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MULTIPLE RATINGS MODEL OF DEFAULTABLE TERM STRUCTURE

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A new approach to modeling credit risk, to valuation of defaultable debt and to pricing of credit derivatives is developed. Our approach, based on the Heath, Jarrow, and Morton (1992) methodology, uses the available information about the credit spreads combined with the available information about the recovery rates to model the intensities of credit migrations between various credit ratings classes. This results in a conditionally Markovian model of credit risk. We then combine our model of credit risk with a model of interest rate risk in order to derive an arbitrage-free model of defaultable bonds. As expected, the market price processes of interest rate risk and credit risk provide a natural connection between the actual and the martingale probabilities.

KEY WORDS: defaultable term structure, credit risk, conditional Markov chain

1. INTRODUCTION

Let $B(t,T)$ and $D_{C_t}(t,T)$ denote time t prices of *default-free* and *default-risky* (or *defaultable*) zero coupon bonds maturing at time T, respectively. The default-free bond pays \$1 at time T. The default-risky bond is assumed to pay at time T an amount called the recovery payment which is less than \$1. (The meaning of the subscript C_t in the notation $D_{C_t}(t,T)$ will be explained later in the text.) For the simplicity of the exposition in this paper we focus on the recovery scheme in which the recovery payment is received by the holder of the defaultable bond at the maturity time of the bond; this is commonly referred to as the *fractional recovery of par.* Of course, if the defaultable bond does not default prior to or on the maturity date, then it pays \$1 at maturity. We are concerned with modeling of the dynamics for the price process $D_{C_t}(t,T)$, as well as with relating $B(t,T)$ and $D_{C_t}(t,T)$. To this end, we first derive a credit risk model—that is, a model for probabilities of credit default, and (i) probabilities of migrations between various credit rating classes. Our credit risk model takes into account available data regarding: (ii) credit spreads and (iii) recovery payments for various credit rating classes. Using this credit risk model we then construct an arbitrage-free model for the price processes $B(t,T)$ and $D_{C_t}(t,T)$.

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There are several existing models for the defaultable term structure in which credit rating classes appear in an explicit way (see, e.g., Arvanitis, Gregory, and Laurent 1999; Duffie and Singleton 1998; Huge and Lando 1998; Jarrow, Lando, and Turnbull 1997; Lando 1998; or Thomas, Allen, and Morkel-Kingsbury 1998). Apparently all the above references fail to account for at least one of the above-itemized characteristics ((i), (ii), or (iii)) of risky debt. In this paper, we make an attempt to create a credit risk model that accounts for all three of these fundamental aspects. Our research was partially motivated by the results of Schönbucher (1998) (see also Duffie 1994 for related studies).

2. MULTIPLE RATINGS CREDIT RISK MODEL

We make the following standing assumptions.

Condition B.1. We are given a d-dimensional standard Brownian motion W, defined on the underlying real-world probability space $(\Omega, \mathbb{F}, \mathbb{P})$ which is endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$.

Condition B.2. For any fixed maturity $T \le T^*$, the default-free instantaneous forward rate $f(t,T)$ satisfies[1]

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\cdot dW_t, \tag{2.1}$$

where α and σ are adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

Conditions B.1 and B.2 are the standard assumptions of the Heath, Jarrow, and Morton (1992) approach to term structure modeling. By definition, the price of a T-maturity default-free zero coupon bond thus equals

$$B(t,T) := \exp\left(-\int_t^T f(t,u)\,du\right), \qquad \forall\, t \in [0,T]. \tag{2.2}$$

2.1. Default-Free Term Structure

For the reader's convenience, we quote the following well-known result (see Heath et al. 1992).

LEMMA 2.1. *The real-world dynamics of the default-free bond price $B(t,T)$ are*

$$dB(t,T) = B(t,T)\left(a(t,T)\,dt + b(t,T)\cdot dW_t\right), \tag{2.3}$$

where

$$a(t,T) = f(t,t) - \alpha^*(t,T) + \tfrac{1}{2}\,|\sigma^*(t,T)|^2, \qquad b(t,T) = -\sigma^*(t,T),$$

and

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)\,du, \qquad \sigma^*(t,T) = \int_t^T \sigma(t,u)\,du.$$

[1] See Heath et al. (1992) for technical assumptions.

We assume, as is customary, that one may also invest in the risk-free savings account B, which corresponds to the short-term interest rate $r_t = f(t,t)$. In view of (2.3), the discounted bond price $Z(t,T) = B_t^{-1}B(t,T)$ satisfies under $\mathbb{P}$

$$dZ(t,T) = Z(t,T)\left(\left(\tfrac{1}{2}|b(t,T)|^2 - \alpha^*(t,T)\right)dt + b(t,T)\cdot dW_t\right).$$

The following condition is known to exclude arbitrage across default-free bonds for all maturities $T \le T^*$, as well as between default-free bonds and the savings account.

