle + \frac{1}{2}\mathrm{tr}\big[u_{xx}(t,x)DD^T\big] - R(x) = 0, \\ u(T,x) = 1. \end{cases} \tag{2.18}$$

Duffie and Kan [5] suggested that

$$u(t,x) = e^{\eta(t) + \langle\, \xi(t), x\,\rangle}, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^2, \tag{2.19}$$

for some deterministic functions $\xi(\cdot)$ and $\eta(\cdot)$. This can be the case if and only if $\xi(\cdot)$ and $\eta(\cdot)$ satisfy the following:

$$\begin{cases} \dot{\xi}(t) = -B^T\xi(t) + e_1, \\ \xi(T) = 0, \end{cases} \tag{2.20}$$

and

$$\begin{cases} \dot{\eta}(t) = -\langle\, b, \xi(t)\,\rangle - \frac{1}{2}|D^T\xi(t)|^2, \\ \eta(T) = 0. \end{cases} \tag{2.21}$$

Further, the following constraints hold (recall $\widehat{T}$ is the maturity corresponding to the yield $\rho(\cdot)$):

$$\eta(T-\widehat{T})=0, \quad \xi(T-\widehat{T})=-\widehat{T}e_2^T, \tag{2.22}$$

where $e_2=(0,1)^T$; these constraints will be used in the next subsection. Finally, based on the above, we are able to prove (see Appendix A and (2.7)) that

$$\begin{cases} \nu(t)=D^T\xi(t), \\ \mu(t)=G^TD^T\xi(t)+e_1, \\ \mu_0(t)=g^TD^T\xi(t). \end{cases} \tag{2.23}$$

To summarize matters at this point, the coefficients c, c_0, and σ for the rolling bond price process in (2.2) must satisfy (2.8), and the discount bond functions $\mu_0(\cdot)$, $\mu(\cdot)$, and $\nu(\cdot)$ in (2.2) are specified by (2.23). It is not hard to see from (2.8), (2.14), (2.20), and (2.23) that if the parameters A, a, D, G, g, c, and c_0 (a total of 19 real numbers) are specified so as to satisfy (2.8), then the model will be specified in a consistent manner that is free of arbitrage opportunities. For instance, if the ten economic factor parameters (A, a, D) of (2.1) and the three rolling horizon bond parameters c and c_0 are specified exogenously (e.g., calibrated from data), then σ is specified by (2.8) and one is left to specify the six real numbers in G and g. However, as shown in the next subsection, some additional constraints need to be imposed, and from these will follow some additional relationships between these parameters.

§2.3. Completeness of the market

We define (note (2.22) and (2.23))

$$N(t):=\begin{pmatrix} \sigma^T \\ \nu(t)^T \end{pmatrix}=\begin{pmatrix} -\widehat{T}e_2^T \\ \xi(t)^T \end{pmatrix}D, \qquad t\in\mathbb{R}. \tag{2.24}$$

By (2.20) we have

$$\xi(t)=-\int_t^T e^{B^T(s-t)}e_1ds=-\int_0^{T-t}e^{B^Ts}e_1ds, \quad t\le T. \tag{2.25}$$

Consequently,

$$\det N(t)=\widehat{T}(\det D)e_1^T\xi(t)=-\widehat{T}(\det D)\int_0^{T-t}e_1^Te^{B^Ts}e_1ds, \qquad t\in\mathbb{R}. \tag{2.26}$$

From (2.26) and Theorem 6.6 in Karatzas and Shreve [8] we see that the market (2.1)–(2.2) is complete if and only if

$$\det D \neq 0 \tag{2.27}$$

and

$$\int_0^t e_1^T e^{B^T s} e_1 ds \neq 0, \text{ for almost every } t \in [0, T]. \tag{2.28}$$

Note that $t \mapsto \int_0^t e_1^T e^{B^T s} e_1 ds$ is analytic and is not identical to zero. Thus (2.28) always holds. On the other hand, (2.27) means that the diffusion appearing in (2.1) has to be nondegenerate. Hereafter, we will keep assumption (2.27).

By (2.14), we see that (2.8) and (2.23) become

$$\begin{cases} \sigma = -\widehat{T} D^T e_2, \quad c = -\widehat{T}(A-B)^T e_2 + e_1, \quad c_0 = -\widehat{T}(a-b)e_2, \\ \nu(t) = D^T \xi(t), \quad \mu(t) = (A-B)^T \xi(t) + e_1, \quad \mu_0(t) = (a-b)^T \xi(t). \end{cases} \tag{2.29}$$

Hence, by (2.25), we know that it suffices for us to determine B and b satisfying these equations (assuming that a, A, D, c and c_0 are specified). To this end, we first note that the solution $\eta(\cdot)$ of (2.21) is given by

$$\begin{aligned} \eta(t) &= -\int_t^T \big[\langle b, \xi(s)\rangle + \frac{1}{2}|D^T \xi(s)|^2\big] ds, \\ &= \int_t^T \big[\langle b, \int_0^{T-s} e^{B^T \tau} e_1 d\tau\rangle - \frac{1}{2}|D^T \int_0^{T-s} e^{B^T \tau} e_1 d\tau|^2\big] ds, \\ &= \int_0^{T-t} \big[\langle b, \int_0^{s} e^{B^T \tau} e_1 d\tau\rangle - \frac{1}{2}|D^T \int_0^{s} e^{B^T \tau} e_1 d\tau|^2\big] ds, \quad t \leq T. \end{aligned} \tag{2.30}$$

Thus, by (2.25) and (2.30), we see that the two conditions in (2.22) are equivalent to the following, respectively:

$$\int_0^{\widehat{T}} e^{B^T s} e_1 ds = \widehat{T} e_2 \tag{2.31}$$

and

$$b^T \int_0^{\widehat{T}} \int_0^t e^{B^T s} e_1 ds dt = \frac{1}{2} \int_0^{\widehat{T}} |D^T \int_0^t e^{B^T s} e_1 ds|^2 dt. \tag{2.32}$$

Note that the equation for c_0 in (2.29) gives the second component of b, while the first component of b should be determined from (2.32). Thus if B is obtained and if we want to determine the first component of b through (2.32) uniquely, then we need (note (2.31))

$$
\begin{aligned}
0 \neq e_1^T \int_0^{\widehat{T}} \int_0^t e^{B^T s} e_1 ds dt &= \int_0^{\widehat{T}} (\widehat{T} - s) e_1^T e^{B^T s} e_1 ds \\
&= - \int_0^{\widehat{T}} s e_1^T e^{B^T s} e_1 ds.
\end{aligned} \tag{2.33}
$$

From the above, one sees that our model is properly and fully specified if one can choose B to be a solution of the following equations (note the first equation here is the same as the one for c in (2.29)):

$$
\begin{cases}
\widehat{T} B^T e_2 = c + \widehat{T} A^T e_2 - e_1, \\
\int_0^{\widehat{T}} e^{B^T s} e_1 ds = \widehat{T} e_2,
\end{cases} \tag{2.34}
$$

while simultaneously satisfying the following constraint:

$$
\int_0^{\widehat{T}} s e_1^T e^{B^T s} e_1 ds \neq 0. \tag{2.35}
$$

The unknown B contains four scalar elements, whereas (2.34) consists of exactly four equations. Thus, roughly speaking, there should be a unique solution solving (2.34). But by adding constraint (2.35), the problem of finding B seems to be over-determined. It is by no means obvious that (2.34)–(2.35) admits a solution. In Appendix B, we will discuss some of the cases for which we do have solutions to (2.34)–(2.35). We note that solving (2.34)–(2.35) is not the main goal of the present paper. Although we do not know whether (2.34)–(2.35) has solutions for every combination of the parameters A, a, D, c, and c_0, we are content knowing solutions exist for at least some combinations of these parameters. Hence hereafter we shall assume that there is a B solving (2.34)–(2.35).

To summarize, once B is determined from (2.34)–(2.35), we can determine b from (2.29)and (2.32); determine $\xi(\cdot)$ by (2.25), and finally determine σ and $(\nu(\cdot), \mu(\cdot), \mu_0(\cdot))$ from (2.29). We may also determine (G, g) from (2.14), since D is required to be invertible. Hence, once $A, a, D, c, c_0, T, \widehat{T}$ are specified (with $\det D \neq 0$), and once we have solved (2.34)-(2.35) for a matrix B, all the parameters of the market will be determined (possibly non-uniquely) so that the market is arbitrage-free and complete, as long as (2.4) holds for the risk premium $\theta(\cdot)$ defined by (2.5) (this issue was discussed at the end of Section 2.1).

§2.4. Trading strategies and wealth processes

We now turn to the wealth (or value) process, which will be denoted by $V(\cdot)$ and will have the initial endowment $V(0) = v > 0$. Let $h_i(t)$ be the proportion of $V(t)$ in the i-th asset, so $h_0(t)+h_1(t)+h_2(t) = 1$. We point out here that since short-selling and borrowing from the bank are allowed, $h_0(\cdot)$, $h_1(\cdot)$, and $h_2(\cdot)$ are not necessarily non-negative, nor are they necessarily bounded.

Under the commonly used self-financing assumption, it is standard that $V(t)$ satisfies:

$$
\begin{aligned}
\frac{dV(t)}{V(t)} &= h_0(t)r(t)dt + h_1(t)\big[(\langle\, c, X(t)\,\rangle + c_0)dt + \langle\, \sigma, dW(t)\,\rangle\big] \\
&\quad + h_2(t)\big[(\langle\, \mu(t), X(t)\,\rangle + \mu_0(t))dt + \langle\, \nu(t), dW(t)\,\rangle\big] \\
&= \big\{r(t) + \langle\, h(t), N(t)[GX(t)+g]\,\rangle\big\}dt + \langle\, h(t), N(t)dW(t)\,\rangle,
\end{aligned}
\tag{2.36}
$$

where

$$
h(\cdot) \triangleq \begin{pmatrix} h_1(\cdot) \\ h_2(\cdot) \end{pmatrix},
$$

and we have used the fact that (note (2.24))

$$
\begin{cases}
\begin{pmatrix} c^T - e_1^T \\ \mu(t)^T - e_1^T \end{pmatrix} = \begin{pmatrix} -\widehat{T} e_2^T (A-B) \\ \xi(t)^T (A-B) \end{pmatrix} = \begin{pmatrix} \sigma^T G \\ \nu(t)^T G \end{pmatrix} = N(t)G, \\
\begin{pmatrix} c_0 \\ \mu_0(t) \end{pmatrix} = \begin{pmatrix} -\widehat{T} e_2^T (a-b) \\ \xi(t)^T (a-b) \end{pmatrix} = \begin{pmatrix} \sigma^T g \\ \nu(t)^T g \end{pmatrix} = N(t)g.
\end{cases}
\tag{2.37}
$$

Note that the above are all analytic in t and therefore uniformly bounded on $[0, T]$. By putting equations (2.1) and (2.36) together, we then obtain the following state equation:

$$
\begin{cases}
dX(t) = [AX(t) + a]dt + DdW(t), \\
dV(t) = V(t)\big\{\big[r(t) + \langle\, N(t)^T h(t), GX(t)+g\,\rangle\big]dt + \langle\, N(t)^T h(t), dW(t)\,\rangle\big\}, \\
X(0) = x, \quad V(0) = v.
\end{cases}
\tag{2.38}
$$

The process $h(\cdot)$ appearing in (2.38) is called a *trading strategy*; it specifies trades in underlying assets in terms of fractions of available capital. Let us denote by $\mathbf{H}[0, T]$ the class of (admissible) trading strategies $h(\cdot)$ (on $[0, T]$) for which, for any $(x, v) \in \mathbb{R}^3$, the state equation (2.38) admits a unique strong solution satisfying some additional technical conditions to be specified in Section 3 below. In Section 3 we shall discuss well-posedness of this state equation and we shall specify the class $\mathbf{H}[0, T]$. It will be convenient to use the following notation: for every $(x, v) \in \mathbb{R}^3$ and for every $h(\cdot) \in \mathbf{H}[0, T]$ the corresponding unique solution of (2.38) is denoted as $(X(\cdot; x, v, h(\cdot)), V(\cdot; x, v, h(\cdot)))$.

§2.5. Utility functions and optimality criteria

The utility function that we are interested in is the so-called HARA utility:

$$U(v;\gamma) \triangleq \begin{cases} \frac{1}{\gamma} v^{\gamma}, & v \geq 0, \\ -\infty, & v < 0, \end{cases} \tag{2.39}$$

where $\gamma < 1$ and $\gamma \neq 0$ is a fixed parameter. For $\gamma = 0$, we define

$$U(v;0) \triangleq \begin{cases} \ln v, & v > 0, \\ -\infty, & v \leq 0. \end{cases} \tag{2.40}$$

We introduce the following payoff functions corresponding to admissible trading strategies $h(\cdot)$ and $\pi(\cdot)$:

$$J_\gamma(x,v;h(\cdot)) \triangleq E\Big[U\big(V(T;x,v,h(\cdot));\gamma\big)\Big]. \tag{2.41}$$

Note again that the above T coincides with the maturity of the discount bond so it, and not the bank account, is the riskless asset for the investor.

Our primary intention is to investigate the following control problem:

Problem ($\mathbf{H}_\gamma$). For given $(x,v) \in \mathbb{R}^3$, find an $\overline{h}(\cdot) \in \mathbf{H}[0,T]$ such that

$$J_\gamma(x,v;\overline{h}(\cdot)) = \max_{h(\cdot)\in\mathbf{H}[0,T]} J_\gamma(x,v;h(\cdot)). \tag{2.42}$$

Any $\overline{h}(\cdot) \in \mathbf{H}[0,T]$ satisfying (2.42) is called an *optimal trading strategy* (or *optimal portfolio*) for Problem (H_γ).

Let us make an observation. Suppose that C is an appropriately integrable nonnegative random variable. Then, for $\gamma \neq 0$, we have

$$\begin{aligned} U^{-1}\Big[E\Big\{U\big(C;\gamma\big)\Big\}\Big] &= \Big[E\{C^\gamma\}\Big]^{\frac{1}{\gamma}} \\ &= 1 + E\big\{\ln C\big\} + \frac{\gamma}{2}\mathrm{Var}\big\{\ln C\big\} + \cdots, \end{aligned} \tag{2.43}$$

where the dots "…" denote the parts of the expansions that depend on the higher order moments of C and on powers of γ of order 2 and higher. Suppose now that $\gamma \leq 0$, in which case both U and the inverse function U^{-1} are monotonically increasing. Then random variables C that make bigger the value of $U^{-1}\Big[E\Big\{U\big(C\big)\Big\}\Big]$ also make bigger the value of

$E\Big\{U(C)\Big\}$, and vice versa. Moreover, in view of (2.43), solving Problem (H_γ) for γ negative and close to zero leads to approximate solutions of related dynamic Markowitz problems for the terminal date T. Recall that the dynamic Markowitz problems associated with the utility U amount to maximizing the terminal mean-variance criteria of the (quadratic) form

$$E\big\{\ln(Y(T;x,y,h(\cdot)))\big\} + \frac{\gamma}{2}\mathrm{Var}\big\{\ln\big[Y(T;x,y,h(\cdot))\big]\big\}. \tag{2.44}$$

It is known that solving the above dynamic Markowitz problems is inherently difficult, and, typically, only necessary conditions for optimality can be stated. It is our belief that solving Problem (H_γ) is simpler. Thus the results of this paper may be helpful for solving the dynamic Markowitz problems stated above.

