A Markov Copulae Approach to Pricing and Hedging of Credit Index Derivatives and Ratings Triggered Step–Up Bonds[*†]

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Abstract

The paper presents selected results from the theory of Markov copulae and some of their applications in finance.

[*] This research was partially supported by NSF Grant DMS-0604789 and Moody's Corporation grant 5-55411.

[†] The authors would like to express their sincere gratitude and appreciation to Matt Woodhams from the GFI Group, for providing us with data relevant to this paper.

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1 Introduction

For many years, the standard approach to valuation and hedging of credit index derivatives (CID, henceforth) was the static copula model (c.f. [9], [10], [6] and references therein). The appealing feature of copulae is that they allow to separate the specification of marginal default probabilities from the dependence structure. However, they have been found to have several shortfalls ([6],[4]), and their static nature is not well suited for hedging and pricing of certain CID.

In recent years, two major families of dynamic models have been developed. One is known as bottom-up and the other is known as top-down. Bottom-up models ([1],[3], [4], [6], [11], [12], [15], [17]) describe the evolution of each individual default process, and are therefore driven by the information generated by the underlying pool of obligations (full information). Top-down models ([2],[8], [16], [13], [22], [24], [25]), on the other hand, describe the evolution of the portfolio loss process (or functionals thereof) and can be viewed as reduced information models, since essentially, only the information about the sum of the defaults, and/or its functionals, such as the cumulative loss process, is used. Top down models proved to be computationally efficient, and well suited for calibration to the term structure of CDO spreads. However, the top down approach fails to consistently incorporate the marginal information (information generated by individual obligors), and thus it cannot produce sensible hedging results.

It is apparent that, in order to capture satisfactorily the relationship between individual default processes and aggregate loss, one cannot avoid working with the full information. The aim of this paper is to combine the advantages of copula models with those of a dynamic bottom up approach. The theory of Markov copulae, which can be loosely regarded as "copulae" for Markov processes, has proved to be useful in this regard.

It also appears that Markov copulae serve as a useful tool for valuation of certain financial products, such a ratings triggered corporate bonds, whose cash flows flows depend on ratings assigned to the issuer by at least two rating agencies.

The paper is organized as follows: in the next section, we provide some relevant mathematical results regarding Markov copulae. Section 3 is devoted to the application of these results to pricing and hedging of CID. In Section 4 we apply Markov copulae to pricing of ratings triggered step up corporate bonds.

2 Multivariate Markov Processes with Given Marginals and Markov Copulae

2.1 Selected Results From the Theory of Markov Copulae

In a nutshell, given a collection of Markov processes $\{X^i\}_{i=1}^d$, a Markov copula allows to construct a multivariate Markov process, $Y = (Y^i)_{i=1}^d$, whose each component Y^i is a Markov process equal in law to X^i. It is clear that process Y must posses some additional structure, besides the Markov property, as it is generally not true that components of a multivariate Markov process are themselves Markovian. The theory of Markov copulae exploits this additional structure, which we shall now briefly discuss. For a comprehensive treatment of the analytical theory of Markov processes, that we use in what follows, we refer to [14].

2.1.1 Markovian Consistency

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with some filtration $\mathbb{F}$, we let $X = (X^1, \dots, X^d)$ be a multivariate $\mathbb{F}$-Markov process taking values in a separable metric product space $E = \Pi_{i=1}^d E_i := E_1 \times \cdots \times E_d$. We require that, for some index set $I \subset \{1, \dots, d\}$, the component $X^I := (X^i,\, i \in I)$ be an $\mathbb{F}$-Markov process, i.e. we require that, for all $f \in B(E_I)$ (bounded, measurable functions), and for all $t, s \geq 0$,

$$\mathbb{E}(f(X^I_{t+s})\big|\mathcal{F}_t) = \mathbb{E}(f(X^I_{t+s})\big|X^I_t), \tag{1}$$

where $E_I = \Pi_{i\in I} E_i$.

Definition 2.1 We say that a Markov process X satisfies *Markovian consistency condition for* X^I if (1) holds. If, in addition, X^I is equal in law to a Markov process Y, taking values in E_I, we say that X satisfies *Markovian consistency condition for* (X^I, Y).

In what follows, we provide sufficient and necessary conditions for Markov consistency to hold. Towards this end, let $L(E) \subset B(E)$ be a given Banach space. For $x = (x_1, \ldots, x_d) \in E$, and for an index set I, we write $x_I = (x_i,\, i \in I)$. Let $(A, \mathcal{D}(A))$ be a linear operator on $L(E)$. We define the following subspaces:

$D^I(E) = \{f \in \mathcal{D}(A) : f(x) = g^f(x_I),\ \text{for some}\, g^f \in B(E_I)\}$,

$D^I(E_I) = \{\mathcal{S}^I(f),\, f \in D^I(E)\}$,

$L^I(E) = \overline{A\big(D^I(E)\big)}$,

$L^I(E_I) = \{\mathcal{S}^I(f),\, f \in L^I(E)\}$,

where $\mathcal{S}^I(f)$ denotes trace of f on E_I, which is defined as

$$\mathcal{S}^I(f) = g^f.$$

The following two propositions are borrowed from [7].

Proposition 2.1 *Let $(A, \mathcal{D}(A))$ be a linear operator on $L(E)$. Suppose that A is the (infinitesimal) generator[1] of an E-valued $\mathbb{F}$-Markov process, $X = (X^1, X^2, \ldots, X^d)$. Assume that the component X^I is $\mathbb{F}$-Markov. Then, for all $f \in D^I(E)$ it holds that:*

$$Af(x) = h^f(x_I),\ \textit{for some}\ h^f\ \textit{in}\ B(E_I). \qquad (i)$$

Proposition 2.2 *Let $(A, \mathcal{D}(A))$ be a linear operator on $L(E)$, s.t. A is the generator of an E-valued $\mathbb{F}$-Markov process, $X = (X^1, X^2, \ldots, X^d)$. Assume that, for every $f \in D^I(E)$, the following holds:*

$$Af(x) = h^f(x_I),\ \textit{for some}\ h^f\ \textit{in}\ B(E_I).$$

Then X^I is an $\mathbb{F}$-Markov process corresponding to the strongly continuous contraction semigroup generated by the operator $(A^I, D^I(E_I))$, defined as:

$$A^I \mathcal{S}^I(f) = \mathcal{S}^I(Af),\ \forall f \in D^I(E). \qquad (ii)$$

Next, we state sufficient conditions on the operator A, so that X^I is $\mathbb{F}$-Markovian with given finite dimensional distributions.

Corollary 2.1 *Let X be an E-valued, $\mathbb{F}$-Markov process with generator A. Let Y be an E_I-valued Markov process, with generator A^Y. Suppose conditions of Proposition 2.2 are satisfied and define A^I by (ii) in Proposition 2.2. Suppose, in addition, that the graphs $\mathfrak{G}(A^I) := (A^I(\mathcal{D}(A^I)), \mathcal{D}(A^I))$ and $\mathfrak{G}(A^Y) := (A^Y(\mathcal{D}(A^Y)), \mathcal{D}(A^Y))$ coincide. Then, X satisfies Markovian consistency condition for (X^I, Y).*

Proof. This immediately follows from Proposition 2.2 and from Proposition 3.1, Chapter 4, in [14]. □

2.1.2 Markov Copulae

We are ready now to introduce the concept of Markov copula. Towards this end, we formulate the following problem:

Given a collection of E_i-valued Markov processes $\{Y^i\}_{\{i=1,\ldots,d\}}$, we want to construct an E-valued process $X = (X^1, X^2, \ldots X^d)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that it is Markov w.r.t. its natural filtration, say $\mathbb{F}^X$, and satisfies Markovian consistency conditions for (X^i, Y^i), $i = 1 \ldots d$.

In the next definition, we denote by $\mathcal{A}$ the collection $\{A^i\}_{\{i=1,\ldots,d\}}$ of generators corresponding to processes Y^i, $i = 1, \ldots, d$.

Definition 2.2 Let $\mathcal{C}^{\mathcal{A}} \subset \mathcal{L}(B(E), B(E))$ be the set of linear operators A satisfying:

i) For all $i = 1 \ldots d$, and for all $f \in D^i(E)$,

$$Af(x) = h^f(x_i)$$

for some $h^f \in B(E_i)$,

[1] All (infinitesimal) generators considered here are taken to be the strong generators. That is why, we shall simplify the notation and say that A is the infinitesimal generator, rather than say that $(A, \mathcal{D}(A))$ is the infinitesimal generator, of a Markov process.

ii) For all $i = 1 \dots d,$ and for all $f \in D^i(E),$ A satisfies

$$A^i \mathcal{S}^i(f) = \mathcal{S}^i(Af),$$

iii) A is the generator of an E-valued Markov process.

Any element $A \in \mathcal{C}^{\mathcal{A}}$ is called a *Markov copula* for processes Y^i, $i = 1 \dots d$.

We now have the following result (cf. [7])

Proposition 2.3 *Let A be a Markov copula for processes Y^i, $i = 1 \dots d$.. Then the canonical Markov process $X = (X^1, \dots, X^d)$ corresponding to the semigroup generated by A satisfies Markovian consistency conditions for (X^i, Y^i), $i = 1, \dots, d$.*

Remark. The results provided above can be extended to time inhomogeneous generators via the standard homogenization argument.

2.2 Examples of Markov Copulae

2.2.1 Diffusion Modulated Multivariate Markov Jump Process

Here we take E to be a compact subset of $\mathbb{R}^d$. Let $J \subset \{1, \dots, d\}$, for any $z \in \mathbb{R}^d$ we denote by z^J an element in $\mathbb{R}^d$ whose i^{th} component is z_i if $i \in J$, and it is 0 otherwise. Next, let $\mathcal{J}$ be the collection of subsets of $\{1, \dots, d\}$ of cardinality of at least 2.

Consider the function spaces $C_0^2(\mathbb{R})$ and $C_0(E_i)$. Let $\widehat{\otimes}$ denote the injective tensor product between Banach spaces, as well as the injective tensor product between linear operators on these spaces (see e.g. [23]). In particular, we have that (see [7]) $C_0^2(\mathbb{R})\widehat{\otimes}C_0(E_i)$ can be identified with $C_0^{2,0}(\mathbb{R} \times E_i)$, and $C_0^2(\mathbb{R})\widehat{\otimes}C_0(E_1)\widehat{\otimes} \dots \widehat{\otimes}C_0(E_d)$ can be identified with $C_0^{2,0\dots0}(\mathbb{R} \times E)$.

