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# THE LINEAR-QUADRATIC CONTROL PROBLEM REVISITED*

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**Abstract.** A long-run, average-cost, stochastic, linear-quadratic control problem that incorporates different time scales is considered. The system dynamics and the cost functional are modeled with the help of a locally square-integrable semimartingale process with independent increments and the corresponding predictable quadratic variation process. The solution of the control problem is given in terms of the solution of certain system of algebraic and differential Riccati equations. The model considered here embodies as particular cases the "traditional" continuous-time and discrete-time linear quadratic control problems, and is applicable, for example, to certain hybrid control problems that cannot be treated using existing control methods.

**Key words.** linear-quadratic control, square-integrable semimartingales, various time scales, hybrid control

**AMS subject classifications.** 60H30, 93E20

## 1. Introduction.

In recent years there has been growing interest in developing a unified approach to control and identification problems for both discrete and continuous-time scales. In Middleton and Goodwin (1990), the unified approach to control and estimation is presented via the so-called "generalized transform." In spite of its many advantages, the method is not capable of handling the problems that "live" in continuous and discrete time simultaneously or stochastic control problems involving the continuous-time scale, for example. This paper provides a way of looking at some of these problems via the stochastic calculus for locally square-integrable semimartingales.

In this paper we consider a long-run, average-cost, stochastic, linear-quadratic control problem that incorporates different time scales. The system dynamics and the cost functional are modeled with the help of a locally square-integrable semimartingale process with independent increments and the corresponding predictable quadratic variation process. The situation considered here is not, of course, "the most general" one. But it is general enough to produce as particular cases the traditional continuous-time (e.g., Davis, 1977) and discrete-time (e.g., Hall and Heyde, 1980) linear-quadratic, stochastic control problems with the average cost per unit of time criterion. The results obtained in the paper follow from an application of the powerful general theory of random processes (Dellacherie and Meyer, 1975, 1980, 1983; Jacod, 1979; Jacod and Shiryayev, 1987; Lipster and Shiryayev, 1989; Protter, 1990, among others). We emphasize that the asymptotic results obtained here are essentially due to the strong law of large numbers type property for semimartingales (see Lipster and Shiryayev, 1989, for example). The $L^2$-ergodic type results for martingales, discussed by Sundar (1989), may be useful in the study of the control problem with the expected long-run average cost, which is not included here.

The solution of the control problem considered in this paper is given in terms of the solution to the system of algebraic and differential Riccati equations (3.1). Theorem 3.1 concerning the existence and uniqueness of the solution for system (3.1) is

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interesting in itself. At the very least it interprets the relationship between the algebraic Riccati equations corresponding to continuous and discrete time, as indicated in Remark 3.2 and in §6. The "classical" relationship between continuous- and discrete-time Riccati equations resulting from time discretization is reconfirmed by limiting analysis of equations (3.1) (see §6).

We were inspired to consider a control problem incorporating different time scales by some work on the semimartingale regression problem (Christopeit, 1986; LeBreton and Musiela, 1988), where the time scales are modeled in terms of the predictable quadratic variation process of a semimartingale. Control problems involving continuous-time semimartingale dynamics were considered before in Foldes (1990), for example. To the best of our knowledge, linear quadratic (LQ) control problems incorporating both continuous- and discrete-time scales in the system dynamics have not been considered in the literature before.

Although we consider here only the ergodic linear-quadratic control problem, the modeling methodology presented in this paper is applicable to a wider spectrum of control problems. As a direct control application we see an application of our methodology to a class of hybrid control problems which are attracting more and more interest (see, e.g., Elliot and Sworder, 1992). A simple example of a hybrid control problem that can be treated by methods presented in this paper is given in §7. In this paper we treat neither a finite-time horizon problem, nor an infinite-time horizon with a discounted cost criterion. These are for future research.

The paper is organized as follows. In §2 we describe the noise process. Section 3 introduces a system of differential-algebraic Riccati equations that plays a central role in characterization of the optimal controls (as expected). The system of those equations reduces to the well-known algebraic Riccati equations corresponding to the continuous-time or discrete-time linear-quadratic control problems under appropriate parametrization. Section 4 formulates a semimartingale driven linear-quadratic control problem and provides a solution to it. In §5 we point out how our control problem relates to some other problems considered before in the literature. Section 6 contains three limiting results. One of them reconfirms the classical relationship between continuous and discrete Riccati equations resulting from time discretization. Moreover the result indicates that our approach allows for a "partial" time discretization, that is, time discretization with respect to only some of the components of the state vector. The other two limiting results analyze the effect on the control system of vanishing discrete components ($k_3 \to 0$) and continuous components ($k_1 \to 0$), respectively. In §7 we provide a simple but illustrative example of a hybrid control problem and solve it by our method. A few final remarks are formulated in §8.

Much of the notation used in the paper is taken from Jacod and Shiryayev (1987). "$T$" denotes the transposition of a matrix.

**2. The noise process.** In this section we shall describe the noise process $Z \cong \{Z_t, t \geq 0\}$ that will be appearing in the dynamics equation of the control model. We begin with the following assumption about $Z$.

*Assumption* A1. $Z$ is an $n$-dimensional locally square-integrable semimartingale (Jacod and Shiryayev (1987), Def. II.2.27) and a process with independent increments. The underlying stochastic basis is $(\Omega, \mathcal{F}, \mathbf{F}, P)$ and it is supposed to satisfy the usual conditions.

Let $J$ denote the set of fixed times of discontinuity of $Z$, that is, $J := \{t \geq 0 : P(\Delta Z_t \neq 0) > 0\}$, where $\Delta Z_t := Z_t - Z_{t-}$ is the jump of $Z$ at time $t$, $(\Delta Z_0 = 0)$. As usual $\{B, C, \nu\}$ will denote a triplet of predictable characteristics of $Z$ with regard to some truncation function $h$. According to Theorem II.4.15 of Jacod and Shiryayev (1987) we also have, under A1, that the characteristics of $Z$ are deterministic processes and $J = \{t \geq 0 : \nu(\{t\} \times R^n) > 0\}$. We will denote the stochastically continuous and stochastically discontinuous components of $Z$ by $\bar{Z}$, and $\bar{\bar{Z}}$ respectively. This means that

$$\bar{\bar{Z}}_t = \sum_{\substack{0<s\leq t \\ s\in J}} \Delta Z_s \text{ and } \bar{Z}_t = Z_t - \bar{\bar{Z}}_t,\ t \geq 0.$$

From Proposition II.1.16 of Jacod and Shiryayev (1987) we know that $J$ is countable. Denote elements of $J$ by $j_n$, $n \in I$, where $I$ is a countable index set. Also let $\epsilon_n := \bar{\bar{Z}}_{j_n}$, $n \in I$, so that $\{\epsilon_n\}_{n\in I}$ is the embedded random sequence. Because of the control problem we will treat in §4 we introduce the following two assumptions.

*Assumption* A2. (a) $I = N^* := \{1, 2, 3, \dots\}$; (b) $j_n = \epsilon n$, $\epsilon > 0$, $n \in N^*$.

*Assumption* A3. Both $\bar{Z}$ and $\{\epsilon_n\}_{n\in N^*}$ have stationary increments.

Assumption A2(b) is not essential for the control-theoretic considerations to follow. It will be used to simplify the presentation.

We will keep the usual notation for the measure of jumps of $Z$ : $\mu^z$. From the above stated assumptions and the results of Jacod and Shiryayev (1987), Chapters I and II, we infer the following.