§3. Well-Posedness of the State Equation

The purpose of this section is to establish the well-posedness of the state equation (2.38). By well-posedness we mean that for any initial state $(x,v) \in \mathbb{R}^2 \times \mathbb{R}$ and admissible control $h(\cdot) \in \mathbf{H}[0,T]$, the state equation (2.38) admits a unique (strong) solution (which belongs to some Banach space) having continuous dependence on the initial state and the control. In our framework, the well-posedness is by no means obvious. To see this, let us make some observations. First, recall that (2.1) admits a unique strong solution $X(\cdot)$ which is given by the following:

$$X(t) = e^{At}x + \int_0^t e^{A(t-s)}a\,ds + \int_0^t e^{A(t-s)}D\,dW(s), \quad t \in [0,T]. \tag{3.1}$$

It is known that the following holds:

$$E\big[\sup_{t\in[0,T]} |X(t)|^{2k}\big] \le C_k(1+|x|^{2k}), \quad \forall k \ge 1, \tag{3.2}$$

for some constant C_k depending on k. On the other hand, it is not hard to see from (3.1) that $X(\cdot)$ is uniformly bounded if and only if $D = 0$, which makes our problem virtually uninteresting. On the other hand, when $D \ne 0$, process $X(\cdot)$ is unbounded, in which case so might be $r(\cdot)$ (a component of $X(\cdot)$). Now if we take $h(\cdot) = 0$ (which means that all the money is put in the bank), then the corresponding solution $V(\cdot)$ of the second equation in (2.38) is given by

$$V(t) = e^{\int_0^t r(\tau)d\tau}v, \qquad t \in [0,T]. \tag{3.3}$$

A natural space to which the process $V(\cdot)$ belongs should be $L^p_{\mathcal F}(0,T;\mathbb R)$ for some $p>0$, where

$$L^p_{\mathcal F}(0,T;\mathbb R)\stackrel{\Delta}{=}\{\varphi:[0,T]\times\Omega\to\mathbb R\,\big|\varphi(\cdot)\text{ is }\{\mathcal F_t\}_{t\ge0}\text{-adapted}, E\int_0^T|\varphi(t)|dt<\infty\}.$$

Also, since we will consider the utility function given by (2.39) (or (2.40)), we might only need $V(T)^\gamma$ to be integrable for some $\gamma<1$. That is why we should at least require $V(\cdot)\in L^p_{\mathcal F}(0,T;\mathbb R)$ for some $p>0$ (instead of $p\ge1$). From (3.3), we see that it is very natural to request the following type of estimate:

$$E\big[e^{\beta\int_0^t r(\tau)d\tau}\big]<\infty, \tag{3.4}$$

for certain values of $\beta\in\mathbb R$. When $r(\cdot)$ is unbounded, estimate (3.4) is not obvious. The goal of the following subsection is to establish such kinds of estimates. More general situations can be found in Yong [18].

§3.1. Some estimates.

We have the following result.

Lemma 3.1. *Let $\Phi(\cdot\,,\cdot)\in L^\infty(0,T;L^2(0,T;\mathbb R^{2\times2}))$. Then*

$$E\Big[e^{\beta\int_0^T\big|\int_0^t\Phi(t,s)dW(s)\big|^\delta dt}\Big]<\infty,\quad\forall\beta>0,\ \delta\in[0,2). \tag{3.5}$$

Furthermore, the above holds for $\delta=2$, provided the following holds for $\beta>0$:

$$2\beta T\operatorname*{esssup}_{t\in[0,T]}\int_0^t|\Phi(t,s)|^2ds<1. \tag{3.6}$$

Proof. By induction, we can show that

$$E\Big|\int_0^t\Phi(t,s)dW(s)\Big|^{2m}\le\frac{(2m)!}{2^m m!}\Big(\int_0^t|\Phi(t,s)|^2ds\Big)^m,\qquad\forall t\in[0,T],\ m\ge1. \tag{3.7}$$

Thus, for any $\beta>0$ and $\delta\in[0,2)$, we have

$$\begin{aligned}E\big[e^{\beta\int_0^T\big|\int_0^t\Phi(t,s)dW(s)\big|^\delta dt}\big]&=\sum_{m=0}^\infty\frac{\beta^m}{m!}E\Big\{\int_0^T\Big|\int_0^t\Phi(t,s)dW(s)\Big|^\delta dt\Big\}^m\\&\le1+\sum_{m=1}^\infty\frac{\beta^m}{m!}E\Big\{T^{m-1}\int_0^T\Big|\int_0^t\Phi(t,s)dW(s)\Big|^{\delta m}dt\Big\}\\&\le1+\sum_{m=1}^\infty\frac{\beta^mT^{m-1}}{m!}\int_0^T\Big\{E\Big|\int_0^t\Phi(t,s)dW(s)\Big|^{2m}\Big\}^{\frac\delta2}dt\\&\le1+\sum_{m=1}^\infty\frac{\beta^mT^{m-1}}{m!}\int_0^T\Big\{\frac{(2m)!}{2^mm!}\Big(\int_0^t|\Phi(t,s)|^2ds\Big)^m\Big\}^{\frac\delta2}dt\\&\le1+\sum_{m=1}^\infty\frac{(\beta T)^m}{T(m!)^{1-\frac\delta2}}\Big\{\frac{(2m)!}{2^m(m!)^2}\Big(\operatorname*{esssup}_{t\in[0,T]}\int_0^t|\Phi(t,s)|^2ds\Big)^m\Big\}^{\frac\delta2}.\end{aligned} \tag{3.8}$$

Let us recall Stirling's formula:

$$\lim_{m\to\infty} \frac{m! e^m}{m^m \sqrt{2\pi m}} = 1. \tag{3.9}$$

Using this formula, we have

$$\frac{(2m)!}{(m!)^2} \sim \frac{(2m)^{2m}\sqrt{4\pi m}e^{2m}}{e^{2m}m^{2m}2\pi m} = \frac{4^m}{\sqrt{\pi m}}\ , \qquad m \gg 1. \tag{3.10}$$

Thus the convergence of the series on the right hand side of (3.8) is equivalent to the following:

$$\sum_{m=1}^{\infty} \frac{1}{(m!)^{1-\frac{\delta}{2}}} \Big\{ 2^{\frac{\delta}{2}} \beta T \Big[\operatorname*{esssup}_{t\in[0,T]} \int_0^t |\Phi(t,s)|^2 ds \Big]^{\frac{\delta}{2}} \Big\}^m \Big(\frac{1}{\sqrt{\pi m}} \Big)^{\frac{\delta}{2}} < \infty, \tag{3.11}$$

which is true for any $t \in [0,T]$, $\delta \in [0,2)$ and $\beta > 0$, proving (3.5).

Now if $\delta = 2$ and $\beta > 0$, then (3.11) becomes

$$\sum_{m=1}^{\infty} \Big\{ 2\beta T \Big[\operatorname*{esssup}_{t\in[0,T]} \int_0^t |\Phi(t,s)|^2 ds \Big] \Big\}^m \Big(\frac{1}{\sqrt{\pi m}} \Big) < \infty, \tag{3.12}$$

which is the case when (3.6) holds. This completes the proof. □

Corollary 3.2. *Let* $X(\cdot) \stackrel{\Delta}{=} (r(\cdot), \rho(\cdot))$ *be the solution of* (2.1). *Then*

$$E\big[e^{\beta \int_0^T |X(t)|^\delta dt}\big] < \infty, \quad \forall \beta > 0,\ \delta \in [0,2). \tag{3.13}$$

Furthermore, the above holds for $\delta = 2$, *provided the following holds for* $\beta > 0$:

$$2\beta T \int_0^T |e^{At} D|^2 dt < 1. \tag{3.14}$$

Proof. Note that

$$\begin{aligned} |X(t)|^\delta &= \Big| e^{At}x + \int_0^t e^{A(t-s)} a ds + \int_0^t e^{A(t-s)} D dW(s) \Big|^\delta \\ &\le 2^{(\delta-1)^+} \Big| e^{At}x + \int_0^t e^{A(t-s)} a ds \Big|^\delta + 2^{(\delta-1)^+} \Big| \int_0^t e^{A(t-s)} D dW(s) \Big|^\delta. \end{aligned} \tag{3.15}$$

Thus if we define

$$\Phi(t,s) = e^{A(t-s)} D, \qquad (t,s) \in [0,T]^2, \tag{3.16}$$

then $\Phi(\cdot\,,\cdot)\in L^\infty(0,T;L^2(0,T;\mathbb{R}^{2\times 2}))$ and

$$\operatorname*{esssup}_{t\in[0,T]}\int_0^t|\Phi(t,s)|^2ds=\int_0^T|e^{As}D|^2ds. \tag{3.17}$$

Applying Lemma 3.1, we obtain our conclusions of this corollary immediately. □

§3.2. Well-posedness of state equation (2.38).

Now we look at the well-posedness of state equation (2.38). For any $v>0$ and $h(\cdot)\in\mathbf{H}[0,T]$ (to be defined below), suppose $V(\cdot)$ is a solution of the second equation in (2.38), and suppose the two integrals defined in (3.19) below are well-defined. Then by Itô's formula we have

$$\begin{aligned}d\big[\ln V(t)\big]=&\big[r(t)+\langle\,N(t)^Th(t),GX(t)+g\,\rangle-\frac12|N(t)^Th(t)|^2\big]dt\\&+\langle\,N(t)^Th(t),dW(t)\,\rangle\,.\end{aligned} \tag{3.18}$$

This implies

$$V(t)=ve^{\int_0^t\big[r(s)+\langle\,N(s)^Th(s),GX(s)+g\,\rangle-\frac12|N(s)^Th(s)|^2\big]ds+\int_0^t\langle\,N(s)^Th(s),dW(s)\,\rangle}, \tag{3.19}$$

which is a well-defined $\{\mathcal{F}_t\}_{t\ge0}$-adapted process. Conversely, if (3.19) is well-defined, then $V(\cdot)$ will be a solution of the second equation in (2.38).

Furthermore, from (3.19), we see that if $V(\cdot)$ is the strong solution of the second equation in (2.38) corresponding to $v>0$, then the following holds:

$$V(t;x,v,h(\cdot))>0,\qquad\forall t\ge0,\ \text{a.s.} \tag{3.20}$$

Thus, as a minimal requirement, we hope that for some $q\in\mathbb{R}$ (we assume $q\ne0$, since otherwise it is trivial) there exists a constant $C_q>0$ such that the following holds:

$$E\big[V(t;x,v,h(\cdot))^q\big]\le C_qv^q,\qquad\forall t\in[0,T],\ v\ge0. \tag{3.21}$$

To ensure (3.21), let us take

$$p_0,p_1,p_2,p_3>1,\quad\frac1{p_0}+\frac1{p_1}+\frac1{p_2}+\frac1{p_3}=1. \tag{3.22}$$

Then, by Corollary 3.2,

$$
\begin{aligned}
E\big[V(t;x,v,h(\cdot)))^q\big] &= v^q E\big[e^{\int_0^t q\left[r(s)+\langle N(s)^T h(s), GX(s)+g\rangle - \frac{1}{2}|N(s)^T h(s)|^2\right]ds} \\
&\qquad\qquad\qquad \cdot e^{\int_0^t q\langle N(s)^T h(s), dW(s)\rangle}\big] \\
&= v^q E\big[e^{\int_0^t q\left[r(s)+\langle N(s)^T h(s), GX(s)+g\rangle + \frac{p_3 q-1}{2}|N(s)^T h(s)|^2\right]ds} \\
&\qquad\qquad \cdot e^{\frac{1}{p_3}\int_0^t \langle p_3 q N(s)^T h(s), dW(s)\rangle - \frac{1}{2p_3}\int_0^t |p_3 q N(s)^T h(s)|^2 ds}\big] \\
&\le v^q \Big\{E\big[e^{p_0 q\int_0^t r(s)ds}\big]\Big\}^{\frac{1}{p_0}} \Big\{E\big[e^{p_1 q\int_0^t \langle N(s)^T h(s), GX(s)+g\rangle ds}\big]\Big\}^{\frac{1}{p_1}} \\
&\qquad \cdot \Big\{E\big[e^{p_2\frac{(p_3 q^2-q)^+}{2}\int_0^t |N(s)^T h(s)|^2 ds}\big]\Big\}^{\frac{1}{p_2}} \\
&\qquad \cdot \Big\{E\big[e^{\int_0^t \langle p_3 q N(s)^T h(s), dW(s)\rangle - \frac{1}{2}\int_0^t |p_3 q N(s)^T h(s)|^2 ds}\big]\Big\}^{\frac{1}{p_3}} \\
&\le v^q \Big\{E\big[e^{p_0|q|\int_0^t |r(s)|ds}\big]\Big\}^{\frac{1}{p_0}} \Big\{E\big[e^{p_1|q|\int_0^t |N(s)^T h(s)||GX(s)+g|ds}\big]\Big\}^{\frac{1}{p_1}} \\
&\qquad \cdot \Big\{E\big[e^{p_2\frac{(p_3 q^2-q)^+}{2}\int_0^t |N(s)^T h(s)|^2 ds}\big]\Big\}^{\frac{1}{p_2}} \le C v^q,
\end{aligned}
\tag{3.23}
$$

with $C > 0$ depending on $x, h(\cdot)$ and q (independent of v), provided

$$
\begin{cases}
E\big[e^{p_2\frac{(p_3 q^2-q)^+}{2}\int_0^t |N(s)^T h(s)|^2 ds}\big] < \infty, \\
E\big[e^{p_1|q|\int_0^t |N(s)^T h(s)||GX(s)+g|ds}\big] < \infty.
\end{cases}
\tag{3.24}
$$