Next, let $I^{(i:j)}, j \geq i$, denote the identity operator on the space $C_0(E_i \times E_{i+1} \times \dots \times E_j)$. Suppose now that we are given a collection of d Markov processes Y^i with values in $\mathbb{R} \times E_i$ and with infinitesimal generators $(A^i, C_0^{2,0}(\mathbb{R}, E_i))$ defined by:

$$A^i f(y, x_i) = \mathcal{L}\widehat{\otimes} I^{(i)} f(y, x_i) + \widetilde{A}^i f_i(y, x_i), \tag{5}$$

where $\mathcal{L}$ is the infinitesimal generator of a diffusion process on $\mathbb{R}$, and where $\widetilde{A}^i$ is defined as

$$\widetilde{A}^i f_i(y, x_i) := \lambda^i(y, x_i) \int_{E_i} (f_i(y, x_i + z_i) - f_i(y, x_i))\, \nu^i(dz_i; y),$$

and where $\lambda^i(y, x_i)$ are bounded, measurable functions, and for every $y \in \mathbb{R}$, $\nu^i(dz_i; y)$ is a probability measures on E_i.

We then have the following result ([7])

Proposition 2.4 *Let $\{A^i\}$, $i = 1 \dots d$ be as in (5) and let*

$$\begin{aligned}
Af(y,x) &:= \mathcal{L}\widehat{\otimes} I^{(1:d)} f(y,x) + \sum_{i=1}^{d} I^{(1:i-1)}\widehat{\otimes}\widetilde{A}^i\widehat{\otimes} I^{(i+1:d)} f(y,x) \\
&+ \sum_{S\in\mathcal{J}} \lambda^S(y,x) \int_{E_S} \Big(f(y, x+z^S) - f(y,x)\Big)\, \nu^S(dz_S; y) \\
&- \sum_{i=1}^{d} \sum_{S\in\mathcal{J}: i\in S} \lambda^S(y,x) \int_{E_S} \Big(f(y, x+z^i) - f(y,x)\Big)\, \nu^S(dz_S; y), && (6)\\
\mathcal{D}(A) &:= C_0^{2,0\dots0}(\mathbb{R}\times E), && (7)
\end{aligned}$$

where,

i) for each $y \in \mathbb{R}$, $\nu^S(dz; y)$ is a probability measure on E_S defined as

$$\nu^S(dz_S; y) := C^S(\nu^i(dz_i; y),\, i \in S) \tag{8}$$

for some copula function C^S,

ii) the non negative bounded functions $\lambda^S(y,x)$ are the intensities of simultaneous jumps of components X_i, $i \in S$, and are chosen so that the following condition holds:

$$\sum_{S \in \mathcal{J}: i \in S} \lambda^S(y,x) \leq \lambda^i(y,x_i), \ \forall x \in E, y \in \mathbb{R}, i \in \{1,\dots,d\}. \tag{9}$$

Then, the operator A is an element of $\mathcal{C}^{\mathcal{A}}$,i.e. A is a Markov copula for Y^i, $i = 1,\dots,d$.

2.2.2 Bivariate Markov Chain

In the case when A^i, $i = 1,\dots,d$ are generators of finite state Markov chains, the set $\mathcal{C}^{\mathcal{A}}$ is determined by the set of positive solutions of a linear algebraic system. For notational ease, we shall only discuss the case $d = 2$. The following proposition is borrowed from [5]

Proposition 2.5 *Consider two finite Markov chains X^1 and X^2, w.r.t. their own filtrations, and with values in $\mathcal{O}^1$ and $\mathcal{O}^2$, respectively. Suppose that their respective generators are $A^1(t) = [\alpha^i_j(t)]_{i,j \in \mathcal{O}^1}$ and $A^2(t) = [\beta^h_k(t)]_{h,k \in \mathcal{O}^2}$. Next, consider the system of equations in the unknowns $\lambda^{ih}_{jk}(t)$, where $i,j \in \mathcal{O}^1$, $h,k \in \mathcal{O}^2$ and $(i,h) \neq (j,k)$:*

$$\sum_{k \in \mathcal{O}^2} \lambda^{ih}_{jk}(t) \quad = \quad \alpha^i_j(t), \ \forall h \in \mathcal{O}^2, \ \forall i,j \in \mathcal{O}^1, \ i \neq j \tag{10}$$

$$\sum_{j \in \mathcal{O}^1} \lambda^{ih}_{jk}(t) \quad = \quad \beta^h_k(t), \ \forall i \in \mathcal{O}^1, \ \forall h,k \in \mathcal{O}^2, \ h \neq k. \tag{11}$$

Then, for any positive solution of the above system, the matrix function $A(t) = [\lambda^{ih}_{jk}(t)]_{i,j \in \mathcal{O}^1, k,h \in \mathcal{O}^2}$, with

$$\lambda^{ih}_{ih}(t) = - \sum_{\substack{(j,k) \in \mathcal{O}^1 \times \mathcal{O}^2, \\ (j,k) \neq (i,h)}} \lambda^{ih}_{jk}(t), \tag{12}$$

is a Markov copula for X^i, $i = 1,2$.

Remark. Note that, typically, system (10) –(11) contains many more unknowns than equations. In fact, given that cardinalities of $\mathcal{O}^1$ and $\mathcal{O}^2$ are K^1 and K^2, respectively, the system consists of $K^1(K^1 - 1) + K^2(K^2 - 1)$ equations in $K^1K^2(K^1K^2 - 1)$ unknowns.

3 Credit Index Derivatives: An efficient bottom up approach

In this section, using the above results, we construct a Markovian market model that will underlie pricing and hedging of CID.

3.1 Markovian Market Model

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be the underlying probability space. On $(\Omega, \mathcal{F}, \mathbb{Q})$ we define the following:

- τ_i, $i = 1,\dots,N$, a collection of positive random variables which represent default times in a pool of N obligors;
- $\widetilde{Z}^i_t$, $i = 1,\dots,N$, a family of processes with $\widetilde{Z}^i_0 = 0$ and $\widetilde{Z}^i_t \in (0,1]$ for $t > 0$. $\widetilde{Z}^i_t$ represents the fractional loss incurred at the default time of the i^{th} obligation in the pool, τ_i;
- $\widetilde{X}_t$, an $\mathbb{R}^d$ valued process of (observable) factors of economic/financial relevance. We assume $\widetilde{X}$ is Markov with generator $\mathcal{L}_t$.

We assume that, for every i, the process $(\widetilde{Z}^i, \widetilde{X})$ is Markov in its natural filtration, which we denote by $\mathbb{F}^i$. In addition, we assume that $(\widetilde{Z}^i, X)$, $i = 1\dots,N$, admit the following, time inhomogeneous, infinitesimal generators:

$$A^i_t f(z_i,x) = (I^i \widehat{\otimes} \mathcal{L}_t) f(z_i,x) + \eta^i(x,t,z_i) \int_{(0,1]} \left(f(z_i + y_i, x) - f(z_i,x)\right) \nu^i(dy_i;x,t), \tag{13}$$

where, $\eta^i(x,t,z_i) = \vartheta^i(x,t)\mathbb{1}_{\{z_i=0\}}$ for some non-negative, measurable and bounded function ϑ, and where $\nu^i(dz_i;x,t)$ is a probability measure on $(0,1]$. Possible specifications for $\nu^i(dy_i;x,t)$ are:

– $\nu^i(dy;x,t) = \delta_{1-R}(dy)$, $R \in [0,1)$, which yields the case of constant loss of size $1-R$ at default,

– $\nu^i(dy;x,t) = \delta_{\zeta(x,t)}(dy)$, for some appropriate function $\zeta : \mathbb{R}^2 \to (0,1]$. This specification yields predictable loss at default,

– $\nu^i(dy;x,t)$ is a probability measure on (0,1], not concentrated on a point mass. In this case, the loss at default ceases to be predictable.

In order to price and hedge credit index derivatives, we need to specify, in particular, the joint dynamics of the loss processes $(\widetilde{Z}^i)_{i=1,\dots,N}$. In analogy to classic copula approaches, we would like to construct a multivariate Markov process with "arbitrary" dependence structure, whose components have desired finite dimensional distributions.

We recall some notation used in the previous section. Let I be a set of indices in $\{1,\dots,N\}$, and $\mathbf{y} = (y_1 \dots y_N)$ be vectors in $[0,1]^N$. We define $\mathbf{y}^I$ to be a vector in $[0,1]^N$, whose i^{th} component is y_i if $i \in I$ and 0 otherwise. Let $\mathbf{Z} = (Z^1,\dots,Z^N)$. We want that process $(\mathbf{Z},X)$ is Markov with prescribed Markovian margins $(\widetilde{Z}^i,X)$, $i=1,2,\dots,N$.

Towards this end, following the Markov copula construction, we define the generator of $(\mathbf{Z},X)$ as follows:

$$\begin{aligned} A_t f(\mathbf{z},x) &= (I^{(1:N)}\widehat{\otimes}\mathcal{L}_t)f(\mathbf{z},x) + \sum_{i=1}^N \eta^i(x,t,z_i)\int_{(0,1]}\Big(f(\mathbf{z}+\mathbf{y}^i,x)-f(\mathbf{z},x)\Big)\,\nu^i(dy_i;x,t) \\ &+ \sum_{I\in\mathcal{J}}\lambda^I(x,t,\mathbf{z})\int_{(0,1]^I}\Big(f(\mathbf{z}+\mathbf{y}^I,x)-f(\mathbf{z},x)\Big)\,\nu^I(d\mathbf{y}_I;x,t) \\ &- \sum_{i=1}^I\sum_{I\in\mathcal{J}:i\in I}\lambda^I(x,t,\mathbf{z})\int_{(0,1]^I}\Big(f(\mathbf{z}+\mathbf{y}^i,x)-f(\mathbf{z},x)\Big)\,\nu^I(d\mathbf{y}_I;x,t), \end{aligned} \tag{14}$$

where,

i) $\nu^I(d\mathbf{y};x,t)$ is a probability measure on $(0,1]^I := \Pi_{\imath\in I}(0,1]$ defined as:

$$\nu^I(d\mathbf{y}_I;x,t) := C^I(\nu^i(dy^i;x,t),\ i\in I) \tag{15}$$

for some copula function C^I,

ii) the non negative bounded functions $\lambda^I(x,t,\mathbf{z})$ are chosen so that the following holds:

$$\sum_{I\in\mathcal{J}:i\in I}\lambda^I(x,t,\mathbf{z}) \le \eta^i(x,t,z^i),\ \forall x,t,\mathbf{z}. \tag{16}$$

In view of Proposition 2.4, $(\mathbf{Z},X)$ is Markov in its natural filtration, which we denote by $\mathbb{F}$, and each component (Z^i,X) is Markov w.r.t. $\mathbb{F}$ and equal in law to $(\widetilde{Z}^i,\widetilde{X})$. We can therefore utilize the process $(\mathbf{Z},X)$ to price and hedge credit basket derivatives on baskets of obligations referencing $(\widetilde{Z}^i)_{i=1,\dots,N}$, or any sub-pool therein. For this purpose we define the following processes:

– The i-th default indicator processes $H^i_t := \mathbb{1}_{\{Z^i_t>0\}}$,

– The cumulative default process $H_t := \sum_i^N \mathbb{1}_{\{Z^i_t>0\}}$,

– The cumulative loss[2], $Z_t := \sum_i^N Z^i_t$.