Condition M.1. There exists an adapted $\mathbb{R}^d$-valued process γ such that

$$\mathbb{E}_{\mathbb{P}}\left\{\exp\left(\int_0^{T^*}\gamma_u \cdot dW_u - \frac{1}{2}\int_0^{T^*}|\gamma_u|^2\,du\right)\right\} = 1,$$

and for any maturity $T \le T^*$ we have

$$\alpha^*(t,T) = \tfrac{1}{2}|\sigma^*(t,T)|^2 - \sigma^*(t,T)\cdot\gamma_t.$$

Let γ be some process satisfying Condition M.1. Then the probability measure $\mathbb{P}^*$, given by the formula

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(\int_0^{T^*}\gamma_u\cdot dW_u - \frac{1}{2}\int_0^{T^*}|\gamma_u|^2\,du\right), \qquad \mathbb{P}\text{ a.s.},$$

is a *spot martingale measure* for the default-free term structure—that is, the equivalent martingale measure corresponding to the choice of the savings account as the numéraire asset. Moreover, if we define a Brownian motion W^* under $\mathbb{P}^*$ by setting

$$W_t^* = W_t - \int_0^t \gamma_u\,du, \qquad \forall\, t \in [0,T^*],$$

then, for any fixed maturity $T \le T^*$, the discounted price of a default-free bond satisfies under $\mathbb{P}^*$

$$dZ(t,T) = Z(t,T)b(t,T)\cdot dW_t^*. \tag{2.4}$$

We shall assume from now on that the process γ is uniquely determined, so that the default-free bonds market is complete.[2] Formally, this means that any default-free contingent claim can be priced through risk-neutral valuation formula under the probability $\mathbb{P}^*$. It should be stressed, however, that this remark does not apply to defaultable claims since they may fail to be tradable securities under the spot martingale measure $\mathbb{P}^*$, in general.

[2] Strictly speaking, this assumption is not required for our further development.

2.2. Predefault Term Structure with Multiple Ratings

We assume that the set of rating classes is $\mathcal{K} = \{1, \ldots, K\}$, where the class K corresponds to the default event. For any $i = 1, \ldots, K$, we write $\delta_i \in [0, 1)$ to denote the corresponding recovery rate. By assumption, δ_i is the fraction of par paid at a bond's maturity for a bond currently in the ith rating class which defaults. This means that if the bond defaults at time $t < T$ when it is in class i, then it pays a fixed [nonrandom] recovery payment $\delta_i < 1$ at time T.

In this section, we consider a default-free term structure, as well as $K - 1$ different defaultable term structures. Notice that the case where $K = 2$ corresponds to a more traditional credit risk modeling where the credit migrations between various credit classes are neglected and only the possibility of default is taken into consideration. We make the following assumption.

Condition B.3. For any fixed maturity $T \leq T^*$, the instantaneous forward rate $g_i(t, T)$, corresponding to the rating class $i = 1, \ldots, K$ satisfies under $\mathbb{P}$

$$dg_i(t, T) = \alpha_i(t, T)\, dt + \sigma_i(t, T) \cdot dW_t, \tag{2.5}$$

where $\alpha_i(\cdot, T)$ and $\sigma_i(\cdot, T)$ are adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We further assume that

$$g_{K-1}(t, T) > g_{K-2}(t, T) > \cdots > g_1(t, T) > f(t, T). \tag{2.6}$$

REMARK 2.1. The inequalities (2.6) characterize various credit classes. For example, in a single-factor framework, if the volatilities $\sigma_i(t, T)$ are equal across the various credit classes, then these inequalities will be enforced by assuming

$$\alpha_{K-1}(t, T) > \alpha_{K-2}(t, T) > \cdots > \alpha_1(t, T) > \alpha(t, T). \tag{2.7}$$

General conditions on the coefficients of the above models of instantaneous forward rates that would imply (2.6) need to be determined.

We set

$$D_i(t, T) := \exp\left(- \int_t^T g_i(t, u)\, du \right) \tag{2.8}$$

for $i = 1, \ldots, K - 1$. Analogously as in the case of the process $B(t, T)$, we obtain the following formulas for the processes $D_i(t, T)$, $i = 1, \ldots, K - 1$,

$$dD_i(t, T) = D_i(t, T)\, (a_i(t, T)\, dt + b_i(t, T) \cdot dW_t)\,, \tag{2.9}$$

where

$$a_i(t, T) = g_i(t, t) - \alpha_i^*(t, T) + \tfrac{1}{2}\, |\sigma_i^*(t, T)|^2, \qquad b_i(t, T) = -\sigma_i^*(t, T), \tag{2.10}$$

and

$$\alpha_i^*(t, T) = \int_t^T \alpha_i(t, u)\, du, \qquad \sigma_i^*(t, T) = \int_t^T \sigma_i(t, u)\, du.$$

Condition M.2. Let γ be the stochastic process introduced in Condition M.1. For $i = 1, \ldots, K-1$, the process λ_i, which is given by the formula

$$\lambda_i(t) := a_i(t,T) - f(t,t) + b_i(t,T) \cdot \gamma_t, \qquad \forall t \in [0,T], \tag{2.11}$$

does not depend on the maturity T.

REMARK 2.2. Note that the above assumption is in the spirit of the Heath et al. (1992) assumption that the market price of the interest rate risk process γ does not depend on T. If we assume that for every i we have $\sigma_i(t,T) = \sigma(t,T)$ then $\lambda_i(t) = g_i(t,t) - f(t,t)$. In this case in view of (2.7) we also have that $\lambda_i(t) > 0$ for $i = 1, \ldots, K$. It is worthwhile to stress, however, that neither the strict positivity of λ_i's nor their independence of maturity T are necessary requirements for our further developments.