In the above we used the facts that

$$
E\big[e^{\int_0^t \langle p_3 q N(s)^T h(s), dW(s)\rangle - \frac{1}{2}\int_0^t |p_3 q N(s)^T h(s)|^2 ds}\big] \le 1, \quad \forall t \in [0,T],
\tag{3.25}
$$

and, by Corollary 3.2 (noting $r(t) = e_1^T X(t)$),

$$
E\big[e^{p_0|q|\int_0^T |r(s)|ds}\big] < \infty.
\tag{3.26}
$$

Now for any $\beta \ge 1$ and $\varepsilon \in (0,1)$, let $\kappa = \frac{\varepsilon}{1+\varepsilon} \in (0, \frac{1}{2})$. By Young's inequality we have

$$
\begin{aligned}
\beta|N(s)^T h(s)||X(s)| &= \big[\varepsilon^{1/2}|N(s)^T h(s)|\big]\big[\varepsilon^{-1/2}\beta|X(s)|\big] \\
&\le \frac{1-\kappa}{2-\kappa}\big[\varepsilon^{1/2}|N(s)^T h(s)|\big]^{\frac{2-\kappa}{1-\kappa}} + \frac{1}{2-\kappa}\big[\varepsilon^{-1/2}\beta|X(s)|\big]^{2-\kappa} \\
&= \frac{1}{2+\varepsilon}\varepsilon^{\frac{2+\varepsilon}{\varepsilon}}|N(s)^T h(s)|^{2+\varepsilon} + \frac{1}{2-\kappa}\big(\varepsilon^{-1/2}\beta\big)^{2-\kappa}|X(s)|^{2-\kappa} \\
&\le \frac{\varepsilon}{2}|N(s)^T h(s)|^{2+\varepsilon} + \frac{\beta^2}{\varepsilon}|X(s)|^{2-\kappa}.
\end{aligned}
\tag{3.27}
$$

Consequently,

$$E\big[e^{\beta\int_0^T|N(s)^Th(s)||X(s)|ds}\big]\le E\big[e^{\frac{\varepsilon}{2}\int_0^T|N(s)^Th(s)|^{2+\varepsilon}ds+\frac{\beta^2}{\varepsilon}\int_0^T|X(s)|^{2-\kappa}ds}\big]$$
$$\le\Big\{E\big[e^{\varepsilon\int_0^T|N(s)^Th(s)|^{2+\varepsilon}ds}\big]\Big\}^{\frac12}\Big\{E\big[e^{\frac{2\beta^2}{\varepsilon}\int_0^T|X(s)|^{2-\kappa}ds}\big]\Big\}^{\frac12}. \tag{3.28}$$

Thus by Corollary 3.2 and a simple calculation (noting the boundedness of $N(\cdot)$), we see that (3.24) holds for all $q\in\mathbb{R}$ if the following is true: for some $\varepsilon\in(0,1)$,

$$E\big[e^{\varepsilon\int_0^T|N(s)^Th(s)|^{2+\varepsilon}ds}\big]<\infty. \tag{3.29}$$

Based on the above observation, we introduce the following:

$$\mathbf{H}[0,T]\stackrel{\Delta}{=}\bigcup_{\varepsilon>0}\{h(\cdot)\in L^1_{\mathcal{F}}(0,T;\mathbb{R}^2)\bigm| E\big[e^{\varepsilon\int_0^T|N(s)^Th(s)|^{2+\varepsilon}ds}\big]<\infty\}. \tag{3.30}$$

We claim that $\mathbf{H}[0,T]$ is a linear space. In fact, if $h_1(\cdot),h_2(\cdot)\in\mathbf{H}[0,T]$, then there exist $\varepsilon_1,\varepsilon_2>0$, such that

$$E\big[e^{\varepsilon_i\int_0^T|N(s)^Th_i(s)|^{2+\varepsilon_i}ds}\big]<\infty,\qquad i=1,2. \tag{3.31}$$

Thus, for any $\lambda_1,\lambda_2\in\mathbb{R}$, we take $0<\varepsilon\le\min\{\varepsilon_1,\varepsilon_2\}$ small enough so that

$$\varepsilon 2^{2+\varepsilon}|\lambda_i|^{2+\varepsilon}\le\varepsilon_i,\qquad i=1,2. \tag{3.32}$$

Then we have

$$\begin{aligned}&E\big[e^{\varepsilon\int_0^T|N(s)^T[\lambda_1h_1(s)+\lambda_2h_2(s)]|^{2+\varepsilon}ds}\big]\\&\le E\big[e^{\varepsilon2^{1+\varepsilon}\int_0^T\big[|\lambda_1|^{2+\varepsilon}|N(s)^Th_1(s)|^{2+\varepsilon}+|\lambda_2|^{2+\varepsilon}|N(s)^Th_2(s)|^{2+\varepsilon}ds}\big]\\&\le\big\{E\big[e^{\varepsilon2^{2+\varepsilon}|\lambda_1|^{2+\varepsilon}\int_0^T|N(s)^Th_1(s)|^{2+\varepsilon}ds}\big]\big\}^{\frac12}\big\{E\big[e^{\varepsilon2^{2+\varepsilon}|\lambda_2|^{2+\varepsilon}\int_0^T|N(s)^Th_2(s)|^{2+\varepsilon}ds}\big]\big\}^{\frac12}\\&\le\big\{E\big[e^{\varepsilon_1\int_0^T|N(s)^Th_1(s)|^{2+\varepsilon_1}ds}\big]\big\}^{\frac12}\big\{E\big[e^{\varepsilon_2\int_0^T|N(s)^Th_2(s)|^{2+\varepsilon_2}ds}\big]\big\}^{\frac12}<\infty.\end{aligned} \tag{3.33}$$

This implies that $\lambda_1h_1(\cdot)+\lambda_2h_2(\cdot)\in\mathbf{H}[0,T]$, proving our claim.

Let us make a simple observation on the space $\mathbf{H}[0,T]$. Denote

$$\mathbf{H}^{\beta,\varepsilon}[0,T]\stackrel{\Delta}{=}\{h(\cdot)\in L^1_{\mathcal{F}}(0,T;\mathbb{R}^2)\bigm| E\big[e^{\beta\int_0^T|N(s)^Th(s)|^{2+\varepsilon}ds}\big]<\infty\},\quad\beta,\varepsilon>0. \tag{3.34}$$

Then

$$\mathbf{H}^{\beta,\varepsilon}[0,T]\subseteq\mathbf{H}^{\bar\beta,\bar\varepsilon}[0,T],\quad\beta\le\bar\beta,\ \varepsilon\le\bar\varepsilon. \tag{3.35}$$

Hence

$$\mathbf{H}[0,T] = \bigcup_{\beta,\varepsilon>0} \mathbf{H}^{\beta,\varepsilon}[0,T]. \tag{3.36}$$

We point out that (3.29) is a little stronger than we need for (3.24). But this stronger assumption ensures that for any $q \in \mathbb{R}$ and $(x, h(\cdot)) \in \mathbb{R}^2 \times \mathbf{H}[0,T]$ the inequality (3.21) holds. On the other hand, from the conditions in (3.24) and the definition of $N(t)$ (see (2.39)), we see that (3.29) is almost the best possible condition. Also, it is clear that

$$L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^2) \subseteq \mathbf{H}[0,T]. \tag{3.37}$$

Further, by (2.24), (2.26) and (2.28), we know that $N(\cdot)$ is analytic on $[0,T]$ with $N(0) = 0$. Thus any $h(\cdot) \in \mathbf{H}[0,T]$ is not necessarily in $L^2_{\mathcal{F}}(0,T;\mathbb{R}^2)$. Finally, since, $N(\cdot)^{-1}$ is bounded on any $[0, T-\delta]$ (with $\delta \in (0,T)$), we have

$$\begin{aligned} \varepsilon E\Big[\int_0^{T-\delta} |h(s)|^{2+\varepsilon} ds\Big] &\le E\Big[e^{\varepsilon \int_0^{T-\delta} |h(s)|^{2+\varepsilon} ds}\Big] \\ &\le E\Big[e^{\varepsilon \|N(\cdot)^{-1}\|_{L^\infty(0,T-\delta;\mathbb{R}^{2\times 2})} \int_0^T |N(s)^T h(s)|^{2+\varepsilon} ds}\Big], \quad \forall \varepsilon > 0,\ \delta \in (0,T). \end{aligned} \tag{3.38}$$

Hence, by the definition of $\mathbf{H}[0,T]$ (see (3.30)), we must have

$$\mathbf{H}[0,T] \subseteq \bigcap_{\delta>0} \bigcup_{p>2} L^p_{\mathcal{F}}(0,T-\delta;\mathbb{R}^2) \equiv L^{2+}_{\mathcal{F}}(0,T-;\mathbb{R}^2). \tag{3.39}$$

We now state and prove the following result, which gives the well-posedness of (2.38).

Proposition 3.3. *Let $X(\cdot)$ be the solution of* (2.1)*. Then for any $(v, h(\cdot)) \in (0,\infty) \times \mathbf{H}[0,T]$, the second equation in* (2.38) *admits a unique solution $V(\cdot)$ on $[0,T]$ such that* (3.20)–(3.21) *hold for all $q \in \mathbb{R}$. Moreover, let $(x, v) \in \mathbb{R}^2 \times (0,\infty)$ and $\gamma \in \mathbb{R} \setminus \{0\}$ be given. Let $h(\cdot), \bar h(\cdot) \in \mathbf{H}[0,T]$ and $V(\cdot), \overline{V}(\cdot)$ be the corresponding solutions of the second equation in* (2.38)*. Then there exists an $\varepsilon > 0$ such that*

$$E\left|\frac{V(t)^\gamma}{\gamma} - \frac{\overline{V}(t)^\gamma}{\gamma}\right| \le C\Big\{E\int_0^T |N(s)^T[h(s) - \bar h(s)]|^{2+\varepsilon} ds\Big\}^{\frac{1}{2+\varepsilon}}, \quad t \in [0,T], \tag{3.40}$$

with $C > 0$ only depending on $x, v, \gamma, \varepsilon$, $h(\cdot)$, and $\bar h(\cdot)$ through

$$\max\Big\{E\big[e^{\varepsilon \int_0^T |N(s)^T h(s)|^{2+\varepsilon} ds}\big], E\big[e^{\varepsilon \int_0^T |N(s)^T \bar h(s)|^{2+\varepsilon} ds}\big]\Big\}. \tag{3.41}$$

Proof. We need only to establish (3.40). To this end, we take arbitrary $h(\cdot), \bar h(\cdot) \in \mathbf{H}[0,T]$. By definition, we can find $\varepsilon > 0$ such that

$$E\big[e^{\varepsilon\int_0^T |N(s)^T h(s)|^{2+\varepsilon}ds}\big] + E\big[e^{\varepsilon\int_0^T |N(s)^T \bar h(s)|^{2+\varepsilon}ds}\big] < \infty. \tag{3.42}$$

Denote

$$\zeta(t) := \frac{V(t)^\gamma}{\gamma} - \frac{\overline V(t)^\gamma}{\gamma}, \quad t \in [0,T]. \tag{3.43}$$

Then, by Itô's formula, we have

$$\begin{aligned} d\zeta(t) = &\Big\{V(t)^\gamma\big[r(t) + \langle\, N(t)^T h(t), GX(t) + g\,\rangle\big] \\ &- \overline V(t)^\gamma\big[r(t) + \langle\, N(s)^T \bar h(t), GX(t) + g\,\rangle\big] \\ &+ \frac{\gamma-1}{2}\big[V(t)^\gamma |N(t)^T h(t)|^2 - \overline V(t)^\gamma |N(t)^T \bar h(t)|^2\big]\Big\}dt \\ &+ \langle\, V(t)^\gamma N(t)^T h(t) - \overline V(t)^\gamma N(t)^T \bar h(t), dW(t)\,\rangle, \\ = &\Big\{\gamma\zeta(t)\big[r(t) + \langle\, N(t)^T h(t), GX(t) + g\,\rangle + \frac{(\gamma-1)}{2}|N(t)^T h(t)|^2\big] \\ &+ \overline V(t)^\gamma \,\langle\, GX(t) + g + \frac{\gamma-1}{2}N(t)^T[h(t) + \bar h(t)], N(t)^T[h(t) - \bar h(t)]\,\rangle\Big\}dt \\ &+ \langle\, \gamma\zeta(t)N(t)^T h(t) + \overline V(t)^\gamma N(t)^T[h(t) - \bar h(t)], dW(t)\,\rangle\,. \end{aligned} \tag{3.44}$$

If we denote

$$\psi_\gamma(t) = e^{\gamma\int_0^t [r(s) + \langle\, N(s)^T h(s), GX(s) + g\,\rangle - \frac12 |N(s)^T h(s)|^2]ds + \gamma\int_0^t \langle\, N(s)^T h(s), dW(s)\,\rangle}, \tag{3.45}$$

then by the variation of constants formula we have

$$\begin{aligned} \zeta(t) = &\int_0^t \psi_\gamma(t)\psi_\gamma(s)^{-1}\overline V(s)^\gamma \,\langle\, GX(s) + g + \frac{\gamma-1}{2}N(s)^T[h(s) + \bar h(s)], \\ &\qquad\qquad N(s)^T[h(s) - \bar h(s)]\,\rangle\, ds \\ &+ \psi_\gamma(t)\int_0^t \psi_\gamma(s)^{-1}\overline V(s)^\gamma \,\langle\, N(s)^T[h(s) - \bar h(s)], dW(s)\,\rangle\,. \end{aligned} \tag{3.46}$$

Similar to (3.23), we can show that for any $q \in \mathbb{R}$ there exists a constant $C_q > 0$ such that

$$E\big\{\big[\psi_\gamma(t)\psi_\gamma(s)^{-1}\big]^q\big\} \le C_q, \qquad \forall 0 \le s < t \le T. \tag{3.47}$$

Consequently, for any $q \in \mathbb{R}$,

$$\begin{cases} E\big\{\big[\psi_\gamma(t)\psi_\gamma(s)^{-1}\overline V(s)^\gamma\big]^q\big\} \le C, & \forall 0 \le s < t \le T, \\ E\big\{\big[\psi_\gamma(s)^{-1}\overline V(s)^\gamma\big]^q\big\} \le C, & \forall s \in [0,T]. \end{cases} \tag{3.48}$$

In (3.47) and (3.48) the constant $C > 0$ depends on x, v, γ, q and (3.41). Next, taking $1 < p < 1 + \varepsilon$ we have

$$
\begin{aligned}
E\big[|\zeta(t)|\big] &\leq CE\int_0^t \psi_\gamma(t)\psi_\gamma(s)^{-1}\overline{V}(s)^\gamma \\
&\qquad \cdot \big(|X(s)| + 1 + |N(s)^T h(s)| + |N(s)^T \bar{h}(s)|\big)|N(s)^T[h(s) - \bar{h}(s)]|ds \\
&\qquad + C\big[E|\psi_\gamma(t)|^2\big]^{\frac{1}{2}}\big\{E\int_0^t \psi_\gamma(s)^{-2}\overline{V}(s)^{2\gamma}|N(s)^T[h(s) - \bar{h}(s)]|^2 ds\big\}^{\frac{1}{2}} \\
&\leq C\Big\{E\!\int_0^t\!\big[|X(s)|^p + |N(s)^T h(s)|^p + |N(s)^T\bar{h}(s)|^p + 1\big]|N(s)^T[h(s) - \bar{h}(s)]|^p ds\Big\}^{\frac{1}{p}} \\
&\qquad + C\big\{E\int_0^t |N(s)^T[h(s) - \bar{h}(s)]|^{2+\varepsilon} ds\big\}^{\frac{1}{2+\varepsilon}}, \\
&\leq C\big\{E\int_0^t |N(s)^T[h(s) - \bar{h}(s)]|^{2+\varepsilon} ds\big\}^{\frac{1}{2+\varepsilon}}.
\end{aligned} \tag{3.49}
$$

The constant $C > 0$ in the above depends on $x, v, \gamma, \varepsilon$ and (3.41). This proves (3.40). □

We now look at a simple consequence of Proposition 3.3. For any $h(\cdot) \in \mathbf{H}[0,T]$ and any scalar $k > 0$, if we let

$$
h^k(t) = \begin{cases} k\dfrac{h(t)}{|h(t)|}, & |h(t)| > k, \\ h(t), & |h(t)| \leq k. \end{cases} \tag{3.50}
$$

then

$$
\begin{cases} E\big[e^{\varepsilon\int_0^T |N(s)^T h^k(s)|^{2+\varepsilon} ds}\big] \leq E\big[e^{\varepsilon\int_0^T |N(s)^T h(s)|^{2+\varepsilon} ds}\big], \\ E\displaystyle\int_0^T |N(s)^T[h(s) - h^k(s)]|^{2+\varepsilon} ds \to 0, \qquad k \to \infty. \end{cases} \tag{3.51}
$$

Thus by Proposition 3.3 and (3.37) we see that

$$
\sup_{h(\cdot)\in\mathbf{H}[0,T]} J_\gamma(x, v; h(\cdot)) = \sup_{h(\cdot)\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^2)} J_\gamma(x, v; h(\cdot)). \tag{3.52}
$$

Since the structure of $L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^2)$ is simpler than that of $\mathbf{H}[0,T]$, the above relation will be useful in studying Problem $(\mathbf{H}_\gamma)$ (see (2.46)).

