INTRICACIES OF DEPENDENCE BETWEEN COMPONENTS OF MULTIVARIATE MARKOV CHAINS: WEAK MARKOV CONSISTENCY AND MARKOV COPULAE

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Abstract. In this paper we examine the problem of existence and construction of a multivariate Markov chain with components that are given Markov chains. In this regard we give sufficient and necessary conditions, in terms of the semimartingale characteristics, for a component of a multivariate Markov chain to be a Markov chain in its own filtration – a property called weak Markov consistency. We also discuss the issue of dependence between the components of a multivariate Markov chain in the context of weak Markovian consistency. Accordingly, we introduce and discuss the concept of weak Markov copulae. Finally, we examine relationships between the concepts of weak Markov consistency and weak Markov copulae, and the concepts of strong Markov consistency and strong Markov copulae that were introduced in our earlier works.

Introduction

Modeling of dependence between stochastic processes is a very important issue arising from many different applications, among others in financial mathematics. By modeling dependence we mean construction of a multivariate stochastic process with prescribed marginal laws. In this paper we focus on Markov chains, and deal with the problem of constructing a multivariate Markov chain such that its components are **given** Markov chains in their own filtrations. It is well known that components of multivariate Markov process are in general not Markovian (in any filtration), so the problem that we study here is by no means a trivial one. We give sufficient and necessary conditions, in terms of the semimartingale characteristics, for a component of a multivariate Markov chain to be a Markov chain in its own filtration.

Our paper continues the study of Markovian consistency and Markov copulae for multivariate Markov processes, initiated in [3], [4], [5] and [6].

Here, we introduce and study the concept of *weak Markovian consistency*, and we relate it to the concept of *strong Markovian consistency* that was explored in the aforementioned papers under the name of *Markovian consistency*. We also continue the study of dependence between Markov processes. Thus, we continue the study of Markov copulae, the concept originally introduced in [3]. Here, we examine Markov copulae with regard to weak Markovian consistency. It turns out that certain unwanted features of Markov copulae, inherent to the framework of strong Markovian consistency, are no longer present in the framework of weak Markovian consistency. This is particularly pleasing in view of applications of Markov copulae in credit risk; more on this later. In order to keep

2010 *Mathematics Subject Classification.* 60J27; 60G55 .

Key words and phrases. multivariate Markov chain; compensator of random measure; dependence; marginal law; Markov consistency; Markov copulae.

Research of T.R. Bielecki was partially supported by NSF Grant DMS-0604789 and NSF Grant DMS-0908099.

Research of J. Jakubowski and M. Niewęgłowski was partially supported by Polish MNiSW grant N N201 547838.

the presentation simple, we confine our discussion, for the most part, to the case of finite Markov chains.[1]

It needs to be noted that problems that we study in the present paper are also connected with lumpability problem for continuous time Markov chains (see Ball and Yeo [1] and discussion there, Burke and Rosenblatt [7]). In [1] necessary and sufficient conditions are provided for intensity matrix so that the marginal component process of a Markov chain is a time homogenous continuous time Markov chain in its natural filtration. If we omit the assumption of time homogeneity and weaken assumption on intensity matrix, then there exist Markov process with marginals being also Markov in their own filtration which does not satisfy conditions from [1] (see Example 3.2.). Moreover assumptions imposed in these papers on intensity matrix exclude Markov chains with absorbing states, a case that can be treated using our methodology.

The paper is organized as follows. In Section 1 we give a sufficient and necessary condition for a multivariate Markov chain to be weakly consistent. Note that a sufficient condition for weak Markovian consistency can be deduced from the result of Rogers and Pitman [12] in which sufficient conditions for a function of a Markov process be a Markov process are given. Our condition for a weak Markovian consistency is not only more explicit, but also necessary. We also study the question when weak Markovian consistency implies strong Markovian consistency. It turns out that this is equivalent to $\mathbb{P}$-immersion between $\mathbb{F}^{X^i}$ and $\mathbb{F}^X$, given that weak Markovian consistency holds. In Section 2 we study weak Markov copulae. In Section 3 we present three simple, but non-trivial examples, that illustrate intricacies of dependence between components of a multivariate Markov chain.

1. Weak Markovian Consistency

As already said, we shall focus on the case of finite Markov chains in this paper. Nevertheless, we shall formulate the concept of weak Markovian consistency in more generality. Towards this end we consider $X = (X^n, n = 1, \ldots, N)$, a multivariate Markov process, defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in $\mathbb{R}^N$.[2] We denote by $\mathbb{F}^X$ the filtration of X, and by $\mathbb{F}^{X^n}$ the filtration of the coordinate X^n of X. It is well known that, in general, the coordinates of X are not Markov with respect to their own filtrations.

Definition 1.1. (i) Let us fix n. We say that the process X satisfies the **weak Markovian consistency condition with respect to the component** X^n if for every $B \in \mathcal{B}(\mathbb{R})$ and all $t, s \geq 0$,

$$\mathbb{P}\left(X^n_{t+s} \in B | \mathcal{F}^{X^n}_t\right) = \mathbb{P}\left(X^n_{t+s} \in B | X^n_t\right), \tag{1}$$

so that the component X^n of X is a Markov process in its own filtration.
(ii) If X satisfies the weak Markovian consistency condition with respect to X^n for each $n = 1, \ldots, N$, then we say that X satisfies the weak Markovian consistency condition.

Previously, in [3], [4], [5] and [6], a stronger concept was studied.

Definition 1.2. (i) Let us fix n. We say that the process X satisfies the **strong Markovian consistency condition with respect to the component** X^n if for every $B \in \mathcal{B}(\mathbb{R})$ and all $t, s \geq 0$,

$$\mathbb{P}\left(X^n_{t+s} \in B | \mathcal{F}^X_t\right) = \mathbb{P}\left(X^n_{t+s} \in B | X^n_t\right) \tag{2}$$

or equivalently

$$\mathbb{P}\left(X^n_{t+s} \in B | X_t\right) = \mathbb{P}\left(X^n_{t+s} \in B | X^n_t\right), \tag{3}$$

[1] The study in the case of general Markov processes (that are semimartingales) is deferred to a follow up paper.
[2] The study presented in this paper carries over to the case of multivariate Markov process taking values in a product of arbitrary (metric) spaces.

so that X^n is a Markov process in the filtration of X.
(ii) If X satisfies the strong Markovian consistency condition with respect to X^n for each $n = 1, \dots, N$, then we say that X satisfies the strong Markovian consistency condition.

Obviously, strong Markovian consistency implies weak Markovian consistency, but not vice versa as will be seen in one of the examples in Section 3. As a matter of fact, it may happen that all components of X are Markovian in their filtrations, but X is not Markovian in its filtration (see e.g. Bielecki et al. [5, Example 2.4.2]).

From now on we assume that $X = (X^1, \dots, X^N)$ is a Markov chain with values in a finite product space, say $\mathcal{X} = \times_{n=1}^N \mathcal{X}^n$, where $\mathcal{X}^n = \{x_1^n, \dots, x_{m_n}^n\} \subseteq \mathbb{R}$. To somewhat simplify the notation we shall consider bivariate processes X only, that is, we put $N = 2$, and we take $\Lambda(t) = [\lambda_y^x(t)]_{x,y\in\mathcal{X}}$ as a generic symbol for the $\mathbb{P}$-infinitesimal generator of X. Thus, $\Lambda(t)$ is an $m \times m$ matrix, where $m = m_1 \cdot m_2$.

