HEDGING OF BASKET CREDIT DERIVATIVES IN CREDIT DEFAULT SWAP MARKET

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Introduction

This paper is the first in a series of works in which we shall conduct a systematic mathematical study of credit derivatives of the swap type. The formal set-up has been chosen here to be relatively simple, so that we can illuminate and explain some non-trivial aspects of the theory of credit related swap contracts without engaging in complicated technical issues that will necessarily transpire in a more realistic model, which will be studied in a follow-up paper.

The topic of this work is a detailed study of stylized credit default swaps within the framework of a generic *reduced-form* credit risk model. By a reduced-form model we mean any model of a single default or several dependent defaults in which we can explicitly identify the distribution of default times. Therefore, the results presented in this work have the potential to cover various alternative approaches, which are usually classified as, for instance, value-of-the-firm approach, intensity-based approach, copula-based approach, etc.

The main goal is to develop general results dealing with the relative valuation of defaultable claims (e.g., basket credit derivatives) with respect to market values of traded credit-risk sensitive securities. As could be expected, we have chosen stylized credit default swaps (CDSs) as liquidly traded assets, so that other credit derivatives are valued with respect to CDS spreads as a benchmark. The tool used to this end is fairly standard. We simply show that a generic defaultable claim (or a generic basket claim, in the case of several underlying credit names) can be replicated by dynamic trading in single-name CDSs. Let us note that in a recent paper by Brasch [6], the author examines a related issue of static hedging of kth-to-default basket claims with other basket claims (in particular, first-to-default claims).

Our approach is based on the assumption that the joint distribution of default times of underlying names under the "pricing measure" is known. Practically speaking, this assumption means that we have chosen some model of dependent defaults and this model has been already calibrated to market data for single name and basket credit derivatives. Hence not only the marginal default distributions under the pricing measure implied by prices of single name CDSs were found, but also the "implied default correlation" was estimated from prices of the most liquid basket credit derivatives.

This work is organized as follows. We start, in Section 1, by dealing with the valuation and trading of a generic defaultable claim. The presentation in this section, although largely based on Section 2.1 in Bielecki and Rutkowski [1], is adapted to our current purposes, and the notation is modified accordingly. We believe that it is more convenient to deal with a generic dividend-paying asset, rather than with any specific examples of credit derivatives, since the fundamental properties of arbitrage prices of defaultable assets, and of related trading strategies, are already apparent in a general set-up.

In Section 2, we provide results concerning the valuation and trading of credit default swaps under the assumption that the default intensity is deterministic and the interest rate is zero. Subsequently, we derive a closed-form solution for replicating strategy for an arbitrary non-dividend paying defaultable claim on a single credit name, in a market in which a bond and a credit default swap are traded. Also, we examine the completeness of such a security market model.

Section 3 deals with hedging of basket credit derivatives using single-name CDSs. We first present results dealing with the case of a first-to-default claim. Subsequently, we show that these results can be adapted to cover the case of a general basket claim. The idea is to show that a general basket claim can be formally seen as a sequence of "conditional" first-to-default claim, where the condition encompasses dates of the past defaults and identities of defaulting names, and a suitably re-defined recovery payoff occurs at the moment of the next default. The paper concludes with few examples of concrete applications of our results to copula-based models of default times.

In a follow-up paper we shall extend results established here to the case of stochastic default intensity. Let us note that hedging under stochastic default intensity covers both default and spread risks. For more general results concerning various alternative techniques for hedging defaultable claims in complete and incomplete models, the interested reader is referred to Blanchet-Scalliet and Jeanblanc [5] and Bielecki et al. [3]-[4].

1 Preliminaries

This section provides some preliminary results and settles the notation used throughout the paper. In a first step, we recall the role of dividends in dynamics of prices.

We write S^i, $i = 1, 2, \ldots, k$ to denote the price processes of k primary securities in an arbitrage-free financial model. We make the standard assumption that the processes S^i, $i = 1, 2, \ldots, k-1$ are semimartingales. In addition, we set $S^k_t = B_t$ where

$$B_t = \exp\left(\int_0^t r_u \, du\right), \quad \forall \, t \in \mathbb{R}_+. \tag{1}$$

so that S^k represents the value process of the savings account. The last assumption is not necessary, however. We can assume, for instance, that S^k is the price of a T-maturity risk-free zero-coupon bond, or choose any other strictly positive price process as numéraire.