PROPOSITION 2.1. *Assume* A1–A3. *Then*

(i) *the canonical decomposition of $Z$ has the form $Z = Z_0 + N_1 + N_2 + N_3 + A$, where $N_1 := Z^c$ is the continuous martingale part of $Z$, $N_2 := (z1_{J^c} * (\mu^z - \nu)$ is the stochastically continuous jump-martingale part of $Z$, $N_3 := (z1_J) * (\mu^z - \nu)$ is the stochastically discontinuous jump-martingale part of $Z$, and $A = B + (z - h(z)) * \nu$ is a deterministic process.*

(ii) *the characteristics of $Z$ are $B_t = bt + (h1_J) * \nu$, $C_t = ct$ and $\nu(\cdot, dt, dz) = dtK_2(dz)1_{J^c}(t) + K_3(dz)1_J(t)$, where $b \in R^n$, $c \in L(R^n, R^n)$ and $c \geq 0$, $K_2$ and $K_3$ are positive measures on $R^n$ satisfying $K_i(\{0\}) = 0$, and $k_i := \int |z|^2 K_i(dz) < +\infty$, $i = 2, 3$.*

(iii) *$N_1$, $N_2$, and $N_3$ are independent and their predictable quadratic variation processes are given by*

$$\begin{aligned}
&\langle N_1^i, N_1^j\rangle = C^{ij},\\
&\langle N_2^i, N_2^j\rangle = (z^i z^j)1_{J^c} * \nu,\\
&\langle N_3^i, N_3^j\rangle = (z^i z^j)1_J * \nu - \sum_{s\leq\cdot} \int z^i \nu(\{s\} \times dz) \int z^j \nu(\{s\} \times dz)
\end{aligned}$$

*for $i, j = 1, 2, \dots, n$.*

*Proof.* (i) The result follows from (2.30) and (2.39), Chapter II of Jacod and Shiryayev (1987).

(ii) The result follows from (2.14), (4.16), and the result analogous to Corollary 4.19 of Jacod and Shiryayev (1987) applied to $\bar{Z}$ and $\bar{\bar{Z}}$, respectively.

(iii) This part of the proposition follows from (2.31) and (4.16) of Jacod and Shiryayev (1987), Chapter II. □

As usual, we let $\langle N_1\rangle :=$ trace $C$, $\langle N_2\rangle :=$ trace$(zz^T)1_{J^c} * \nu$ and $\langle N_3\rangle :=$ trace$(zz^T)1_J * \nu$ denote the scalar predictable quadratic variation processes of $N_1$, $N_2$, and $N_3$, respectively.

*Remark* 2.1. From now on we will assume (without loss of generality) that $A \equiv 0$

and $Z_0 \equiv 0$.

COROLLARY 2.1. *Under conditions of Proposition* 2.1 *we have*

$$\begin{aligned}\langle N_1\rangle_t &= t \cdot \text{ trace } c,\\ \langle N_2\rangle_t &= t \cdot k_2 \text{ for } t \geq 0, \text{ and}\\ \langle N_3\rangle_t &= n \cdot k_3,\ t \in ]\epsilon n, \epsilon(n+1)],\ n \geq 0,\\ \langle N_3\rangle_0 &= 0.\end{aligned}$$

*Remark* 2.2. In fact, $N_1$ is a Wiener process.

**3. The Riccati equations.** In this section we let $A, B \in L(R^n, R^n)$, $E \in L(R^m, R^n)$, and $F \in L(R^k, R^n)$. Also let $Q_1, Q_2 \in L(R^n, R^n)$, $R_1 \in L(R^m, R^m)$, $R_2 \in L(R^k, R^k)$, and $Q_1, Q_2 \geq 0$, $R_1, R_2 > 0$.

In Definition 3.1 below we recall the concepts of Hurwitz and Schur stability of a matrix, which we shall respectively call $c$- and $d$-stability with reference to "continuous" and "discrete" time.

DEFINITION 3.1. *A quadratic matrix $M$ is called d-stable iff its spectrum is contained in an open unit disk. A quadratic matrix $N$ is called c-stable iff its spectrum is contained in the complex left open half-plane.*

DEFINITION 3.2.

(a) *A pair $(B, F)$ is called d-stabilizable iff there exists $H \in L(R^n, R^k)$ so that $B + FH$ is d-stable. A pair $(B, F)$ is said to be $d^T$-stabilizable iff there exists $H \in L(R^k, R^n)$ so that $B + HF$ is d-stable.*

(b) *A four-tuple $(A, E, B, F)$ is called cd-stabilizable iff there exist $H_1 \in L(R^n, R^m)$ and $H_2 \in L(R^n, R^k)$ so that $\mathcal{A}(H_1, H_2)$ is d-stable, where*

$$\mathcal{A}(H_1, H_2) := e^{A+EH_1} \cdot (B + FH_2).$$

DEFINITION 3.3. *A four-tuple $(A, Q_1, B, Q_2)$ is called cd-detectable iff $(Be^A, \sqrt{B^T e^{A^T} Q_1 e^A B + Q_2})$ is $d^T$-stabilizable.*

*Remark* 3.1. Note that if $(e^A,\ \sqrt{e^{A^T} Q_1 e^A})$ is $d^T$-stabilizable then $(e^A,\ \sqrt{Q_1})$, is $d^T$-stabilizable and consequently $(\sqrt{Q_1},\ A)$ is $c$-detectable, which means that there is a matrix $H$ such that $A^T + \sqrt{Q_1^T}\ H$ is $d^T$-stable.

*Proof.* This follows from the fact that $\text{Ker}(\sqrt{M}) = \text{Ker}(M)$ for any symmetric, nonnegative semidefinite matrix $M$ and from Proposition 3.1 in Wonham (1979). □

In what follows we will require more notation. Let $\epsilon > 0$. Let $P_t : [0, \epsilon] \rightarrow L_+(R^n, R^n)$ be a continuous function, where "+" denotes nonnegative semidefiniteness. Next define

$$\begin{aligned}\mathcal{P}(\epsilon) &:= \{P_t, t \in [0, \epsilon]\},\\ L_t(\epsilon) &:= -R_1^{-1} E^T P_{\epsilon - t}, \quad L_1(\mathcal{P}(\epsilon), \epsilon) := \int_0^\epsilon L_t(\epsilon) dt,\\ \mathcal{D}_1(\mathcal{P}(\epsilon), \epsilon) &:= \epsilon A + EL_1(\mathcal{P}(\epsilon), \epsilon), \quad \mathcal{D}_{1,t}(\epsilon) := A + EL_t(\epsilon),\\ L_2(\mathcal{P}(\epsilon), \epsilon) &:= -(F^T P_\epsilon F + R_2)^{-1} F^T P_\epsilon B,\\ \mathcal{D}_2(\mathcal{P}(\epsilon), \epsilon) &:= B + FL_2(\mathcal{P}(\epsilon), \epsilon),\\ \mathcal{S}(\mathcal{P}(\epsilon), \epsilon) &:= \int_0^\epsilon e^{\int_s^\epsilon \mathcal{D}_{1,\epsilon - t}^T(\epsilon) dt} [Q_1^T + L_{\epsilon - s}^T(\epsilon) R_1 L_{\epsilon - s}(\epsilon)] e^{\int_s^\epsilon \mathcal{D}_{1,\epsilon - t}^T(\epsilon) dt} ds,\\ \mathcal{B}_{(\mathcal{P}(\epsilon), \epsilon)} &:= \mathcal{D}_2^T(\mathcal{P}(\epsilon), \epsilon) \mathcal{S}(\mathcal{P}(\epsilon), \epsilon) \mathcal{D}_2(\mathcal{P}(\epsilon), \epsilon)) + Q_2 + L_2^T(\mathcal{P}(\epsilon), \epsilon) R_2 L_2(\mathcal{P}(\epsilon), \epsilon),\end{aligned}$$

and

$$\mathcal{A}_{(\mathcal{P}(\epsilon),\epsilon)} : L(R^n, R^n) \to L(R^n, R^n)$$

given by

$$\mathcal{A}_{(\mathcal{P}(\epsilon),\epsilon)}(K) := \mathcal{D}_2^T(\mathcal{P}(\epsilon),\epsilon)e^{\mathcal{D}_1^T(\mathcal{P}(\epsilon),\epsilon)}Ke^{\mathcal{D}_1(\mathcal{P}(\epsilon),\epsilon)}\mathcal{D}_2(\mathcal{P}(\epsilon),\epsilon).$$