1.1. **Semimartingale characterization of a finite Markov chain.** Let us consider a càdlàg process V defined on $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in a finite set $\mathcal{V} \subset \mathbb{R}^N$.

For any two distinct states $v, w \in \mathcal{V}$, we define an $\mathbb{F}^V$-optional random measure N_{vw} on $[0,\infty)$ by

(4)
$$N_{vw}((0,t]) = \sum_{0<s\le t} \mathbb{1}_{\{V_{s-}=v, V_s=w\}}.$$

We shall simply write $N_{vw}(t)$ in place of $N_{vw}((0,t])$. Manifestly, $N_{vw}(t)$ represents the number of jumps from state v to state w that the process V executes over the time interval $(0,t]$. Let us denote by ν_{vw} the dual predictable projection (the *compensator*) with respect to $\mathbb{F}^V$ of the random measure N_{vw}.

Next, let us define a deterministic matrix valued function Λ on $[0,\infty)$ by

(5)
$$\Lambda(t) = [\lambda_w^v(t)]_{v,w\in\mathcal{V}},$$

where λ_w^v's are real valued, locally integrable functions on $[0,\infty)$ such that for $t \in [0,\infty)$ and $v, v \in \mathcal{V}$, $v \neq w$, we have

$$\lambda_w^v(t) \ge 0$$

and

$$\lambda_v^v(t) = -\sum_{w\neq v} \lambda_w^v(t).$$

Then we have the following result, which is a version of Lemma 5.1 in [4]:[3]

Proposition 1.1. *A process V is a Markov chain (with respect to $\mathbb{F}^V$) with infinitesimal generator $\Lambda(t)$ iff the compensators with respect to $\mathbb{F}^V$ of the counting measures $N_{vw}(dt)$, $v, w \in \mathcal{V}$, are of the form*

(6)
$$\nu_{vw}((0,t]) = \int_0^t \mathbb{1}_{\{V_s=v\}} \lambda_w^v(s) ds.$$

Remarks 1.1. *A finite Markov chain V with a locally integrable generator $\Lambda(t)$ is a semimartingale (see, e.g., Elliott et al.* [10, Chapter 7.2]*). The jump measure of V, say μ^V, can be expressed in terms of summation of the jump measures N_{vw}. Thus, in view of Proposition 1.1 the infinitesimal characteristic of V (with respect to an appropriate truncation function), which is the compensator of μ^V, denoted as ν^V, is given in terms of summation of the compensators ν_{vw}. One can easily check that if we define a truncation function h by*

$$h(x) := x\mathbb{1}_{\{|x|\le d\}}, \quad \textit{where} \quad d := \frac{1}{2} \min\{|v-w| : v \neq w, v \in \mathcal{V}, w \in \mathcal{V}\},$$

[3] It can be shown that the left hand limits X_{t-} used in Lemma 5.1 in [4] can be replaced with X_t.

then $(0, 0, \nu^V)$ *is the local characteristic of* V*, where*

$$\nu^V(dx, dt) = \sum_{v,w\in\mathcal{V}:v\neq w} \delta_{w-v}(dx)\nu_{vw}(dt),$$

and δ *denotes the Dirac measure.*

1.2. **Necessary and sufficient conditions for weak Markovian consistency in terms of semimartingale characteristics.** Let us recall that we consider bivariate processes. We take $n = 1$ and we study the weak Markovian consistency of X with respect to X^1. A completely analogous discussion can be carried out with respect to X^2.

For any two states $x^1, y^1 \in \mathcal{X}^1$ such that $x^1 \neq y^1$, we define the following $\mathbb{F}^X$-optional random measure on $[0, \infty)$:

(7) $$N^1_{x^1y^1}((0, t]) = \sum_{0<s\leq t} \mathbb{1}_{\{X^1_{s-}=x^1, X^1_s=y^1\}}.$$

We shall write $N^1_{x^1y^1}(t)$ in place of $N^1_{x^1y^1}((0, t])$, and we shall denote by $\nu^1_{x^1y^1}$ the dual predictable projection (the compensator) with respect to $\mathbb{F}^X$ of the random measure $N^1_{x^1y^1}$.

Next, for any two states $x = (x^1, x^2), y = (y^1, y^2) \in \mathcal{X}$ such that $x \neq y$, we define an $\mathbb{F}^X$-optional random measure on $[0, \infty)$ by

(8) $$N_{xy}((0, t]) = \sum_{0<s\leq t} \mathbb{1}_{\{(X^1_{s-}=x^1, X^2_{s-}=x^2), (X^1_s=y^1, X^2_s=y^2)\}}.$$

We shall write $N_{xy}(t)$ in place of $N_{xy}((0, t])$, and we shall denote by ν_{xy} the compensator of N_{xy} with respect to $\mathbb{F}^X$.

It is easy to see that

(9) $$N^1_{x^1y^1}(t) = \sum_{x^2,y^2\in\mathcal{X}^2} N_{(x^1,x^2),(y^1,y^2)}(t),$$

and consequently (due to uniqueness of compensators)

(10) $$\nu^1_{x^1y^1}((0, t]) = \sum_{x^2,y^2\in\mathcal{X}^2} \nu_{(x^1,x^2),(y^1,y^2)}((0, t]).$$

In view of Proposition 1.1, we see that for any two distinct states $x = (x^1, x^2), y = (y^1, y^2) \in \mathcal{X}$,

(11) $$\nu_{(x^1,x^2),(y^1,y^2)}(dt) = \mathbb{1}_{\{(X^1_t, X^2_t)=(x^1,x^2)\}}\lambda^{x^1x^2}_{y^1y^2}(t)dt.$$

Let us denote by $\widehat{\nu}^1_{x^1y^1}$ the compensator of the measure $N^1_{x^1y^1}$ with respect to $\mathbb{F}^{X^1}$.

Lemma 1.1. *Assume that* X *is a Markov chain with respect to its own filtration. The* $\mathbb{F}^{X^1}$*-compensator of* $N^1_{x^1,y^1}$ *has the form*

(12) $$\widehat{\nu}^1_{x^1y^1}(dt) = \mathbb{1}_{\{X^1_t=x^1\}} \sum_{x^2,y^2\in\mathcal{X}^2} \lambda^{x^1x^2}_{y^1y^2}(t)\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{X^2_t=x^2\}}|\mathcal{F}^{X^1}_t)dt.$$

Proof. It follows from Lemma 4.3 in [4] that

(13) $$\begin{aligned}\widehat{\nu}^1_{x^1y^1}(dt) &= \sum_{x^2,y^2\in\mathcal{X}^2} \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{(X^1_t, X^2_t)=(x^1,x^2)\}}\lambda^{x^1x^2}_{y^1y^2}(t)|\mathcal{F}^{X^1}_{t-})dt \\ &= \sum_{x^2,y^2\in\mathcal{X}^2} \lambda^{x^1x^2}_{y^1y^2}(t)\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{X^1_t=x^1\}}\mathbb{1}_{\{X^2_t=x^2\}}|\mathcal{F}^{X^1}_{t-})dt.\end{aligned}$$

The process X is quasi-left continuous, since it is a Markov chain. Hence, X^1 is also quasi-left continuous, so its natural filtration $\mathbb{F}^{X^1}$ is quasi-left continuous and hence $\mathcal{F}_t^{X^1} = \mathcal{F}_{t-}^{X^1}$ (see Rogers and Williams [13, III.11]). Thus by (13) we have (12). □

Using Lemma 1.1 and Proposition 1.1 we obtain the following important result.