For the sake of convenience, we assume that S^i, $i = 1, 2, \ldots, k-1$ are non-dividend-paying assets, and we introduce the discounted price processes S^{i*} by setting $S^{i*}_t = S^i_t / B_t$. All processes are assumed to be given on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{Q})$ where $\mathbb{Q}$ is interpreted as the real-life (i.e., statistical) probability measure. We assume that our market model is arbitrage-free, meaning that it admits a *spot martingale measure* $\mathbb{Q}^*$ equivalent to $\mathbb{Q}$ (not necessarily unique), which is associated with the choice of B as a numéraire. We say that $\mathbb{Q}^*$ is a *spot martingale measure* if the discounted price S^{i*} of any non-dividend paying traded security follows a $\mathbb{Q}^*$-martingale with respect to $\mathbb{G}$.

Let us now assume that we have an additional traded contract that pays (negative or positive) *dividends* during its lifespan, assumed to be the time interval $[0, T]$, according to a process of finite variation D with $D_0 = 0$. Let us stress that the contract expires at time T, so that no dividend payments occur after this date. We make the standing assumption that the random variable $\int_{]t,T]} B_u^{-1} dD_u$ is $\mathbb{Q}^*$-integrable for any $t \in [0, T]$. The proposition below is standard and it follows, for instance, from Duffie [10] (see Chapter 6, Section 11 therein) or Bielecki and Rutkowski [1] (see Section 2.1 therein). Note that $\mathbb{Q}^*$ represents here some arbitrarily selected spot martingale measure.

Proposition 1.1 *The ex-dividend price process S of a contract expiring at T and paying dividends according to a process $D_t, t \in [0, T]$, equals, for every $t \in [0, T]$,*

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t\Big). \tag{2}$$

Remarks. (i) Under the assumption of uniqueness of a spot martingale measure $\mathbb{Q}^*$, any $\mathbb{Q}^*$-integrable contingent claim is *attainable* (i.e., it can be replicated) and thus the valuation formula established above can be supported by standard replication arguments.
(ii) If, however, a spot martingale measure $\mathbb{Q}^*$ is not uniquely determined then the right-hand side of (2) depends on the choice of $\mathbb{Q}^*$, in general. In that case, the process S defined by formula (2), for an arbitrarily chosen spot martingale measure $\mathbb{Q}^*$, can be taken to be the ex-dividend price of a T-maturity contract. Finally, if a T-maturity contract (a defaultable claim, say) with the dividend process D is among traded assets then the right-hand side of (2) does not depend on the choice of a spot martingale measure $\mathbb{Q}^*$.
(iii) If a defaultable claim is attainable in a given market model, so that the pricing formula (2) is supported by replication arguments, we refer to S as the *replication price*. Otherwise, the process S given by (2) is qualified as the *risk-neutral value*. It is worth noting that most papers on valuation of defaultable claims are restricted to risk-neutral valuation. The main goal of this work is to show that risk-neutral valuation of defaultable claims can be supported by replication in a market model in which some liquidly traded contracts (single name CDSs, in our case) are assumed to be traded.

The following auxiliary concept will be useful.

Definition 1.1 The *cumulative price* process $\widehat{S}$ of a T-maturity contract with the dividend process D is given by the formula, for every $t \in [0,T]$,

$$\widehat{S}_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big) = S_t + \widehat{D}_t \tag{3}$$

where $\widehat{D}_t$ equals

$$\widehat{D}_t = B_t \int_{]0,t]} B_u^{-1} \, dD_u, \quad \forall \, t \in [0,T],$$

so that it represents the current value at time t of all dividend payments occurring during the period $]0,t]$ under the convention that they were immediately reinvested in the savings account.

The next result shows that the discounted cumulative price has a convenient martingale property.

Corollary 1.1 *The discounted cumulative price* $\widehat{S}^*_t = B_t^{-1}\widehat{S}_t$, $t \in [0,T]$, *of a* T*-maturity contract is a* $\mathbb{Q}^*$*-martingale with respect to* $\mathbb{G}$.