From now on, we make standing assumptions M.1 and M.2. It is easily seen that under a martingale measure $\mathbb{P}^*$ processes $Z_i(t,T) = B_t^{-1} D_i(t,T)$ satisfy

$$dZ_i(t,T) = Z_i(t,T)\left(\lambda_i(t)\,dt + b_i(t,T) \cdot dW_t^*\right). \tag{2.12}$$

Let us observe that the processes $D_i(t,T)$ do not represent price processes of traded securities. $D_i(t,T)$ should be interpreted as a conditional price at time t of the defaultable zero coupon bond, given that the bond has not defaulted yet (by the time t) and is currently in the credit rating class i. In particular, the processes $Z_i(t,T)$ may fail to be (local) martingales under the measure $\mathbb{P}^*$. Our goal is to construct a genuine price process for a defaultable bond. Such a process will naturally take account of jumps between various rating classes. On the other hand, the jump intensities will appropriately account for credit spreads $\gamma_i(t,T) := g_i(t,T) - f(t,T)$, through the dynamics (2.3) and (2.9) as well as the recovery rates. We achieve this goal by enlarging the underlying probability space and, in particular, by constructing a new probability measure, denoted by $\mathbb{Q}^*$, under which both the discounted price process of the default-free bond and the discounted price process of the defaultable bond follow martingales. An important role in our derivation is played by a conditionally Markov process which we interpret as the credit migration process. We shall now briefly discuss relevant properties of this process (we refer to Bielecki and Rutkowski 1999 for proofs and the technical details).

2.3. Credit Migration Process

We introduce a conditionally Markov chain, denoted C^1, on the state space $\mathcal{K} = \{1, \ldots, K\}$. The state K represents the default. The formal construction of C^1 is done by means of enlargement of the underlying probability space $(\Omega, \mathbb{F}, \mathbb{P}^*)$. Suitable extensions of $\mathbb{F}$ and $\mathbb{P}^*$ are denoted by $\tilde{\mathbb{F}}$ and $\mathbb{Q}^*$, respectively. It should be stressed that the $\mathbb{P}^*$-Wiener process W^* follows a standard Wiener process under $\mathbb{Q}^*$ with respect to the filtration $\tilde{\mathbb{F}}$.

The conditional infinitesimal generator under $\mathbb{Q}^*$ of C^1 at time t, given the σ-field $\mathcal{F}_t$, is

$$\Lambda_t = \begin{pmatrix} \lambda_{1,1}(t) & \cdots & \lambda_{1,K}(t) \\ \cdot & \cdots & \cdot \\ \lambda_{K-1,1}(t) & \cdots & \lambda_{K-1,K}(t) \\ 0 & \cdots & 0 \end{pmatrix}, \tag{2.13}$$

where $\lambda_{i,i}(t) = -\sum_{j\neq i}\lambda_{i,j}(t)$ for $i = 1, \ldots, K-1$, and where $\lambda_{i,j}$ are adapted, strictly positive processes (satisfying some mild boundedness and integrability conditions). This means that if for a real-valued function f on $\mathcal{K}$ we denote

$$\Lambda_t f(i) = \sum_{j\in\mathcal{K}} \lambda_{i,j}(t) f(j), \tag{2.14}$$

then we have the following result (see Bielecki and Rutkowski 1999).

PROPOSITION 2.1. *For every real-valued function f on $\mathcal{K}$ the process $\mathcal{M}^f$, given by the formula*

$$\mathcal{M}_t^f = f(C_t^1) - \int_0^t \Lambda_u f(C_u^1)\,du, \qquad \forall\, t \in \mathbb{R}_+,$$

is an $\tilde{\mathbb{F}}$-martingale.

Let us denote $H_i(t) = \mathbb{1}_{\{C_t^1 = i\}}$ for $i = 1, \ldots, K$. For any $i = 1, \ldots, K-1$ and $j \neq i$, we introduce an $\tilde{\mathbb{F}}$-adapted process

$$M_{i,j}(t) := H_{i,j}(t) - \int_0^t \lambda_{i,j}(u) H_i(u)\,du, \qquad \forall\, t \in [0, T], \tag{2.15}$$

where $H_{i,j}(t)$ represents the number of transitions from i to j by C^1 over the time interval $(0, t]$. It follows from Proposition 2.1 (see Bielecki and Rutkowski 1999) that $M_{i,j}(t)$ follows a local martingale on the enlarged probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \mathbb{Q}^*)$. This property, combined with the consistency condition (2.23) that is imposed below on the intensities $\lambda_{i,j}$, will provide our model of defaultable term structure with arbitrage-free features.

REMARK 2.3. As demonstrated in Bielecki and Rutkowski (1999), for any fixed t the transition probability matrix $\hat{P}(t,u)$, $u \geq t$, for the migration process C^1 satisfies the forward Kolmogorov equation

$$\frac{d\hat{P}(t,u)}{du} = \hat{P}(t,u)\mathbb{E}_{\mathbb{Q}^*}\left(\Lambda_u \,\middle|\, \mathcal{F}_t \vee \sigma(C_t^1)\right)$$

with the initial condition $\hat{P}(t,t) = \mathrm{Id}$, where Id is the identity matrix.

We find it useful (and of practical importance) to introduce a process C^2 representing the previous rating of the defaultable bond. Formally, we set $C_t^2 = C_{u(t)-}^1$, where $u(t) = \sup\{u \leq t : C_u^1 \neq C_t^1\}$ (by convention, $\sup\emptyset = 0$, therefore $C_t^2 = C_t^1$ if $C_u^1 = C_0^1$ for every $u \in [0, t]$). In words, $u(t)$ is the time of the last jump of C^1 before (and including) time t, so that C_t^2 represents the last state of C^1 before the current state C_t^1. The price process of the defaultable bond will be defined in terms of the process $C = (C^1, C^2)$ and the predefault term structure by means of the following formula:

$$D_{C_t}(t,T) := \mathbb{1}_{\{C_t^1 \neq K\}} D_{C_t^1}(t,T) + \delta_{C_t^2}\mathbb{1}_{\{C_t^1 = K\}} B(t,T). \tag{2.16}$$

The above formula appears to be quite intuitive. The mathematical details behind its formal derivation, as well as some of its important consequences, are given in the next section.