Theorem 1.1. *Assume that X is a Markov chain. The process X^1 is a Markov chain with respect to its own filtration if and only if*

(14)
$$\mathbb{1}_{\{X_t^1=x^1\}} \sum_{x^2,y^2\in\mathcal{X}^2} \lambda_{y^1y^2}^{x^1x^2}(t)\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{X_t^2=x^2\}}|\mathcal{F}_t^{X^1}\right) = \mathbb{1}_{\{X_t^1=x^1\}}\lambda_{x^1y^1}^1(t) \quad dt\otimes d\mathbb{P}\text{-a.s. } \forall x^1,y^1\in\mathcal{X}^1, x^1\neq y^1$$

for some locally integrable functions $\lambda_{x^1y^1}^1$. The generator of X^1 is $\Lambda^1(t) = [\lambda_{x^1y^1}^1(t)]_{x^1,y^1\in\mathcal{X}^1}$ with $\lambda_{x^1x^1}^1$ given by

$$\lambda_{x^1x^1}^1(t) = -\sum_{y^1\in\mathcal{X}^1, y^1\neq x^1} \lambda_{x^1y^1}^1(t) \qquad \forall x^1\in\mathcal{X}^1.$$

Proof. Assume that (14) holds. Since X is a Markov chain, for each $x^1, y^1 \in \mathcal{X}^1$, the $\mathbb{F}^{X^1}$ compensator of N_{x^1,y^1}^1 has, by Lemma 1.1 and (14), the form

$$\widehat{\nu}_{x^1y^1}^1(dt) = \mathbb{1}_{\{X_t^1=x^1\}}\lambda_{x^1y^1}^1(t)dt$$

for some locally integrable deterministic function $\lambda_{x^1y^1}^1$. Then, by Proposition 1.1, X^1 is a Markov chain with generator $\Lambda^1(t) = [\lambda_{x^1y^1}^1(t)]_{x^1,y^1\in\mathcal{X}^1}$. Conversely, assume that X^1 is a Markov chain with respect to its natural filtration with generator $\Lambda^1(t) = [\lambda_{x^1y^1}^1(t)]_{x^1,y^1\in\mathcal{X}^1}$. Then (14) follows from Lemma 1.1 and Proposition 1.1. □

Remarks 1.2. *Note that* (14) *implies that*

(15)
$$\mathbb{1}_{\{X_t^1=x^1\}} \sum_{x^2,y^2\in\mathcal{X}^2} \lambda_{y^1y^2}^{x^1x^2}(t)\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{X_t^2=x^2\}}|X_t^1 = x^1\right) = \mathbb{1}_{\{X_t^1=x^1\}}\lambda_{x^1y^1}^1(t) \quad dt\otimes d\mathbb{P}\text{-a.s. } \forall x^1,y^1\in\mathcal{X}^1, x^1\neq y^1$$

Thus, condition (15) *is necessary for the weak Markovian consistency of X with respect to X^1.*

Now, let us recall condition (M) from [4]:[4]

Condition (M): *The generator matrix $\Lambda(t)$ satisfies*

(M1)
$$\sum_{y^2\in\mathcal{X}^2} \lambda_{y^1y^2}^{x^1x^2}(t) = \sum_{y^2\in\mathcal{X}^2} \lambda_{y^1y^2}^{x^1\bar{x}^2}(t), \ \forall x^2,\bar{x}^2\in\mathcal{X}^2, \ \forall x^1,y^1\in\mathcal{X}^1, \ x^1\neq y^1.$$

and

(M2)
$$\sum_{y^1\in\mathcal{X}^1} \lambda_{y^1y^2}^{x^1x^2}(t) = \sum_{y^1\in\mathcal{X}^1} \lambda_{y^1y^2}^{\bar{x}^1x^2}(t), \ \forall x^1,\bar{x}^1\in\mathcal{X}^1, \ \forall x^2,y^2\in\mathcal{X}^2, \ x^2\neq y^2.$$

Next, consider the functions $\lambda_{x^1y^1}^1$ given by

(16)
$$\lambda_{x^1y^1}^1(t) = \sum_{y^2\in\mathcal{X}^2} \lambda_{y^1y^2}^{x^1x^2}(t), \quad x^1,y^1\in\mathcal{X}^1, \ x^1\neq y^1, \quad \lambda_{x^1x^1}^1(t) = -\sum_{y^1\in\mathcal{X}^1, y^1\neq x^1} \lambda_{x^1y^1}^1(t), \quad \forall x^1\in\mathcal{X}^1.$$

Under condition (M1), the functions $\lambda_{x^1y^1}^1$ are well defined and locally integrable. It is straightforward to verify that they satisfy (14), so that weak Markovian consistency holds with respect to X^1.

[4]Since condition (M) has been originally formulated for both components of X, we state it here for both components as well, even if for the purpose of this section only the part (M1) of this condition suffices.

Remarks 1.3. *Theorem 1.1 provides a necessary and sufficient condition for the Markov process X to satisfy the weak Markovian consistency condition with respect to its first component X^1. The analogous condition with respect to X^2 reads: The process X^2 is a Markov chain with respect to its own filtration if and only if*

(17)
$$\mathbb{1}_{\{X_t^2=x^2\}}\sum_{x^1,y^1\in\mathcal{X}^1}\lambda^{x^1x^2}_{y^1y^2}(t)\mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{X_t^1=x^1\}}|\mathcal{F}_t^{X^2}\right)=\mathbb{1}_{\{X_t^2=x^2\}}\lambda^2_{x^2y^2}(t)\quad dt\otimes d\mathbb{P}\text{-a.s. }\forall x^2,y^2\in\mathcal{X}^1,x^2\neq y^2$$

for some locally integrable functions $\lambda^2_{x^2y^2}$. Then the generator of X^2 is $\Lambda^2(t)=[\lambda^2_{x^2y^2}(t)]_{x^2,y^2\in\mathcal{X}^2}$ with $\lambda^2_{x^2x^2}$ given by

$$\lambda^2_{x^2x^2}(t)=-\sum_{y^2\in\mathcal{X}^2,y^2\neq x^2}\lambda^2_{x^2y^2}(t),\quad\forall x^2\in\mathcal{X}^2.$$

If we define

(18)
$$\lambda^2_{x^2y^2}(t)=\sum_{y^1\in\mathcal{X}^1}\lambda^{x^1x^2}_{y^1y^2}(t),\quad x^2,y^2\in\mathcal{X}^2,\,x^2\neq y^2,\quad\lambda^2_{x^2x^2}(t)=-\sum_{y^2\in\mathcal{X}^2,y^2\neq x^2}\lambda^2_{x^2y^2}(t),\quad\forall x^2\in\mathcal{X}^2,$$

then under condition (M2) the functions $\lambda^2_{x^2y^2}$ are well defined and locally integrable. It is straightforward to verify that they satisfy (17)*, so that weak Markovian consistency holds with respect to X^2.*

Remarks 1.4. *a) It was shown in* [4] *that, in fact, condition (M) implies not only weak, but also strong Markovian consistency.*

b) Ball and Yeo [1] *considered time homogeneous Markov chains with intensity matrix Λ satisfying some additional assumptions (cf.* [1, Condition 2.2]*). In* [1, Theorem 3.1]*, it is proved that the marginal process X^1 of time a homogenous Markov chain X is a time homogenous Markov chain in its natural filtration if and only if a condition equivalent to Condition (M1) holds. However, if we omit the assumption of time homogeneity, then* [1, Theorem 3.1] *does not hold; see our Example 3.2 below. Moreover, assumptions imposed in* [1] *on Λ exclude Markov chains with absorbing states.*

We shall see in Section 3 that there exist Markov chains that are weakly Markovian consistent, but not strongly Markovian consistent.