Consider the following system of Riccati equations, which we will call a *cd*-Riccati equation:

$$\text{(3.1)} \qquad \begin{cases} \mathcal{A}_{(\mathcal{P}(\epsilon),\epsilon)}(R) + \mathcal{B}_{(\mathcal{P}(\epsilon),\epsilon)} = R, \\ \dot{P}_t = Q_1 + A^T P_t + P_t A - P_t E R_1^{-1} E^T P_t, \\ P_0 = R, \ t \in [0,\epsilon]. \end{cases}$$

Observe that

$$\text{(3.2)} \qquad P_\epsilon = e^{\mathcal{D}_1^T(\mathcal{P}(\epsilon),\epsilon)}Re^{\mathcal{D}_1(\mathcal{P}(\epsilon),\epsilon)} + \mathcal{S}(\mathcal{P}(\epsilon),\epsilon).$$

Therefore the first equation in (3.1) can equivalently be written as

$$\text{(3.3)} \quad (B+FL_2(\mathcal{P}(\epsilon),\epsilon))^T P_\epsilon (B+FL_2(\mathcal{P}(\epsilon),\epsilon)) + Q_2 + L_2^T(\mathcal{P}(\epsilon),\epsilon)R_2L_2(\mathcal{P}(\epsilon),\epsilon) = R.$$

*Remark* 3.2.

(a) If we assume that $B = I$, or $B = -I$, $F = 0$, $Q_2 = 0$, and $P_t =$ const, $t \in [0,\epsilon]$, then (3.1) reduces to the following algebraic Riccati equation;

$$\text{(c-ARE)} \qquad \begin{aligned} 0 &= Q_1 + A^T R + RA - RER_1^{-1}E^T R, \\ P_t &= R, \ t \in [0,\epsilon]. \end{aligned}$$

We call the above equation c-ARE because it is related to the continuous-time linear-quadratic control problem (see Davis (1977), p. 185).

(b) If we assume that $A = 0$, $E = 0$, $Q_1 = 0$, and $\epsilon = 1$ then (3.1) reduces to the following algebraic Riccati equation,

$$\text{(d-ARE)} \qquad \begin{aligned} R &= B^T[R - RF(F^T RF + R_2)^{-1}F^T R]B + Q_2, \\ P_t &= R, \ t \in [0,1]. \end{aligned}$$

We call the above equation d-ARE since it is related to the discrete-time linear-quadratic control problem (see Bertsekas (1976), p. 355).

In Theorem 3.1 below we shall consider equations (3.1) for $\epsilon = 1$ only. The result is true for any $\epsilon > 0$, as can be easily deduced from the proof of the theorem. We will use a simplified notation by omitting $\epsilon = 1$ from the above definitions. So, for example, we will write $\mathcal{P}$ instead of $\mathcal{P}(1)$, $L_1$ instead of $L_1(\mathcal{P}(1),1)$, $\mathcal{A}_\mathcal{P}$ instead $\mathcal{A}_{(\mathcal{P}(1),1)}$, etc.

We note that Theorem 12.2 of Wonham (1979) and Proposition on page 75 of Bertsekas (1977) are special cases of Theorem 3.1.

THEOREM 3.1. *Let* $\epsilon = 1$. *Assume that* $(A, E, B, F)$ *is cd-stabilizable and* $(A, Q_1, B, Q_2)$ *is cd-detectable. Then there exists the unique solution* $(\bar{R}, \bar{\mathcal{P}})$ *to* (3.1) *such that* $\bar{R} \geq 0$, $\bar{P}_t \geq 0$ *for* $t \in [0,1]$, *and* $\mathcal{A}(\bar{L}_1, \bar{L}_2)$ *is d-stable, where* $\bar{L}_i := L_i(\bar{\mathcal{P}})$, $i = 1,2$.

*Proof.* See Appendix 1.

**4. Linear-quadratic stochastic control problem.** We begin with introducing the dynamics of the controlled process first:

$$
\begin{aligned}
dx_t &= (\widetilde{A}x_{t-} + \widetilde{E}v_{t-})d\langle M\rangle_t + (\widetilde{B}x_{t-} + \widetilde{F}u_{t-})d\langle N\rangle_t \\
&\quad + dZ_t,\ x_0 = x,\ t \geq 0,
\end{aligned} \tag{4.1}
$$

where $M = N_1 + N_2$, $N = N_3$. The admissible control processes $u. := \{u_t, t \geq 0\}$ and $v. := \{v_t, t \geq 0\}$ are supposed to satisfy the following conditions:

- They are non-anticipating w.r.t. $Z$,
- There exists a weak semimartingale solution to (4.1) in the sense of Jacod (1979) Chap. XIV,
- $\lim_{t\to\infty} \frac{\|x_t\|^2}{t} = 0$, a.s.,
- $\overline{\lim}_{t\to\infty} \frac{1}{t}\int_0^t (\|x_s\|^2 + \|u_s\|^2 + \|v_s\|^2)ds < +\infty$, a.s.

The class of admissible controls is denoted by $\mathcal{U}_{ad}$. The cost functional will be given in terms of ($T \geq 0$)

$$
\begin{aligned}
\mathbf{C}_T(v., u., x) = \int_{0^+}^T &[\langle \widetilde{Q}_1 x_{t-}, x_{t-}\rangle \\
&+ \langle \widetilde{R}_1 v_{t-}, v_{t-}\rangle]d\langle M\rangle_t \\
+ \int_{0^+}^T &[\langle \widetilde{Q}_2 x_{t-}, x_{t-}\rangle \\
&\quad + \langle \widetilde{R_2} u_{t-}, u_{t-}\rangle]d\langle N\rangle_t.
\end{aligned}
$$

We want to show the existence and characterization of optimal controls, that is, admissible controls $u^0.$ and $v^0.$ such that for all $v., u. \in \mathcal{U}_{ad}$ and $x \in R^n$ it holds that

$$
\mathbf{C}(v^0., u^0., x) \leq \mathbf{C}(v., u., x),
$$

where

$$
\mathbf{C}(v., u., x) := \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbf{C}_T(v., u., x).
$$

In the above description of the control problem we have supposed $\widetilde{A}, \widetilde{B} \in L(R^n, R^n)$, $\widetilde{E} \in L(R^m, R^n)$, $\widetilde{F} \in L(R^k, R^n)$, $\widetilde{Q}_1$ and $\widetilde{Q}_2$ are in $L(R^n, R^n)$, and $\widetilde{Q}_1, \widetilde{Q}_2 \geq 0$, $\widetilde{R}_1 \in L(R^m, R^m)$, $\widetilde{R}_2 \in L(R^k, R^k)$, and $\widetilde{R}_1, \widetilde{R} > 0$. Throughout this section we let $k_1 := k_2 + \text{trace } c$, $A := k_1\widetilde{A}$, $E := k_1\widetilde{E}$, $B := k_3\widetilde{B} + I$, $F := k_3\widetilde{F}$, $Q_1 := k_1\widetilde{Q}_1$, $Q_2 := k_3\widetilde{Q}_2$, $R_1 := k_1\widetilde{R}_1$, and $R_2 := k_3\widetilde{R}_2$. The following assumptions will be used.

*Assumption* A4. $(A, E, B, F)$ is *cd*-stabilizable.

*Assumption* A5. $(A, Q_1, B, Q_2)$ is *cd*-detectable.