Proof. It suffices to observe that

$$\widehat{S}^*_t = \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \quad \forall \, t \in [0,T].$$

Note also that $S^*_t = \widehat{S}^*_t - \widehat{D}^*_t$ where

$$\widehat{D}^*_t = \int_{]0,t]} B_u^{-1} \, dD_u, \quad \forall \, t \in [0,T],$$

and thus S^* is not a $\mathbb{G}$-martingale, unless the process D is null (in that case the claim is trivial). □

2 Hedging of Single Name Credit Derivatives

We shall now apply the general theory to a particular class of contracts, namely, to credit default swaps. We do not need to specify the underlying market model at this stage, but we make the following standing assumptions.

Assumptions (A). We assume throughout that:
(i) $\mathbb{Q}^*$ is a spot martingale measure on $(\Omega, \mathcal{G}_T)$,
(ii) the interest rate $r = 0$, so that the price of a savings account $B_t = 1$ for every $t \in \mathbb{R}_+$.

For the sake of simplicity, these restrictions are maintained in Section 3 of the present work, but they will be relaxed in a follow-up paper [4].

2.1 Defaultable Claims

A strictly positive random variable τ defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ is termed a *random time*. In view of its interpretation, it will be later referred to as a *default time*. We introduce the *default indicator process* $H_t = \mathbb{1}_{\{\tau \le t\}}$ associated with τ and we denote by $\mathbb{H}$ the filtration generated by this process. We augment it with all sets that are subsets of $\sigma(\tau)$-measurable sets of zero probability. Reasoning analogously as in Appendix A in Wong [15], one can show that $\mathbb{H}$ is a right-continuous filtration and thus it satisfies the so-called 'usual conditions'.

In this work, we shall analyze the valuation and trading credit default swaps in a simple model of default risk in which the information flow is modeled by the filtration $\mathbb{H}$. It is worth stressing that most results of this paper can be extended to the case of a more general filtration. However, such extension is by no means trivial; it will be studied in detail in a follow-up paper [2].

Definition 2.1 A *defaultable claim* expiring at time T is a T-maturity contract given by a quadruple (X, A, Z, τ) where X is a constant, A is a function of finite variation with $A(0) = 0$, Z is some function, and τ is a random time. The *dividend process* D of a defaultable claim maturing at T equals, for every $t \in [0, T]$,

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T]}(t) + \int_{]0,t]} (1 - H_u)\, dA(u) + \int_{]0,t]} Z(u)\, dH_u.$$

The financial interpretation of D justifies the following terminology: X is the *promised payoff* at maturity T, A represents the *promised dividends*, and Z specifies the *recovery payoff* at default. Note that the cash payment (premium) at time 0 is not included in the dividend process D associated with a defaultable claim.

It is clear that the dividend process D is a process of finite variation on $[0, T]$. Since

$$\int_{]0,t]} (1 - H_u)\, dA(u) = \int_{]0,t]} \mathbb{1}_{\{\tau > u\}}\, dA(u) = A(\tau-) \mathbb{1}_{\{\tau \le t\}} + A(t) \mathbb{1}_{\{\tau > t\}},$$

it is also apparent that if default occurs at some date t, the *promised dividend* $A(t) - A(t-)$ that is due to be received or paid at this date is forfeited. We assume that a function Z is right-continuous with finite left-hand limits. If we denote $\tau \wedge t = \min(\tau, t)$ then we have

$$\int_{]0,t]} Z(u)\, dH_u = Z(\tau \wedge t) \mathbb{1}_{\{\tau \le t\}} = Z(\tau) \mathbb{1}_{\{\tau \le t\}}.$$

Let us stress that the process $D_u - D_t$, $u \in [t, T]$, represents all cash flows from a defaultable claim received by an investor who purchases it at time t. Of course, the process $D_u - D_t$ may depend on the past behavior of the claim (e.g., through some intrinsic parameters, such as credit spreads) as well as on the history of the market prior to t. The past dividends are not valued by the market, however, so that the current market value at time t of a claim (i.e., the ex-dividend price S_t at which it trades at time t) reflects only on future dividends to be paid or received over the time interval $]t, T]$.