§4. Feasibility and Accessibility.

Let us recall the utility functions (2.39)–(2.40), and denote

$$
\widetilde{J}_\gamma = \sup_{v>0} U(v;\gamma) \equiv \begin{cases} 0, & \gamma < 0, \\ \infty, & \gamma \geq 0. \end{cases} \tag{4.1}
$$

We now introduce the following notions concerning Problem $(\mathbf{H}_\gamma)$.

Definition 4.1. Problem $(\mathbf{H}_\gamma)$ is said to be

(i) *feasible* at $(x,v)\in\mathbb{R}^3$ if there exists an $h(\cdot)\in\mathbf{H}[0,T]$ such that

$$J_\gamma(x,v;h(\cdot))\quad\text{is well-defined.}\tag{4.2}$$

Any $h(\cdot)\in\mathbf{H}[0,T]$ satisfying (4.2) is called a *feasible portfolio*. The set of all feasible portfolios is denoted by $\mathbf{H}_\gamma^{(x,v)}[0,T]$, which depends on (x,v) and γ.

(ii) *accessible* at $(x,v)\in\mathbb{R}^3$ if

$$\sup_{h(\cdot)\in\mathbf{H}_\gamma^{(x,v)}[0,T]} J_\gamma(x,v;h(\cdot))<\widetilde{J}_\gamma.\tag{4.3}$$

(iii) (*uniquely*) *solvable* at $(x,v)\in\mathbb{R}^3$ if there exists a (unique) $\bar{h}(\cdot)\in\mathbf{H}_\gamma^{(x,v)}[0,T]$ such that

$$J_\gamma(x,v;\bar{h}(\cdot))=\max_{h(\cdot)\in\mathbf{H}_\gamma^{(x,v)}[0,T]} J_\gamma(x,v;h(\cdot)).\tag{4.4}$$

Any $\bar{h}(\cdot)\in\mathbf{H}_\gamma^{(x,v)}[0,T]$ satisfying (4.4) is called an *optimal portfolio.*

It is clear that the following implications hold:

$$\text{solvability}\quad\Rightarrow\quad\text{accessibility}\quad\Rightarrow\quad\text{feasiblity.}\tag{4.5}$$

Also, it is not hard to see that the above three notions are not equivalent, in general. The notion of accessibility is a little more general than the so-called finiteness introduced by Chen and Yong [3], where the corresponding $\widetilde{J}_\gamma=\infty$ (thus the name of finiteness seemed to be natural there). Since $\widetilde{J}_\gamma$ might be finite itself (see (4.1)), the name "accessible" seems more suitable here. The following proposition easily follows from Proposition 3.3.

Proposition 4.2. *Problem* $(\mathbf{H}_\gamma)$ *is feasible at any* $(x,v)\in\mathbb{R}^2\times(0,\infty)$, *and any* $h(\cdot)\in\mathbf{H}[0,T]$ *is a feasible portfolio for this* (x,v).

Note that in the above proposition we claimed that

$$\mathbf{H}_\gamma^{(x,v)}[0,T]=\mathbf{H}[0,T],\qquad\forall(x,v)\in\mathbb{R}^2\times(0,\infty),\ \gamma<1.\tag{4.6}$$

The following result gives a sufficient condition for Problem $(\mathbf{H}_\gamma)$ to be accessible.

Proposition 4.3. *Let $\gamma < 1$. Then Problem $(\mathbf{H}_\gamma)$ is accessible at any $(x,v)\in\mathbb{R}^2\times(0,\infty)$ if the following condition holds:*

$$T\gamma\int_0^T |Ge^{As}D|^2 ds < (1-\sqrt{|\gamma|})^2. \tag{4.7}$$

In particular, this is the case if $\gamma \le 0$.

Proof. We first consider the case $\gamma\in(0,1)$. Similar to (3.23), for any $0<\gamma<1$, and $1<p<\frac{1}{\gamma}$, we have (note $|r(s)|\le|X(s)|$)

$$\begin{aligned}
E\big[V(T;x,v,h(\cdot)))^\gamma\big] &= v^\gamma E\Big[e^{\int_0^T \gamma\big[r(s)+\langle N(s)^T h(s), GX(s)+g\rangle-\frac{1}{2}|N(s)^T h(s)|^2\big]ds} \\
&\qquad\cdot e^{\int_0^T \gamma\langle N(s)^T h(s), dW(s)\rangle}\Big] \\
&= v^\gamma E\Big[e^{\int_0^T \gamma\big[r(s)+\langle N(s)^T h(s), GX(s)+g\rangle+\frac{p\gamma-1}{2}|N(s)^T h(s)|^2\big]ds} \\
&\qquad\cdot e^{\frac{1}{p}\int_0^T \langle p\gamma N(s)^T h(s), dW(s)\rangle-\frac{1}{2p}\int_0^T |p\gamma N(s)^T h(s)|^2 ds}\Big] \\
&\le v^\gamma\Big\{E\Big[e^{\int_0^T \frac{p\gamma}{p-1}\big[|X(s)|-\frac{1-p\gamma}{2}|N(s)^T h(s)-\frac{GX(s)+g}{1-p\gamma}|^2+\frac{|GX(s)+g|^2}{2(1-p\gamma)}\big]ds}\Big]\Big\}^{\frac{p-1}{p}} \\
&\qquad\cdot\Big\{E\Big[e^{\int_0^T \langle p\gamma N(s)^T h(s), dW(s)\rangle-\frac{1}{2}\int_0^t |p\gamma N(s)^T h(s)|^2 ds}\Big]\Big\}^{\frac{1}{p}} \\
&\le v^\gamma\Big\{E\Big[e^{\int_0^T \frac{p\gamma}{p-1}\big[|X(s)|+\frac{|GX(s)+g|^2}{2(1-p\gamma)}\big]ds}\Big]\Big\}^{\frac{p-1}{p}}.
\end{aligned} \tag{4.8}$$

Since the right hand side of the above is independent of $h(\cdot)$, the accessibility of Problem $(\mathbf{H}_\gamma)$ will follow if we can show that the right hand side of (4.8) is finite. By Hölder's inequality and Lemma 3.1, we see that the right hand side of (4.8) is finite if the following holds:

$$E\Big[e^{\int_0^T (1+\varepsilon)\frac{p\gamma}{2(p-1)(1-p\gamma)}|GX(s)|^2 ds}\Big] < \infty, \tag{4.9}$$

for some $\varepsilon > 0$, arbitrarily small. By Lemma 3.1 again (noting (3.1)), (4.9) holds if

$$(1+\varepsilon)\frac{Tp\gamma}{(p-1)(1-p\gamma)}\int_0^T |Ge^{As}D|^2 ds < 1. \tag{4.10}$$

Since $\varepsilon > 0$ is arbitrarily small and since a direct computation shows that

$$\min_{1<p<\frac{1}{\gamma}} \frac{p}{(p-1)(1-p\gamma)} = \frac{1}{(1-\sqrt{\gamma})^2}, \tag{4.11}$$

we see that (4.10) can be replaced by (4.7).

Next, we note that for $\gamma \le 0$, (4.7) holds automatically. Thus we need to show that for $\gamma \le 0$, Problem $(\mathbf{H}_\gamma)$ is automatically accessible. To show this, we first look at the case $\gamma < 0$. Note that for any $p > 1$ and any positive integrable random variables ζ_1, ζ_2, by Hölder's inequality, we have

$$E[\zeta_1\zeta_2] \ge \{E[\zeta_1^{\frac{1}{1-p}}]\}^{1-p}\{E[\zeta_2^{\frac{1}{p}}]\}^p, \tag{4.12}$$

provided all the terms involved make sense. Using the above with

$$\begin{cases} \zeta_1 = e^{\int_0^T \gamma\left[r(s)+\langle N(s)^T h(s), GX(s)+g\rangle + \frac{\gamma-p}{2p}|N(s)^T h(s)|^2\right]ds}, \\ \zeta_2 = e^{p\int_0^T \langle \frac{\gamma}{p}N(s)^T h(s), dW(s)\rangle - \frac{p}{2}\int_0^t |\frac{\gamma}{p}N(s)^T h(s)|^2 ds}, \end{cases} \tag{4.13}$$

we have the following:

$$\begin{aligned} E\big[V(T;x,v,h(\cdot))^\gamma\big] &= v^\gamma E\big[e^{\int_0^T \gamma\left[r(s)+\langle N(s)^T h(s),GX(s)+g\rangle - \frac{1}{2}|N(s)^T h(s)|^2\right]ds} \\ &\qquad\qquad \cdot e^{\int_0^T \gamma\langle N(s)^T h(s), dW(s)\rangle}\big] \\ &= v^\gamma E\big[e^{\int_0^T \gamma\left[r(s)+\langle N(s)^T h(s),GX(s)+g\rangle + \frac{\gamma-p}{2p}|N(s)^T h(s)|^2\right]ds} \\ &\qquad \cdot e^{p\int_0^T \langle \frac{\gamma}{p}N(s)^T h(s), dW(s)\rangle - \frac{p}{2}\int_0^T |\frac{\gamma}{p}N(s)^T h(s)|^2 ds}\big] \\ &\ge v^\gamma\Big\{E\big[e^{\int_0^T \frac{\gamma}{1-p}\left[r(s)+\frac{\gamma-p}{2p}|N(s)^T h(s)+\frac{p[GX(s)+g]}{\gamma-p}|^2 - \frac{p|GX(s)+g|^2}{2(\gamma-p)}\right]ds}\big]\Big\}^{1-p} \\ &\qquad \cdot \Big\{E\big[e^{\int_0^T \langle \frac{\gamma}{p}N(s)^T h(s), dW(s)\rangle - \frac{1}{2}\int_0^t |\frac{\gamma}{p}N(s)^T h(s)|^2 ds}\big]\Big\} \\ &= v^\gamma\Big\{E\big[e^{\int_0^T \frac{\gamma}{1-p}\left[r(s)+\frac{\gamma-p}{2p}|N(s)^T h(s)+\frac{p[GX(s)+g]}{\gamma-p}|^2 - \frac{p|GX(s)+g|^2}{2(\gamma-p)}\right]ds}\big]\Big\}^{1-p} \\ &\ge v^\gamma\Big\{E\big[e^{\int_0^T \frac{\gamma}{1-p}\left[r(s)-\frac{p|GX(s)+g|^2}{2(\gamma-p)}\right]ds}\big]\Big\}^{1-p}. \end{aligned} \tag{4.14}$$

Here we should note that for any $h(\cdot) \in \mathbf{H}[0,T]$ the process $\frac{\gamma}{p}N(\cdot)^T h(\cdot)$ satisfies Novikov's condition. Thus the last equality holds in (4.14). Hence Problem $(\mathbf{H}_\gamma)$ is accessible if

$$E\big[e^{\int_0^T \frac{\gamma}{1-p}\left[r(s)-\frac{p|GX(s)+g|^2}{2(\gamma-p)}\right]ds}\big] < \infty. \tag{4.15}$$

Similar to the proof of case $\gamma \in (0,1)$, we see that (4.15) holds if (note $\gamma < 0$ and $p > 1$)

$$\frac{T\gamma p}{(1-p)(p-\gamma)}\int_0^T |Ge^{As}D|^2 ds = \frac{T|\gamma|p}{(p-1)(p+|\gamma|)}\int_0^T |Ge^{As}D|^2 ds < 1. \tag{4.16}$$

It is clear that by choosing $p > 1$ large enough, we have the last inequality in (4.16), which leads to

$$\begin{aligned} J_\gamma(x,v;h(\cdot)) &\equiv \frac{1}{\gamma}E\big[V(T;x,v,h(\cdot))^\gamma\big] \\ &\le \frac{v^\gamma}{\gamma}\Big\{E\big[e^{\int_0^T \frac{\gamma}{1-p}\left[r(s)-\frac{p|GX(s)+g|^2}{2(\gamma-p)}\right]ds}\big]\Big\}^{1-p} < 0. \end{aligned} \tag{4.17}$$

This gives the accessibility of Problem $(\mathbf{H}_\gamma)$ at $(x,v)\in\mathbb{R}^2\times(0,\infty)$.