1.3. **Operator interpretation of necessary conditions for weak Markovian consistency, and of condition (M) for strong Markovian consistency.** For $i=1,2$ and $t\geq 0$, we define an operator Q_t^i, acting on any function f on $\mathcal{X}=\mathcal{X}^1\times\mathcal{X}^2$, by

$$(Q_t^i f)(x^i)=\mathbb{E}_{\mathbb{P}}(f(X_t)|X_t^i=x^i),\quad\forall x^i\in\mathcal{X}^i.\tag{19}$$

We also introduce an extension operator $C^{i,*}$ as follows: for any function f^i on $\mathcal{X}^i$ the function $C^{i,*}f^i$ is defined on $\mathcal{X}$ by

$$(C^{i,*}f^i)(x)=f^i(x^i),\quad\forall x=(x^1,x^2)\in\mathcal{X}.$$

We have the following proposition, which will be important in the next section in the context of weak Markov copulae.

Theorem 1.2. *Fix $i\in\{1,2\}$. The condition*

$$Q_t^i\Lambda(t)C^{i,*}=\Lambda^i(t),\ t\geq 0,\tag{20}$$

is necessary for weak Markovian consistency with respect to X^i.

Proof. We give the proof for $i=1$. It is enough to observe that (15) is equivalent to (20). Indeed, first note that (20) is equivalent to the equality

$$(Q_t^1\Lambda(t)C^{1,*}g)(x^1)=\sum_{y^1\in\mathcal{X}^1}\lambda^1_{x^1y^1}(t)g(y_1)\tag{21}$$

for an arbitrary function g on $\mathcal{X}^1$ and $x^1 \in \mathcal{X}^1$. Now, we rewrite the left hand side:

$$\begin{aligned}
(Q_t^1 \Lambda(t) C^{1,*} g)(z^1) &= \mathbb{E}\left(\sum_{(x^1,x^2)\in\mathcal{X}} \mathbb{1}_{\{X_t^1=x^1, X_t^2=x^2\}} \sum_{(y^1,y^2)\in\mathcal{X}} \lambda_{y^1 y^2}^{x^1 x^2}(t) g(y^1) \middle| X_t^1 = z^1 \right) \\
&= \sum_{x^2\in\mathcal{X}^2} \left(\mathbb{E}\left(\mathbb{1}_{\{X_t^2=x^2\}} \middle| X_t^1 = z^1 \right) \sum_{(y^1,y^2)\in\mathcal{X}} \lambda_{y^1 y^2}^{z^1 x^2}(t) g(y^1)) \right) \\
&= \sum_{y^1\in\mathcal{X}^1} \left(\sum_{x^2\in\mathcal{X}^2} \sum_{y^2\in\mathcal{X}^2} \mathbb{E}\left(\mathbb{1}_{\{X_t^2=x^2\}} \middle| X_t^1 = z^1 \right) \lambda_{y^1 y^2}^{z^1 x^2}(t) \right) g(y^1).
\end{aligned}$$

Since g is arbitrary, (21) is equivalent to

$$\lambda_{x^1 y^1}^1(t) = \sum_{x^2\in\mathcal{X}^2} \sum_{y^2\in\mathcal{X}^2} \mathbb{E}\left(\mathbb{1}_{\{X_t^2=x^2\}} \middle| X_t^1 = z^1 \right) \lambda_{y^1 y^2}^{z^1 x^2}(t),$$

which is exactly (15). $\square$

Now we consider an operator interpretation of condition (M) for strong Markovian consistency.

Proposition 1.2. *Condition* (16) *is equivalent to*

$$C^{1,*} \Lambda^1(t) = \Lambda(t) C^{1,*}, \tag{22}$$

and condition (18) *is equivalent to*

$$C^{2,*} \Lambda^2(t) = \Lambda(t) C^{2,*}. \tag{23}$$

Proof. We only prove the first equivalence. The proof of the other one is analogous.
We note that (22) is equivalent to the equality

$$(C^{1,*} \Lambda^1(t) g)(x^1, x^2) = (\Lambda(t) C^{1,*} g)(x^1, x^2), \quad \forall x^1 \in \mathcal{X}^1, \tag{24}$$

for an arbitrary function g on $\mathcal{X}^1$. By definition, the right hand side of (24) is

$$\begin{aligned}
(\Lambda(t) C^{1,*} g)(x^1, x^2) &= \sum_{(y^1,y^2)\in\mathcal{X}} \lambda_{y^1 y^2}^{x^1 x^2}(t) (C^{1,*} g)(y^1, y^2) = \sum_{(y^1,y^2)\in\mathcal{X}} \lambda_{y^1 y^2}^{x^1 x^2}(t) g(y^1) \\
&= \sum_{y^1\in\mathcal{X}^1} \left(\sum_{y^2\in\mathcal{X}^2} \lambda_{y^1 y^2}^{x^1 x^2}(t) \right) g(y^1),
\end{aligned}$$

and the left hand side of (24) is given by

$$(C^{1,*} \Lambda^1(t) g)(x^1, x^2) = \sum_{y^1\in\mathcal{X}^1} \lambda_{x^1 y^1}^1(t) g(y^1).$$

Since g is arbitrary, we obtain

$$\lambda_{x^1 y^1}^1(t) = \sum_{y^2\in\mathcal{X}^2} \lambda_{y^1 y^2}^{x^1 x^2}(t) \qquad \forall x^1, y^1 \in \mathcal{X}^1, \forall x^2 \in \mathcal{X}^2 \tag{25}$$

Hence, using the fact that Λ is the generator of a Markov chain we see that (16) is equivalent to (25), so to (22). $\square$

Proposition 1.3. *Condition* (22) *implies* (20) *for* $i = 1$ *and condition* (23) *implies* (20) *for* $i = 2$.

Proof. Since $Q_t^i C^{i,*} = \mathrm{Id}$ for $i = 1, 2$, we have

$$Q_t^i \Lambda(t) C^{i,*} = Q_t^i C^{i,*} \Lambda^i(t) = \Lambda^i(t), \ t \geq 0, \quad i = 1, 2.$$

$\square$

Remarks 1.5. *Another proof of Proposition 1.3 is the following: Conditions* (22) *and* (23) *are sufficient for strong Markovian consistency of X(see Remark 1.4), which implies weak Markovian consistency of X, for which* (20) *is a necessary condition.*

Remarks 1.6. *In the case of time homogeneous Markov processes, conditions analogous to* (22) *and* (23) *have been previously studied in* [3] *and* [15], *and it has been shown that they are sufficient for strong Markovian consistency. So* (22) *and* (23) *imply that each coordinate of the Markov process in question is a Markov process with respect to $\mathbb{F}^X$. It is worth noting that* (22) *and* (23) *agree with (10.60) of Dynkin* [9], *if the latter is applied to f being a component projection function.*

Remarks 1.7. *The operator conditions* (22) *and* (23) *for strong Markovian consistency can be interpreted in the context of martingale characterization of Markov chains.*