Note that (14) can be re-written as (we let $\mathcal{I} := 2^{\{1,\dots,d\}}$):

$$A_t f(\mathbf{z},x) = \mathcal{L}_t f(\mathbf{z},x) + \sum_{I\in\mathcal{I}}\lambda^I(x,t,\mathbf{z})\int_{(0,1]^I}\Big(f(\mathbf{z}+\mathbf{y}^I,x)-f(\mathbf{z},x)\Big)\,\nu^I(d\mathbf{y}_I;x,t) \tag{17}$$

where

$$\lambda^i(x,t,\mathbf{z}) = \eta^i(x,t,z_i) - \sum_{I\in\mathcal{J}:i\in I}\lambda^I(x,t,\mathbf{z}) \qquad \text{for } i=1,\dots,N, \tag{18}$$

where $\mathbf{z} = (z_1,\dots,z_N)$. Henceforth, we shall use the more concise notation in (51) for the generator A_t.

[2]We assume without loss of generality that the notional of each obligor is 1.

Remark. Observe that

$$\eta^i(x,t,z_i) = \sum_{I \in \mathcal{I}: i \in I} \lambda^I(x,t,\mathbf{z}) \tag{19}$$

Remark. It is clear that, for N large, the sum over all possible sets of at least two indices in $\{1,\ldots,N\}$, contains an unmanageable number of terms. In applications one does not need to consider all possible jump sizes, and can select (preferably using financial or economical reasoning) sub-pools of obligors that are more likely to suffer from the *frailty* effect[3], in the sense that they are prone to simultaneous default.

3.2 Pricing Credit Index Derivatives

The primary securities, underlying the CID market, are the individual vanilla credit default swaps. For this reason we begin by discussing pricing of CDSs within our model.

3.2.1 Credit Default Swaps

In what follows we shall interpret the probability measure $\mathbb{Q}$ as a pricing measure corresponding to the discount factor $\beta_t = e^{-\int_0^t r_s\,ds}$, where r is the spot interest rate process. We shall assume that process r is the first coordinate, say X^1, of the factor process X, so that $\beta_t = e^{-\int_0^t X_s^1\,ds}$. The time-$t$ fair spread of the i-th CDS contract maturing at T is defined as:

$$\kappa_t^i := \frac{\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \frac{\beta_u}{\beta_t} dZ_u^i \Big| \mathcal{F}_t\Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \frac{\beta_u}{\beta_t}(1-H_u^i)du \Big| \mathcal{F}_t\Big)} = \frac{\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \frac{\beta_u}{\beta_t} d\widetilde{Z}_u^i \Big| \mathcal{F}_t\Big)}{\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \frac{\beta_u}{\beta_t}(1-\widetilde{H}_u^i)du \Big| \mathcal{F}_t\Big)}, \tag{20}$$

although we shall use a more convenient representation.

Lemma 3.1 *The spread in (20) can be represented as:*

$$\kappa_t^i = \frac{\mathbb{E}_{\mathbb{Q}}\left(\frac{\beta_T}{\beta_t} Z_T^i - Z_t^i + \int_t^T X_u^1 \frac{\beta_u}{\beta_t} Z_u^i\,du \Big| \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\int_t^T \frac{\beta_u}{\beta_t}(1-H_u^i)\,du \Big| \mathcal{F}_t\right)}. \tag{21}$$

Proof. The result follows from integration by parts formula. □

The expectations in (20) can be evaluated by solving related PDEs, as shown in the next proposition.

Proposition 3.1 *Let A_t^i be as in (13). Then*

$$\kappa_t^i = \frac{\phi^i(Z_t^i, X_t, t) - Z_t^i}{\rho^i(Z_t^i, X_t, t)}, \tag{22}$$

where functions ϕ^i and ρ^i satisfy, for[4] $x \in \mathcal{O} \subset \mathbb{R}^d$, $z^i \in [0,1]$:

$$\begin{aligned} &\partial_t \phi^i(z_i,x,t) + A_t^i \phi^i(z_i,x,t) - x_1 \phi^i(z_i,x,t) = -x_1 z_i, \quad t \in [0,T) \\ &\phi(z_i,x,T) = z_i \end{aligned} \tag{23}$$

and, letting $h(z_i) = \mathbb{1}_{\{z_i>0\}}$,

$$\begin{aligned} &\partial_t \rho^i(z_i,x,t) + A_t^i \rho^i(z_i,x,t) - x_1 \rho^i(z_i,x,t) = 1 - h(z_i), \quad t \in [0,T) \\ &\rho^i(z_i,x,T) = 0. \end{aligned} \tag{24}$$

Proof. By the Feynman-Kac formula, (23) and (24) imply:

$$\phi^i(Z_t^i, X_t, t) = \frac{1}{\beta_t}\mathbb{E}_{\mathbb{Q}}\left(\beta_T Z_T^i + \int_t^T \beta_u Z_u^i r\,du \Big| \mathcal{F}_t\right) \tag{25a}$$

$$\rho^i(Z_t^i, X_t, t) = \frac{1}{\beta_t}\mathbb{E}_{\mathbb{Q}}\left(\int_t^T \beta_u (1-H_u^i) du \Big| \mathcal{F}_t\right), \tag{25b}$$

and the result follows from (21). □

[3] Please see the concluding remarks section for some discussion of so called contagion effect

[4] The domain $\mathcal{O}$ depends on a particular choice of factors X. In the remainder of this paper we will omit the specification of the domains for the pricing PDEs, when it is clear from the context.

3.2.2 Credit Index Derivatives

We are interested in valuation and hedging of CID, such as index swaps, tranche swaps, or n^{th}–to–default swaps.

Let $\Pi(\cdot)$ be a general payoff function, then the fair spread of a credit index derivative maturing at T, and whose payoff is determined by Π, can be expressed as:

$$\kappa_t = \frac{\mathbb{E}_{\mathbb{Q}}\left(\int_t^T \frac{\beta_u}{\beta_t}\, d\Pi_u^\Gamma \Big| \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\int_t^T \frac{\beta_u}{\beta_t}\left(\Pi(\mathcal{N}) - \Pi(\Psi_u)\right) du \Big| \mathcal{F}_t\right)}, \tag{26}$$

where $\Pi_t^\Gamma = \Pi\big(\Gamma_t\big)$, and where constant $\mathcal{N}$ and processes Γ and Ψ are determined by the covenants of the swap; as usually we let $\frac{0}{0} = 0$. For example,

(IS): In case of a credit index swap (IS) we have

$$\Pi(x) = x,$$

$$\mathcal{N} = N, \quad \Gamma_t = Z_t, \quad \Psi_t = H_t.$$

Note that here $\mathcal{N}$ represents the aggregate notional of the basket. Also, note that here $\kappa_t^{IS} = 0$ on the event $H_t = N$.

(TS): In case of a synthetic CDO tranche swap (TS) with attachments $L \in (0,1)$ and $U \in (0,1)$, $L < U$, we have[5]:

$$\Pi(x) = (x - A)\mathbb{1}_{[A,B]}(x) + (B - A)\mathbb{1}_{(B,\mathcal{N}]}(x) = (x - B)^+ - (x - A)^+,$$

where $\mathcal{N} = N$, $A = \mathcal{N}L$ and $B = \mathcal{N}U$. In addition, we have

$$\Gamma_t = \Psi_t = Z_t.$$

Also, note that here $\kappa_t^{TS} = 0$ on the event $Z_t \geq B$.

(nTDS): In case of a n^{th}-to-default swap (nTDS) we have

$$\Pi(x) = x,$$

$$\mathcal{N} = n, \quad \Gamma_t = Z_{t\wedge\tau^{(n)}}, \quad \Psi_t = H_t \wedge \mathcal{N},$$

where $\tau^{(n)}$ indicates the time of the n^{th} default. Also, note that here $\kappa_t^{nTDS} = 0$ on the event $t \geq \tau^{(n)}$.

Remark. Integrating by parts, (26) can be written as:

$$\kappa_t = \frac{\mathbb{E}_{\mathbb{Q}}\left(\frac{\beta_T}{\beta_t}\Pi\big(\Gamma_T\big) - \Pi\big(\Gamma_t\big) + \frac{1}{\beta_t}\int_t^T X_u^1 \beta_u \Pi\big(\Gamma_u\big) du \Big| \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\int_t^T \frac{\beta_u}{\beta_t}\left(\Pi(\mathcal{N}) - \Pi(\Psi_u)\right) du \Big| \mathcal{F}_t\right)}. \tag{27}$$

In what follows, we let $g(\mathbf{z}) = \sum_{i=1}^N z_i$, and $h(\mathbf{z}) = \sum_{i=1}^N \mathbb{1}_{\{z_i>0\}}$. In addition, we shall consider functions $\gamma(\mathbf{z})$ and $\psi(\mathbf{z})$ that indicate levels of processes Γ and Ψ, respectively. The forms of γ and ψ depend on a particular application; for example, in case of the index swap (IS) we have

$$\gamma(\mathbf{z}) = g(\mathbf{z}), \ \psi(\mathbf{z}) = h(\mathbf{z}),$$

and in case of the tranche swap (TS) we have

$$\gamma(\mathbf{z}) = \psi(\mathbf{z}) = g(\mathbf{z}).$$

Recalling that A_t denotes the infinitesimal generator of $(\mathbf{Z}, X)$, we have the following proposition:

[5] The definitions provided here apply to all tranches but the equity tranche, which is quoted in a very specific way (cf. [6]) Straightforward adjustments need to be made so to deal with the equity tranche.