Fix $\epsilon > 0$. Let $(\bar{\bar{R}}, \bar{\bar{P}})$ denote the solution to (3.1) with $t$ substituted with $k_3 t$ and $\dot{P}_t$ changed to $k_3\dot{P}_{k_3 t}$, $t \in [0, \epsilon]$. Define $\Pi_t : [0, \infty) \to L(R^n, R^n)$ by

$$
\Pi_{n+s} = \bar{\bar{P}}_{(\epsilon - s)k_3},\ s \in [0, \epsilon),\ n = 0, 1, 2, \ldots .
$$

*Remark* 4.1. Note that

$$
\begin{aligned}
\dot{\Pi}_t &= -k_3\dot{\bar{\bar{P}}}_{(\epsilon n - t)k_3} = -Q_1 - A^T\bar{\bar{P}}_{(\epsilon n - t)k_3} - \bar{\bar{P}}_{(\epsilon n - t)k_3}A \\
&\quad + \bar{\bar{P}}_{(\epsilon n - t)k_3}ER_1^{-1}E^T\bar{\bar{P}}_{(\epsilon n - t)}k_3 \\
&= -Q_1 - A^T\Pi_t - \Pi_t A + \Pi_t E R_1^{-1}E^T\Pi_t
\end{aligned}
$$

for $t \in [\epsilon n, \epsilon(n+1))$, $n = 0, 1, 2, \ldots$.

Also let $\widetilde{P} := \epsilon^{-1} \int_0^1 \Pi_t dt = \epsilon^{-1} \int_0^1 \bar{\bar{P}}_{tk_3} dt$. Define controls $(u^0., v^0.)$ by

$$v_t^0 := \bar{\Lambda}_t x_t, \; u_t^0 := \Lambda_2 x_t, \tag{4.2}$$

where

$$\bar{\Lambda}_t := -R_1^{-1} E^T \Pi_t, \; \Lambda_2 := -(F^T \Pi_0 F + R_2)^{-1} F^T \Pi_0 B$$

for $t \geq 0$.

We will need one more assumption.

*Assumption* A6. $\int |z|^4 K_2(dz) < +\infty$.

THEOREM 4.1. *Suppose assumptions* A1–A6 *are satisfied. Then we have the following:*

(a) *The definitions* (4.2) *are correct: there exists a unique, strong, semimartingale solution to* (4.1) *with* $(v^0., u^0.)$ *in place of* $(v., u.)$,

(b) *The controls* $(v^0., u^0.)$ *are optimal,*

(c) $\mathbf{C}(v^0., u^0., x) = \epsilon^{-1} \mathit{trace}(c + k_2 I)\widetilde{P} + \epsilon^{-1} \mathit{trace}\; k_3 \Pi_0$ *for all* $x \in R^n$.

*Proof.* (a). It is enough to note that $\langle M \rangle$ and $\langle N \rangle$ are special semimartingales and apply Theorem V.3.7 of Protter (1990).

(b). We will use the standard comparison method.

*Step* 1. Let us first observe the following:

- For all $x \in R^n$ and $t \geq 0$

$$\min_{v \in R^m} [x^T \Pi_t E v + v^T E^T \Pi_t x + v^T R_1 v] = x^T \Pi_t E R_1^{-1} E^T \Pi_t x = -x^T \Pi_t E \bar{\Lambda}_t x,$$

  where the minimum is realized by

$${}^0 v_t := \bar{\Lambda}_t x;$$

- For each $x \in R^n$

$$\begin{aligned} \min_{u \in R^k} & [u^T F^T \Pi_0 B x + x^T B^T \Pi_0 F u + u^T F^T \Pi_0 F u + u^T R_2 u] \\ &= x^T B^T \Pi_0 F (F^T \Pi_0 F + R_2)^{-1} F^T \Pi_0 B x \\ &= x^T \Lambda_2^T (F^T \Pi_0 F + R_2) \Lambda_2 x \end{aligned}$$

  and the minimum is realized by

$${}^0 u := \Lambda_2 x.$$

*Step* 2. Now let $(v., u.) \in \mathcal{U}_{ad}$ be an arbitrary pair of admissible controls. From Lemma A.3.1 of Appendix 3 it follows that $(v^0., u^0.) \in \mathcal{U}_{ad}$. Consider the function $V : [0, \infty) \times R^n \to R$ given by

$$V(t, x) := x^T \Pi_t x.$$

Upon application of Ito's rule for semimartingales (Jacod and Shiryayev (1987), Thm. I.4.57) to $V$ we obtain, for $t \geq 0$,

(4.3)

$$\begin{aligned} x_t^T \Pi_t x_t - x^T \Pi_0 x = \int_{0+}^t & [x_{s-}^T \Pi_{s-} \widetilde{A} x_{s-} \\ & + x_{s-}^T \Pi_{s-} \widetilde{E} v_{s-} + x_{s-}^T \widetilde{A}^T \Pi_{s-} x_{s-} + v_{s-}^T \widetilde{E}^T \Pi_{s-} x_{s-}] d\langle M \rangle_s \end{aligned}$$

$$
\begin{aligned}
&+\int_{0+}^{t}[x_{s-}^T\Pi_{s-}\widetilde{B}x_{s-}+x_{s-}^T\Pi_{s-}\widetilde{F}u_{s-}+x_{s-}^T\widetilde{B}^T\Pi_{s-}x_{s-}\\
&\qquad +u_{s-}^T\widetilde{F}^T\Pi_{s-}x_s]d\langle N\rangle_s\\
&+2\int_{0+}^{t}x_{s-}^T\Pi_{s-}dZ_s+\int_{0+}^{t}x_{s-}^T d\Pi_s x_{s-}\\
&+\int_{0+}^{t}\text{trace }\Pi_{s-}cds\\
&+\sum_{\substack{0<s\le t\\ s\in J^c}}\{x_s^T\Pi_s x_s-x_{s-}^T\Pi_{s-}x_{s-}-\Delta x_s^T\Pi_{s-}x_{s-}\\
&\qquad -x_{s-}^T\Pi_{s-}\Delta x_s-x_{s-}^T\Delta\Pi_s x_s\}\\
&+\sum_{0<\epsilon n\le t}\{(Bx_{\epsilon n-}+Fu_{\epsilon n-}+\Delta N_{\epsilon n})^T\Pi_{\epsilon n}(Bx_{\epsilon n-}+Fu_{\epsilon n-}+\Delta N_{\epsilon n})\\
&\qquad -x_{\epsilon n-}^T\Pi_{\epsilon n-}x_{\epsilon n-}-(Bx_{\epsilon n-}+Fu_{\epsilon n-})^T\Pi_{\epsilon n-}x_{\epsilon n-}\\
&\qquad -x_{\epsilon n-}^T\Pi_{\epsilon n-}(Bx_{\epsilon n-}+Fu_{\epsilon n-})-x_{\epsilon n-}^T\Delta\Pi_{\epsilon n}x_{\epsilon n-}\}\\
&=-\int_{0+}^{t}[x_{s-}^T\widetilde{Q}_1x_{s-}-v_{s-}^T\widetilde{R}_1v_{s-}]d\langle M\rangle_s\\
&+\int_{0+}^{t}[x_{s-}^T\Pi_{s-}\widetilde{E}v_{s-}+v_{s-}^T\widetilde{E}^T\Pi_{s-}x_{s-}+v_{s-}^T\widetilde{R}_1v_{s-}\\
&\qquad +x_{s-}^T\Pi_{s-}\widetilde{E}\widetilde{R}_1^{-1}\widetilde{E}^T\Pi_{s-}x_{s-}]d\langle M\rangle_s\\
&+\sum_{\substack{0<s\le t\\ s\in J^C}}\{x_s^T\Pi_s x_s-x_{s-}^T\Pi_{s-}x_{s-}-\Delta x_s^T\Pi_{s-}x_{s-}\\
&\qquad -x_{s-}^T\Pi_{s-}\Delta x_s\}+\int_{0+}^{t}\text{trace }\Pi_{s-}c\,ds\\
&-\int_{0+}^{t}[x_{s-}^T\widetilde{Q}_2x_{s-}+u_{s-}^T\widetilde{R}_2u_{s-}]d\langle N\rangle_s\\
&+\int_{0+}^{t}[x_{s-}^TB^T\Pi_sFu_{s-}+u_{s-}^TF^T\Pi_sBx_{s-}+u_{s-}^TF^T\Pi_sFu_{s-}\\
&\qquad +u_{s-}^TR_2u_s-x_{s-}^T\Lambda_2^T(F^T\Pi_sF+R_2)\Lambda_2x_s]\frac{d\langle N\rangle_s}{k^3}\\
&+2\int_{0+}^{t}[x_{s-}^TB^T\Pi_s+u_{s-}^TF^T\Pi_s]dN_s+2\int_{0+}^{t}x_{s-}^T\Pi_{s-}dZ_s\\
&+\int_{0+}^{t}dN_s^T\Pi_s dN_s\\
&=\sum_{i=1}^{9}I_t^i,\quad\text{a.s.}
\end{aligned}
$$