Let C^i, $i = 1, 2$, be the projection from $\mathcal{X}^1 \times \mathcal{X}^2$ on the ith component. Fix $i \in \{1, 2\}$ and $0 \le s \le t$. Since X is a Markov chain, for any function f^i on $\mathcal{X}^i$ we have the representation

$$C^{i,*} f^i(X_t) = C^{i,*} f^i(X_s) + \int_s^t (\Lambda(u)(C^{i,*} f^i))(X_u) du + M_t^{C^{i,*},f^i} - M_s^{C^{i,*},f^i}, \tag{26}$$

where $M^{C^{i,*},f^i}$ is a martingale with respect to $\mathbb{F}^X$. Thus,

$$f^i(C^i X_t) = f^i(C^i X_s) + \int_s^t (\Lambda(u)(C^{i,*} f^i))(X_u) du + M_t^{C^{i,*},f^i} - M_s^{C^{i,*},f^i}. \tag{27}$$

If conditions (22) *and* (23) *hold then we may rewrite* (27) *as*

$$f^i(X_t^i) = f^i(X_s^i) + \int_s^t (\Lambda^i(u) f^i)(X_u^i) du + M_t^{C^{i,*},f^i} - M_s^{C^{i,*},f^i}, \tag{28}$$

which shows that X^i is a Markov chain with respect to $\mathbb{F}^X$.

1.4. **When Does Weak Markov Consistency Imply Strong Markov Consistency?** It is well known that if a process X is a $\mathbb{P}$-Markov chain with respect to a filtration $\mathbb{F}$, and if it is adapted with respect to a filtration $\hat{\mathbb{F}} \subset \mathbb{F}$, then X is a $\mathbb{P}$-Markov chain with respect to $\hat{\mathbb{F}}$. However, the converse is not true in general. Nevertheless, if X is a $\mathbb{P}$-Markov chain with respect to $\hat{\mathbb{F}}$, and $\hat{\mathbb{F}}$ is $\mathbb{P}$-*immersed* in $\mathbb{F}$ [5], then we can deduce from the martingale characterization of Markov chains that X is also a $\mathbb{P}$-Markov chain with respect to $\mathbb{F}$.

Thus, if $\mathbb{F}^{X^i}$ is $\mathbb{P}$-immersed in $\mathbb{F}^X$, then weak Markovian consistency of X with respect to X^i will imply strong Markovian consistency of X with respect to X^i. In the following theorem we demonstrate that in fact this property is equivalent to $\mathbb{P}$-immersion between $\mathbb{F}^{X^i}$ and $\mathbb{F}^X$, given that weak Markovian consistency holds.

Theorem 1.3. *Assume that X satisfies the weak Markovian consistency condition with respect to X^i. Then X satisfies the strong Markovian consistency condition if and only if $\mathbb{F}^{X^i}$ is $\mathbb{P}$-immersed in $\mathbb{F}^X$.*

Proof. " $\Longrightarrow$ " We give a proof in the case of $i = 1$. By Proposition 1.1 the process

$$M_{x^1y^1}^1(t) := N_{x^1y^1}^1(t) - \int_{(0,t]} \widehat{\nu}_{x^1y^1}^1(ds)$$

is an $\mathbb{F}^{X^1}$-martingale for every $x^1 \neq y^1$ since X^1 is a Markov process with respect to its own filtration. By Jeanblanc, Yor and Chesney [8, Proposition 5.9.1.1] it is sufficient to show that every $\mathbb{F}^{X^1}$-square integrable martingale Z is also an $\mathbb{F}^X$-square integrable martingale under $\mathbb{P}$. Using the martingale representation theorem (see Rogers and Williams [13, Theorem 21.15]) we have

$$Z_t = Z_0 + \sum_{x^1 \neq y^1} \int_{(0,t]} g(s, x^1, y^1, \omega)(N_{x^1y^1}^1(ds) - \widehat{\nu}_{x^1y^1}^1(ds)) \tag{29}$$

[5] We say that a filtration $\hat{\mathbb{F}}$ is $\mathbb{P}$-*immersed* in a filtration $\mathbb{F}$ if $\hat{\mathbb{F}} \subset \mathbb{F}$ and every $(\mathbb{P}, \hat{\mathbb{F}})$-local-martingale is a $(\mathbb{P}, \mathbb{F})$-local-martingale.

for some function $g:(0,\infty)\times\mathcal{X}^1\times\mathcal{X}^1\times\Omega\to\mathbb{R}$, such that for every x^1,y^1 the mapping $(t,\omega)\mapsto g(t,x^1,y^1,\omega)$ is $\mathbb{F}^{X^1}$-predictable and $g(t,x^1,x^1,\omega)=0$, $\mathbb{P}$-a.s.. The $\mathbb{F}^{X^1}$-angle bracket of $M^1_{x^1y^1}$ (i.e. the $\mathbb{F}^{X^1}$-compensator of $(M^1_{x^1,y^1})^2$) is equal to $(\int_0^t \widehat{\nu}^1_{x^1y^1}(ds))_{t\geq 0}$, and therefore g satisfies the integrability condition

$$\mathbb{E}\left(\sum_{x^1\neq y^1}\int_{(0,T]}|g(s,x^1,y^1)|^2\widehat{\nu}^1_{x^1y^1}(ds)\right)<\infty \qquad \forall\, T>0. \tag{30}$$

From the assumption that weak Markovian consistency implies strong Markovian consistency we infer that X^1 is a Markov chain with respect to $\mathbb{F}^X$, and therefore $M^1_{x^1y^1}$ are $\mathbb{F}^X$-martingales for every $x^1\neq y^1$. Moreover, the $\mathbb{F}^X$-angle bracket of $M^1_{x^1y^1}$ is also equal to $(\int_0^t \widehat{\nu}^1_{x^1y^1}(ds))_{t\geq 0}$, and obviously for every x^1,y^1 the mapping $(t,\omega)\to g(t,x^1,y^1,\omega)$ is $\mathbb{F}^X$-predictable. Hence using (29) and (30) we deduce that Z is also an $\mathbb{F}^X$-square integrable martingale.

" $\Longleftarrow$ " Assume that $\mathbb{F}^{X^i}$ is immersed in $\mathbb{F}^X$. Weak Markovian consistency for X^1 implies that the process $M^1_{x^1y^1}$ is an $\mathbb{F}^{X^1}$-martingale for every $x^1\neq y^1$. By immersion we know that $M^1_{x^1y^1}$ are $\mathbb{F}^X$-martingales for every $x^1\neq y^1$ and therefore Proposition 1.1 implies that X^1 is a Markov process with respect to $\mathbb{F}^X$.

□

2. Weak Markov copulae

We now turn to the problem of constructing a multivariate finite Markov chain whose components are finite univariate Markov chains with given generator matrices.

This problem was previously studied in [4] and [5], for example, in the context of strong Markovian consistency. This meant that the components of the multivariate Markov chain constructed were Markovian both in their own filtrations and in the filtration of the entire chain. Thus, essentially, these references dealt with constructing of what we shall term here *strong Markov copulae.*

In this paper, we shall additionally be concerned with *weak Markov copulae* in the context of finite Markov chains. It will be seen that any strong Markov copula is also a weak Markov copula.