Proposition 3.2 *We have that for $t \in [0,T]$*

$$\kappa_t = \frac{\phi(\mathbf{Z}_t, X_t, t) - \Pi(\gamma(\mathbf{Z}_t))}{\rho(\mathbf{Z}_t, X_t, t)}, \tag{28}$$

where functions ϕ and ρ solve, respectively,

$$\begin{aligned} &\partial_t \phi(\mathbf{z}, x, t) + A_t \phi(\mathbf{z}, x, t) - x_1 \phi(\mathbf{z}, x, t) = -x_1 \Pi(\gamma(\mathbf{z})) \\ &\phi(\mathbf{z}, x, T) = \Pi(\gamma(\mathbf{z})) \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\partial_t \rho(\mathbf{z}, x, t) + A_t \rho(\mathbf{z}, x, t) - x_1 \rho(\mathbf{z}, x, t) = \Pi(\mathcal{N}) - \Pi(\psi(\mathbf{z})) \\ &\rho(\mathbf{z}, x, t) = 0. \end{aligned} \tag{30}$$

Proof. By the Feynman-Kac formula, (29) and (30) imply that:

$$\phi(\mathbf{Z}_t, X_t, t) = \frac{1}{\beta_t} \mathbb{E}_{\mathbb{Q}}\big(\beta_T \Pi(\Gamma_T) + \int_t^T \beta_u \Pi(\Gamma_u) X_u^1 \, du \Big| \mathcal{F}_t\big) \tag{31a}$$

$$\rho(\mathbf{Z}_t, X_t, t) = \frac{1}{\beta_t} \mathbb{E}_{\mathbb{Q}}\big(\int_t^T \beta_u \left(\Pi(\mathcal{N}) - \Pi(\Psi_u)\right) du \Big| \mathcal{F}_t\big), \tag{31b}$$

The result follows from (27). □

Remark. For small pool sizes, N, the above PDEs can be solved numerically, for example by finite difference approximation. Otherwise, one has to resort to Monte Carlo simulation to compute the expectations in (31a) and (31b), as we shall discuss later.

3.3 Hedging CID with CDS

In this section we provide formulas for the hedging ratios of the CIDs with respect to the individual CDSs.[6]

It is common practice to manage risky positions in basket swaps by holding portfolios of individual CDS contracts. The portfolio weights are chosen so as to offset some of the risks associated with the basket product. This approach, usually referred to as "delta" hedging, requires computation of certain ratios such as $d\widetilde{P}/d\widetilde{P^i}$ and $d\widetilde{P}/dH^i$, where $\widetilde{P}$ and $\widetilde{P^i}$ denote the *cumulative* price of a CID, and the *cumulative* price of a single name CDS, respectively.

In this section, we derive formulas for the hedge ratios, under the following assumptions:

1. $\mathcal{L}_t f(x) := b(x,t)\partial_x f(x) + a(x,t)\partial_{xx} f(x)$,
2. $\nu(dy_i; x, t) := \delta_{1-R}(dy_i)$, i.e. the jump size at default is deterministic.

Under this set of assumptions, the infinitesimal generator of $(\mathbf{Z}, X)$ in (51) is of the form:

$$A_t f(\mathbf{z}, x) = \mathcal{L}_t f(\mathbf{z}, x) + \sum_{I \in \mathcal{I}} \lambda^I(x, t, \mathbf{z}) \int_{(0,1]^I} \Big(f(\mathbf{z} + \mathbf{y}^I, x, t) - f(\mathbf{z}, x, t)\Big) \delta_{\mathbf{1}-\mathbf{R}}(\mathbf{dy}_I) \tag{32}$$

where $\delta_{\mathbf{1}-\mathbf{R}}(\mathbf{dy}_I) := \Pi_{\imath \in I} \delta_{1-R}(dy_i)$.

For ease of notation, for an index set in $\{1, \ldots, N\}$, we define the following operator:

$$\Delta^I f(\mathbf{z}, x, t) := \int_{(0,1]^I} \Big(f(\mathbf{z} + \mathbf{y}^I, x, t) - f(\mathbf{z}, x, t)\Big) \; \delta_{\mathbf{1}-\mathbf{R}}(d\mathbf{y}_I). \tag{33}$$

[6]Using the same techniques, we can compute hedging ratios of CID with respect to sub-baskets of credit default swaps.

3.3.1 Computing the Hedging Ratios

Let[7] $\tau^I := \inf\{t \geq 0 : Z_t^i - Z_{t-}^i > 0, \forall i \in I\}$, and denote by $M_t^I := \mathbb{1}_{\{\tau^I \leq t\}}$ the corresponding indicator process. Note that

$$\sum_{I \in \mathcal{I}: i \in I} M_t^I = H_t^i. \tag{34}$$

We define the following auxiliary processes:

$$\widehat{X}_t := X_t - \int_0^t b(X_s, s)ds \tag{35a}$$

$$\widehat{H}_t^i := H_t^i - \int_0^t \eta^i(X_s, s, Z_s^i)ds \tag{35b}$$

$$\widehat{M}_t^I := M_t^I - \int_0^t \lambda^I(X_s, s, \mathbf{Z}_s)ds \tag{35c}$$

where τ^I is the random time of the simultaneous jump of all components $(Z^i, i \in I)$. In view of (19) and (34) we have that

$$\widehat{H}_t^i = \sum_{I \in \mathcal{I}: i \in I} \widehat{M}_t^I. \tag{36}$$

We also have the following important result:

Lemma 3.2 *The processes in (35) are $\mathbb{F}$-local martingales.*

Proof. See the Appendix. □

We proceed with deriving a martingale representation for the *cumulative price* process of the i^{th} CDS contract initiated at time $t = 0$, at the contracted spread κ^i; we shall denote this price process by $\widetilde{P}_t^i(\kappa^i)$, $t \in [0, T]$. By an application of integration by parts formula, $\widetilde{P}_t^i(\kappa^i)$ can be written as:

$$\begin{aligned} \widetilde{P}_t(\kappa^i) &= \frac{1}{\beta_t}\Bigg(\int_0^t \beta_u Z_u^i X_u^1\, du - \kappa^i \int_0^t \beta_u(1 - H_u)\, du \\ &+ \mathbb{E}_{\mathbb{Q}}\left(\beta_T Z_T^i + \int_t^T \beta_u Z_u^i X_u^1\, du|\mathcal{F}_t\right) - \kappa^i \mathbb{E}_{\mathbb{Q}}\left(\int_t^T \beta_u(1 - H_u^i)du\Big|\mathcal{F}_t\right)\Bigg). \end{aligned} \tag{37}$$

Proposition 3.3 *Assume A_t^i is as in (13), then the discounted cumulative price process of the i^{th} CDS in the pool is a local martingale with the following predictable representation:*

$$d\Big(\beta\widetilde{P}^i(\kappa^i)\Big)_t = \Big(\partial_x\phi_t^i - \kappa^i\partial_x\rho_t^i\Big)d\widehat{X}_t + \Big(\Delta^i\phi_t^i - \kappa^i\Delta^i\rho_t^i\Big)d\widehat{H}_t^i, \tag{38}$$

where ϕ_t^i and ρ_t^i are defined by (25a) and (25b), respectively.

Proof.

By Itô formula,

$$\begin{aligned} d\Big(\beta\widetilde{P}^i(\kappa^i)\Big)_t &= Z_t^i\beta_t X_t^1\, dt - \kappa^i\beta_t(1 - H_t^i)dt + \beta_t(d\phi_t^i - \kappa^i d\rho_t^i) - (\phi_t^i - \kappa^i\rho_t^i)\beta_t X_t^1\, dt \\ &= -A_t^i\phi_t^i\, dt + \partial_x\phi_t^i dX_t + \frac{1}{2}\partial_{xx}\phi_t^i d[X]_t + \Delta^i\phi_t^i dH_t^i \\ &- \kappa^i\Big(-A_t^i\rho_t^i\, dt + \partial_x\rho_t^i dX_t + \frac{1}{2}\partial_{xx}\rho_t^i d[X]_t + \Delta^i\rho_t^i dH_t^i\Big), \end{aligned}$$

where we used (23) and (24). The result (38) follows after some straightforward simplifications. □

Let us now consider a general CID with payoff function Π, contracted at time $t = 0$ for the spread κ and maturing at time T. Similarly to (37), its *cumulative* price can be written as:

$$\begin{aligned} \widetilde{P}_t(\kappa) &= \frac{1}{\beta_t}\Bigg(\int_0^t \beta_u\Pi(Z_u)r\, du - \kappa\int_0^t \beta_u(\Pi(\mathcal{N}) - \Pi(Z_u))\, du \\ &+ \mathbb{E}_{\mathbb{Q}}\left(\beta_T\Pi(Z_T) + \int_t^T \beta_u\Pi(Z_u)X_u^1\, du\Big|\mathcal{F}_t\right) - \kappa\mathbb{E}_{\mathbb{Q}}\left(\int_t^T \beta_u\left(\Pi(\mathcal{N}) - \Pi(Z_u)\right)du\Big|\mathcal{F}_t\right)\Bigg). \end{aligned} \tag{39}$$

[7] By definition, $Z_{0-} = Z_0$.

The proof of the next result follows along the same lines as the proof of Proposition 3.3, and therefore will be omitted.

Proposition 3.4 *The cumulative price of a CID whose payoff is determined by function* Π *is a* $\mathbb{Q}$ *local martingale and admits the following predictable representation:*

$$d\big(\beta\widetilde{P}(\kappa)\big)_t = \Big(\partial_x\phi_t - \kappa\partial_x\rho_t\Big)d\widehat{X}_t + \sum_{I\in\mathcal{I}}\Big(\Delta^I\phi_t - \kappa\Delta^I\rho_t\Big)d\widehat{M}^I_t, \tag{40}$$

where ϕ_t *and* ρ_t *are defined by (31a) and (31b), respectively.*

The next proposition yields explicit formulae for the desired hedging ratios.