2.2 Stylized Credit Default Swap

A stylized T-maturity credit default swap is formally introduced through the following definition.

Definition 2.2 A *credit default swap* (CDS) with a constant rate κ and *recovery at default* is a defaultable claim $(0, A, Z, \tau)$ where $Z(t) = \delta(t)$ and $A(t) = -\kappa t$ for every $t \in [0, T]$. A function $\delta : [0, T] \to \mathbb{R}$ represents the *default protection*, and κ is the *CDS rate* (also termed the *spread, premium* or *annuity* of a CDS).

We denote by F the cumulative distribution function of the default time τ under $\mathbb{Q}^*$, and we assume that F is a continuous function, with $F(0) = 0$ and $F(T) < 1$. Also, we write $G = 1 - F$ to denote the *survival probability function* of τ, so that $G(t) > 0$ for every $t \in [0, T]$.

Since we start with only one tradeable asset in our model (the savings account), it is clear that any probability measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{H}_T)$ equivalent to $\mathbb{Q}$ can be chosen as a spot martingale measure. The choice of $\mathbb{Q}^*$ is reflected in the cumulative distribution function F (in particular, in the default intensity if F admits a density function). In practical applications of reduced-form models, the choice of F is done by calibration.

2.3 Pricing of a CDS

Since the ex-dividend price of a CDS is the price at which it is actually traded, we shall refer to the ex-dividend price as the *price* in what follows. Recall that we also introduced the so-called *cumulative price*, which encompasses also past dividends reinvested in the savings account.

Let $s \in [0,T]$ stands for some fixed date. We consider a stylized T-maturity CDS contract with a constant rate κ and default protection function δ, initiated at time s and maturing at T. The dividend process of a CDS equals

$$D_t = \int_{]0,t]} \delta(u)\, dH_u - \kappa \int_{]0,t]} (1 - H_u)\, du \tag{4}$$

and thus, in view of (2), the price of this CDS is given by the formula

$$S_t(\kappa, \delta, T) = \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\le T\}} \delta(\tau) \,\Big|\, \mathcal{H}_t \Big) - \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\}} \kappa\big((\tau \wedge T) - t\big) \,\Big|\, \mathcal{H}_t \Big) \tag{5}$$

where the first conditional expectation represents the current value of the *default protection stream* (or the *protection leg*), and the second is the value of the *survival annuity stream* (or the *fee leg*). To alleviate notation, we shall write $S_t(\kappa)$ instead of $S_t(\kappa, \delta, T)$ in what follows.

Lemma 2.1 *The price at time $t \in [s,T]$ of a credit default swap started at s, with rate κ and protection payment $\delta(\tau)$ at default, equals*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \left(- \int_t^T \delta(u)\, dG(u) - \kappa \int_t^T G(u)\, du \right). \tag{6}$$

Proof. We have, on the set $\{t < \tau\}$,

$$\begin{aligned} S_t(\kappa) &= -\frac{\int_t^T \delta(u)\, dG(u)}{G(t)} - \kappa \left(\frac{-\int_t^T u\, dG(u) + TG(T)}{G(t)} - t \right) \\ &= \frac{1}{G(t)} \left(- \int_t^T \delta(u)\, dG(u) - \kappa \Big(TG(T) - tG(t) - \int_t^T u\, dG(u) \Big) \right). \end{aligned}$$

Since

$$\int_t^T G(u)\, du = TG(T) - tG(t) - \int_t^T u\, dG(u), \tag{7}$$

we conclude that (6) holds. □

The *pre-default price* is defined as the unique function $\widetilde{S}(\kappa)$ such that we have (see Lemma 3.1 with $n = 1$)

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa), \quad \forall\, t \in [0,T]. \tag{8}$$