2.4. Defaultable Term Structure

In this section we use the migration process C^1 about which we shall additionally assume that the intensities $\lambda_{i,j}$'s satisfy the consistency conditions (2.23).

For the reader's convenience, we first examine the case when $K = 3$. Recall that this means that the state $i = 3$ is the state of default, and that there are two distinct credit classes labeled as 1 and 2. We assume that the defaultable bond is in any of these two credit classes at the initial time $t = 0$; that is, $C_0^1 = 1$ or $C_0^1 = 2$. Without loss of generality we also assume that $C_0^1 = C_0^2$. Thus, we assume that $C_0 = (C_0^1, C_0^2) \in \{(1,1),(2,2)\}$, so that $H_1(0) + H_2(0) = \mathbb{1}_{\{C_0^1=1\}} + \mathbb{1}_{\{C_0^1=2\}} = 1$. We also observe that for $i, j = 1, 2$, $i \neq j$, and for all $t \in [0, T]$ we have

$$H_i(t) = H_i(0) + H_{j,i}(t) - H_{i,j}(t) - H_{i,3}(t) \tag{2.17}$$

and

$$H_{i,3}(t) = \mathbb{1}_{\{C_t^1=3,\, C_t^2=i\}}. \tag{2.18}$$

Next, we define an auxiliary process $\hat{Z}(t,T)$, which follows an $\tilde{\mathbb{F}}$-local martingale under $\mathbb{Q}^*$, by setting

$$\begin{aligned} d\hat{Z}(t,T) := {} & (Z_2(t,T) - Z_1(t,T))\, dM_{1,2}(t) + (Z_1(t,T) - Z_2(t,T))\, dM_{2,1}(t) \\ & + (\delta_1 Z(t,T) - Z_1(t,T))\, dM_{1,3}(t) + (\delta_2 Z(t,T) - Z_2(t,T))\, dM_{2,3}(t) \\ & + (H_1(t) Z_1(t,T) b_1(t,T) + H_2(t) Z_2(t,T) b_2(t,T)) \cdot dW_t^* \\ & + \big(\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)\big)\, Z(t,T) b(t,T) \cdot dW_t^* \end{aligned}$$

with the initial condition

$$\hat{Z}(0,T) = H_1(0) Z_1(0,T) + H_2(0) Z_2(0,T). \tag{2.19}$$

Using (2.15), we arrive at the following representation for the dynamics of $\hat{Z}(t,T)$

$$\begin{aligned} d\hat{Z}(t,T) = {} & Z_1(t,T)\big(dH_{2,1}(t) - dH_{1,2}(t) - dH_{1,3}(t)\big) + H_1(t)\, dZ_1(t,T) \\ & + Z_2(t,T)\big(dH_{1,2}(t) - dH_{2,1}(t) - dH_{2,3}(t)\big) + H_2(t)\, dZ_2(t,T) \\ & + Z(t,T)(\delta_1\, dH_{1,3}(t) + \delta_2\, dH_{2,3}(t)) \\ & + \big(\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)\big)\, dZ(t,T) \\ & - [\lambda_{1,2}(t)\,(Z_2(t,T) - Z_1(t,T)) + \lambda_{1,3}(t)\,(\delta_1 Z(t,T) - Z_1(t,T)) \\ & \qquad + \lambda_1(t) Z_1(t,T)] H_1(t)\, dt \\ & - [\lambda_{2,1}(t)\,(Z_1(t,T) - Z_2(t,T)) + \lambda_{2,3}(t)\,(\delta_2 Z(t,T) - Z_2(t,T)) \\ & \qquad + \lambda_2(t) Z_2(t,T)] H_2(t)\, dt. \end{aligned}$$

To construct an arbitrage-free model of the defaultable term structure consistent with the dynamics (2.12) of predefault bond prices, it is indispensable to specify the matrix Λ in a judicious way. We postulate that the entries of Λ are chosen in such a way that the equalities

$$
\left\{
\begin{aligned}
&\lambda_{1,2}(t)\,(Z_2(t,T) - Z_1(t,T)) \\
&\qquad + \lambda_{1,3}(t)\,(\delta_1 Z(t,T) - Z_1(t,T)) + \lambda_1(t) Z_1(t,T) = 0, \\
&\lambda_{2,1}(t)\,(Z_1(t,T) - Z_2(t,T)) \\
&\qquad + \lambda_{2,3}(t)\,(\delta_2 Z(t,T) - Z_2(t,T)) + \lambda_2(t) Z_2(t,T) = 0,
\end{aligned}
\right. \tag{2.20}
$$

are satisfied for all $t \in [0, T]$.