Proposition 3.5 *Assume* $\Delta^i\phi^i_t - \kappa^i\Delta^i\rho^i_t \neq 0, \forall i$. *Then the discounted price of a CID whose payoff is determined by function* Π *admits the following representation:*

$$\begin{aligned} d\big(\beta\widetilde{P}(\kappa)\big)_t &= \left(\partial_x\phi_t - \kappa\partial_x\rho_t - \sum_{i=1}^{N}\frac{\Big(\partial_x\phi^i_t - \kappa^i\partial_x\rho^i_t\Big)\Big(\Delta^i\phi_t - \kappa\Delta^i\rho_t\Big)}{\Delta^i\phi^i_t - \kappa^i\Delta^i\rho^i_t}\right)d\widehat{X}_t \qquad (41)\\ &+ \sum_{i=1}^{N}\left(\frac{\Delta^i\phi_t - \kappa\Delta^i\rho_t}{\Delta^i\phi^i_t - \kappa^i\Delta^i\rho^i_t}\right)d(\beta\widetilde{P}^i(\kappa^i))_t \\ &+ \sum_{I\in\mathcal{J}}\left((\Delta^I\phi_t - \sum_{i\in I}\Delta^i\phi_t) - \kappa(\Delta^I\rho_t - \sum_{i\in I}\Delta^i\rho_t)\right)d\widehat{M}^I_t. \end{aligned}$$

Proof. From (38) we derive the expression for $d\widehat{H}^i$:

$$d\widehat{H}^i_t = \frac{1}{\Big(\Delta^i\phi^i_t - \kappa^i\Delta^i\rho^i_t\Big)}d\Big(\beta\widetilde{P^i}(\kappa^i)\Big)_t - \frac{\Big(\partial_x\phi^i_t - \kappa^i\partial_x\rho^i_t\Big)}{\Big(\Delta^i\phi^i_t - \kappa^i\Delta^i\rho^i_t\Big)}d\widehat{X}_t, \tag{42}$$

which is well defined by assumption. From (3.4) we have:

$$\begin{aligned} d\big(\beta\widetilde{P}(\kappa)\big)_t &= \Big(\partial_x\phi_t - \kappa\partial_x\rho_t\Big)d\widehat{X}_t + \sum_{I\in\mathcal{I}}\Big(\Delta^I\phi_t - \kappa\Delta^I\rho_t\Big)d\widehat{M}^I_t \\ &= \Big(\partial_x\phi_t - \kappa\partial_x\rho_t\Big)d\widehat{X}_t + \sum_{i=1}^{N}\Big(\Delta^i\phi_t - \kappa\Delta^i\rho_t\Big)d\widehat{M}^i_t + \sum_{I\in\mathcal{J}}\Big(\Delta^I\phi_t - \kappa\Delta^I\rho_t\Big)d\widehat{M}^I_t \\ &+ \sum_{i=1}^{N}\sum_{I\in\mathcal{J}:i\in I}\Big(\Delta^i\phi_t - \kappa\Delta^i\rho_t\Big)d\widehat{M}^I_t - \sum_{i=1}^{N}\sum_{I\in\mathcal{J}:i\in I}\Big(\Delta^i\phi_t - \kappa\Delta^i\rho_t\Big)d\widehat{M}^I_t \\ &= \Big(\partial_x\phi_t - \kappa\partial_x\rho_t\Big)d\widehat{X}_t + \sum_{i=1}^{N}\Big(\Delta^i\phi_t - \kappa\Delta^i\rho_t\Big)d\widehat{H}^i_t \\ &+ \sum_{I\in\mathcal{J}}\Big(\Delta^I\phi_t - \kappa\Delta^I\rho_t\Big)d\widehat{M}^I_t - \sum_{i=1}^{N}\sum_{I\in\mathcal{J}:i\in I}\Big(\Delta^i\phi_t - \kappa\Delta^i\rho_t\Big)d\widehat{M}^I_t \\ &= \Big(\partial_x\phi_t - \kappa\partial_x\rho_t\Big)d\widehat{X}_t + \sum_{i=1}^{N}\Big(\Delta^i\phi_t - \kappa\Delta^i\rho_t\Big)d\widehat{H}^i_t \\ &+ \sum_{I\in\mathcal{J}}\Big(\Delta^I\phi_t - \kappa\Delta^I\rho_t - \sum_{i\in I}(\Delta^i\phi_t - \kappa\Delta^i\rho_t)\Big)d\widehat{M}^I_t, \end{aligned}$$

where the last step can be easily shown by an induction argument on N. Plugging (42) into the above, we obtain the desired predictable representation (41). $\square$

3.4 Simulation Algorithm

For small basket sizes conditional expectations that need to be computed for the purpose of valuation and hedging of basket derivatives can be computed by (numerically) solving relevant (systems of) PDEs or IPDEs. However, for large basket sizes such (quasi) analytical solutions are infeasible within our framework. Thus, we apply Monte Carlo simulation to carry these computations. It turns our that this simulation approach is extremely effective in the framework of Markovian copulae.

We consider here simulations of sample paths of $(\mathbf{Z}, X)$ over the time interval, $[t_1, t_2]$. given $(\mathbf{Z}_{t_1}, X_{t_1}) = (\mathbf{z}, x)$. It is clear that simulating $\mathbf{Z}$ is equivalent to simulating times of individual jumps, times of common jumps, and jump sizes. However, we need to stress here that in applications we do not consider all $I \in \mathcal{J}$ as far as common jumps are concerned, but rather we use judgment and economic information regarding to what I's to select, so to make the simulation procedure efficient.

Generating one sample path will, in general, involve the following steps:

Step 1: simulate a sample path of the factor process X. Typically X is a diffusion or jump diffusion, and standard simulation procedures for this type of processes are discussed, for instance, in Kloeden and Platen [18]). We denote by $\widetilde{X}$ the simulated sample path of X.

Step 2: generate a sample path of $\mathbf{Z}$ on the interval $[t_1, t_2]$ as follows:

Step 2.1: simulate the 1^{st} jump time of $\mathbf{Z}$ in the time interval $[t_1, t_2]$. Towards this end, draw a unit exponential random variable, which we denote by η. The simulated value of the first jump time, τ, is then given by:

$$\tau = \inf\Big\{ t > t_1 \ : \ \int_{t_1}^{t} \lambda(\mathbf{z}, \widetilde{X}_u, u)\, du \geq \eta \Big\},$$

where

$$\lambda(\mathbf{z}, \widetilde{X}_t, t) := \sum_{I \in \mathcal{I}} \lambda^I(\mathbf{z}, X_t, t)$$

If $\tau > t_2$ return to step 1, otherwise go to Step 2.2.

Step 2.2: simulate which one of the M^I jumps at τ, by drawing from the conditional distribution:

$$\mathbb{Q}(\Delta M^I_\tau = 1) = \frac{\lambda^I(\mathbf{z}, \widetilde{X}_{\tau-}, \tau)}{\lambda(\mathbf{z}, \widetilde{X}_{\tau-}, \tau)}.$$

Step 2.3: given that, in step 2.2 we obtained that $\Delta M^J_\tau = 1$ for some $J \in \mathcal{I}$, simulate the size of the loss for each obligor in the index set J by drawing from the multivariate distribution $\nu^J(\cdot\,; \widetilde{X}_{\tau-}, \tau-)$ [8].

Step 2.4: update the state of $\mathbf{Z}$ and set $t_1 = \tau$. Repeat Steps 2.1-2.3.

Step 3: calculate the simulated value of a relevant functional.

3.5 Model Calibration

In the previous sections we assumed a risk neutral pricing measure as given. Arbitrage free pricing, in fact, requires existence of a risk neutral measure, under which the price processes in the underlying market are martingales.

In our market model, relevant assets are single name CDS contracts composing the credit indices, the indices themselves, and the related derivative products, such as CDOs, CDO2s, etc. . It is a standing assumption that financial markets are arbitrage free, and a risk neutral measure can thus be inferred from the prevailing market prices.

3.5.1 Calibration by simulation procedure

Choosing a risk-neutral probability measure such as to reproduce the prices of traded derivative products is known as model calibration. Since the dynamics of $(\mathbf{Z}, X)$ are specified via a Markov copula, calibration of the risk neutral parameters of the model, that is, the parameters corresponding to the risk neutral measure, can be split into three separate problems [9]:

Step i) calibration of the dynamics of the factor process X,

Step ii) calibration of the infinitesimal generators of the processes (Z^i, X), $i = 1, \ldots, N$,

Step iii) calibration of the infinitesimal generator of the process $(\mathbf{Z}, X)$.

[8] For some treatment of simulation from copulae we refer to [19] and references therein.

[9] The market data used in calibration was courteously provided by GFI

This considerably reduces the dimensionality of the optimization problem embedded in model calibration. It is in fact clear from (14) that, once the intensities $\eta^i(x,t,z_i)$ and the jump size distributions $\nu^i(dy;,x,t)$ have been fitted to individual name CDS market spread data, it suffices to calibrate the common jump intensities $\lambda^I(x,t), \forall I \in \mathcal{J}$ (see however the Remark at the end of Section 3.1), along with the parameters of the copula function $C(\cdot)$, which determines the jump size distribution $\nu^C(d\mathbf{y},x,t)$. In fact, since steps *i)* and *ii)* can be executed at extremely low computational cost, we can regard the probability law of the marginal processes (Z^i, X) as given, and devote our attention to step *iii)*. The jump intensities $\lambda^I(x,t)$ and the copula function, $C(\cdot)$ are fitted to market data for relevant CIDs, such as CDO and CDS Index.

To perform *Step iii)* of our calibration procedure we need to compute theoretical values for various spreads according to formulae provided in Proposition 3.2. Since computing the expectations in (31a) and (31b) requires Monte Carlo simulation then, for the purpose of solving the optimization problem embedded in our calibration procedure, it is best to use algorithms that do not require computation of gradients . In particular, we suggest using the downhill simplex method (also known as the Nelder-Mead algorithm) or Powell method to perform the minimization. We refer to [6] for a more detailed description of calibration by simulation procedure that underlies what we do here.