2.1. Strong Markov copulae.

The key observation leading to the concept of strong Markov copula is the following: Let there be given two generator functions $\Lambda^1(t)=[\lambda^1_{x^1y^1}(t)]$ and $\Lambda^2(t)=[\lambda^2_{x^2y^2}(t)]$, and suppose that there exists a valid generator matrix function $\Lambda(t)=[\lambda^{x^1x^2}_{y^1y^2}(t)]_{x^1,y^1\in\mathcal{X}^1,x^2,y^2\in\mathcal{X}^2}$ satisfying (16) for every $x^2\in\mathcal{X}^2$, and satisfying (18) for every $x^1\in\mathcal{X}^1$. Then, Condition (M) is clearly satisfied, so that (cf. Remark 1.4) strong Markovian consistency holds for the Markov chain, X generated by $\Lambda(t)$.

Note that, typically, system (16) and (18), considered as a system with given $\Lambda^1(t)=[\lambda^1_{x^1y^1}(t)]$ and $\Lambda^2(t)=[\lambda^2_{x^2y^2}(t)]$ and with unknown $\Lambda(t)=[\lambda^{x^1x^2}_{y^1y^2}(t)]_{x^1,y^1\in\mathcal{X}^1,x^2,y^2\in\mathcal{X}^2}$, contains many more unknowns (i.e., $\lambda^{x^1x^2}_{y^1y^2}(t)$, $x^1,y^1\in\mathcal{X}^1,x^2,y^2\in\mathcal{X}^2$) than it contains equations. In fact, given that the cardinalities of $\mathcal{X}^1$ and $\mathcal{X}^2$ are K_1 and K_2, respectively, the system consists of $K_1(K_1-1)+K_2(K_2-1)$ equations in $K_1K_2(K_1K_2-1)$ unknowns.

Thus, in principle, one can create several bivariate Markov chains X with margins X^1 and X^2 that are Markovian in the filtration of X, and such that the law of X^i agrees with the law of a given Markov chain Y^i, $i=1,2$. Thus, indeed, the system (16) and (18) essentially serves as a "copula"[6] between the Markovian margins Y^1, Y^2 and the bivariate Markov chain X. This observation leads to the following definition,

[6] We use the term "copula" in analogy to classical copulae for probability distributions of finite-dimensional random variables (cf. e.g. [11]).

Definition 2.1. Let Y^1 and Y^2 be two Markov chains with values in $\mathcal{X}^1$ and $\mathcal{X}^2$, and with generators $\Lambda^1(t) = [\lambda^1_{x^1y^1}(t)]$ and $\Lambda^2(t) = [\lambda^2_{x^2y^2}(t)]$. A Strong Markov Copula between the Markov chains Y^1 and Y^2 is any solution to (16) and (18) such that the matrix function $\Lambda(t) = [\lambda^{x^1x^2}_{y^1y^2}(t)]_{x^1,y^1\in\mathcal{X}^1,x^2,y^2\in\mathcal{X}^2}$, with $\lambda^{x^1x^2}_{x^1x^2}(t)$ given as

$$\lambda^{x^1x^2}_{x^1x^2}(t) = -\sum_{(z^1,z^2)\in\mathcal{X}^1\times\mathcal{X}^2,\, z^i\neq x^i,\, i=1,2} \lambda^{x^1x^2}_{z^1z^2}(t), \tag{31}$$

correctly defines the infinitesimal generator function of a Markov chain with values in $\mathcal{X}^1\times\mathcal{X}^2$.

Thus, any strong Markov copula between Markov chains Y^1 and Y^2 produces a bivariate Markov chain, say $X = (X^1, X^2)$, such that

- the components X^1 and X^2 are Markovian in the filtration of X,
- the law of X^i is the same as the law of Y^i, $i = 1, 2$.

In the terminology of [6], the process X satisfies the strong Markovian consistency condition relative to Y^1 and Y^2.

It is clear that there exists at least one solution to (16) and (18) such that the matrix function $\Lambda(t) = [\lambda^{x^1x^2}_{y^1y^2}(t)]_{x^1,y^1\in\mathcal{X}^1,x^2,y^2\in\mathcal{X}^2}$ is a valid generator matrix. This solution correspond to the case of independent processes X^1 and X^2. In this case we have $\Lambda(t) = I^1\hat{\otimes}\Lambda^2(t)+\Lambda^1(t)\hat{\otimes}I^2$ where $A\hat{\otimes}B$ denotes tensor product of operators A and B (see Ryan [14]), and where I^i is identity operator on $\mathcal{X}^i$. Matrix $\Lambda(t)$ that corresponds to independent process can be also written more explicitly

$$\lambda^{x^1x^2}_{y^1y^2}(t) = \begin{cases} \lambda^1_{x^1x^1}(t)+\lambda^2_{x^2x^2}(t), & y^1=x^1, y^2=x^2,\\ \lambda^1_{x^1y^1}(t), & y^1\neq x^1, y^2=x^2,\\ \lambda^2_{x^2y^2}(t), & y^2\neq x^2, y^1=x^1,\\ 0, & \text{otherwise.}\end{cases}$$

2.2. **Weak Markov Copulae.** The concept of weak Markov copula corresponds to the concept of weak Markovian consistency. We do not have any clear analytical characterization of the latter property, analogous to condition (M) that is sufficient for strong Markovian consistency.

Consequently, the concept of weak Markov copula is much more intricate than that of strong Markov copula, because it involves both probabilistic and analytical (indeed, algebraic in our case) characterizations.

Definition 2.2. Let Y^1 and Y^2 be two Markov chains with values in $\mathcal{X}^1$ and $\mathcal{X}^2$, and with generators $\Lambda^1(t) = [\lambda^1_{x^1y^1}(t)]$ and $\Lambda^2(t) = [\lambda^2_{x^2y^2}(t)]$, respectively. A Weak Markov Copula between Y^1 and Y^2 is any matrix function $\Lambda(t) = [\lambda^{x^1x^2}_{y^1y^2}(t)]_{x^1,y^1\in\mathcal{X}^1,x^2,y^2\in\mathcal{X}^2}$ that satisfies the following conditions:

- **(WMC1):** $\Lambda(t)$ correctly properly defines the infinitesimal generator of a bivariate Markov chain, say $X = (X^1, X^2)$, with values in $\mathcal{X}^1\times\mathcal{X}^2$,
- **(WMC2):** Conditions (14) and (17) are satisfied, so that X is weakly Markovian consistent.

Thus, any weak Markov copula between the Markov chains Y^1 and Y^2 produces a bivariate Markov chain, say $X = (X^1, X^2)$, such that

- the components X^1 and X^2 are Markovian in their own filtrations, but not necessarily Markovian in the filtration of X, and
- the law of X^i is the same as the law of Y^i, $i = 1, 2$.

If a process X is produced as above, then we say that it satisfies the weak Markovian consistency condition relative to Y^1 and Y^2.

It is clear that any strong Markov copula between Y^1 and Y^2 is also a weak Markov copula between Y^1 and Y^2.

The issue of constructing weak Markov copulae that are not strong Markov copulae is important and difficult. It is important since in the context of credit risk weak Markov copulae allow for modeling of default contagions between individual obligors and the rest of the credit pool (cf. [2] for a discussion); this kind of contagion is precluded in the context of strong Markov copulae. It is difficult since conditions (14) and (17) are much more difficult to handle than condition (M).

A possible way of constructing a weak Markov copula which is not a strong Markov copula, is to start with the necessary condition (20) and to find a generator matrix $\Lambda(t)$ that satisfies this condition with given $\Lambda^1(t)$ and $\Lambda^2(t)$. If we are lucky, then the matrix $\Lambda(t)$ found will generate a Markov chain satisfying the weak Markovian consistency condition relative to the Markov chains Y^1 and Y^2 generated by $\Lambda^1(t)$ and $\Lambda^2(t)$, respectively. This approach will be illustrated in Example 3.2 below.