Note that from Step 1 it follows that $I_t^2\ge 0$ and $I_t^6\ge 0$, $t\ge 0$. As in Theorem 3.6.1 of Lipster and Shiryayev (1989) we have

$$I_t^3=(z^T\Pi z)1_{J^c}*(\mu^z-\nu)_t+(z^T\Pi z)1_{J^c}*\nu_t$$

and note that in view of A6 the process $\zeta_t:=(z^T\Pi z)1_{J^c}*(\mu^z-\nu)_t$ is a locally square-integrable martingale with predictable quadratic variation process

$$\langle \zeta \rangle_t = \int_{0+}^{t} \int_{R^n} (z^T \Pi_s z)^2 1_{J^c} ds K_2(dz).$$

Taking the above remarks into account we obtain from (4.2) the following, $t \geq 0$,

$$\begin{aligned} x_t^T \Pi_t x_t - x^T \Pi_0 x + \mathbf{C}_t(v., u., x) &\\ \geq (z^T \Pi z) 1_{J^c} * \nu_t + \int_{0+}^{t} \text{trace } \Pi_{s-} c \, ds &\\ + \int_{0+}^{t} dN_s^T \Pi_s dN_s + \zeta_t + \rho_t + \xi_t, \text{ a.s.}, & \end{aligned} \tag{4.4}$$

where

$$\rho_t := 2 \int_{0+}^{t} [x_{s-}^T B^T \Pi_s + u_{s-}^T F^T \Pi_s] dN_s$$

and $\xi_t := 2 \int_{0+}^{t} x_{s-}^T \Pi_{s-} dz_s$ are locally square-integrable martingales. Since $(\Pi_t)_{t \geq 0}$ is periodic we have $\lim_{t\to\infty} \frac{1}{t}(z^T \Pi z) 1_{J^c} * \nu_t = \epsilon^{-1} k_2 \text{trace } \widetilde{P}$ and $\lim_{t\to\infty} \frac{1}{t} \int_{0+}^{t} \text{trace } \Pi_{s-} c ds = \epsilon^{-1}$ trace $\widetilde{P}c$. Applying ergodic theorem to $\varphi_t := \int_{0+}^{t} dN_s^T \Pi_s dN_s$ we get $\lim_{t\to\infty} \frac{1}{t}\varphi_t = \epsilon^{-1} k_3$ trace $\Pi_0$, a.s. Also, it follows from the results of Lipster and Shiryayev (1989), §2.6, that $\lim_{t\to\infty} \frac{1}{t}\zeta_t = \lim_{t\to\infty} \frac{1}{t}\rho_t = \lim_{t\to\infty} \frac{1}{t}\xi_t = 0$, a.s. Therefore from (4.4) we conclude that

$$\mathbf{C}(v., u., x) \geq \epsilon^{-1} \text{trace } (c + k_2 I)\widetilde{P} + \epsilon^{-1} k \Pi_0, \text{ a.s.} \tag{4.5}$$

Using considerations analogous to the ones above, it is straightforward to show that

$$\mathbf{C}(v^0., u^0., x) = \epsilon^{-1} \text{trace } (c + k_2 I)\widetilde{P} + \epsilon^{-1} k \Pi, \text{ a.s.} \tag{4.6}$$

This concludes the proof of (b).

(c). This result follows from (4.5) and (4.6). ∥

**5. Some special cases.** In this section we will shortly demonstrate that Theorem 4.1 encompasses solutions to some "classical" stochastic linear-quadratic control problems.

*Case* 1 (continuous time system driven by Wiener process). Using our notation this case corresponds to

$$k_2 = 0, \ k_3 = 0.$$

For a problem of this type see, for example, Davis (1977).

*Case* 2 (continuous time system driven by Wiener and Poisson processes). This corresponds to

$$k_3 = 0, \ N_2 \text{ equivalent to a Poisson process.}$$

Note that in case of a Poisson process Assumption A6 is automatically satisfied. For a problem of this type see, for example, Wonham (1970).

*Case* 3 (continuous time system driven by a Poisson process). This case corresponds to

$$c = 0 \text{ and } k_3 = 0, \ N_2 \text{ equivalent to a Poisson process.}$$

For a more general model of this type (including the multiplicative noise components) see, for example, Li and Blankenship (1986).

*Case* 4 (discrete time system driven by a sequence of independent random variables). In our terminology this case corresponds to

$$c = 0, \quad k_2 = 0.$$

For a problem of this type see, for example, Hall and Heyde (1980).

**6. Three limiting results.** Let us consider equations (3.1) with $A, B, E, F, Q_1, Q_2, R_1, R_2$ as in §4. We also require that $k_3$ is changed to $\epsilon k_3$ in the definitions of $B, F, Q_2$, and $R_2$, that time index $t$ is substituted with $k_3 t$, and that $\dot{P}_t$ is changed to $k_3 \dot{P}_{k_3 t}$.

In this section we shall analyze the behavior of equations (3.1) in the present setting when (i) $\epsilon$ tends to 0, (ii) $k_3$ tends to zero, and (iii) $k_1$ tends to zero. Note that the first case corresponds to "increasing frequency of the discrete time component." A "classical" prototype of it has been considered before in the context of approximating of a continuous-time linear-quadratic problem with a sequence of discrete-time linear-quadratic problems (see Whittle, 1983, Ex. 1, p. 209, for example). The second case corresponds to vanishing of the discrete-time component, and the third case corresponds to vanishing of the continuous-time component of the system.

*Case* i. Assume A4 and A5. Also assume that $(A + k_3\widetilde{B}, [E\ F])$ and $(A^T + k_3\widetilde{B}^T, \sqrt{Q_1 + Q_2})$ are $c$-stabilizable pairs. Denote by $(R(\epsilon, k_1, k_3), \mathcal{P}(\epsilon, k_1, k_3))$ the solution to (3.1). Then, using (3.1)–(3.3) and some algebra, it can be shown that

$$\lim_{\epsilon \to 0} (R(\epsilon, k_1, k), \mathcal{P}(\epsilon, k_1, k)) = (P(0, k_1, k_3), P(0, k_1, k_3)) \ , \tag{6.1}$$

where $P(0, k_1, k_3)$ is the solution to

$$\begin{aligned} &Q_1 + Q_2 + (A + k_3\widetilde{B})^T P + P(A + k_3\widetilde{B}) \\ &-P[E\ F]\begin{pmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{pmatrix}[E\ F]^T P = 0 \ . \end{aligned} \tag{6.2}$$

*Example* (partial time discretization). Here $n = 2$, $m = k = k_1 = k_3 = 1$. We also let

$$\widetilde{A} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \ \widetilde{E} = \begin{pmatrix} e \\ 0 \end{pmatrix}, \ \widetilde{B} = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}, \ \widetilde{F} = \begin{pmatrix} 0 \\ f \end{pmatrix},$$
$$\widetilde{Q}_1 = \begin{pmatrix} q_1 & 0 \\ 0 & 0 \end{pmatrix}, \ \widetilde{Q}_2 = \begin{pmatrix} 0 & 0 \\ 0 & q_2 \end{pmatrix},$$
$$\widetilde{R}_1 = r_1, \ \widetilde{R}_2 = r_2 \ .$$

This parametrization corresponds, for example, to a partial time discretization, with time step $\epsilon$, of the following control problem (here we are using notation $x(t)$ and $u(t)$ instead of $x_t$ and $u_t$):

(6.3) Minimize

$$\overline{\lim_{T \to \infty}}\, T^{-1} \int_0^T \left\{ (x_1(t), x_2(t)) \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + (u_1(t), u_2(t)) \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \right\} dt$$

subject to

$$\begin{aligned} dx_1(t) &= (ax_1(t) + bx_2(t))dt + eu_1(t)dt + dw_1(t), \\ dx_2(t) &= (cx_1(t) + dx_2(t))dt + fu_2(t)dt + dw_2(t), \end{aligned}$$

where $w_1$ and $w_2$ are standard one-dimensional Brownian motions.