3.5.2 Toy Model

We implement a "toy" model to test the performance of our framework. In particular, we assume a constant interest rate r, constant recovery $R = .4$ and marginal generators of the form

$$A_t^i f(t,z_i) = \eta^i(t,z_i)\int_{(0,1]} f(z_i+y) - f(z_i)\delta_{1-R}(dy),$$

where $\eta^i(t,z_i) = (a_i + b_i t)\mathbb{1}_{\{z_i=0\}}$. Under these assumptions, the individual time-$t=0$ spreads are given by the formulae (here we make explicit dependence on maturity by writing $\kappa_0^i(T)$):

$$\kappa_0^i(T) = \frac{\int_0^T re^{-rs}(1-e^{-a_i s-\frac{b_i}{2}s^2})ds + e^{-rT}(1-e^{-a_i T-\frac{b_i}{2}T^2})}{\int_0^T e^{-rs}e^{-a_i s-\frac{b_i}{2}s^2}ds}(1-R)10000 \tag{43}$$

which can be computed in terms of the *erf*. The (positive) parameters a_i and b_i are fitted so as to match the 5 and 10 year spreads of the iTraxx S6 constituents as of December 1, 2006[10].

Recall that a key to numerical efficiency of our simulation procedure, and thus of our calibration by simulation, is a judicious choice of groups of obligors who may default simultaneously. Here, towards this end, we let $\mathcal{R} = \{10, 20, 40, 125\}$ and, for $n \in \mathcal{R}$, we define I_n as the set containing the indices of the n riskiest obligors[11]. In particular, we have $I_{125} = \{1, 2, \ldots, 125\}$, and $I_{10} \subset I_{20} \subset I_{40} \subset I_{125}$.

We construct the generator of process $\mathbf{Z}$ as:

$$\begin{aligned} A_t f(\mathbf{z}) &= \sum_{i=1}^{125} \eta^i(z_i,t)\int \Big(f(\mathbf{z}+\mathbf{y}^i) - f(\mathbf{z})\Big)\,\delta_{(1-R)}(dy_i) \\ &+ \sum_{n\in\mathcal{R}}\sum_{I\subset I_n} \lambda^I(\mathbf{z},t)\int \Big(f(\mathbf{z}+\mathbf{y}^I) - f(\mathbf{z})\Big)\,\pi(d\mathbf{y}_I) \\ &- \sum_{i=1}^{125}\sum_{n\in\mathcal{R}}\sum_{I\subset I_n: i\in I} \lambda^I(\mathbf{z},t)\int \Big(f(\mathbf{z}+\mathbf{y}^i) - f(\mathbf{z})\Big)\,\pi(d\mathbf{y}_I), \end{aligned} \tag{44}$$

where, for $I \subset I_n$, $\lambda^I(\mathbf{z},t) = (\bar{a}_n + \bar{b}_n t)\mathbb{1}_{\{z_i=0\ \forall i\subset I,\ z_i>0\ \forall i\in I_n\setminus I\}}$, with $\bar{a}_n = \alpha_n \min_{\{i\in I_n\}} a_i$, $\bar{b}_n = \alpha_n \min_{\{i\in I_n\}} b_i$ and $\pi(d\mathbf{y}_I) = \bigotimes_{i\in I}\delta_{1-R}(dy_i)$. In words, the form of the above generator implies that, at every time instant, either each alive obligor can default individually, or all the surviving names whose indices are in the set I_n, $n \in \mathcal{R}$ can default simultaneously. In particular,

[10] Since the 10y spread data is available to us only for 65 out of 125 names, we infer the missing 10y spreads by linear regression.

[11] We measure the risk associated to each name in the pool by the magnitude of the spread of the corresponding five year CDS.

note that the double summation in (44) contains at most four non-zero terms, whereas the triple summation in (44) contains at most 125×4 non-zero terms. This makes the simulation of process $\mathbf{Z}$ very fast, as we essentially need to simulate at most $125 + 4$ jump indicator processes. Moreover, we only need to calibrate four parameters, namely α_n with $n \in \mathcal{R}$.

3.5.3 Calibration Results for the Toy Model

We calibrate the parameters to the 5y CDO tranche spreads, and test the performance of the calibrated model against 7y and 10y market spreads. The results are shown in the Table 1 below.

Tranche	5y market	5y model	7y market	7y model	10y market	10y model
0-3%	12.375	12.2828	27.125	27.7419	41.5	44.6305
3-6%	56.5	57.2899	135	114.855	333	318.309
6-9%	15.5	15.0546	38	32.4026	99	83.0893
9-12%	6	6.26695	18.75	13.3415	42	28.221
12-22%	2.5	2.42193	6.25	4.21169	13.5	7.51529
22-100%	.875	0.81408	1.25	1.30499	2.875	1.53266

Table 1: *Model fit to term structure of iTraxx S6 CDO tranches, Dec 2006.*

The fit is very good, especially in consideration of the fact that half of the 10y individual CDS spreads, that are needed for calibration of the individual default intensities, were the result of a linear regression, as we did not have the respective market data.

For this simple model and set of data, CPU time was approximately 1 sec. per 50,000 simulations for 5 year maturity, 2 sec. per 50,000 simulations for 7 year maturity, and 3 sec.per 50,000 simulations for 10 year maturity (using a non-optimized C++ code). The model calibration takes approximately 2-3 minutes on a Pentium D 2.8 GHz machine.

4 Ratings Triggered Corporate Step-Up Bonds

Here we shall apply results on Markov copulae to the problem of valuation of ratings triggered corporate step-up bonds. Similarly as in the case of CID, in case of step-up bonds we shall also use simulation techniques for pricing. Here however, the main reason for using simulation based approach is that the payoffs of these bonds have, in general, quite complicated path dependent structure, which essentially prohibits the use if analytical or quasi-analytical methodologies.

4.1 Description of Ratings Triggered Step-Up Bonds

These bonds were issued by some European telecom companies in the recent 5-6 years. As of now, to our knowledge, these products are not traded in baskets, however they are of interest because they offer protection against credit events other than defaults. In particular, ratings triggered corporate step-up bonds (step-up bonds for short) are corporate coupon issues for which the coupon payment depends on the issuer's credit quality: in principle, the coupon payment increases when the credit quality of the issuer declines. In practice, for such bonds, credit quality is reflected in credit ratings assigned to the issuer by at least one credit ratings agency (Moody's-KMV or Standard&Poor's). The provisions linking the cash flows of the step-up bonds to the credit rating of the issuer have different step amounts and different rating event triggers. In some cases, a step-up of the coupon requires a downgrade to the trigger level by both rating agencies. In other cases, there are step-up triggers for actions of each rating agency. Here, a downgrade by one agency will trigger an increase in the coupon regardless of the rating from the other agency. Provisions also vary with respect to step-down features which, as the name suggests, trigger a lowering of the coupon if the company regains its original rating after a downgrade. In general, there is no step-down below the initial coupon for ratings exceeding the initial rating.

Next, we give a brief summary of the most common provisions characterizing the payoff of a step-up bond (typically, a step-up bond is subject to a selection of the provisions listed below):

(i) Step-up: The coupon increases if the rating decreases and hits the rating-trigger.

(ii) Step-down: The coupon decreases if the rating increases over the rating-trigger after the trigger level was previously hit.

(iii) One-off: The coupon increases only once, even if the rating falls further below the rating-trigger; for bonds that are not one-off, each further decrease in the rating, causes a further increase in the coupon.

(iv) And/or: Determines whether the coupon is adjusted if both Moody's and S&P ratings hit the trigger, or whether the adjustment occurs if either Moody's or S&P ratings hit the trigger level.

(v) Accrual: the coupon increases may be enforced either starting from the next coupon payment or immediately following a rating action.

Let R_t stand for some indicator of credit quality at time t (note that in this case, the process R may be composed of two, or more, distinct rating processes). Assume that $t_i,\ i = 1, 2, \dots, n$ are coupon payment dates. In this paper we assume the convention that coupon paid at date t_n depends only on the rating history through date t_{n-1}, that is: $c_n = c(R_t,\ t \leq t_{n-1})$ are the coupon payments. In other words, we assume that no accrual convention is in force.

Assuming that the bond's notional amount is 1, the cumulative discounted cash flow of the step-up bond is (as usual we assume that the current time is 0):

$$(1 - H_T)\beta_T + \int_{(0,T]} (1 - H_u)\beta_u \, dC_u + \beta_\tau Z_\tau H_T, \tag{45}$$

where $C_t = \sum_{t_i \leq t} c_i$, τ is the bond's default time, $H_t = \mathbb{1}_{\tau \leq t}$, and where Z_t is a (predictable) recovery process.

4.2 Pricing Ratings Triggered Step-Up Bonds via Simulation

Here, using our results on Markov copulae, we shall apply a simulation approach to pricing ratings triggered step-up bonds.

Let us consider a ratings triggered step-up bond issued by an obligor XYZ. Recall that, typically, cash-flows associated with a step-up bond depend on ratings assigned to XYZ by both Moody‘s Investors Service (Moody's in what follows) and Standard & Poor's (S&P in what follows). Thus, a straightforward way to model joint credit migrations would be to consider a credit migration process K such that $R_t = (M_t, SP_t)$, where M_t and SP_t denote the time t credit rating assigned to XYZ by Moody's and SP_t, respectively. We assume that process M is a time-homogeneous Markov chain w.r.t. its natural filtration, under the statistical probability $\mathbb{P}$, and that its state space is $\mathcal{K} = \{1, 2, \dots, K\}$. Likewise, we assume that process SP is a time-homogeneous Markov chain w.r.t. its natural filtration, under the statistical probability $\mathbb{P}$, and that its state space is $\mathcal{K} = \{1, 2, \dots, K\}$.

4.2.1 Credit ratings dynamics and Markov copula under the statistical probability

Typically, we are only provided with individual statistical characteristics of each of the processes M and SP. Thus, in a sense, we know the marginal distributions of the joint process R under the measure $\mathbb{P}$ (where M and SP are considered as the "univariate" margins). The crucial issue is thus the appropriate modeling of dependence between processes M and SP. In particular, we want to model dependence, under $\mathbb{P}$, between M and SP so that the joint process R is a time-homogeneous Markov chain, and so that the components M and SP are time-homogeneous Markov chains with given $\mathbb{P}$-generators, say A^M and A^{SP}, respectively. Thus, essentially, we need to model a $\mathbb{P}$-generator matrix, say A^R, so that process R is a time-homogeneous Markov chain with $\mathbb{P}$-generator A^R and that processes M and SP are time-homogeneous Markov chains with $\mathbb{P}$-generators A^M and A^{SP}. We can of course deal with this problem using the theory of Markov copulae.