REMARK 2.4. Suppose that the recovery rates $\delta_1 = \delta_2 = 0$. In this case, we postulate that the entries of Λ satisfy

$$
\left\{
\begin{aligned}
&\lambda_{1,2}(t)(1 - D_{21}(t)) + \lambda_{1,3}(t) = \lambda_1(t), \\
&\lambda_{2,1}(t)(1 - D_{12}(t)) + \lambda_{2,3}(t) = \lambda_2(t),
\end{aligned}
\right.
$$

where we set $D_{ij}(t) = Z_i(t,T)/Z_j(t,T) = D_i(t,T)/D_j(t,T)$. Notice that the coefficients $\lambda_{i,j}(t)$ are not uniquely determined. We may take, for instance, $\lambda_{1,2}(t) = \lambda_{2,1}(t) = 0$ (no migrations between classes 1 and 2) to obtain $\lambda_{1,3}(t) = \lambda_1(t)$ and $\lambda_{2,3}(t) = \lambda_2(t)$, but other choices are also possible. Notice also that we cannot set $\lambda_{1,3}(t) = \lambda_{2,3}(t) = 0$ (no default possible) since we would then have either $\lambda_{1,2}(t) < 0$ or $\lambda_{2,1}(t) < 0$. Suppose, on the contrary, that $\delta_1 + \delta_2 > 0$. In this case, we have

$$
\left\{
\begin{aligned}
&\lambda_{1,2}(t)(1 - D_{21}(t)) + \lambda_{1,3}(t)(1 - \delta_1 d_{31}(t)) = \lambda_1(t), \\
&\lambda_{2,1}(t)(1 - D_{12}(t)) + \lambda_{2,3}(t)(1 - \delta_2 d_{32}(t)) = \lambda_2(t),
\end{aligned}
\right.
$$

where $d_{ij}(t) = Z(t,T)/Z_j(t,T) = B(t,T)/D_j(t,T)$.

REMARK 2.5. Calibrating our model to market data, which in particular involves specification of the matrix Λ, is currently under investigation.

Let us return to the analysis of the process $\hat{Z}(t,T)$. Under (2.20), $\hat{Z}(t,T)$ satisfies

$$
\begin{aligned}
d\hat{Z}(t,T) := {}& (Z_2(t,T) - Z_1(t,T))\, dH_{1,2}(t) + (Z_1(t,T) - Z_2(t,T))\, dH_{2,1}(t) \\
& + (\delta_1 Z(t,T) - Z_1(t,T))\, dH_{1,3}(t) + (\delta_2 Z(t,T) - Z_2(t,T))\, dH_{2,3}(t) \\
& + H_1(t)\, dZ_1(t,T) + H_2(t)\, dZ_2(t,T) + \big(\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)\big)\, dZ(t,T)
\end{aligned}
$$

with the initial condition (2.19). This representation of the process $\hat{Z}(t,T)$, combined with (2.17) and (2.18), results in the following important formula:

$$
\hat{Z}(t,T) = \mathbb{1}_{\{C_t^1 = 1\}} Z_1(t,T) + \mathbb{1}_{\{C_t^1 = 2\}} Z_2(t,T) + \big(\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)\big) Z(t,T).
$$

Put another way,

$$
\hat{Z}(t,T) = \mathbb{1}_{\{C_t^1 \neq 3\}} Z_{C_t^1}(t,T) + \delta_{C_t^2} \mathbb{1}_{\{C_t^1 = 3\}} Z(t,T). \tag{2.21}
$$

Finally, we introduce the price process of a T-maturity defaultable bond by setting (compare with (2.16))

$$D_{C_t}(t,T) := B_t \hat{Z}(t,T) = \mathbb{1}_{\{C_t^1 \neq 3\}} D_{C_t^1}(t,T) + \delta_{C_t^2} \mathbb{1}_{\{C_t^1 = 3\}} B(t,T). \tag{2.22}$$

We are in a position now to examine the general case. For any $K \geq 3$, we define the process $\hat{Z}(t,T)$ by setting

$$\begin{aligned} d\hat{Z}(t,T) := & \sum_{i,j=1,\, i \neq j}^{K-1} \big(Z_j(t,T) - Z_i(t,T)\big)\, dM_{i,j}(t) \\ & + \sum_{i=1}^{K-1} (\delta_i Z(t,T) - Z_i(t,T))\, dM_{i,K}(t) \\ & + \sum_{i=1}^{K-1} H_i(t) Z_i(t,T) b_i(t,T) \cdot dW_t^* \\ & + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t) Z(t,T) b(t,T) \cdot dW_t^* \end{aligned}$$

with the initial condition

$$\hat{Z}(0,T) = \sum_{i=1}^{K-1} H_i(0) Z_i(0,T).$$

We shall now generalize the consistency condition (2.20).

Condition M.3. The following equalities are satisfied for all $i, j = 1, \ldots, K-1, i \neq j$, and for every $t \in [0,T]$:

$$\begin{aligned} & \lambda_{i,j}(t) \big(Z_j(t,T) - Z_i(t,T)\big) \\ & \quad + \lambda_{i,K}(t) (\delta_i Z(t,T) - Z_i(t,T)) + \lambda_i(t) Z_i(t,T) = 0. \end{aligned} \tag{2.23}$$

Under the assumption above, the process $\hat{Z}(t,T)$ is easily seen to satisfy

$$\begin{aligned} d\hat{Z}(t,T) = & \sum_{i,j=1,\, i \neq j}^{K-1} \big(Z_j(t,T) - Z_i(t,T)\big)\, dH_{i,j}(t) \\ & + \sum_{i=1}^{K-1} (\delta_i Z(t,T) - Z_i(t,T))\, dH_{i,K}(t) \\ & + \sum_{i=1}^{K-1} H_i(t)\, dZ_i(t,T) + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t)\, dZ(t,T). \end{aligned}$$

The following lemma can be proved along lines similar to those for the case of $K = 3$, therefore its proof is omitted.