3. Examples

As before, we take $n = 2$ in the examples below. We shall present examples illustrating

- Construction of a strong Markov copula (Example 3.1), i.e., a construction of a two dimensional Markov chain $X = (X^1, X^2)$ with components X^1 and X^2 that are Markovian in the filtration of X, and such that the law of X^i agrees with the law of a given Markov chain Y^i, $i = 1, 2$.
- Construction of a weak Markov copula (Example 3.2), i.e., a construction of a two dimensional Markov chain $X = (X^1, X^2)$ with the components X^1 and X^2 that are Markovian in their own filtrations, but are not Markovian in the filtration of X, and such that the law of X^i agrees with the law of a given Markov chain Y^i, $i = 1, 2$.
- Existence of a Markov chain for which weak Markovian consistency does not hold, that is, a Markov chain that can't serve as a weak Markov copula (Example 3.3). In this example, component X^2 of Markov chain $X = (X^1, X^2)$ is shown to be not Markovian in its own filtration.

Example 3.1. Let us consider two processes, Y^1 and Y^2, that are time-homogeneous Markov chains, each taking values in the state space $\{0, 1\}$, with respective generators

$$\Lambda^1 = \begin{pmatrix} -(a+c) & a+c \\ 0 & 0 \end{pmatrix} \tag{32}$$

and

$$\Lambda^2 = \begin{pmatrix} -(b+c) & b+c \\ 0 & 0 \end{pmatrix}, \tag{33}$$

for $a, b, c \geq 0$.

We shall first consider the system of equations (22) and (23) for this example. In this case we identify $C^{i,*}$, $i = 1, 2$, with the matrices

$$C^{1,*} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C^{2,*} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{34}$$

It can be easily checked that the matrix Λ below satisfies (22) and (23):

$$\Lambda = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\ \begin{pmatrix} -(a+b+c) & b & a & c \\ 0 & -(a+c) & 0 & a+c \\ 0 & 0 & -(b+c) & b+c \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{35}$$

Thus, according to the theory of Section 2, Λ is a strong Markov copula between Y^1 and Y^2. Nevertheless, it will be instructive to verify this directly. Towards this end, let us consider the bivariate Markov chain $X = (X^1, X^2)$ on the state space

$$E = \{(0,0), (0,1), (1,0), (1,1)\}$$

generated by the matrix Λ given by (35). We first compute the transition probability matrix for X, for $t \geq 0$:

$$P(t) = \begin{pmatrix} e^{-(a+b+c)t} & e^{-(a+c)t}(1-e^{-bt}) & e^{-(b+c)t}(1-e^{-at}) & e^{-(a+b+c)t} - e^{-(b+c)t} - e^{-(a+c)t} + 1 \\ 0 & e^{-(a+c)t} & 0 & 1 - e^{-(a+c)t} \\ 0 & 0 & e^{-(b+c)t} & 1 - e^{-(b+c)t} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, for any $t \geq 0$,

$$\lim_{h \to 0} \frac{P(X^2_{t+h} = 0 | X^2_t = 0) - 1}{h} = -(b + c).$$

Similarly, for any $t \geq 0$,

$$\lim_{h \to 0} \frac{P(X^1_{t+h} = 0 | X^1_t = 0) - 1}{h} = -(a + c).$$

It is clear that X^1 and X^2 are Markov chains in their own filtrations (as both chains are absorbed in state 1). From the above calculations we see that the generator of X^i is Λ^i, $i = 1, 2$.

To verify that Λ is a strong Markov copula between Y^1 and Y^2, it remains to show that components X^1 and X^2 are Markovian in the filtration of X. This can also be verified by direct computations: indeed,

$$\lim_{h \to 0} \frac{P(X^1_{t+h} = 0 | X^1_t = 0, X^2_t = 0) - 1}{h} = \lim_{h \to 0} \frac{P(X^1_{t+h} = 0 | X^1_t = 0, X^2_t = 1) - 1}{h}$$
$$= -(a + c) = \lim_{h \to 0} \frac{P(X^1_{t+h} = 0 | X^1_t = 0) - 1}{h},$$

or, equivalently,

$$P(X^1_{t+h} = 0 | X^1_t = 0, X^2_t = 0) = P(X^1_{t+h} = 0 | X^1_t = 0, X^2_t = 1) = P(X^1_{t+h} = 0 | X^1_t = 0) = e^{-(a+c)h},$$

so that condition (2) is satisfied for X^1, and similarly for X^2.

Finally, note that in accordance with the concept of strong Markovian consistency, the transition intensities and transition probabilities for X^1 do not depend on the state of X^2:

- No matter what the state of X^2 is, whether 0 or 1, the intensity of transition of X^1 from 0 to 1 is equal to $a + c$.
- The transition probability of X^1 from 0 to 1 in t units of time, no matter what the state of X^2 is, is equal to
$$e^{-(b+c)t}(1 - e^{-at}) + e^{-(a+b+c)t} - e^{-(b+c)t} - e^{-(a+c)t} + 1 = 1 - e^{-(a+c)t}.$$

An analogous observation holds for X^2.

Example 3.2. Let us consider two processes, Y^1 and Y^2, that are Markov chains, each taking values in the state space $\{0, 1\}$, with respective generator functions

$$\Lambda^1(t) = \begin{pmatrix} -(a+c) + \alpha(t) & a + c - \alpha(t) \\ 0 & 0 \end{pmatrix}$$

and

$$\Lambda^2(t) = \begin{pmatrix} -(b+c) + \beta(t) & b + c - \beta(t) \\ 0 & 0 \end{pmatrix},$$

where

$$\alpha(t) = c \cdot \frac{e^{-at}(1 - e^{-(b+c)t})\frac{b}{b+c}}{e^{-(a+b+c)t} + e^{-at}(1 - e^{-(b+c)t})\frac{b}{b+c}}, \quad \beta(t) = c \cdot \frac{e^{-bt}(1 - e^{-(a+c)t})\frac{a}{a+c}}{e^{-(a+b+c)t} + e^{-bt}(1 - e^{-(a+c)t})\frac{a}{a+c}},$$

for $a, b, c \geq 0$.

Here we shall seek a weak Markov copula for Y^1 and Y^2. Thus we shall investigate the necessary condition (20). Towards this end we first note that in this example the matrix representation of the operator Q^1_t takes the form

$$Q^1(t)=\begin{pmatrix} P(X^1_t=0,X^2_t=0|X^1_t=0) & P(X^1_t=0,X^2_t=1|X^1_t=0) & P(X^1_t=1,X^2_t=0|X^1_t=0) & P(X^1_t=1,X^2_t=1|X^1_t=0) \\ P(X^1_t=0,X^2_t=0|X^1_t=1) & P(X^1_t=0,X^2_t=1|X^1_t=1) & P(X^1_t=1,X^2_t=0|X^1_t=1) & P(X^1_t=1,X^2_t=1|X^1_t=1) \end{pmatrix},$$

and similarly for $Q^2(t)$. It turns out that a solution to the necessary condition (20) is a valid generator matrix

$$\Lambda = \begin{pmatrix} -(a+b+c) & b & a & c \\ 0 & -a & 0 & a \\ 0 & 0 & -b & b \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

where $a, b, c \geq 0$. Verification of this is straightforward, but computationally intensive, and can be obtained from the authors on request.