Towards this end, we fix an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. On this space we consider two univariate Markov chains M and SP, with given infinitesimal $\mathbb{P}$-generators $A^M = [a^M_{ij}]$ and $A^{SP} = [a^{SP}_{hk}]$, respectively. Next, we consider the system equations in variables

$$\sum_{k\in\mathcal{K}} a^R_{ih,jk} = a^M_{ij}, \ \forall i,j \in \mathcal{K}, i\neq j, \ \forall h \in \mathcal{K}, \tag{46}$$

$$\sum_{j\in\mathcal{K}_1} a^R_{ih,jk} = a^{SP}_{hk}, \ \forall h,k, \in \mathcal{K}, h\neq k, \ \forall i \in \mathcal{K}. \tag{47}$$

Now, provided that the system (46) –(47) has a positive solution, then it follows from Proposition 2.5 that resulting matrix[12] $A^R = [a^R_{ih,jk}]_{i,j\in\mathcal{K}_1,\, h,k\in\mathcal{K}_2}$ satisfies conditions for a $\mathbb{P}-$generator matrix of a bivariate time-homogenous Markov chain, say $R = (R^1, R^2)$ whose components take values in finite state spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ with cardinalities K_1 and K_2, respectively, and, more importantly, they are Markov chains with the same distributions as M and SP under under $\mathbb{P}$. Thus, indeed, the system (46)–(47) essentially serves as a Markov copula between the Markovian margins M, SP and the bivariate Markov chain R.

Note that, typically, the system (46)–(47) contains many more variables than equations. Thus, one can create several bivariate Markov chains R with the given margins M and SP. In financial applications this feature leaves a lot of room for various modeling options and for calibration of the model. For example, as observed by Lando and Mortensen [20] although the ratings assigned by S&P and Moody's to the same company do not necessarily coincide, split ratings are rare and are usually only observed in short time intervals. This feature can easily be modelled using the Markovian copula system (46) –(47) via imposing side constraints for the unknowns $a^R_{ih,jk}$'s. In order to model such observed behavior of the joint rating process, we thus impose additional constraints on the variables in the system (46) –(47). Specifically, we postulate that

$$a^R_{ih,jk} = \begin{cases} 0, & \text{if } i\neq j \text{ and } h\neq k \text{ and } j\neq k, \\ \alpha \min(a^M_{ij}, a^{SP}_{hk}), & \text{if } i\neq j \text{ and } h\neq k \text{ and } j = k, \end{cases} \tag{48}$$

where $\alpha \in [0,1]$ is a modelling parameter. Using constraint (48) we can easily solve system (46) –(47) (in this case the system actually becomes fully decoupled) and we can obtain the generator of the joint process. The interpretation of constraint (48) is the following: The components M and SP of the process R migrate according to their marginal laws, but they tend to join, that is, they tend to both take the same values. The strength of such tendency is measured by the parameter α. When $\alpha = 0$ then, in fact, the two components are independent processes; when $\alpha = 1$ the intensity of both components migrating simultaneously to the same rating category is maximum (given the specified functional form for the intensities of common jumps).

4.2.2 Markovian Changes of Measure

For pricing purposes the statistical probability measure is changed to the EMM. Typically, the Radon-Nikodym density is chosen in such a way that the resulting (risk-neutral) default probabilities are consistent with the term structure of CDS spreads. In addition, we require that the process R, which is Markovian under the statistical measure, is also Markovian under the pricing measure. As a consequence, such change of measure must be chosen with some care. We briefly state some facts concerning Markovian changes of measure. Let X_t be an E valued Markov process under $\mathbb{P}$ with extended generator A (see [21]). In addition define the process

$$M^f_t := \frac{f(X_t)}{f(X_0)} \exp\left(-\int_0^t \frac{Af(X_s)}{f(X_s)} ds\right). \tag{49}$$

Definition 4.1 We say that a strictly positive function $f \in \mathcal{D}(A)$ is a good function if M^f_t is a genuine martingale with $\mathbb{E}_{\mathbb{P}}(M^f_t) = 1$.

Let $f \in \mathcal{D}(A)$ and h be a good function in $C(E)$ or $\mathcal{M}_b(E)$ and define the operator

$$A^h f = h^{-1}A(fh) - fA(h). \tag{50}$$

In view of Definition 4.1, process M^h may play the rôle a the Radon-Nikodym density between measure $\mathbb{P}$ and the resulting measure, say $\mathbb{Q}^h$. We have the following result (cf. [21])

[12] System (46) –(47) does not include diagonal elements of A^R. These elements are obtained as $a^R_{ih,ih} = -\sum_{(j,k)\in\mathcal{K}} a^R_{ih,jk}$.

Theorem 4.1 *Let $\mathbb{Q}^h$ be the probability measure associated to the density process M_t^h. Then X_t is a Markov process under $\mathbb{Q}^h$ with extended generator $(A^h, \mathcal{D}(A))$.*

In the case of a finite state Markov chain, Theorem 4.1 yields the following corollary (cf. [21])

Corollary 4.1 *Let X_t be a finite state Markov chain on $\mathcal{K}$ with cardinality K and generator $A = a_{ij}$. In addition let $h = (h_1, \dots, h_K)$ be a positive vector. Then X_t is a Markov process under $\mathbb{Q}^h$ with generator $A^h = [a_{ij}h_j h_i^{-1}]$.*

Going back to the problem at hand, we recall that $A^R = [a^R_{ih,jk}]$ is the generator of R under the statistical measure $\mathbb{P}$. In view of Corollary 4.1, given a vector $h = [h_{11}, \cdots, h_{KK}] \in \mathbb{R}^{K^2}$, we can change statistical measure $\mathbb{P}$ to an equivalent "risk-neutral" measure $\mathbb{Q}$ in such a way that process R is a time-homogeneous Markov chain under $\mathbb{Q}$, and its $\mathbb{Q}$-infinitesimal generator is given by

$$\widetilde{A}^R = [\widetilde{a}_{ih,jk}],$$

where $\widetilde{a}_{ih,jk} = a_{ih,jk}\frac{h_{jk}}{h_{ih}}$ for $ih \neq jk$ and $\widetilde{a}_{ih,jk} = -\sum_{jk\neq ih} a_{ih,jk}\frac{h_{jk}}{h_{ih}}$ for $ih = jk$.

Remark. Not that, although the change of measure preserves Markov property of the joint process R, its components may not be Markov (in their natural filtration) under the new probability measure. This however is not an issue for us.

An arbitrary choice of vector h may lead to a heavy parametrization of the pricing model. We suggest that the vector h_{ij} be chosen as follows:

$$h_{ij} = \exp(\alpha_1 i + \alpha_2 j), \quad \forall i, j \in \mathcal{K},$$

where α_1 and α_2 are parameters to be calibrated. It turns out, as the calibration results provided in the next section indicate, that this is a good choice.

4.2.3 Model Calibration and Pricing

The model is fully specified by three parameters, namely α, α_1, α_2, which are calibrated to market data.

Let us consider a vanilla bond, which is equivalent[13] to the given step–up bond. One would presume, then, that the price of a step–up bond is equal to the price of the equivalent vanilla bond plus the (positive) value of the step–up provision. In general, equivalent vanilla bonds are not traded on the market. However, their price can be synthesized by applying a standard bootstrapping-interpolation procedure to the market prices of traded vanilla bonds. Surprisingly, the value of the step–up provision is often negligible or even negative. This was already noted by some recent empirical literature (cf. eg. [20]), which provides strong evidence that the market typically "underprices" step-up bonds. These findings suggest that step–up bond investors are more risk averse than vanilla bond investors. In particular, on the theoretical level, this means that the pricing kernel implied by step–up bonds prices should be different from that implied by vanilla bonds. For calibration purposes, this implies that the model parameters, or at least those relative to credit migrations, should not be calibrated to vanilla bond prices. Nevertheless, such data provides useful information. In particular, under the assumptions given below, vanilla bond prices can be used to compute a term structure of firm-specific, liquidity adjusted, discount factors (risk–free rate + liquidity spread).

Our first assumption is that the vanilla bond market assesses likelihood of the default event in the same way as the CDS (Credit Default Swap) market[14]. Our second assumption is that liquidity risk is priced identically by the step–up and vanilla bond markets.

Given the above, we can apply a standard bootstrapping-interpolation procedure to a pool of reference bonds[15] to obtain a term-structure of firm specific, liquidity adjusted, zero-coupons.

[13]By equivalent, we mean a coupon bearing bond, backed by the same company, whose all provisions, other than the step-up provision, are identical to those of the given step–up bond. That is, maturity and coupon dates are the same, and the coupons of the equivalent bond are equal to the fixed coupons of the step–up bond. In addition, credit risk is the same and liquidity risk is comparable. The term vanilla means that the step-up provision is not present.

[14]This is not necessary since default risk can be inferred from yield spreads in the bond market, but the higher liquidity of the CDS market makes it a preferable choice.

[15]We adopt here terminology from [20] to denote vanilla bonds of several maturities which have comparable liquidity and are issued by the same company as the relevant step-up bond.

The straightforward procedure is briefly described below. We are given a set of J reference bonds with associated cash-flows $CF^j_{t^j_i}$, $j = 1, \ldots, J$, and coupon dates $t^j_0 = 0, \ldots, t^j_N = T^j$ such that $T^1 < T^2 < \cdots < T^J$. The cash-flows are then adjusted by the default probability implied by the CDS spreads. Let τ denote the default time of the relevant obligor, the default adjusted cash-flows are $\widetilde{CF}^j_{t^j_i} = CF^j_{t^j_i}\mathbb{Q}(\tau > t^j_i)$. The interpolation-bootstrapping procedure is now applied to the reference bonds with default-risk adjusted cash flows, so that the resulting discount factors account only for the firm specific liquidity spread[16]. At this point, the price of an arbitrary step–up bond can be computed by simulating the evolution of the joint rating process and the relative discounted cash-flows[17]. The model parameters, α, α_1, α_2 are calibrated to step-up bond prices.

Calibration Results

We shall present now some calibration results. The bond data, obtained from Bloomberg's Corporate Bonds section, is relative to mid market quotes on April 5, 2006.