LEMMA 2.2. *Under* (2.23), *the process* $\hat{Z}(t, T)$ *satisfies*

$$\hat{Z}(t, T) = \sum_{i=1}^{K-1} \left(\mathbb{1}_{\{C_t^1 = i\}} Z_i(t, T) + \delta_i H_{i,K}(t) Z(t, T) \right),$$

or, equivalently,

$$\hat{Z}(t, T) = \mathbb{1}_{\{C_t^1 \neq K\}} Z_{C_t^1}(t, T) + \delta_{C_t^2} \mathbb{1}_{\{C_t^1 = K\}} Z(t, T). \tag{2.24}$$

Moreover, the process $\hat{Z}(t, T)$ *is the unique solution of the SDE,*

$$\begin{aligned} d\hat{Z}(t, T) = & \sum_{i,j=1, i \neq j}^{K-1} \left(Z_j(t, T) - H_i(t) \hat{Z}(t-, T) \right) dM_{i,j}(t) \\ & + \sum_{i=1}^{K-1} \left(\delta_i Z(t, T) - H_i(t) \hat{Z}(t-, T) \right) dM_{i,K}(t) \\ & + \sum_{i=1}^{K-1} H_i(t) \hat{Z}(t, T) b_i(t, T) \cdot dW_t^* + H_K(t) \hat{Z}(t, T) b(t, T) \cdot dW_t^* \end{aligned}$$

with the initial condition $\hat{Z}(0, T) = \sum_{i=1}^{K-1} H_i(0) Z_i(0, T)$.

As expected, to define the price of a T-maturity defaultable bond we set

$$D_{C_t}(t, T) := B_t \hat{Z}(t, T) = \mathbb{1}_{\{C_t^1 \neq K\}} D_{C_t^1}(t, T) + \delta_{C_t^2} \mathbb{1}_{\{C_t^1 = K\}} B(t, T). \tag{2.25}$$

The following result is thus an immediate consequence of the properties of the auxiliary process $\hat{Z}(t, T)$.

PROPOSITION 2.2. *The dynamics of the price process* $D_{C_t}(t, T)$ *under the risk-neutral probability* $\mathbb{Q}^*$ *are*

$$\begin{aligned} dD_{C_t}(t, T) = & \sum_{i,j=1, i \neq j}^{K-1} \left(D_j(t, T) - D_i(t, T) \right) dH_{i,j}(t) \\ & + \sum_{i=1}^{K-1} (\delta_i B(t, T) - D_i(t, T)) \, dH_{i,K}(t) \\ & + \sum_{i=1}^{K-1} H_i(t) \, dD_i(t, T) + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t) \, dB(t, T) + r_t D_{C_t}(t, T) \, dt, \end{aligned}$$

where the differentials $dB(t, T)$ *and* $dD_i(t, T)$ *are given by the formulas*

$$dB(t, T) = B(t, T) \left(r_t \, dt + b(t, T) \cdot dW_t^* \right)$$

and

$$dD_i(t, T) = D_i(t, T) \left((r_t + \mathfrak{l}_i(t)) \, dt + b_i(t, T) \cdot dW_t^* \right).$$

The next proposition shows that the process $D_{C_t}(t, T)$, formally introduced through (2.25), can be given an intuitive interpretation in terms of default time and recovery rate. To this end, we make the following technical assumption.

Condition M.4. The process $\hat{Z}(t, T)$, given by (2.24), follows an $\tilde{\mathbb{F}}$-martingale (as opposed to a local martingale) under $\mathbb{Q}^*$.

The main result of this section holds under assumptions B.1–B.3 and M.1–M.4.

THEOREM 2.1. *For any $i = 1, \dots, K-1$, let $\delta_i \in [0, 1)$ be the recovery rate for a defaultable bond that belongs to the ith rating class at time of default. The price process $D_{C_t}(t, T)$ of a T-maturity defaultable bond equals, for any $t \in [0, T]$,*

$$D_{C_t}(t, T) = \mathbb{1}_{\{C_t^1 \neq K\}} \exp\left(-\int_t^T g_{C_t^1}(t, u)\, du\right) + \delta_{C_t^2} \mathbb{1}_{\{C_t^1 = K\}} \exp\left(-\int_t^T f(t, u)\, du\right), \tag{2.26}$$

or, equivalently,

$$D_{C_t}(t, T) = B(t, T) \left\{ \mathbb{1}_{\{C_t^1 \neq K\}} \exp\left(-\int_t^T \gamma_{C_t^1}(t, u)\, du\right) + \delta_{C_t^2} \mathbb{1}_{\{C_t^1 = K\}} \right\}, \tag{2.27}$$

where $\gamma_i(t, u) = g_i(t, u) - f(t, u)$ is the ith credit spread. Moreover, $D_{C_t}(t, T)$ satisfies the following version of the risk-neutral valuation formula:

$$D_{C_t}(t, T) = B_t \, \mathbb{E}_{\mathbb{Q}^*} \left(\delta_{C_T^2} B_T^{-1} \mathbb{1}_{\{T \geq \tau\}} + B_T^{-1} \mathbb{1}_{\{T < \tau\}} \mid \tilde{\mathcal{F}}_t \right), \tag{2.28}$$

where τ is the default time; that is, $\tau = \inf\{t \in \mathbb{R}_+ : C_t^1 = K\}$. The last formula can also be rewritten as

$$D_{C_t}(t, T) = B(t, T) \, \mathbb{E}_{\mathbb{Q}_T} \left(\delta_{C_T^2} \mathbb{1}_{\{T \geq \tau\}} + \mathbb{1}_{\{T < \tau\}} \mid \tilde{\mathcal{F}}_t \right), \tag{2.29}$$

where $\mathbb{Q}_T$ is the T-forward measure associated with $\mathbb{Q}^*$ through