In this case "partial time discretization" means time discretization with respect to the second state component $x_2(t)$. The partially time-discretized problem is

Minimize

$$\overline{\lim_{T\to\infty}}\, T^{-1}\int_0^T \left(q_1 x_{1,\epsilon}^2(t) + r_1 u_1(t)^2\right) dt + \epsilon^{-1}\overline{\lim_{N\to\infty}}\, N^{-1}\sum_{n=1}^{N}\left(\epsilon q_2 x_{2,\epsilon}^2(n) + \epsilon r_2 u_{2,\epsilon}^2(n)\right)$$

subject to

$$\begin{aligned} dx_{1,\epsilon}(t) &= (ax_{1,\epsilon}(t) + bx_{2,\epsilon}(n))dt + eu_1(t)dt + dw_1(t),\\ &\quad t\in[\epsilon n, \epsilon(n+1)),\\ x_{2,\epsilon}(n+1) &= \epsilon c x_{1,\epsilon}(\epsilon n) + (\epsilon+1)dx_{2,\epsilon}(n) + \epsilon u_{2,\epsilon}(n) + (w_2(\epsilon(n+1)) - w_2(\epsilon n))\\ &\quad n = 0,1,2,\dots. \end{aligned}$$

Here, the limiting equation (6.2) coincides with the algebraic Riccati equation corresponding to the original problem (6.3).

We believe that our methodology will allow for time discretization of continuous-time control problems using various time steps for various components of state vector, if necessary.

*Case* ii. Assume A4 and A5. Then

$$\lim_{k_3\to 0}(R(\epsilon,k_1,k_3),\mathcal{P}(\epsilon,k_1,k_3)) = (P(\epsilon,k_1,0),\mathcal{P}(\epsilon,k_1,0))\ , \tag{6.4}$$

where $\mathcal{P}(\epsilon,k_1,0)=\{P_t = P(\epsilon,k_1,0), t\in[0,\epsilon]\}$ and $P(\epsilon,k_1,0)$ is the solution of

$$0 = Q_2 + A^T P + PA - PER^{-1}E^T P\ . \tag{6.5}$$

*Case* iii. Assume A4 and A5. Then

$$\lim_{k_1\to 0}(R(\epsilon,k_1,k_3),\mathcal{P}(\epsilon,k_1,k_3)) = (P(\epsilon,0,k_3),\mathcal{P}(\epsilon,0,k_3))\ , \tag{6.6}$$

where $\mathcal{P}(\epsilon,0,k_3)=\{P_t = P(\epsilon,0,k_3), t\in[0,\epsilon]\}$ and $P(\epsilon,0,k_3)$ is the solution to

$$P = B^T[P - PF(F^T PF + R_2)^{-1}F^T P]B + Q_2\ . \tag{6.7}$$

**7. A simple hybrid control problem.** Consider the following special form of the control problem considered in §4.

System dynamics

$$\begin{aligned} dx_t &= y_n dt + dw_t,\ t\in[n,n+1),\\ y_{n+1} &= y_n + u_n + e_n,\ n=0,1,2,\dots\\ x_0 &= x,\ y_0 = y, \end{aligned}$$

where $x_t, y_n \in R^1$, $(w_t)_{t\geq 0}$ is a standard Brownian motion in $R^1$, and $(e_n)_{n=0}^{\infty}$ is an independently and identically distributed (i.i.d.) sequence of Gaussian random variables with mean zero and variance one.

Cost functional

$$\overline{C}(\overline{u}_\cdot,(x,y)) = \overline{\lim_{T\to\infty}}\, T^{-1}\int_0^T (x_t^2 + \overline{u}_t^2)\, dt\ ,$$

where $\overline{u}_t = u_n$, $t\in[n,n+1)$.

The point here is that a continuous-time subsystem corresponding to $x_t$ is controlled via a discrete-time subsystem corresponding to $y_n$. Using results of §4 we compute optimal controls

$$u_n^o = -\frac{1+\sqrt{13}}{2+\sqrt{13}} y_n, \quad n = 0, 1, 2, \ldots,$$

and the optimal cost

$$\overline{C}(\overline{u}^o, (x, y)) = \frac{29 + 3\sqrt{13}}{18}\ .$$

**8. Concluding remarks.** We refer to the control problem considered in §4 as to the "backward problem." The "forward problem" for which the cost functional is given in terms of

$$\left\{ \int_0^t [x_s^t \widetilde{Q}_1 x_s + v_s^t \widetilde{R}_1 v_s] d\langle M \rangle_s + \int_0^t [x_s^t \widetilde{Q}_s x_s + u_s^t \widetilde{R}_2 u_s] d\langle N \rangle_s \right\}$$

can be studied in the similar way as the "backward problem."

Our formulation of the linear-quadratic stochastic control problem does not allow for a direct consideration of a deterministic linear-quadratic control problem, one of the reasons being that the time scales in (4.1) would vanish for $k_1 = 0$ and/or $k_3 = 0$. An obvious reparametrization will allow for including a deterministic situation in the model (4.1) as well. We have not done that in order to keep the calculations easy. Note that equations (3.1) are serving both deterministic and stochastic situations, as in the "classical" case.

It is still an open question under what nontrivial conditions on the parameters there exists a stationary distribution for $(x_t^0)_{t \geq 0}$. We have some preliminary results for the noncontrolled case corresponding to the one considered by Zabczyk (1983).

In a subsequent paper we shall consider implications of the approach taken here for control and identification of general (multiple time scales) ARMA models represented via a certain integral transform that is given in terms of the predictable quadratic variation process of the driving semimartingale noise.

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**Appendix 1.** In this appendix we prove Theorem 3.1. We will need the following three technical results, which are counterparts of Theorem 3.6 ii) and Lemmas 12.1 and 12.2 of Wonham (1979).