We calibrated the model parameters to a DT (Deutsche Telecom) step-up issue described in the table below:

ISIN	$XS0132407957$
Maturity	07/11/11
Coupon	$6\frac{5}{8}$ Annual
Step provision	$\begin{cases} +50 \text{ bps}, & \text{if both downgraded below single Aaa3/A-;} \\ -50 \text{ bps}, & \text{if both subsequently upgraded above Baa1/BBB+.} \end{cases}$

Table 2: *DT step-up issue on April 5, 2006.*

Given the default probability implied by the 5-y CDS spread of DT (46 bps), the liquidity adjusted discount rates are obtained using the above mentioned bootstrapping-interpolation procedure from the following pool of reference bonds:

ISIN	Maturity	Coupon	Mid-Price
XS0141544691	01/22/07	$5\frac{1}{4}$	1.015698
DE0002317807	05/20/08	$5\frac{1}{4}$	1.031821
XS0242840345	02/02/09	3	0.979798
XS0217817112	04/22/09	3	0.978352
XS0210319090	01/19/10	$3\frac{1}{4}$	0.976716
XS0210318795	01/19/15	4	0.960349

Table 3: *Reference bonds pool on April 5, 2006.*

The calibration results are given in the following table:

	Model Price	Market Price
Bond Price	1.11705	1.11705
Step-up provision	.00574	-

Table 4: *Calibration results*

We remark that, since our calibration problem is overdetermined (three parameters are calibrated to one piece of data), the value of the step-up provision is not uniquely defined. This problem can be easily overcome by calibrating the model to more step-up issues of different maturities and/or provisions.

[16]Plus market risk spreads other than credit spread.

[17]Simulation seems to be the only feasible computation technique, because of certain path dependencies in the payoff structure, induced by the step–down provision present in most step–up issues. Such path dependency is well explained in [20].

Valuation of Step-up Bonds

Using the calibrated model, we price selected issues of DT step-up bonds; we refer to Tables 5 and 6 for the description of the bonds.

ISIN	$XS0113709264$
Maturity	07/06/10
Coupon	$6\frac{5}{8}$ Annual
Step provision	$\begin{cases} +50 \text{ bps}, & \text{if both downgraded below single Aaa3/A-;} \\ -50 \text{ bps}, & \text{if both subsequently upgraded above Baa1/BBB+.} \end{cases}$

Table 5: *DT step-up issue* $XS0113709264$ *on April 5, 2006.*

ISIN	$XS0155788150$
Maturity	10/07/09
Coupon	$6\frac{1}{2}$ Annual
Step provision	$\begin{cases} +50 \text{ bps}, & \text{if both downgraded below Baa1/BBB+;} \\ -50 \text{ bps}, & \text{if both subsequently upgraded above Baa2/BBB.} \end{cases}$

Table 6: *DT step-up issue* $XS0155788150$ *on April 5, 2006.*

Table 7 presents the pricing results as well as the corresponding market quotes. The results are very satisfactory, indicating that the model is robust and prices consistently across maturities and step–up provisions.

ISIN	XS0113709264	XS0155788150
	Mkt Price/Model Price	Mkt Price/Model Price
Bond Price	1.10105/1.103546	1.08435/1.08685
Step-up provision	- /.003752	- /.00215

Table 7: *Pricing results using calibrated model*

5 Appendix: proof of Lemma 3.2

We prove here Lemma 3.2. Towards this end we first define an operator $\widehat{A}$ as

$$\begin{aligned} \widehat{A}_t f(\mathbf{z}, \widehat{\mathbf{z}}, x) &= (I^{(1:N)} \widehat{\otimes} \mathcal{L}_t) f(\mathbf{z}, \widehat{\mathbf{z}}, x) + \sum_{i=1}^{N} \eta^i(x, t, z_i) \int_{(0,1]} f(\mathbf{z} + \mathbf{y}^i, \mathbf{z}, x) - f(\mathbf{z}, \widehat{\mathbf{z}}, x) \nu^i(dy_i; x, t) \\ &+ \sum_{I \in \mathcal{J}} \lambda^I(x, t, \mathbf{z}) \int_{(0,1]^I} f(\mathbf{z} + \mathbf{y}^I, \mathbf{z}, x) - f(\mathbf{z}, \widehat{\mathbf{z}}, x) \nu^I(d\mathbf{y}_I; x, t) \\ &- \sum_{i=1}^{N} \sum_{I \in \mathcal{J} : i \in I} \lambda^I(x, t, \mathbf{z}) \int_{(0,1]^I} f(\mathbf{z} + \mathbf{y}^i, \mathbf{z}, x) - f(\mathbf{z}, \widehat{\mathbf{z}}, x) \nu^I(d\mathbf{y}_I; x, t). \end{aligned} \tag{51}$$

It is rather clear that $\widehat{A}$ is the generator of an $\mathbb{F}$-Markov process $(\mathbf{Z}, \widehat{\mathbf{Z}}, X)$, where $\widehat{\mathbf{Z}}$ is the process s.t. $Z_0 = 0$, which represents the latest state visited by process $\mathbf{Z}$ prior to its most recent jump. Now, it is well known that for any function in the domain of $\widehat{A}$, the following process,

$$M_t^f = f(Z_t, \widehat{Z}_t, X_t) - \int_0^t \widehat{A}_s f(Z_s, \widehat{Z}_s, X_s)\, ds \tag{52}$$

is an $\mathbb{F}$-local martingale.

Thus, in order to verify that $\widehat{X}$ in (35c) is an $\mathbb{F}$-local martingale it is enough to take $f(\mathbf{z}, \widehat{\mathbf{z}}, x) = x$.

Next, observe that for all I we have that

$$M_t^I = \Pi_{i\in I}\,\mathbb{1}_{\{Z_t^i>0, Z_{t-}^i=0\}} = \Pi_{i\in I}\,\mathbb{1}_{\{Z_t^i>0, \widehat{Z}_t^i=0\}}.$$

Thus, in order to verify that $\widehat{X}$ in (35a) is an $\mathbb{F}$-local martingale it is enough to take $f(\mathbf{z}, \widehat{\mathbf{z}}, x) = \Pi_{i\in I}\,\mathbb{1}_{\{z_t^i>0, \widehat{z}_t^i=0\}}$. This, together with (34) implies that $\widehat{H}^i$ in (35b) is an $\mathbb{F}$-local martingale for each i. □

References

[1] C. Albanese, J. Campolieti, O. Chen, A. Zavidonov (2003) "Credit barrier model," *Risk*, 16(6).

[2] N. Bennani (2005) "The forward loss model: a dynamic term structure approach for the pricing of portfolio credit derivatives," *working paper.*

[3] T.R. Bielecki and M. Rutkowski (2002) "Intensity-based valuation of basket credit derivatives," In: *Mathematical Finance*, J. Yong, ed. World Scientific, Singapore, 2002, 12-27.

[4] T.R. Bielecki, S. Crepey, M. Jeanblanc and M. Rutkowski (2006) "Valuation of basket credit derivatives in the credit migrations environment," *Handbook on Financial Engineering*, J. Birge and V. Linetsky eds., Elsevier, forthcoming.

[5] T.R. Bielecki, J. Jakubowski, A. Vidozzi and L. Vidozzi (2007) "Study of Dependence for Some Classes of Stochastic Processes," *work in progress.*

[6] T.R. Bielecki, A. Vidozzi and L. Vidozzi (2006) "An efficient approach to valuation of credit basket products and ratings triggered step-up bonds," *working paper.*

[7] T.R. Bielecki, A. Vidozzi and L. Vidozzi (2007) "Markov Copulae," *work in progress.*

[8] D. Brigo, A. Pallavicini and R. Torresetti (2006) " Calibration of CDO tranches with the generalized Poisson–Loss model," *working paper.*

[9] X. Burtschell, J. Gregory and J.P. Laurent (2005) "A comparative analysis of CDO pricing models," *preprint.*

[10] X. Burtschell, J. Gregory and J.P. Laurent (2005) "Beyond the Gaussian Copula: Stochastic and Local Correlation," *preprint.*

[11] L. Chen and D. Filipović (2005) "Simple model for credit migration and spread curves," *Finance and Stochastics,* 9(2).

[12] R. Douady and M. Jeanblanc (2002) "A rating-based model for credit derivatives," *European Investment Review* 1, 17-29.

[13] E. Errais, K. Giesecke and L. Goldberg (2005) "Pricing credit from the top down with affine point processes," *working paper.*

[14] S.N. Ethier and T. Kurtz (2005) *Markov Processes. Characterization and Convergence*, John Wiley & Sons, Inc., Hoboken.

[15] R. Frey and J. Backhaus (2006) "Portfolio credit risk models with interacting default intensities: a Markovian approach," *working paper.*

[16] K. Giesecke and L. Goldberg (2005) "A top–down approach to multi-name credit," *working paper.*

[17] R.A. Jarrow and F. Yu (2001) "Counterparty risk and the pricing of defaultable securities," *Journal of Finance* 56, pp. 1765-1799.

[18] P.E. Kloeden and E. Platen (2000) " Numerical Solution of Stochastic Differential Equations," Springer.

[19] W. Kang and P. Shahabuddin (2005) "FAST SIMULATION FOR MULTIFACTOR PORTFOLIO CREDIT RISK IN THE t-COPULA MODEL", *Proceedings of the 2005 Winter Simulation Conference*, M. E. Kuhl, N. M. Steiger, F. B. Armstrong, and J. A. Joines, eds.

[20] D. Lando and A. Mortensen (2005) "On the pricing of step-up bonds in the European telecom sector," *J. Credit Risk*, 1(1), pp. 71–110.

[21] Z. Palmowski and T. Rolski (2002) "A technique for exponential change of measure for Markov processes," *Bernoulli* 8(6), pp. 767-785.

[22] C. Pedersen (2003) "Valuation of portfolio credit default swaptions," *working paper.*

[23] R. Ryan (2002) *Introduction to Tensor Products of Banach Spaces*, Springer.

[24] J. Sidenius, V. Piterbarg and L. Andersen (2004) "A new framework for dynamic credit portfolio loss modelling," *working paper.*

[25] P. Schönbucher (2005) "Portfolio losses and the term structure of loss transition rates: a new methodology for pricing portfolio credit derivatives," *working paper.*