Finally, we look at the case $\gamma=0$. By definition we have

$$
\begin{aligned}
J_0(x,v;h(\cdot)) &\equiv E\big\{\ln\big[V(T;x,v,h(\cdot))\big]\big\}\\
&=\ln v+E\int_0^T\big[r(s)+\langle\, N(s)^Th(s),GX(s)+g\,\rangle-\frac{1}{2}|N(s)^Th(s)|^2\big]ds\\
&=\ln v+E\int_0^T\big[r(s)-\frac{1}{2}|N(s)^Th(s)-GX(s)-g|^2+\frac{1}{2}|GX(s)+g|^2\big]ds\\
&\le \ln v+E\int_0^T\big[r(s)+\frac{1}{2}|GX(s)+g|^2\big]ds.
\end{aligned}
\tag{4.18}
$$

The right hand side of the above is independent of $h(\cdot)$. This gives the accessibility of Problem $(\mathbf{H}_0)$ at $(x,v)\in\mathbb{R}^2\times(0,\infty)$, proving the proposition. □

§5. Solvability of Problem $(\mathbf{H}_\gamma)$.

In this section we shall study the solvability of Problem $(\mathbf{H}_\gamma)$. We begin with the solvability of the Problem $(\mathbf{H}_0)$. We recall from Proposition 4.3 that Problem $(\mathbf{H}_0)$ is accessible. By (4.19) we have the following:

$$
\begin{aligned}
J_0(x,v;h(\cdot)) &= E\Big[\ln V(T;x,v,h(\cdot))\Big]\\
&=\ln v+E\int_0^T\big[r(s)-\frac{1}{2}|N(s)^Th(s)-GX(s)-g|^2-\frac{1}{2}|GX(s)+g|^2\big]ds.
\end{aligned}
\tag{5.1}
$$

It is clear that

$$
\overline{h}(\cdot)=[N(\cdot)^T]^{-1}[GX(\cdot)+g]\in\mathbf{H}[0,T]
\tag{5.2}
$$

is the unique optimal portfolio with the optimal expected utility

$$
J_0(x,v;\overline{h}(\cdot))=\ln v+E\int_0^T\big[r(s)-\frac{1}{2}|GX(s)+g|^2\big]ds.
\tag{5.3}
$$

The optimal wealth process $\overline{V}(\cdot)$ is given by the solution of the following:

$$
\begin{cases}
d\overline{V}(t)=\overline{V}(t)\big\{[e_1^TX(t)+|GX(t)+g|^2]dt+\langle\, GX(t)+g,dW(t)\,\rangle\big\}, & t\in[0,T],\\
\overline{V}(0)=v,
\end{cases}
\tag{5.4}
$$

with $X(\cdot)$ being the solution of (2.1). We summarize the above in the following proposition.

Proposition 5.1. *For any $(x,v)\in\mathbb{R}^2\times(0,\infty)$, Problem $(\mathbf{H}_0)$ is uniquely solvable with the optimal portfolio $\overline{h}(\cdot)\in\mathbf{H}[0,T]$ given by (5.2).*

Next we shall deal with Problems ($\mathbf{H}_\gamma$) for $\gamma < 1$ and $\gamma \neq 0$. We are going to use the Bellman Dynamic Programming Principle for this.

Let $(t, x, v) \in [0, T) \times \mathbb{R}^2 \times (0, \infty)$ and consider Problem ($\mathbf{H}_\gamma$) on the time interval $[t, T]$. To this end, we consider the state equations on $[t, T]$:

$$(5.5) \qquad \begin{cases} dX(s) = [AX(s) + a]ds + DdW(s), \\ dV(s) = V(s)\big\{\big[e_1^T X(s) + \langle N(s)^T h(s), GX(s) + g \rangle\big]ds \\ \qquad\qquad + \langle N(s)^T h(s), dW(s) \rangle \big\}, \\ X(t) = x, \quad V(t) = v. \end{cases}$$

We introduce the following:

$$(5.6) \qquad \mathbf{H}[t, T] \stackrel{\Delta}{=} \bigcup_{\varepsilon > 0} \{h(\cdot) \in L^1_{\mathcal{F}}(t, T; \mathbb{R}^n) \mid E\big[e^{\varepsilon \int_t^T |h(s)|^{2+\varepsilon} ds}\big] < \infty\}.$$

For any $(t, x, v) \in [0, T) \times \mathbb{R}^2 \times (0, \infty)$, and any $h(\cdot) \in \mathbf{H}[t, T]$, the solution to (5.5) is denoted by $(X(\cdot\,; t, x, v), V(\cdot\,; t, x, v, h(\cdot)))$. Note that $X(\cdot\,; t, x, v)$ is independent of $h(\cdot)$. Next we define

$$(5.7) \qquad J_\gamma(t, x, v; h(\cdot)) \stackrel{\Delta}{=} E\Big\{U(Y(T; t, x, v, h(\cdot)); \gamma)\Big\}.$$

Then we can pose the following problems.

Problem ($\mathbf{H}_\gamma[t, T]$). For given $(t, x, v) \in [0, T] \times \mathbb{R}^2 \times (0, \infty)$, find a $\overline{h}(\cdot) \in \mathbf{H}[t, T]$ such that

$$(5.8) \qquad J_\gamma(t, x, v; \overline{h}(\cdot)) = \max_{h(\cdot) \in \mathbf{H}[t, T]} J_\gamma(t, x, v; h(\cdot)) \stackrel{\Delta}{=} J^\gamma(t, x, v).$$

We see that for $t = 0$, Problem ($\mathbf{H}_\gamma[t, T]$) and coincides with Problem ($\mathbf{H}_\gamma$). We call $J^\gamma(t, x, y)$ the value function of Problem ($\mathbf{H}_\gamma$). Similar to (3.52), we know that

$$(5.9) \qquad J^\gamma(t, x, v) = \sup_{h(\cdot) \in \mathbf{H}[t, T]} J_\gamma(t, x, v; h(\cdot)) = \sup_{h(\cdot) \in L^\infty_{\mathcal{F}}(t, T; \mathbb{R}^2)} J_\gamma(t, x, v; h(\cdot)).$$

Hence, by some relevant arguments found in Yong and Zhou [19] and by using the Bellman Principle of Optimality, we know that if $J^\gamma(\cdot\,, \cdot\,, \cdot)$ is smooth, then the following holds (assuming $J^\gamma_{vv} < 0$, and note $y > 0$):

$$
\begin{aligned}
0 &= J^\gamma_t + \sup_{h\in\mathbb{R}^2} \Big\{ \langle J^\gamma_x, Ax + a \rangle + J^\gamma_v \big[(e_1^T x)v + v \langle N^T h, Gx + g \rangle \big] \\
&\quad + \frac{1}{2}\mathrm{tr}\Big[\begin{pmatrix} J^\gamma_{xx} & J^\gamma_{xv} \\ (J^\gamma_{xv})^T & J^\gamma_{vv} \end{pmatrix} \begin{pmatrix} D \\ vN^T h \end{pmatrix} \begin{pmatrix} D^T & vh^T N \end{pmatrix} \Big] \Big\} \\
&= J^\gamma_t + \langle J^\gamma_x, Ax + a \rangle + v(e_1^T x) J^\gamma_v + \frac{1}{2} D^T J^\gamma_{xx} D \\
&\quad + \sup_{h\in\mathbb{R}^2} \Big\{ \frac{v^2}{2} J^\gamma_{vv} |N^T h|^2 + v \langle D^T J^\gamma_{xv} + (Gx+g) J^\gamma_v, N^T h \rangle \Big\} \\
&= J^\gamma_t + \langle J^\gamma_x, Ax + a \rangle + v(e_1^T x) J^\gamma_v + \frac{1}{2}\mathrm{tr}[DD^T J^\gamma_{xx}] \\
&\quad + \frac{1}{2} J^\gamma_{vv} \sup_{h\in\mathbb{R}^2} \Big\{ v^2 |N^T h|^2 + 2v \langle \frac{D^T J^\gamma_{xv} + (Gx+g)J^\gamma_v}{J^\gamma_{vv}}, N^T h \rangle \Big\} \\
&= J^\gamma_t + \langle J^\gamma_x, Ax + a \rangle + v(e_1^T x) J^\gamma_v + \frac{1}{2}\mathrm{tr}[DD^T J^\gamma_{xx}] - \frac{|D^T J^\gamma_{xv} + (Gx+g)J^\gamma_v|^2}{2J^\gamma_{vv}} \\
&\quad + \frac{1}{2} J^\gamma_{vv} \inf_{h\in\mathbb{R}^2} \Big\{ \Big| vN^T h + \frac{D^T J^\gamma_{xv} + (Gx+g)J^\gamma_v}{j^\gamma_{vv}} \Big|^2 \Big\} \\
&= J^\gamma_t + \langle J^\gamma_x, Ax + a \rangle + v(e_1^T x) J^\gamma_v + \frac{1}{2}\mathrm{tr}[DD^T J^\gamma_{xx}] - \frac{|D^T J^\gamma_{xv} + (Gx+g)J^\gamma_v|^2}{2J^\gamma_{vv}}.
\end{aligned} \tag{5.10}
$$

Here we should note that in view of our assumption about market completeness (see (2.26)–(2.28)) the matrix $N(t)$ is nondegenerate for every $t \in [0, T)$. Thus the infimum above is attained by

$$
\overline{h}(t, x, v; \gamma) = -\frac{1}{v}\Big(N(t)^T\Big)^{-1} \frac{D^T J^\gamma_{xv}(t,x,v) + (Gx+g)J^\gamma_v(t,x,v)}{J^\gamma_{vv}(t,x,v)}. \tag{5.11}
$$

Finally, we end up with the following Bellman equation for the value function $J^\gamma(t, x, v)$:

$$
\begin{cases}
J^\gamma_t + \langle J^\gamma_x, Ax + a \rangle + v(e_1^T x) J^\gamma_v + \frac{1}{2}\mathrm{tr}[DD^T J^\gamma_{xx}] - \frac{|D^T J^\gamma_{xv} + (Gx+g)J^\gamma_v|^2}{2J^\gamma_{vv}} = 0, \\
\qquad\qquad\qquad\qquad (t, x, v) \in [0, T) \times \mathbb{R}^2 \times (0, \infty), \\
J^\gamma\big|_{t=T} = \frac{1}{\gamma} v^\gamma, \qquad\qquad (x, v) \in \mathbb{R}^2 \times (0, \infty).
\end{cases} \tag{5.12}
$$

Now if we introduce the function $p^\gamma(t, x)$ by setting

$$
J^\gamma(t, x, v) = \frac{1}{\gamma} e^{p^\gamma(t,x)} v^\gamma, \qquad (t, x, v) \in [0, T] \times \mathbb{R}^2 \times (0, \infty), \tag{5.13}
$$

then one has (suppressing (t,x) in $p^\gamma(t,x)$)

$$
\begin{aligned}
0 &= J^\gamma_t + \langle\, J^\gamma_x, Ax+a\,\rangle + v(e_1^Tx)J^\gamma_v + \frac{1}{2}\mathrm{tr}[DD^TJ^\gamma_{xx}] - \frac{|D^TJ^\gamma_{xv} + (Gx+g)J^\gamma_v|^2}{2J^\gamma_{vv}} \\
&= e^{p^\gamma}\Big\{\frac{1}{\gamma}p^\gamma_t v^\gamma + \langle\, \frac{1}{\gamma}p^\gamma_x v^\gamma, Ax+a\,\rangle + (e_1^Tx)v^\gamma + \frac{v^\gamma}{2\gamma}\mathrm{tr}\Big[DD^Tp^\gamma_{xx} + DD^Tp^\gamma_x(p^\gamma_x)^T\Big] \\
&\qquad + \frac{|D^Tp^\gamma_x v^{\gamma-1} + (Gx+g)v^{\gamma-1}|^2}{2(1-\gamma)v^{\gamma-2}}\Big\} \qquad (5.14) \\
&= \frac{v^\gamma}{\gamma}e^{p^\gamma}\Big\{p^\gamma_t + \langle\, p^\gamma_x, Ax+a\,\rangle + \gamma(e_1^Tx) + \frac{1}{2}\mathrm{tr}[DD^Tp^\gamma_{xx} + DD^Tp^\gamma_x(p^\gamma_x)^T] \\
&\qquad + \frac{\gamma|D^Tp^\gamma_x + Gx + g|^2}{2(1-\gamma)}\Big\}.
\end{aligned}
$$

Hence $J^\gamma(\cdot\,,\cdot\,,\cdot)$ defined by (5.13) satisfies (5.12) if and only if $p^\gamma(\cdot\,,\cdot)$ satisfies the following quasi-linear PDE:

$$
\begin{cases}
p^\gamma_t + \frac{1}{2}\mathrm{tr}[DD^Tp^\gamma_{xx}] + \frac{1}{2}|D^Tp^\gamma_x|^2 + \langle\, Ax+a, p^\gamma_x\,\rangle + \gamma\langle\, e_1, x\,\rangle \\
\qquad + \dfrac{\gamma|D^Tp^\gamma_x + Gx + g|^2}{2(1-\gamma)} = 0, \qquad (t,x)\in[0,T)\times\mathbb{R}^2, \\
p^\gamma\big|_{t=T} = 0, \qquad x\in\mathbb{R}^2.
\end{cases} \qquad (5.15)
$$

We now construct a solution $p^\gamma(\cdot\,,\cdot)$ to the above equation. To this end, we assume that $p^\gamma(\cdot\,,\cdot)$ has the following form:

$$
p^\gamma(t,x) \stackrel{\Delta}{=} \langle\, K^\gamma(t)x, x\,\rangle + 2\langle\, k^\gamma(t), x\,\rangle + k^\gamma_0(t), \quad (t,x)\in[0,T]\times\mathbb{R}^2, \qquad (5.16)
$$

with $K^\gamma(\cdot)\in C^1([0,T];\mathcal{S}^2)$, $k^\gamma(\cdot)\in C^1([0,T];\mathbb{R}^2)$ and $k^\gamma_0(\cdot)\in C^1([0,T];\mathbb{R})$. Plugging (5.16) into the equation in (5.15), we have (suppressing γ in $(K^\gamma(\cdot), k^\gamma(\cdot), k^\gamma_0(\cdot))$)

$$
\begin{aligned}
0 &= \langle\, \dot K(t)x, x\,\rangle + 2\langle\, \dot k(t), x\,\rangle + \dot k_0(t) \\
&\quad + \mathrm{tr}[DD^TK(t)] + 2|D^T[K(t)x + k(t)]|^2 + 2\langle\, Ax+a, K(t)x + k(t)\,\rangle \\
&\quad + \gamma\langle\, e_1, x\,\rangle + \frac{\gamma|[2D^TK(t) + G]x + 2D^Tk(t) + g|^2}{2(1-\gamma)} \\
&= \langle\, \big\{\dot K(t) + 2K(t)DD^TK(t) + K(t)A + A^TK(t) \\
&\qquad + \frac{\gamma}{2(1-\gamma)}(2K(t)D + G^T)(2D^TK(t) + G)\big\}x, x\,\rangle \qquad (5.17) \\
&\quad + \langle\, 2\dot k(t) + 4K(t)DD^Tk(t) + 2A^Tk(t) + 2K(t)a + \gamma e_1 \\
&\qquad + \frac{\gamma}{1-\gamma}(2K(t)D + G^T)(2D^Tk(t) + g), x\,\rangle \\
&\quad + \dot k_0(t) + \mathrm{tr}[DD^TK(t)] + 2|D^Tk(t)|^2 + 2\langle\, a, k(t)\,\rangle + \frac{\gamma|2D^Tk(t) + g|^2}{2(1-\gamma)}.
\end{aligned}
$$