Since condition (20) is just a necessary condition for weak Markovian consistency, the matrix Λ in (36) may not be a weak Markov copula for Y^1 and Y^2. This has to be verified by direct inspection.

Let us consider the bivariate Markov chain $X = (X^1, X^2)$ on the state space

$$E = \{(0,0), (0,1), (1,0), (1,1)\}$$

generated by the matrix Λ given by (36).

Arguing as in the previous example, it is clear that the components X^1 and X^2 are Markovian in their own filtrations. We shall show that:

- X^1 and X^2 are NOT Markovian in the filtration $\mathbb{F}^X$, and
- the generators of X^1 and X^2 are given by (37) and (38), respectively.

We first compute the transition probability matrix for X, for $t \geq 0$:

$$P(t)=\begin{pmatrix} e^{-(a+b+c)t} & e^{-at}(1-e^{-(b+c)t})\frac{b}{b+c} & e^{-bt}(1-e^{-(a+c)t})\frac{a}{a+c} & 1+e^{-(a+b+c)t}(\frac{a}{a+c}-\frac{c}{b+c})-\frac{a}{a+c}e^{-bt}-\frac{b}{b+c}e^{-at} \\ 0 & e^{-at} & 0 & 1-e^{-at} \\ 0 & 0 & e^{-bt} & 1-e^{-bt} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} P(X^1_{t+h}=0|X^1_t=0, X^2_t=0) &= e^{-(a+b+c)t} + e^{-at}(1-e^{-(b+c)t})\frac{b}{b+c} \\ &\neq P(X^1_{t+h}=0|X^1_t=0, X^2_t=1) = e^{-at} \end{aligned}$$

unless $c = 0$, which is the case of independent X^1 and X^2. Thus, in general, X^1 is NOT a Markov process in the full filtration. Similarly for X^2.

We shall now compute the generator function for X^2. As in the previous example, for any $t \geq 0$,

$$\lim_{h\to 0} \frac{P(X^2_{t+h}=0|X^2_t=0)-1}{h} = -(b+c) + c\frac{P(X^1_t=1, X^2_t=0)}{P(X^2_t=0)}.$$

Similarly, for any $t \geq 0$,

$$\lim_{h\to 0} \frac{P(X^1_{t+h}=0|X^1_t=0)-1}{h} = -(a+c) + c\frac{P(X^1_t=0, X^2_t=1)}{P(X^1_t=0)}.$$

Thus, both X^1 and X^2 are time-inhomogeneous Markov chains with generator functions, respectively,

$$A^1(t) = \begin{pmatrix} -(a+c) + c\frac{P(X_t^1=0,X_t^2=1)}{P(X_t^1=0)} & a+c-c\frac{P(X_t^1=0,X_t^2=1)}{P(X_t^1=0)} \\ 0 & 0 \end{pmatrix} \tag{37}$$

and

$$A^2(t) = \begin{pmatrix} -(b+c) + c\frac{P(X_t^1=1,X_t^2=0)}{P(X_t^2=0)} & b+c-c\frac{P(X_t^1=1,X_t^2=0)}{P(X_t^2=0)} \\ 0 & 0 \end{pmatrix}. \tag{38}$$

It is easily checked that $A^1(t) = \Lambda^1(t)$ and $A^2(t) = \Lambda^2(t)$, as claimed. Consequently, the matrix Λ in (36) is a weak Markov copula for Y^1 and Y^2, but it is not a strong Markov copula for Y^1 and Y^2.

Finally, note that the transition intensities and transition probabilities for X^1 do depend on the state of X^2:

- When X^2 is in state 0, the intensity of transition of X^1 from 0 to 1 is equal to a; when X^2 is in state 1, the intensity of transition of X^1 from 0 to 1 is equal to a.
- When X^2 is in state 0, the transition probability of X^1 from 0 to 1 in t units of time is
$$e^{-bt}(1-e^{-(a+c)t})\frac{a}{a+c} + 1 + e^{-(a+b+c)t}\left(\frac{a}{a+c} - \frac{c}{b+c}\right) - \frac{a}{a+c}e^{-bt} - \frac{b}{b+c}e^{-at};$$
when X^2 is in state 1, the transition probability of X^1 from 0 to 1 in t units of time is
$$1 - e^{-at}.$$

An analogous observation holds for X^2, that is, the transition intensities and transition probabilities for X^2 do depend on the state of X^1.

Example 3.3. Here we give an example of a bivariate Markov chain which is not weakly Markovian consistent.

Let us consider the bivariate Markov chain $X = (X^1, X^2)$ on the state space

$$E = \{(0,0), (0,1), (1,0), (1,1)\}$$

generated by the matrix

$$A = \begin{pmatrix} -(a+b+c) & b & a & c \\ 0 & -(d+e) & d & e \\ 0 & 0 & -f & f \\ 0 & 0 & g & -g \end{pmatrix}. \tag{39}$$

We denote by $H^2_{0,1}$ the process that counts the number of jumps of the component X^2 from state 0 to state 1. The $\mathbb{F}^X$-intensity of such jumps is

$$\mathbb{1}_{\{X_t^1=0,X_t^2=0\}}(b+c) + \mathbb{1}_{\{X_t^1=1,X_t^2=0\}}f, \tag{40}$$

so the optional projection of this intensity on $\mathbb{F}^{X^2}$ has the form

$$(b+c)\mathbb{P}(X_t^1=0, X_t^2=0|\mathcal{F}_t^{X^2}) + f\mathbb{P}(X_t^1=1, X_t^2=1|\mathcal{F}_t^{X^2}). \tag{41}$$

Since $\{X_t^2=0, X_{t/2}^2=1\} \subseteq \{X_t^1=1\}$, on the set $\{X_t^2=0, X_{t/2}^1=1\}$ we have

$$\mathbb{P}(X_t^1=0, X_t^2=0|X_t^2=0, X_{t/2}^2=1) = 0, \quad \mathbb{P}(X_t^1=1, X_t^2=0|X_t^2=0, X_{t/2}^2=1) = 1. \tag{42}$$

Therefore the above optional projection, on the set $\{X_t^2=0, X_{t/2}^2=1\}$, is equal to

$$f\mathbb{P}(X_t^1=1, X_t^2=0|X_t^2=0, X_{t/2}^2=1) = f. \tag{43}$$

However, on $\{X_t^2=0\}$ the above optional projection is equal to

$$\begin{aligned}(b+c)\mathbb{P}(X_t^1=0, X_t^2=0|X_t^2=0) &+ f\mathbb{P}(X_t^1=1, X_t^2=0|X_t^2=0) \\ &= (b+c-f)\mathbb{P}(X_t^1=0, X_t^2=0|X_t^2=0) + f.\end{aligned}$$

Assuming that the process X starts from $(0,0)$ at time $t=0$, it can be shown that $\mathbb{P}(X^1_t=0, X^2_t=0|X^2_t=0)>0$. Verification of this is straightforward, but computationally intensive, and can be obtained from the authors on request. Thus, if $b+c\neq f$, then the optional projection on $\mathcal{F}^{X^2}_t$ of the $\mathbb{F}^X$ intensity of $H^2_{0,1}$ depends on the trajectory of X^2 until time t, and not just on the state of X^2 at time t. Thus, X^2 is not Markovian in its own filtration. It is obviously not Markovian in the filtration of the entire process X either.

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