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}^*} = \frac{1}{B(0, T) B_T}, \qquad \mathbb{Q}^* \text{ a.s.} \tag{2.30}$$

Proof. The first formula (2.26) is an immediate consequence of (2.25) combined with (2.2) and (2.8). For the second, notice first that in view of the second equality in (2.25) and the definition of τ, the process $D_{C_t}(t, T)$ satisfies the terminal condition $D_{C_T}(T, T) = \delta_{C_T^2} \mathbb{1}_{\{T \geq \tau\}} + \mathbb{1}_{\{T < \tau\}}$. Furthermore, using the first equality in (2.25), we deduce the discounted process $B_t^{-1} D_{C_t}(t, T)$ equals $\hat{Z}(t, T)$, so that it follows an $\tilde{\mathbb{F}}$-martingale under $\mathbb{Q}^*$. Equality (2.28) is thus obvious. Formula (2.29) is also easy to establish. □

2.5. Market Prices of Interest Rate and Credit Risk

Let us fix a horizon date T^*. We shall now change, using the general version of Girsanov's theorem, the measure $\mathbb{Q}^*$ to the equivalent probability measure $\mathbb{Q}$. In financial interpretation, the probability measure $\mathbb{Q}$ plays the role of the real-world probability in our model. For this reason, we postulate that the restriction of $\mathbb{Q}$ to the original probability space Ω necessarily coincides with the underlying probability $\mathbb{P}$. To this end, we set

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^*}\Big|_{\tilde{\mathcal{F}}_t} = L_t,$$

where the $\mathbb{Q}^*$-local positive martingale L is given by the formula

$$dL_t = -L_t\gamma_t \cdot dW_t^* + L_{t-}dM_t,$$

where in turn the $\mathbb{Q}^*$-local martingale M equals

$$dM_t = \sum_{i\neq j}(\phi_{i,j}(t) - 1)\, dM_{i,j}(t) = \sum_{i\neq j}(\phi_{i,j}(t) - 1)\, \big(dH_{i,j}(t) - \lambda_{i,j}(t)H_i(t)\, dt\big),$$

and, for any $i \neq j$, we denote by $\phi_{i,j}$ an arbitrary nonnegative $\mathbb{F}$-predictable process such that

$$\int_0^{T^*} \phi_{i,j}(t)\lambda_{i,j}(t)\, dt < \infty, \qquad \mathbb{Q}^* \text{ a.s.}$$

We assume that $\mathbb{E}_{\mathbb{Q}^*}(L_{T^*}) = 1$, so that the probability measure $\mathbb{Q}$ is well defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}_{T^*})$. It can be verified that under the probability measure $\mathbb{Q}$ the migration process C^1 is still a conditionally Markov process, and it has under $\mathbb{Q}$ the infinitesimal generator $\bar{\Lambda}_t$ with the entries $\bar{\lambda}_{i,j}(t) = \phi_{i,j}(t)\lambda_{i,j}(t)$ for every $i \neq j$ and every $t \in [0, T^*]$. The process γ ($\phi_{i,j}$, resp.) is referred to as the market price of interest rate risk (credit risk, resp.).

In particular, if the market price for credit risk depends only on the current rating i (and not on the rating j after a jump) so that $\phi_{i,j} = \phi_{i,i} =: \phi_i$ for every j, the relationship between the intensity matrices under $\mathbb{Q}$ and $\mathbb{Q}^*$ is the following: $\tilde{\Lambda}_t = \Phi\Lambda_t$, where $\Phi = \mathrm{diag}[\phi_i]$ is the diagonal matrix (such a relationship has already been postulated in Jarrow et al. 1997).

3. APPLICATIONS

3.1. Valuation of Defaultable Coupon Bonds

Consider a default-prone coupon bond with the face value F that matures at time T and promises to pay coupons c_i at times T_i ($T_i < T$), $i = 1, 2, \ldots, n$. The coupon payments are only made prior to default. For simplicity we also assume that the recovery payment is made at maturity T, in case the bond defaults before or at the maturity. Arbitrage valuation of such a bond is a straightforward consequence of the results obtained earlier in this paper. Note that the intensity matrix of the migration process C^1 may depend both on the maturity T and on the recovery rates δ_i, $i \in \mathcal{I} := \{1, 2, \ldots, K-1\}$. We shall emphasize this (possible) dependence by writing $C_t(T, \delta_{\mathcal{I}})$. In case of zero recovery we shall write $C_t(T, 0)$. Similarly, we find it convenient to emphasize the dependence of the defaultable bond's value on the recovery rates by writing $D^{\delta_{\mathcal{I}}}_{C_t(T,\delta_{\mathcal{I}})}(t, T)$ (or $D^0_{C_t(T,0)}(t, T)$, in case of zero recovery). We postulate that the arbitrage price $B_c(t, T)$ of the coupon bond considered here is given by

$$B_c(t, T) = \sum_{i=1}^{n} c_i D^0_{C_t(T_i,0)}(t, T_i) + F D^{\delta_{\mathcal{I}}}_{C_t(T,\delta_{\mathcal{I}})}(t, T), \tag{3.1}$$

with the usual convention that $D^0_{C_t(T_i,0)}(t, T_i) = 0$ for $t > T_i$. Notice that the defaultable bond covenants described above do not necessarily hold (unless a certain monotonicity of default times is imposed). Also, each zero coupon component of a defaultable coupon bond has its own ratings process. This means that a defaultable coupon bond is treated as a portfolio of defaultable zero coupon bonds. An alternative way would be to consider a particular defaultable coupon bond as a nondivisible asset, and to introduce its own ratings process.