Hence, $K(\cdot)$, $k(\cdot)$, and $k_0(\cdot)$ should be the solutions of the following ODEs, respectively (noting the terminal condition in (5.15)):

$$
(5.18)\qquad \begin{cases} \dot K(t)+\dfrac{2}{1-\gamma}K(t)DD^TK(t)+K(t)(A+\dfrac{\gamma}{1-\gamma}DG)+(A+\dfrac{\gamma}{1-\gamma}DG)^TK(t) \\ \qquad\qquad +\dfrac{\gamma}{2(1-\gamma)}G^TG=0, \\ K(T)=0, \end{cases}
$$

$$
(5.19)\qquad \begin{cases} \dot k(t)+\big[\dfrac{2}{1-\gamma}K(t)DD^T+A^T+\dfrac{\gamma}{1-\gamma}G^TD^T\big]k(t) \\ \qquad\qquad +K(t)a+\dfrac{1}{2}\gamma e_1+\dfrac{\gamma}{2(1-\gamma)}(2K(t)D+G^T)g=0, \\ k(T)=0, \end{cases}
$$

$$
(5.20)\qquad \begin{cases} \dot k_0(t)+\mathrm{tr}[DD^TK(t)]+2|D^Tk(t)|^2+2\,\langle\, a,k(t)\,\rangle+\dfrac{\gamma|2D^Tk(t)+g|^2}{2(1-\gamma)}=0, \\ k_0(T)=0. \end{cases}
$$

It is clear that once (5.18) admits a unique solution $K(\cdot)\in C^1([0,T];\mathcal{S}^2)$, we can obtain unique solutions $k(\cdot)\in C^1([0,T];\mathbb{R}^2)$ and $k_0(\cdot)\in C([0,T];\mathbb{R})$ of (5.19) and (5.20), respectively. Then we obtain a solution $p^\gamma(\cdot\,,\cdot)$ to (5.15).

Let us now look at (5.18), which is a Riccati equation. If $K(\cdot)$ is a solution of (5.18) and we set

$$
(5.21)\qquad \begin{cases} P(t)=-K(t), \qquad t\in[0,T], \\ A_\gamma=A+\dfrac{\gamma}{1-\gamma}DG, \quad Q_\gamma=-\dfrac{\gamma}{1-\gamma}G^TG, \quad R_\gamma=\dfrac{1-\gamma}{2}, \end{cases}
$$

then $P(\cdot)$ satisfies

$$
(5.22)\qquad \begin{cases} \dot P(t)+P(t)A_\gamma+A_\gamma^TP(t)-P(t)DR_\gamma^{-1}D^TP(t)+Q_\gamma=0, \\ P(T)=0. \end{cases}
$$

This is the Riccati equation of the following LQ problem (see Yong and Zhou [19]):

$$
(5.23)\qquad \begin{cases} \dot x(t)=A_\gamma x(t)+Du(t), \\ J(u(\cdot))=\displaystyle\int_0^T\big\{\,\langle\, Q_\gamma x(t),x(t)\,\rangle+\langle\, R_\gamma u(t),u(t)\,\rangle\,\big\}dt. \end{cases}
$$

Clearly, in the case that $\gamma < 0$, both Q_γ and R_γ are positive definite. By standard LQ theory (from Yong and Zhou [19], say), the LQ problem admits a unique optimal control and Riccati equation (5.23) admits a unique solution $P(\cdot)$. Consequently, Problem $(\mathbf{H}_\gamma)$ is solvable.

Recall from Proposition 4.4 that Problem $(\mathbf{H}_\gamma)$ is accessible for any $\gamma \le 0$. From this we obtain unique solvability for all $\gamma \le 0$. The case when $\gamma \in (0,1)$ is much more subtle because in this case, although R_γ is still positive definite, Q_γ is negative definite. The corresponding LQ problem is solvable only if some conditions are satisfied (see Yong and Zhou [19]). We state the following result, which is found in Yong and Zhou [19] (stated in terms of LQ problem (5.23)).

Proposition 5.2. *If there exists a* $\delta > 0$ *such that*

$$\begin{aligned}\frac{1-\gamma}{2}\int_s^T |u(t)|^2 dt - \frac{\gamma}{1-\gamma}\int_s^T \Big|\int_s^t G e^{A_\gamma(t-\tau)} D u(\tau) d\tau\Big|^2 dt \\ \ge \delta \int_s^T |u(t)|^2 dt, \qquad \forall u(\cdot) \in L^2(s,T;\mathbb{R}^2),\ s \in [0,T),\end{aligned} \tag{5.24}$$

then LQ problem (5.23) *is uniquely solvable.*

According to the above analysis, condition (5.24) also gives a sufficient condition for the solvability of Problem $(\mathbf{H}_\gamma)$. Let us make some further manipulations of (5.24). Since

$$\begin{aligned}\int_s^T \Big|\int_s^t G e^{A_\gamma(t-\tau)} D u(\tau) d\tau\Big|^2 dt &\le \int_s^T \Big\{\int_s^t |G e^{A_\gamma(t-\tau)} D|^2 d\tau \int_s^t |u(\tau)|^2 d\tau\Big\} dt \\ &\le (T-s)\int_s^T |G e^{A_\gamma t} D|^2 dt \int_s^T |u(t)|^2 dt,\end{aligned} \tag{5.25}$$

it follows that (5.24) holds if

$$\gamma T \int_0^T |G e^{A_\gamma t} D|^2 dt < \frac{(1-\gamma)^2}{2}. \tag{5.26}$$

This condition has a very similar nature to that of (4.8). Note that (5.26) holds automatically for $\gamma \le 0$. Hence we may summarize the above analysis to state the following theorem.

Theorem 5.3. *For any* $\gamma \in (-\infty, 1)$ *satisfying* (5.26), *Problem* $(\mathbf{H}_\gamma)$ *is solvable at any* $(x,v) \in \mathbb{R}^2 \times (0,\infty)$ *with the optimal portfolio given by*

$$\bar{h}(t) = \frac{[N(t)^T]^{-1}[(2D^T K(t) + G)X(t) + 2D^T k(t) + g]}{1-\gamma}, \quad t \in [0,T). \tag{5.27}$$

We note that when $\gamma = 0$, the unique solution of (5.18) is $K(\cdot) = 0$, and thus from (5.19) we have $k(\cdot) = 0$ as well. Then (5.27) coincides with (5.2).

References

[1] I. Bajeux-Besnainou, J.V. Jordan, and R. Portait, *Dynamic allocation of stocks, bonds, and cash for long horizons, Journal of Business,* forthcoming.

[2] T. Bielecki and S.R. Pliska, *A risk sensitive intertemporal CAPM, with application to fixed income management,* preprint.

[3] S. Chen and J. Yong, *Stochastic linear quadratic optimal control problems, Appl. Math. Optim.,* 43 (2001), 21–45.

[4] J.C. Cox, J.E. Ingersoll, and S.A. Ross, *A theory of the term structure of interest rates, Econometrica* 53 (1985), 385-408.

[5] G. Deelstra, M. Grasselli, and P.-F. Koehl, *Optimal investment strategies in a CIR framework, J. Appl. Probability,* 37 (2000), 936–946.

[6] D. Duffie and R. Kan, *A yield-factor model of interest rates, Math. Finance,* 6 (1996), 379–406.

[7] J.M. Harrison and S.R. Pliska, *Martingales and stochastic integrals in the theory of continuous trading, Stochastic Processes and their Applications*, 11 (1981), 215–260.

[8] I. Karatzas and S.E. Shreve, *Methods of Mathematical Finance,* Applications of Math. 39, Springer-Verlag, Berlin, 1998.

[9] J. Liu, *Portfolio selection in stochastic environments, Stanford University Ph.D. dissertation,* (1999).

[10] J. Ma and J. Yong, *Forward-Backward Stochastic Differential Equations and Their Applications,* Lecture Notes in Math., Vol. 1702, Springer-Verlag, Berlin, 1999.

[11] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modeling,* Applications of Math. 36, Springer-Verlag, Berlin, 1997,

[12] E. Pardoux and S. Peng, *Adapted solution of a backward stochastic differential equation, Systems Control Lett.,* 14 (1990), 55–61.

[13] S.R. Pliska, *A stochastic model of continuous trading: optimal portfolios, Mathematics of Operations Research,* 11 (1986), 371-384.

[14] M. Rutkowski, *Self-financing trading strategy for sliding, rolling-horizon, and consol bonds, Math. Finance,* 9 (1999), 361–385.

[15] C. Sorensen, *Dynamic asset allocation and fixed income management, J. Financial and Quantitative Analysis,* 34 (2002), 513-531.

[16] O.A. Vasicek, *An equilibrium characterization of the term structure, J. Financial Economics,* 5 (1977), 177-188.

[17] J. Yong, *Optimal portfolios in an incomplete market, Ann. Econ. Finance,* 1 (2000), 359–381.

[18] J. Yong, *Some estimates on exponentials of solutions to stochastic differential equations,* preprint.

[19] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations,* Springer-Verlag, New York, 1999.

Appendix A

In this appendix we shall use some ideas of Duffie and Kan [5] and Ma and Yong [10] to determine the discount bond's volatility $\nu(\cdot)$, thereby filling in some details omitted from subsection 2.2.

First of all, let $\theta(\cdot)$ be given by (2.5) satisfying (2.3)–(2.4). We define the risk neutral probability measure $\widetilde{\mathbf{P}}$ and the corresponding standard Brownian motion $\widetilde{W}(t)$ as in (2.11) and (2.12), respectively. Substituting (2.5), (2.7), and (2.8) into (2.2) yields

$$\begin{cases} dS_1(t) = S_1(t)\big[e_1^T X(t)dt + \langle\, \sigma, d\widetilde{W}(t)\,\rangle\big], \\ dS_2(t) = S_2(t)\big[e_1^T X(t)dt + \langle\, \nu(t), d\widetilde{W}(t)\,\rangle\big]. \end{cases} \tag{A.1}$$

Next we observe (recall (2.1) and (2.5))

$$\begin{aligned} dX(t) &= [AX(t)+a]dt + DdW(t) \\ &= [AX(t)+a]dt + D[d\widetilde{W}(t) - \theta(t)dt] \\ &= [(A-DG)X(t) + (a-Dg)]dt + Dd\widetilde{W}(t) \\ &\stackrel{\Delta}{=} [BX(t)+b]dt + Dd\widetilde{W}(t). \end{aligned} \tag{A.2}$$

Note we have defined the matrix B and the vector b as in (2.14), and so if a, A, and D are specified, then determining g and G is equivalent to determining b and B. Moreover, (2.1) becomes (2.13).

Now suppose that for any times t and τ the market price at time t of a zero-coupon bond maturing at $t+\tau$ is given by $F(\tau, X(t))$ for some smooth function $F : [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}$. Define

$$R(x) \stackrel{\Delta}{=} \lim_{\tau\downarrow 0} \frac{-\log F(\tau, x)}{\tau}, \qquad \forall x \in \mathbb{R}^2. \tag{A.3}$$

Thus the short rate is given by

$$r(t) = R(X(t)), \qquad t \in [0,\infty). \tag{A.4}$$

Consider a discounted zero-coupon bond maturing at a fixed time $T > 0$ and whose price process is denoted by $Y(\cdot)$, and suppose the short interest rate is $R(X(t))$. Then $Y(\cdot)$ satisfies the backward stochastic differential equation (2.15). If $(Y(\cdot), Z(\cdot))$ is the adapted solution of (2.15), then

$$Y(t) = e^{-\int_t^T R(X(s))ds} - \int_t^T e^{-\int_s^T R(X(\tau))d\tau} \langle Z(s), d\widetilde{W}(s) \rangle, \quad t \in [0,T], \tag{A.5}$$

which implies (2.16).

Next we use the idea of Ma and Yong [10] to assume (2.17), that is,

$$Y(t) = u(t, X(t)), \qquad t \in [0,T].$$

Then by Itô's formula we have (note (2.13))

$$\begin{aligned} & R(X(t))u(t,X(t))dt + \langle Z(t), d\widetilde{W}(t) \rangle = dY(t) = d[u(t,X(t))] \\ & = \big\{u_t(t,X(t)) + \langle u_x(t,X(t)), BX(t) + b \rangle + \frac{1}{2}\mathrm{tr}\big[u_{xx}(t,X(t))DD^T\big]\big\}dt \\ & \quad + \langle u_x(t,X(t)), Dd\widetilde{W}(t) \rangle . \end{aligned} \tag{A.6}$$

Consequently, we should choose $u(\cdot\,,\cdot)$ to be the solution of the partial differential equation (2.18). By (2.16) and (2.17) we also have

$$u(t,X(t)) = E_{\widetilde{\mathbf{P}}}\Big[e^{-\int_t^T R(X(s))ds} \,\Big|\, \mathcal{F}_t\Big], \qquad t \in [0,T]. \tag{A.7}$$

To determine the solution of (2.18) we follow Duffie and Kan [5] and suppose that $u(t,x)$ has the exponential form (2.19), that is,

$$u(t,x) = e^{\eta(t)+\langle \xi(t), x \rangle}, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^2,$$

for some deterministic functions $\xi(\cdot)$ and $\eta(\cdot)$. Then by the terminal condition in (2.18) we have

$$\eta(T)=0, \qquad \xi(T)=0. \tag{A.8}$$

Take $R(x)=\langle e_1, x\rangle$, so by the equation in (2.18) one has

$$\begin{aligned} 0 &= \dot\eta(t)+\langle\dot\xi(t),x\rangle+\langle\xi(t),Bx+b\rangle+\frac{1}{2}|D^T\xi(t)|^2-\langle e_1,x\rangle \\ &= \dot\eta(t)+\langle\xi(t),b\rangle+\frac{1}{2}|D^T\xi(t)|^2+\langle\dot\xi(t)+B^T\xi(t)-e_1,x\rangle, \\ &\qquad \forall (t,x)\in[0,\infty)\times\mathbb{R}^2. \end{aligned} \tag{A.9}$$

Hence we end up with equations (2.20) and (2.21) for $\eta(\cdot)$ and $\xi(\cdot)$. In other words, if $\xi(\cdot)$ and $\eta(\cdot)$ are solutions of (2.20) and (2.21), respectively, then the function $u(\cdot\,,\cdot)$ defined by (2.19) is the solution of (2.18). In particular, the solution $\xi(\cdot)$ of (2.20) is given by

$$\xi(t)=-\int_t^T e^{B^T(s-t)}e_1ds=-\int_0^{T-t}e^{B^Ts}e_1ds, \quad t\le T, \tag{A.10}$$

and the solution $\eta(\cdot)$ of (2.21) is given by

$$\begin{aligned} \eta(t) &= -\int_t^T\big[\langle b,\xi(s)\rangle+\frac{1}{2}|D^T\xi(s)|^2\big]ds, \\ &= \int_t^T\big[\langle b,\int_0^{T-s}e^{B^T\tau}e_1d\tau\rangle-\frac{1}{2}|D^T\int_0^{T-s}e^{B^T\tau}e_1d\tau|^2\big]ds, \\ &= \int_0^{T-t}\big[\langle b,\int_0^{s}e^{B^T\tau}e_1d\tau\rangle-\frac{1}{2}|D^T\int_0^{s}e^{B^T\tau}e_1d\tau|^2\big]ds, \quad t\le T. \end{aligned} \tag{A.11}$$

Next we consider the price at (the possibly negative) time $T-\widehat T$ of a zero-coupon, discount bond that matures at time T. Then under the no-arbitrage condition, the risk-free yield should be the same as $\rho(T-\widehat T)\equiv e_2^TX(T-\widehat T)$. By (2.19), and looking at the price of this bond at $T-\widehat T$, we have

$$e^{-\rho(T-\widehat T)\widehat T}=u(T-\widehat T,X(T-\widehat T))=e^{\eta(T-\widehat T)+\langle\xi(T-\widehat T),X(T-\widehat T)\rangle}. \tag{A.12}$$

Hence we should have the additional conditions (2.22) for $\xi(\cdot)$ and $\eta(\cdot)$ (note $X(\cdot)=(r(\cdot),\rho(\cdot))^T$).