LEMMA A.1.1. *If $Q \geq 0$ and $B$ is d-stable then the equation*

$$B^T R B + Q = R$$

*has a unique solution $R$ and $R \geq 0$.*

*Proof.* $R = \sum_{k=0}^{\infty} (B^T)^k Q\ B^k$. □

LEMMA A.1.2 ($d$-Liapunov criterion). *Suppose $R \geq 0$, $Q \geq 0$, $(B, \sqrt{Q})$ is $d^T$-stabilizable and $B^T R B + Q = R$. Then $B$ is d-stable.*

*Proof.* We have

$$R = (B^T)^k R B^k + \sum_{i=0}^{k-1} (B^T)^i Q B^i, \quad k \geq 0.$$

Assume $B$ is not $d$-stable and let $\lambda$ be an eigenvalue of $B$ with $|\lambda| \geq 1$, and $X$ the corresponding eigenvector. We have $X^T R X = |\lambda|^{2k} X^T R X + \sum_{i=0}^{k-1} |\lambda|^{2i} \|\sqrt{Q}\, X\|^2$. This means that $|\lambda| = 1$ and $\sqrt{Q} X = 0$. Let $\widetilde{K}$ be a matrix such that $B + \widetilde{K}\sqrt{Q}$ is $d$-stable. We see that $\lambda$ and $X$ are respectively an eigenvalue and corresponding eigenvectors of $B + \widetilde{K}\sqrt{Q}$. This is a contradiction. □

LEMMA A.1.3. *Assume that the four-tuple $(A, Q_1, B, Q_2)$ is cd-detectable. Let $G : [0,1] \to L(R^n, R^n)$ be continuous. Let $G := \int_0^1 G_s ds$ and $M > 0$, $N > 0$, $\Phi \geq 0$, $L$ be arbitrary matrices of appropriate dimensions. Then the pair*

$$\Bigg( (B+L)e^{(A+G)},$$

$$\sqrt{Q_2 + L^T N L + \Phi + (B+L)^T \left[ \int_0^1 e^{\int_t^1 (A+G_{1-s})^T ds} (Q_1 + G_{1-t}^T M G_{1-t}) e^{\int_0^1 (A+G_{1-s}) ds} dt \right]}$$

$$\sqrt{\cdot (B+L)} \Bigg)$$

*is $d^T$-stabilizable.*

*Proof.* It will be shown in Lemma A.2.1 of Appendix 2 that

$$\begin{aligned} \mathrm{Ker} \int_0^1 & e^{\int_0^t (A+G_s)^T ds} [Q_1 + G_t^T M G_t] e^{\int_0^t (A+G_s) ds} dt \\ & \subset \mathrm{Ker}\, e^{A^T} \int_0^1 (Q_1 + G_t^T M G_t) dt e^A. \end{aligned}$$

Therefore we have the following chain of inclusions,

$$\begin{aligned} & \mathrm{Ker} \sqrt{Q_2 + L^T N L + \Phi} \\ & \sqrt{+ (B+L)^T \left[ \int_0^1 e^{\int_t^1 (A+G_{1-s})^T ds} (Q_1 + G_{1-t}^T M G_{1-t}) e^{\int_t^1 (A+G_{1-s}) ds} dt \right] (B+L)} \\ & \subset \mathrm{Ker} \sqrt{Q_2 + L^T N L + B^T e^{A^T} Q_1 e^A B + (B+L)^T e^{A^T} \int_0^1 G_t^T M G_t dt e^A (B+L)} \\ & \subset \mathrm{Ker} \sqrt{Q_2 + B^T e^{A^T} Q_1 e^A B} \cap \mathrm{Ker}(-L) \cap \mathrm{Ker}(e^A B - e^{A+G} B) \\ & \subset \mathrm{Ker}(\sqrt{Q_2 + B^T e^{A^T} Q_1 e^A B} - e^{A+G} L + e^A B - e^{A+G} B). \end{aligned}$$

Let $K$ be such that $e^A B + K\sqrt{Q_2 + B^T e^{A^T} Q_1 e^A B}$ is $d$-stable. In view of the above inclusions we conclude that there exists matrix $\widetilde{K}$ such that

$$\begin{aligned} & e^{A+G}(B+L) \\ & + \widetilde{K} \sqrt{Q_2 + L^T N L + \Phi} \\ & \sqrt{+ (B+L)^T \left[ \int_0^1 e^{\int_t^1 (A+G_{1-s})^T ds} (Q_1 + G_{1-t}^T M G_{1-t}) e^{\int_t^1 (A+G_{1-s} ds} dt \right] (B+L)} \\ & = e^A B + K \sqrt{Q_2 + B^T e^{A^T} Q_1 e^A B}. \qquad □ \end{aligned}$$

*Proof of Theorem* 3.1.

*Step* 1. Let $R \geq 0$. Consider the control problem

$$J(x_0, R) = \min_{u,v.} J(x_0, u, v., R)$$

where

$$J(x_0, u, v., R) := x_0^T Q_2 x_0 + u^T R_2 u + y_1^T R y_1 + \int_0^1 (y_s^T Q_1 y_s + v_s^T R_1 v_s) ds$$

subject to

$$\begin{cases} x_1 &= Bx_0 + Fu, \\ \dot{y}_t &= Ay_t + Ev_t, \\ y_0 &= x_1, \ t \in [0,1]. \end{cases}$$

It can be easily verified that

$$J(x_0, R) = x_0^T [B^T P_1 B + Q_2 - B^T P_1 F (F^T P_1 F + R_2)^{-1} F^T P_1 B] x_0,$$

and the optimal controls are

$$\begin{aligned} v_t^* &:= L_t(\mathcal{P}) y_t, \ t \in [0,1], \\ u^* &:= L_2(\mathcal{P}) x_0, \end{aligned}$$

where $\mathcal{P}$ corresponds to $R$ in the sense that $\mathcal{P} = P_t, t \in [0,1]$ solves the differential equation in (3.1) with $P_0 = R$.

*Step* 2. (a) Choose $L_0^1 := \{\widetilde{L}_t^1, \ t \in [0,1]\}$ and $L_2^1$ so that $\mathcal{A}(L_1^1, L_2^1)$ is $d$-stable. Here $L_1^1 := \int_0^1 \widetilde{L}_t^1 dt$.

(b) Having chosen $(L_0^1, L_2^1), \ldots, (L_0^k, L_2^k)$, obtain $R^k \geq 0$ from (use Lemma A.1.1)

$$\mathcal{A}^T(L_1^k, L_2^k) R^k \mathcal{A}(L_1^k, L_2^k) + \int_0^1 e^{\int_s^1 (A + E\widetilde{L}_{1-t}^k)^T dt} \cdot [Q_1 + (\widetilde{L}_{1-s}^k)^T R_1 \widetilde{L}_{1-s}^k] e^{\int_s^1 (A + E\widetilde{L}_{1-t}^k) dt} ds + Q_2 + (L_2^k)^T R_2 L_2^k = R^k.$$

In the above $L_1^k := \int_0^1 \widetilde{L}_t^k dt$. Note that with regard to the control problem of Step 1 we have $x_0^T R^k x_0 = J(x_0, u^k, v^k., R^k)$, where

$$u^k := L_2^k x_0, \ v_t^k := \widetilde{L}_t^k y_t, \ t \in [0,1].$$

(c) Obtain $\mathcal{P}^{k+1} := \{P_t^{k+1}, t \in [0,1]\}$ from

$$\text{(A.1.1)} \qquad \begin{cases} \dot{P}_1^{k+1} &= Q_1 + A^T P_t^{k+1} + P_t^{k+1} A - P_t^{k+1} E R_1^{-1} E^T P_t^{k+1}, \\ P_0^{k+1} &= R^k, \quad t \in [0,1] \end{cases}$$

and define

$$\begin{aligned} \widetilde{L}_t^{k+1} &:= -R_1^{-1} E^T P_{t-1}^{k+1}, \ t \in [0,1], \\ L_2^{k+1} &:= -(F^T P_1^{k+1} F + R_2)^{-1} F^T P_1^{k+1} B. \end{aligned}$$

*Remark* A.1.1. In case $B = I$, $Q_2 = 0$, $F = 0$, and $P_t =$ const, $t \in [0,1]$, consider (A.1.1) as

$$0 = Q_1 + (A + EL_1^k)^T R^k + R^k (A + EL_1^k) + (L_1^k)^T R_1 L_1^k$$

and $\widetilde{L}_s^k = L_1, \ s \in [0,1]$.