Combining (6) with (8), we find that the pre-default price of the CDS equals, for $t \in [s,T]$,

$$\widetilde{S}_t(\kappa) = \frac{1}{G(t)} \left(- \int_t^T \delta(u)\, dG(u) - \kappa \int_t^T G(u)\, du \right) = \widetilde{\delta}(t,T) - \kappa \widetilde{A}(t,T) \tag{9}$$

where

$$\widetilde{\delta}(t,T) = -\frac{1}{G(t)} \int_t^T \delta(u)\, dG(u)$$

is the pre-default price at time t of the protection leg, and

$$\widetilde{A}(t,T) = \frac{1}{G(t)} \int_t^T G(u)\, du$$

represents the pre-default price at time t of the fee leg for the period $[t,T]$ per one unit of spread κ. We shall refer to $\widetilde{A}(t,T)$ as the *CDS annuity*. Note that $\widetilde{S}(\kappa)$ is a continuous function, under our assumption that G is continuous.

2.4 Market CDS Rate

A CDS that has null value at its inception plays an important role as a benchmark CDS, and thus we introduce a formal definition, in which it is implicitly assumed that a recovery function δ of a CDS is given, and that we are on the event $\{\tau > s\}$.

Definition 2.3 A *market CDS started at s* is the CDS initiated at time s whose initial value is equal to zero. The T-maturity *market CDS rate* (also known as the *fair CDS spread*) at time s is the fixed level of the rate $\kappa = \kappa(s,T)$ that makes the T-maturity CDS started at s valueless at its inception. The market CDS rate at time s is thus determined by the equation $\widetilde{S}_s(\kappa(s,T)) = 0$ where $\widetilde{S}_s(\kappa)$ is given by (9).

Under the present assumptions, by virtue of (9), the T-maturity market CDS rate $\kappa(s,T)$ equals, for every $s \in [0,T]$,

$$\kappa(s,T) = \frac{\widetilde{\delta}(s,T)}{\widetilde{A}(s,T)} = -\frac{\int_s^T \delta(u)\,dG(u)}{\int_s^T G(u)\,du}\,. \tag{10}$$

Example 2.1 Assume that $\delta(t) = \delta$ is constant, and $F(t) = 1 - e^{-\gamma t}$ for some constant default intensity $\gamma > 0$ under $\mathbb{Q}^*$. In that case, the valuation formulae for a CDS can be further simplified. In view of Lemma 2.1, the ex-dividend price of a (spot) CDS with rate κ equals, for every $t \in [0,T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\delta\gamma - \kappa)\gamma^{-1}\Big(1 - e^{-\gamma(T-t)}\Big).$$

The last formula (or the general formula (10)) yields $\kappa(s,T) = \delta\gamma$ for every $s < T$, so that the market rate $\kappa(s,T)$ is here independent of s. As a consequence, the ex-dividend price of a market CDS started at s equals zero not only at the inception date s, but indeed at any time $t \in [s,T]$, both prior to and after default. Hence this process follows a trivial martingale under $\mathbb{Q}^*$. As we shall see in what follows, this martingale property the ex-dividend price of a market CDS is an exception, in the sense so that it fails to hold if the default intensity varies over time.

In what follows, we fix a maturity date T and we assume that credit default swaps with different inception dates have a common recovery function δ. We shall write briefly $\kappa(s)$ instead of $\kappa(s,T)$. Then we have the following result, in which the quantity $\nu(t,s) = \kappa(t) - \kappa(s)$ represents the *calendar CDS market spread* (for a given maturity T).

Proposition 2.1 *The price of a market CDS started at s with recovery δ at default and maturity T equals, for every $t \in [s,T]$,*

$$S_t(\kappa(s)) = \mathbb{1}_{\{t<\tau\}}\,(\kappa(t) - \kappa(s))\,\widetilde{A}(t,T) = \mathbb{1}_{\{t<\tau\}}\,\nu(t,s)\widetilde{A}(t,T). \tag{11}$$

Proof. To establish (11), it suffices to observe that $S_t(\kappa(s)) = S_t(\kappa(s)) - S_t(\kappa(t))$ since $S_t(\kappa(t)) = 0$, and to use (9) with $\kappa = \kappa(t)$ and $\kappa = \kappa(s)$. □

Note that formula (11) can be extended to any value of κ, specifically, we have that

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\kappa(t) - \kappa)\widetilde{A}(t,T), \tag{12}$$

assuming that the CDS with rate κ was initiated at some date $s \in [0,t]$. The last representation shows that the price of a CDS can take negative values. The negative value occurs whenever the current market spread is lower than the contracted spread.