For the discount bond's volatility $\nu(\cdot)$, by the definitions of $S_2(\cdot)$ and $u(\cdot\,,\cdot)$ we must have

$$S_2(t)=u(t,X(t)), \qquad t\in[0,T]. \tag{A.13}$$

Then by Itô's formula we obtain (note (A.1) and (2.19)–(2.21))

$$
\begin{aligned}
&S_2(t)\big[\langle e_1, X(t)\rangle\, dt + \langle \nu(t), d\widetilde{W}(t)\rangle\big] = dS_2(t) = d[u(t, X(t))] \\
&= u(t, X(t))\big[\dot{\eta}(t) + \langle \dot{\xi}(t), X(t)\rangle + \langle \xi(t), BX(t) + b\rangle + \frac{1}{2}|D^T\xi(t)|^2\big]dt \\
&\qquad + u(t, X(t))\,\langle \xi(t), Dd\widetilde{W}(t)\rangle \\
&= S_2(t)\big[\langle e_1, X(t)\rangle\, dt + \langle D^T\xi(t), d\widetilde{W}(t)\rangle\big].
\end{aligned} \tag{A.14}
$$

Hence by (2.7) we obtain the equations in (2.23).

Appendix B

As discussed in Subsection 2.3, for a consistent model that is complete, free of arbitrage opportunities, and calibrated to market data, we would like to solve the following system of equations for B:

$$
\begin{cases}
\widehat{T}B^T e_2 = c + \widehat{T}A^T e_2 - e_1, \\
\displaystyle\int_0^{\widehat{T}} e^{B^T t} e_1 dt = \widehat{T} e_2,
\end{cases} \tag{B.1}
$$

subject to the following constraint:

$$
\int_0^{\widehat{T}} t e_1^T e^{B^T t} e_1 dt \neq 0, \tag{B.2}
$$

for given A, c, and $\widehat{T}$ (satisfying suitable conditions). In this appendix we study the existence of solutions to this system.

We denote a possible solution of (B.1)–(B.2) to be

$$
B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. \tag{B.3}
$$

It is clear that the first equation in (B.1) is equivalent to the following:

$$
\begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix} = B^T e_2 = \widehat{T}^{-1}(c - e_1) + A^T e_2. \tag{B.4}
$$

Hence we need only to determine b_{11} and b_{12}, or, equivalently, $B^T e_1$, from the second equation in (B.1) and the constraint (B.2), because $B^T e_2$ is given by (B.4).

We start with some necessary conditions for the solutions. The following is our first result.

Proposition B.1. *Suppose* (B.1) *admits a solution* $B^T = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$. *Then* A, c *and* $\widehat{T}$ *must satisfy the following necessary condition:*

$$e_1^T(c + \widehat{T}A^T e_2) \neq 1, \tag{B.5}$$

and

$$b_{12} \neq 0. \tag{B.6}$$

Proof. By (B.4), we see that (B.5) is the same as $b_{21} \neq 0$. Now suppose either $b_{12} = 0$, or $b_{21} = 0$. Then B^T is upper or lower triangle matrix, and thus

$$e^{B^T s} = \begin{pmatrix} e^{b_{11}s} & * \\ * & e^{b_{22}s} \end{pmatrix}, \quad s \in \mathbb{R}, \tag{B.7}$$

where $*$ represents some entries that are irrelevant. Consequently, noting the second equation in (B.1),

$$0 = e_1^T(\widehat{T}e_2) = \int_0^{\widehat{T}} e_1^T e^{B^T s} e_1 ds = \int_0^{\widehat{T}} e^{b_{11}s} ds > 0, \tag{B.8}$$

which is a contradiction, proving (B.5) and (B.6). □

Now we would like to draw some further necessary conditions for a solution B of (B.1).

Proposition B.2. *For any* $B \in \mathbb{R}^{2\times 2}$,

$$e^{B^T t} = \varphi_0(t)I + \varphi_1(t)B^T, \qquad \forall t \in \mathbb{R}, \tag{B.9}$$

with $(\varphi_0(\cdot), \varphi_1(\cdot))^T$ *being the solution of the following:*

$$\begin{cases} \begin{pmatrix} \dot\varphi_0(t) \\ \dot\varphi_1(t) \end{pmatrix} = \begin{pmatrix} 0 & -(\det B) \\ 1 & (\mathrm{tr} B) \end{pmatrix} \begin{pmatrix} \varphi_0(t) \\ \varphi_1(t) \end{pmatrix}, \\ \begin{pmatrix} \varphi_0(0) \\ \varphi_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{cases} \tag{B.10}$$

If I *and* B *are linearly independent, which is the case when* (B.5)–(B.6) *holds, then representation* (B.9) *is unique.*

Proof. First of all, for $B \in \mathbb{R}^{2\times 2}$, by the Cayley-Hamilton theorem one has

$$B^2 - (\mathrm{tr} B)B + (\det B)I = 0. \tag{B.11}$$

Therefore, if $(\varphi_0(\cdot), \varphi_1(\cdot))^T$ is the solution of (B.9), then

$$
\begin{aligned}
&\frac{d}{dt}\big[\varphi_0(t)I + \varphi_1(t)B^T\big] = \dot{\varphi}_0(t)I + \dot{\varphi}_1(t)B^T \\
&= -(\det B)\varphi_1(t)I + [\varphi_0(t) + (\mathrm{tr}B)\varphi_1(t)]B^T \\
&= \varphi_0(t)B^T + \varphi_1(t)[-(\det B)I + (\mathrm{tr}B)B^T] \\
&= \varphi_0(t)B^T + \varphi_1(t)(B^T)^2 = B^T[\varphi_0(t)I + \varphi_1(t)B^T].
\end{aligned} \tag{B.12}
$$

This together with $\varphi_0(0) = 1$ and $\varphi_1(0) = 0$ means that $e^{B^T t}$ admits representation (B.9). If I and B are linearly independent, it is easy to prove that the representation (B.9) is unique. □

Corollary B.3. *Let B be a solution of* (B.1)*. Then*

$$\int_0^{\widehat{T}} \varphi_1(t)dt \neq 0, \tag{B.13}$$

and

$$\begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} \equiv B^T e_1 = \Big(\int_0^{\widehat{T}} \varphi_1(t)dt\Big)^{-1}\Big[\widehat{T}e_2 - \Big(\int_0^{\widehat{T}} \varphi_0(t)dt\Big)e_1\Big]. \tag{B.14}$$

Proof. By (B.9) and the second equation in (B.1) we have

$$\widehat{T}e_2 = \int_0^{\widehat{T}} e^{B^T t}e_1 dt = \Big(\int_0^{\widehat{T}} \varphi_0(t)dt\Big)e_1 + \Big(\int_0^{\widehat{T}} \varphi_1(t)dt\Big)B^T e_1. \tag{B.15}$$

Since $\widehat{T} > 0$ and e_1 and e_2 are linearly independent, we must have (B.13). Then (B.14) follows from (B.15). □

The next result gives some solutions to (B.1)–(B.2) (under certain conditions).

Proposition B.4. *Suppose B is a solution of* (B.1)–(B.2) *such that*

$$\det B = 0. \tag{B.16}$$

Then A, c, and $\widehat{T}$ satisfy the following necessary conditions:

$$(e_1 + e_2)^T(c + \widehat{T}A^T e_2) = 1, \tag{B.17}$$

$$e_2^T(c + \widehat{T}A^T e_2) > 1, \tag{B.18}$$

and B must be of the following form:

$$B = \begin{pmatrix} -b_{22} + \widehat{T}^{-1} z_0 & b_{22} - \widehat{T}^{-1} z_0 \\ -b_{22} & b_{22} \end{pmatrix}, \tag{B.19}$$

where b_{22} is given by (B.4) *and z_0 is determined in the following way:*

(i) *In the case that*

$$b_{22}\widehat{T} \equiv e_2^T(c + \widehat{T} A^T e_2) = 2, \tag{B.20}$$

$z_0 = 0$, $\mathrm{tr}B = 0$, *and*

$$B = \frac{2}{\widehat{T}} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{B.21}$$

(ii) *In the case that*

$$b_{22}\widehat{T} \equiv e_2^T(c + \widehat{T} A^T e_2) \neq 2, \tag{B.22}$$

z_0 is the unique non-zero solution of

$$f(z) \stackrel{\Delta}{=} (z - b_{22}\widehat{T})(e^z - 1 - z) + z^2 = 0 \tag{B.23}$$

and $\mathrm{tr}B \neq 0$.

Moreover, (B.19) *and* (B.21) *do give a solution to* (B.1) *satisfying* (B.2).

Proof. Suppose B is a solution of (B.1)–(B.2) satisfying (B.16). Then, by (B.5)–(B.6), we know that

$$b_{11}b_{22} = (\det B) + b_{12}b_{21} = b_{12}b_{21} \neq 0. \tag{B.24}$$

Now, under (B.16), system (B.10) gives:

$$\varphi_0(t) \equiv 1, \qquad t \in \mathbb{R}, \tag{B.25}$$

and

$$\varphi_1(t) = \begin{cases} \dfrac{e^{(\mathrm{tr}B)t} - 1}{\mathrm{tr}B}, & \mathrm{tr}B \neq 0, \\ t, & \mathrm{tr}B = 0. \end{cases} \tag{B.26}$$

Consequently, (B.14) becomes

$$\begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} \equiv B^T e_1 = \widehat{T}\Big(\int_0^{\widehat{T}} \varphi_1(t)dt\Big)^{-1}(e_2 - e_1). \tag{B.27}$$

This implies

$$b_{11} + b_{12} = 0. \tag{B.28}$$

Then it follows from (B.24) that

$$(e_1 + e_2)^T B^T e_2 = b_{21} + b_{22} = 0, \tag{B.29}$$

which implies (B.17) (by using (B.4)).

Now we separate two cases:

Case 1: If $\mathrm{tr}B \equiv b_{11} + b_{22} = 0$, then equation (B.27) is equivalent to

$$-\widehat{T} = b_{11}\int_0^{\widehat{T}} \varphi_1(t)dt = b_{11}\int_0^{\widehat{T}} tdt = \frac{b_{11}\widehat{T}^2}{2}. \tag{B.30}$$

Hence, by (B.28)–(B.29) together with $\mathrm{tr}B = 0$, we conclude that in this case B has the form (B.21). This also implies that $b_{22}\widehat{T} = 2$, which is the same as (B.20).

Case 2: If $\mathrm{tr}B \neq 0$, then, instead of (B.30), we have to solve

$$\begin{aligned} -\widehat{T} &= b_{11}\int_0^{\widehat{T}} \varphi_1(t)dt = \frac{b_{11}}{b_{11}+b_{22}}\int_0^{\widehat{T}} \big[e^{(b_{11}+b_{22})t} - 1\big]dt \\ &= \frac{b_{11}}{b_{11}+b_{22}} \frac{e^{(b_{11}+b_{22})\widehat{T}} - 1 - (b_{11}+b_{22})\widehat{T}}{b_{11}+b_{22}}. \end{aligned} \tag{B.31}$$

Note that b_{22} is given by (B.4). Thus the above is an equation for b_{11}. To solve this equation, we introduce a variable

$$z = (b_{11} + b_{22})\widehat{T}. \tag{B.32}$$

Then some simple calculation show that one has to solve equation (B.23). To this end, we observe the following.

$$\begin{cases} f'(z) = ze^z - (b_{22}\widehat{T} - 1)(e^z - 1), \\ f''(z) = e^z[z - (b_{22}\widehat{T} - 2)]. \end{cases} \tag{B.33}$$

Thus

(B.34) $$f'(0) = 0, \quad f'(\infty) = \infty, \quad f'(-\infty) = b_{22}\widehat{T} - 1,$$

and

(B.35) $$f''(z) \begin{cases} > 0, & z > b_{22}\widehat{T} - 2, \\ = 0, & z = b_{22}\widehat{T} - 2, \\ < 0, & z < b_{22}\widehat{T} - 2. \end{cases}$$

Hence we have the following several subcases:

(a) $b_{22}\widehat{T} > 2$: In this case, $f(\cdot)$ is convex on $(b_{22}\widehat{T}-2,\infty)$ and concave on $(-\infty, b_{22}\widehat{T}-2)$, with the unique stationary point $z = b_{22}\widehat{T} - 2$, and with $z = 0$ being a local strict maximum. Consequently, $f(\cdot)$ is strictly increasing on $(-\infty, 0)$ and on $(\bar{z},\infty)$ for some $\bar{z} \in (0, b_{22}\widehat{T} - 2)$, and strictly decreasing on $(0, \bar{z})$ (thus, this $\bar{z}$ is a local strict minimum of $f(\cdot)$). Hence $f(\cdot)$ admits a unique non-zero root $z_0 \in (b_{22}\widehat{T} - 2, \infty)$.