3.2. Valuation of Credit Derivatives

3.2.1. *Default Swaps.* Consider first a *default swap*, as described, for instance, in Duffie (1999). The contingent payment X is triggered by the default event $\{C^1_t = K\}$. It is settled at time τ, and equals

$$X = \left(1 - \delta_{C^2_T} B(\tau, T)\right) \mathbb{1}_{\{\tau \leq T\}}.$$

Notice the dependence of the payment X on the initial rating C^1_0 through default time τ and recovery rate $\delta_{C^2_T}$. We consider two cases. Either (i) the buyer pays a lump sum at contract's inception (such a contract is referred to as the *default option*), or (ii) the buyer pays an annuity at the fixed time instants t_i, $i = 1, 2, \ldots, m$ (*default swap*). In case (i), the value at time 0 of a default option is given by the risk-neutral valuation formula

$$\pi_0(X) = \mathbb{E}_{\mathbb{Q}^*}\left(B^{-1}_\tau \left(1 - \delta_{C^2_T} B(\tau, T)\right) \mathbb{1}_{\{\tau \leq T\}}\right).$$

In case (ii), the annuity κ satisfies

$$\pi_0(X) = \kappa\, \mathbb{E}_{\mathbb{Q}^*}\left(\sum_{i=1}^{m} B^{-1}_{t_i} \mathbb{1}_{\{t_i < \tau\}}\right).$$

3.2.2. *Total Rate of Return Swaps.* Another important example of a credit derivative is the *total rate of return swap* (see Das 1998). We take as a *reference asset* the coupon bond described above, with the promised cash flows c_i at times T_i. We assume that its price process is given by equality (3.1). Suppose the contract maturity is $\tilde{T} \leq T$. Further suppose that the *reference rate* payments are made by the investor at fixed scheduled times $t_i \leq \tilde{T}$, $i = 1, 2, \ldots, m$. The owner of a total rate of return swap is entitled not only to all coupon payments during the life of the contract, but also to the change in the value of the underlying bond paid as a lump sum at the contract's termination. Then, the reference rate ρ to be paid by the investor should be computed from

$$\rho\,\mathbb{E}_{\mathbb{Q}^*}\left(\sum_{i=1}^{m} B_{t_i}^{-1}\,\mathbb{1}_{\{C^1_{t_i}(T,\delta_{\mathcal{I}})\neq K\}}\right) = \sum_{i=1}^{n} c_i\,D^0_{C_0(T_i,0)}(0,T_i)\,\mathbb{1}_{\{T_i\leq\tilde{T}\}}$$
$$+\mathbb{E}_{\mathbb{Q}^*}\left(B_{\tilde{\tau}}^{-1}\,(B_c(\tilde{\tau},T)-B_c(0,T))\right),$$

where $\tilde{\tau}=\tau\wedge\tilde{T}$, and $\tau=\inf\{t\geq 0\,:\,C^1_t(T,\delta_{\mathcal{I}})=K\}$. For simplicity, in the left-hand side of the valuation formula above, as well as in the second term in the right-hand side, the default time of the underlying coupon bond was assumed to be represented by the default time of its face value component.

The important issue of replicating strategies for credit derivatives will be treated elsewhere.

4. CONCLUDING REMARKS

Let us summarize the content of this work:

(i) Our main objective has been to construct an arbitrage-free model for the prices of default-free and defaultable debt in the case of multiple credit rating classes.

(ii) As the primitive objects (or data) in our approach we have chosen the default-free and defaultable instantaneous forward rates, as well as the credit recovery rates. Our arbitrage-free pricing model for defaultable bonds is constructed so that it supports these data.

(iii) Our fundamental pricing equation (2.27) represents the price process of the defaultable debt $D_{C_t}(t,T)$ in terms of: (a) the price process $B(t,T)$ of default-free debt, (b) the credit recovery rates δ_i's, (c) the credit migration process C_t, and (d) the credit spread processes $\gamma_i(t,u)$'s. This equation is intuitively clear.

(iv) Other useful representations of the price process $D_{C_t}(t,T)$ have also been derived. Most notably, the equation (2.28) provides a version of the risk-neutral valuation formula. The process $D_{C_t}(t,T)$ is represented here in terms of: (a) the price process $B(t,T)$ of default-free debt, (b) the credit recovery rates δ_i's, (c) the credit migration process C_t, (d) the equivalent martingale measure $\mathbb{Q}^*$ and the enlarged filtration $\tilde{\mathbb{F}}$. Notice that the measure $\mathbb{Q}^*$ and the filtration $\tilde{\mathbb{F}}$ encompass both the market risk and the credit risk.

(v) The arbitrage-free property of the model is provided by our consistency condition (2.23). The credit migration intensities $\lambda_{i,j}$'s are not uniquely determined by this condition, in general. Calibration of the model to the statistical data on credit migrations is supposed to pick up the right solution. Let us finally observe that the model developed in this section can be accurately referred to in the following cumbersome fashion: the credit-spreads-based HJM-type arbitrage-free term structure model with multiple ratings.

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