Note that with regard to the control problem of Step 1 we have

$x_0^T(B^TP_1^{k+1}B+Q_2-B^TP_1^{k+1}F(F^TP_1^{k+1}F+R_2)^{-1}F^TP_1^{k+1}B)x_0 = J(x_0,R^k) \le x_0^TR^kx_0.$

After letting $\mathcal{K}^i := B^TP_1^iB + Q_2 - B^TP_1^iF(F^TP_1^iF + R_2)^{-1}F^TP_1^iB$ we then have

$$\mathcal{K}^{k+1} \le R^k.$$

Now note that

$$\mathcal{A}_{\mathcal{P}^{k+1}}(R^k) + \mathcal{B}_{\mathcal{P}^{k+1}} = \mathcal{K}^{k+1} - R^k + R^k.$$

Therefore, in view of Lemmas A.1.2 and A.1.3 we conclude that $\mathcal{A}(L_1^{k+1}, L_2^{k+1})$ is $d$-stable. Obtain $R^{k+1}$ from

$$\mathcal{A}_{\mathcal{P}^{k+1}}(R^{k+1}) + \mathcal{B}_{\mathcal{P}^{k+1}} = R^{k+1}$$

and note that

$$\mathcal{A}^T(L_1^{k+1}, L_2^{k+1})(R^{k+1} - R^k)\mathcal{A}(L_1^{k+1}, L_2^{k+1}) + \mathcal{K}^{k+1} - R^k = R^{k+1} - R^k.$$

Henceforth we have

$$0 \le R^{k+1} \le R^k.$$

Thus there exist limits

$$0 \le \bar{R} = \lim_{k\to\infty} R^k,$$
$$0 \le \bar{P}_t = \lim_{k\to\infty} P_t^k,\ t \in [0,1],$$

and the pair $(\bar{R}, \bar{\mathcal{P}})$ satisfies (3.1), where $\bar{\mathcal{P}} := \{\bar{P}_t,\ t \in [0,1]\}$. Again by Lemmas A.1.2 and A.1.3, we conclude that $\mathcal{A}(\bar{L}_1, \bar{L}_2)$ is $d$-stable.

Uniqueness of $(\bar{R}, \bar{\mathcal{P}})$ follows from the following argument. Let $(\widetilde{R}, \widetilde{\mathcal{P}})$ be another solution to (3.1) such that $\widetilde{R} \ge 0$ and $\widetilde{\mathcal{P}}_t \ge 0$, $t \in [0,1]$. In view of the control problem analogous to the one considered in Step 1 we have

$$\mathcal{A}_{\widetilde{\mathcal{P}}}(\bar{R} + \mathcal{B}_{\widetilde{\mathcal{P}}}) \le \bar{R}.$$

Therefore it holds that

$$\mathcal{A}_{\widetilde{\mathcal{P}}}(\bar{R} - \widetilde{R}) \le \bar{R} - \widetilde{R},$$

and since $\mathcal{A}(\widetilde{L}_1, \widetilde{L}_2)$ is $d$-stable (Lemmas A.1.2 and A.1.3 again) we obtain that $\bar{R} - \widetilde{R} \ge 0$. Similarly we get that $\mathcal{A}_{\bar{\mathcal{P}}}(\widetilde{R} - \bar{R}) \le \widetilde{R} - \bar{R}$ and therefore $\widetilde{R} - \bar{R} \ge 0$. □

## Appendix 2.

LEMMA A.2.1. *In the notation of Lemma* A.1.3 *it holds that*

$$\operatorname{Ker} \int_0^1 e^{\int_0^t (A+G_s)^T ds}[Q_1 + G_t^T M G_t] e^{\int_0^t (A+G_s)^T ds} dt$$
$$\subset \operatorname{Ker} e^{A^T} \int_0^1 (Q_1 + G_t^T M G_t) dt\ e^A.$$

*Proof.* Consider the dynamic system

$$\dot{x}_t = (A + G_t)x_t,$$
$$x_0 = 0,\ t \in [0,1],$$

and the functional

$$I = \int_0^1 x_s^T (Q_1 + G_s^T M G_s) x_s ds.$$

Note that $I = 0$ implies $G_s x_s = 0$ for almost all $s \in [0,1]$ and therefore

$$0 = x_0^T \int_0^1 e^{tA^T} (Q_1 + G_t^T M G_t) e^{tA} dt \; x_0.$$

This implies that

$$x_0^T e^{tA^T} Q_1 (t^k A^k) x_0 = x_0^T e^{tA^T} G_t M G_t (t^k A^k) x_0 = 0$$

for $k \geq 0$ and almost all $t \in [0,1]$. Thus we obtain

$$0 = x_0^T e^{A^T} \int_0^1 (Q_1 + G_t^T M G_t) dt e^A x_0. \qquad \square$$

**Appendix 3.**

LEMMA A.3.1. *Under the assumptions of Theorem* 4.1 *the pair of controls* $(v^0., u^0.)$ *defined in* (4.2) *is admissible.*

*Proof.* Let $(x_t^0)_{t\geq 0}$ be the unique, strong, semimartingale solution to (4.1), under $(v^0., u^0.)$, that exists according to Theorem 4.1(a). It remains to show

(A.3.1) $$\overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t \|x_s^0\|^2 ds < +\infty, \text{ a.s.}$$

and

(A.3.2) $$\lim_{t\to\infty} \frac{1}{t} \|x_t^0\|^2 = 0, \text{ a.s.}$$

It is enough to consider the stochastic sequence $\{y_n\}_{n\geq 1}$ where $y_n := x_{n^-}^0$, $n \geq 1$. Note that

$$y_{n+1} = \mathcal{A}(\Lambda_1, \Lambda_2) y_n + e_n,$$
$$y_1 = x_{1^-}^0, \; n \geq 1,$$

where $\Lambda_1 := \int^1 \bar{\Lambda}_t dt$ and

$$e_n := \int_n^{n+1} e^{\int_s^{n+1}(A+E\bar{\Lambda}_t)dt} dM_s + e^{A+E\Lambda_1} \Delta N_n, \; n \geq 1.$$

Since $\mathcal{A}(\Lambda_1, \Lambda_2)$ is $d$-stable (Theorem 3.1) we conclude, upon applying the strong law of large numbers for martingales (Lipster and Shiryayev (1989), Thm. 2.6.1), that

$$\lim_{n\to\infty} \frac{\|y_n\|^2}{n} = 0, \text{ a.s.},$$

which in turn implies (A.3.2).

Since $\mathcal{A}(\Lambda_1, \Lambda_2)$ is $d$-stable then there exists a positive definite matrix $G$ such that

$$\mathcal{A}^T(\Lambda_1, \Lambda_2) G \mathcal{A}(\Lambda_1, \Lambda_2) + 2I = G$$

(compare Lemma A.1.1). Next note that, for $n \geq 1$,

$$\begin{aligned} y_{n+1}^T G y_{n+1} &= y_n^T \mathcal{A}^T(\Lambda_1, \Lambda_2) G \mathcal{A}(\Lambda_1, \Lambda_2) y_n \\ &\quad + 2 y_n^T \mathcal{A}^T(\Lambda_1, \Lambda_2) G e_n + e_n^T G e_n \\ &\leq y_n^T G y_n - \|y_n\|^2 + 2 y_n^T \mathcal{A}^T(\Lambda_1, \Lambda_2) G e_n \\ &\quad + e_n^T G e_n, \quad \text{a.s.} \end{aligned}$$

Therefore we have

$$\begin{aligned} y_{n+1}^T G y_{n+1} + \sum_{k=1}^{n} \|y_k\|^2 &\leq y_1^T G y_1 \\ &\quad + 2 \sum_{k=1}^{n} y_k^T \mathcal{A}(\Lambda_1, \Lambda_2) G e_k \\ &\quad + \sum_{k=1}^{n} e_n^T G e_n, \quad \text{a.s.} \end{aligned}$$

Invoking the law of large numbers for martingales again, we finally obtain

$$\overline{\lim_{n \to \infty}} \frac{1}{n} \sum_{k=1}^{n} \|y_k\|^2 < +\infty, \quad \text{a.s.},$$

from which (A.3.1) follows. □

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