(b) $1 < b_{22}\widehat{T} < 2$: In this case, $f(\cdot)$ is also convex on $(b_{22}\widehat{T} - 2, \infty)$ and concave on $(-\infty, b_{22}\widehat{T} - 2)$, with the unique stationary point $z = b_{22}\widehat{T} - 2$. But now, $z = 0$ is a local strict minimum. Consequently, $f(\cdot)$ is strictly increasing on $(-\infty, \bar{z})$ and on $(0,\infty)$ for some $\bar{z} \in (b_{22}\widehat{T} - 2, 0)$, and strictly decreasing on $(\bar{z}, 0)$ (thus, this $\bar{z}$ is a local strict maximum of $f(\cdot)$). Hence $f(\cdot)$ admits a unique non-zero root $z_0 \in (-\infty, b_{22}\widehat{T} - 2)$.

(c) $b_{22}\widehat{T} = 2$: In this case, $f(\cdot)$ is strictly increasing on $\mathbb{R}$. Thus $f(\cdot)$ only admits the zero solution.

(d) $b_{22}\widehat{T} \leq 1$: In this case, $f(\cdot)$ is strictly increasing on $(0,\infty)$ and strictly decreasing on $(-\infty, 0)$. Thus $f(\cdot)$, again, only admits the zero solution.

Now, in subcases (c) and (d), $f(\cdot)$ only admits the zero solution. This corresponds to the case $\mathrm{tr}B = 0$ discussed above. By the analysis there, we see that $b_{22}\widehat{T} = 2$ has to be true. Hence subcase (d) is not possible. In other words, in order for the solution B to satisfy $\det B = 0$, one must have $b_{22}\widehat{T} > 1$. This gives (B.18). The subcase (c) merges into Case 1 above.

Next we look at subcases (a) and (b). For these subcases we let z_0 be the unique non-zero solution of (B.23). Then

(B.36) $$b_{11} = -b_{22} + \widehat{T}^{-1}z_0, \qquad b_{12} = b_{22} - \widehat{T}^{-1}z_0.$$

Hence B has to be of the form (B.18).

Finally, we need to check if constraint (B.2) is satisfied. To this end, we note by (B.25) that

$$\begin{aligned}\int_0^{\widehat{T}} te_1 e^{B^T t} e_1 dt &= \int_0^{\widehat{T}} te_1^T [I + \varphi_1(t) B^T] e_1 dt \\ &= \frac{\widehat{T}^2}{2} + b_{11} \int_0^{\widehat{T}} t\varphi_1(t) dt \\ &= \frac{\widehat{T}^2}{2} + b_{11}\widehat{T} \int_0^{\widehat{T}} \varphi_1(t) dt - b_{11} \int_0^{\widehat{T}} \int_0^t \varphi_1(s) ds dt \\ &= -\frac{\widehat{T}^2}{2} - b_{11} \int_0^{\widehat{T}} \int_0^t \varphi_1(s) ds dt.\end{aligned} \tag{B.37}$$

Thus, in the case $\mathrm{tr}B = 0$, we have

$$\int_0^{\widehat{T}} te_1^T e^{B^T t} e_1 dt = -\frac{\widehat{T}^2}{2} + \frac{2}{\widehat{T}} \int_0^{\widehat{T}} \int_0^t s\, ds\, dt = -\frac{\widehat{T}^2}{2} + \frac{\widehat{T}^2}{3} = -\frac{\widehat{T}^2}{6} \neq 0. \tag{B.38}$$

On the other hand, in the case $\mathrm{tr}B \neq 0$ (i.e., we are in the subcase (a) or (b); see the earlier proof), from equation (B.10) one has

$$\begin{aligned}\int_0^{\widehat{T}} \varphi_1(t) dt &= \int_0^{\widehat{T}} \int_0^t \varphi_0(s) ds dt + (\mathrm{tr}B) \int_0^{\widehat{T}} \int_0^t \varphi_1(s) ds dt \\ &= \frac{\widehat{T}^2}{2} + (\mathrm{tr}B) \int_0^{\widehat{T}} \int_0^t \varphi_1(s) ds dt.\end{aligned} \tag{B.39}$$

Hence, by (B.27),

$$-\widehat{T} = b_{11} \int_0^{\widehat{T}} \varphi_1(t) dt = \frac{b_{11}\widehat{T}^2}{2} + (\mathrm{tr}B) b_{11} \int_0^{\widehat{T}} \int_0^t \varphi_1(s) ds dt. \tag{B.40}$$

Consequently, by (B.37),

$$\begin{aligned}\int_0^{\widehat{T}} te_1 e^{B^T t} e_1 dt &= -\frac{\widehat{T}^2}{2} - b_{11} \int_0^{\widehat{T}} \int_0^t \varphi_1(s) ds dt = -\frac{\widehat{T}^2}{2} + \frac{1}{\mathrm{tr}B} \Big(\widehat{T} + \frac{b_{11}\widehat{T}^2}{2} \Big) \\ &= -\frac{\widehat{T}^2}{2} + \frac{\widehat{T}}{z_0} \Big(\widehat{T} + \frac{-b_{22}\widehat{T}^2 + z_0\widehat{T}}{2} \Big) = \frac{\widehat{T}^2}{2z_0} (2 - b_{22}\widehat{T}) \neq 0.\end{aligned} \tag{B.41}$$

This completes the proof. □

We now consider the case where $\det B \neq 0$. In this case, by applying B^T to the second equation in (B.1), one obtains (noting (B.4))

$$(e^{B^T\widehat{T}} - I)e_1 = \widehat{T}B^Te_2 = c - e_1 + \widehat{T}A^Te_2. \tag{B.42}$$

This leads to

$$e^{B^T\widehat{T}}e_1 = c + \widehat{T}A^Te_2 = \begin{pmatrix} b_{21}\widehat{T}+1 \\ b_{22}\widehat{T} \end{pmatrix} \equiv \begin{pmatrix} \bar{c}_1 \\ \bar{c}_2 \end{pmatrix}, \tag{B.43}$$

which is equivalent to the second equation in (B.1) (because B is invertible). Keep in mind that we only need to solve for b_{11} and b_{12} from the above two (scalar) equations.

To solve (B.43) we shall use the following:

Proposition B.5. *Denote*

$$\Delta \triangleq (\mathrm{tr}B)^2 - 4(\det B). \tag{B.44}$$

(i) *If $\Delta > 0$, then B has two different real eigenvalues:*

$$\lambda_1 = \frac{\mathrm{tr}B + \sqrt{\Delta}}{2}, \qquad \lambda_2 = \frac{\mathrm{tr}B - \sqrt{\Delta}}{2}, \tag{B.45}$$

and

$$\begin{cases} \varphi_0(t) = \dfrac{-\lambda_2 e^{\lambda_1 t} + \lambda_1 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \\ \varphi_1(t) = \dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}. \end{cases} \tag{B.46}$$

(ii) *If $\Delta = 0$, then B has one real eigenvalue of multiplicity 2:*

$$\lambda = \frac{\mathrm{tr}B}{2}, \tag{B.47}$$

and

$$\begin{cases} \varphi_0(t) = e^{\lambda t}(1 - \lambda t), \\ \varphi_1(t) = te^{\lambda t}. \end{cases} \tag{B.48}$$

(iii) *If $\Delta < 0$, then B has a pair of complex eigenvalues:*

$$\lambda = \frac{\mathrm{tr}B \pm i\sqrt{-\Delta}}{2} \triangleq \alpha \pm i\beta, \tag{B.49}$$

and

(B.50)
$$\begin{cases} \varphi_0(t) = \dfrac{e^{\alpha t}}{\beta}\big[\beta\cos\beta t - \alpha\sin\beta t\big], \\ \varphi_1(t) = -\dfrac{e^{\alpha t}\sin\beta t}{\beta}. \end{cases}$$

Proof. (i). In this case, B has two different real eigenvalues given by (B.45). Thus

$$\begin{cases} \varphi_0(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \\ \varphi_1(t) = -\dfrac{\dot\varphi_0(t)}{\det B} = -\dfrac{1}{\det B}\big[\lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t}\big]. \end{cases}$$

By the initial conditions in (B.10), we obtain

$$\begin{cases} C_1 + C_2 = 1, \\ \lambda_1 C_1 + \lambda_2 C_2 = 0. \end{cases}$$

Thus
$$C_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1}, \qquad C_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2}.$$
Hence, noting $\det B = \lambda_1\lambda_2$, we have (B.46).

(ii). In this case, B has one real eigenvalue of multiplicity 2 given by (B.47). Thus

$$\begin{cases} \varphi_0(t) = e^{\lambda t}(C_1 + C_2 t), \\ \varphi_1(t) = -\dfrac{1}{\det B} e^{\lambda t}\big[\lambda(C_1 + C_2 t) + C_2\big]. \end{cases}$$

By the initial condition in (B.10), we obtain

$$C_1 = 1, \qquad C_2 = -\lambda.$$

Hence, noting $\det B = \lambda^2$, we obtain (B.48).

(iii). In this case, B has a pair of complex eigenvalues given by (B.49). Thus

$$\begin{cases} \varphi_0(t) = e^{\alpha t}\big[C_1\cos\beta t + C_2\sin\beta t\big], \\ \varphi_1(t) = -\dfrac{1}{\det B} e^{\alpha t}\big\{\alpha\big[C_1\cos\beta t + C_2\sin\beta t\big] + \beta\big[-C_1\sin\beta t + C_2\cos\beta t\big]\big\}. \end{cases}$$

By the initial conditions, we have

$$C_1 = 1, \qquad C_2 = -\frac{\alpha}{\beta}.$$

Hence, noting $\det B = \alpha^2 + \beta^2$, we have (B.50). $\square$

Now we return to equation (B.43). By representation (B.9), we have

$$e^{B^T \widehat{T}} e_1 = \big[\varphi_0(\widehat{T}) I + \varphi_1(\widehat{T}) B^T\big] e_1 = \begin{pmatrix} \varphi_0(\widehat{T}) + \varphi_1(\widehat{T}) b_{11} \\ \varphi_1(\widehat{T}) b_{12} \end{pmatrix}. \tag{B.51}$$

On the other hand, recall from (B.5) (which means $b_{21} \neq 0$), we have

$$\begin{cases} b_{11} = (\mathrm{tr} B) - b_{22}, \\ b_{12} = \dfrac{b_{11} b_{22} - (\det B)}{b_{21}} = \dfrac{-b_{22}^2 + b_{22}(\mathrm{tr} B) - (\det B)}{b_{21}}. \end{cases} \tag{B.52}$$

Note here, again, that b_{21} and b_{22} are determined by (B.4). Thus they are supposed to be known here. Now we can write (B.51) as follows:

$$\begin{cases} \varphi_0(\widehat{T}) + \varphi_1(\widehat{T}) \big[(\mathrm{tr} B) - b_{22}\big] = \bar{c}_1, \\ \varphi_1(\widehat{T}) \dfrac{-b_{22}^2 + b_{22}(\mathrm{tr} B) - (\det B)}{b_{21}} = \bar{c}_2. \end{cases} \tag{B.53}$$

We separate three cases.

Case 1: Suppose B has two distinct real eigenvalues λ_1 and λ_2. In this case, (B.53) becomes (note $(\mathrm{tr} B) = \lambda_1 + \lambda_2$ and $(\det B) = \lambda_1 \lambda_2$)

$$\begin{cases} \dfrac{-\lambda_2 e^{\lambda_1 \widehat{T}} + \lambda_1 e^{\lambda_2 \widehat{T}}}{\lambda_1 - \lambda_2} + \dfrac{e^{\lambda_1 \widehat{T}} - e^{\lambda_2 \widehat{T}}}{\lambda_1 - \lambda_2} \big[\lambda_1 + \lambda_2 - b_{22}\big] = \bar{c}_1, \\ \dfrac{e^{\lambda_1 \widehat{T}} - e^{\lambda_2 \widehat{T}}}{\lambda_1 - \lambda_2} \dfrac{-b_{22}^2 + b_{22}(\lambda_1 + \lambda_2) - \lambda_1 \lambda_2}{b_{21}} = \bar{c}_2. \end{cases} \tag{B.54}$$

This is a system of equations in λ_1 and λ_2. Once a solution $(\zeta_1, \zeta - 2)$ to (B.54) is obtained, from (B.52), we can obtain b_{11} and b_{12} immediately.

Case 2: Suppose B has a real eigenvalue λ with multiplicity 2. In this case, (B.53) becomes (note $\mathrm{tr} B = 2\lambda$ and $\det B = \lambda^2$):

$$\begin{cases} e^{\lambda \widehat{T}} (1 - \lambda \widehat{T}) + \widehat{T} e^{\lambda \widehat{T}} \big[2\lambda - b_{22}\big] = \bar{c}_1, \\ \widehat{T} e^{\lambda \widehat{T}} \dfrac{-b_{22}^2 + 2 b_{22} \lambda - \lambda^2}{b_{21}} = \bar{c}_2. \end{cases} \tag{B.55}$$

This is a system of two equations for just one known λ. Thus some necessary conditions have to be satisfied among the coefficients in order to have a solution λ. Then, similar to the above, by using (B.52), we can obtain b_{11} and b_{12} as long as λ is obtained.

Case 3: Suppose B has a pair of complex eigenvalues $\alpha \pm \beta i$. In this case, (B.53) becomes (note $\text{tr}B = 2\beta$ and $\det B = \alpha^2 + \beta^2$):

$$\text{(B.56)} \qquad \begin{cases} \dfrac{e^{\alpha\widehat{T}}}{\beta}\big[\beta\cos\beta\widehat{T} - \alpha\sin\beta\widehat{T}\big] - \dfrac{e^{\alpha\widehat{T}}\sin\beta\widehat{T}}{\beta}\big[2\beta - b_{22}\big] = \bar{c}_1, \\ -\dfrac{e^{\alpha\widehat{T}}\sin\beta\widehat{T}}{\beta}\dfrac{-b_{22}^2 + 2b_{22}\beta - (\alpha^2+\beta^2)}{b_{21}} = \bar{c}_2. \end{cases}$$

This is a system of equations for unknowns α and β. Again, if we can solve the above, we will be able to obtain b_{11} and b_{12} from (B.52). For example, if we suppose that $b_{22} = \bar{c}_2 = 0$, then the above implies that $\sin\beta\widehat{T} = 0$, so that $\beta\widehat{T} = k\pi$, $k = 0, \pm1, \pm2, \ldots$. Now if $\bar{c}_1 = 0$, then (B.56) does not admit any solution α. If $\bar{c}_1 > 0$, then we can only have $\beta\widehat{T} = 2k\pi$, $k = 0, \pm1, \pm2, \ldots$, in which case $\alpha = \frac{\ln\bar{c}_1}{\widehat{T}}$. Similarly, if $\bar{c}_1 < 0$, then we can only have $\beta\widehat{T} = (2k+1)\pi$, $k = 0, \pm1, \pm2, \ldots$, in which case $\alpha = \frac{\ln(-\bar{c}_1)}{\widehat{T}}$.