2.5 Price Dynamics of a CDS

In the remainder of Section 2, we assume that

$$G(t) = \mathbb{Q}^*(\tau > t) = \exp\left(-\int_0^t \gamma(u)\,du\right), \quad \forall\, t \in [0,T],$$

where the default intensity $\gamma(t)$ under $\mathbb{Q}^*$ is a strictly positive deterministic function. It is then well known (see, for instance, Lemma 4.2.1 in [1]) that the process M, given by the formula

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\, du, \quad \forall\, t \in [0, T], \tag{13}$$

is an $\mathbb{H}$-martingale under $\mathbb{Q}^*$.

We first focus on dynamics of the price of a CDS with rate κ started at some date $s < T$.

Lemma 2.2 (i) *The dynamics of the price $S_t(\kappa)$, $t \in [s, T]$, are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt. \tag{14}$$

(ii) *The cumulative price process $\widehat{S}_t(\kappa)$, $t \in [s, T]$, is an $\mathbb{H}$-martingale under $\mathbb{Q}^*$, specifically,*

$$d\widehat{S}_t(\kappa) = \big(\delta(t) - S_{t-}(\kappa)\big)\, dM_t. \tag{15}$$

Proof. To prove (i), it suffices to recall that

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa) = (1 - H_t)\widetilde{S}_t(\kappa)$$

so that the integration by parts formula yields

$$dS_t(\kappa) = (1 - H_t)\, d\widetilde{S}_t(\kappa) - \widetilde{S}_{t-}(\kappa)\, dH_t.$$

Using formula (6), we find easily that

$$d\widetilde{S}_t(\kappa) = \gamma(t)\widetilde{S}_t(\kappa)\, dt + (\kappa - \delta(t)\gamma(t))\, dt. \tag{16}$$

In view of (13) and the fact that $S_{\tau-}(\kappa) = \widetilde{S}_{\tau-}(\kappa)$ and $S_t(\kappa) = 0$ for $t \geq \tau$, the proof of (14) is complete.

To prove part (ii), we note that (2) and (3) yield

$$\widehat{S}_t(\kappa) - \widehat{S}_s(\kappa) = S_t(\kappa) - S_s(\kappa) + D_t - D_s. \tag{17}$$

Consequently,

$$\begin{aligned}
\widehat{S}_t(\kappa) - \widehat{S}_s(\kappa) &= S_t(\kappa) - S_s(\kappa) + \int_s^t \delta(u)\, dH_u - \kappa \int_s^t (1 - H_u)\, du \\
&= S_t(\kappa) - S_s(\kappa) + \int_s^t \delta(u)\, dM_u - \int_s^t (1 - H_u)(\kappa - \delta(u)\gamma(u))\, du \\
&= \int_s^t \big(\delta(u) - S_{u-}(\kappa)\big)\, dM_u
\end{aligned}$$

where the last equality follows from (14). □

Equality (14) emphasizes the fact that a single cash flow of $\delta(\tau)$ occurring at time τ can be formally treated as a dividend stream at the rate $\delta(t)\gamma(t)$ paid continuously prior to default. It is clear that we also have

$$dS_t(\kappa) = -\widetilde{S}_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt. \tag{18}$$

2.6 Dynamic Replication of a Defaultable Claim

Our goal is to show that in order to replicate a general defaultable claim, it suffices to trade dynamically in two assets: a CDS maturing at T, and the savings account B, assumed here to be constant.

Since one may always work with discounted values, the last assumption is not restrictive. Moreover, it is also possible to take a CDS with any maturity $U \geq T$.

Let ϕ^0, ϕ^1 be $\mathbb{H}$-predictable processes and let $C : [0,T] \to \mathbb{R}$ be a function of finite variation with $C(0) = 0$. We say that $(\phi, C) = (\phi^0, \phi^1, C)$ is a *self-financing trading strategy with dividend stream* C if the wealth process $V(\phi, C)$, defined as

$$V_t(\phi, C) = \phi^0_t + \phi^1_t S_t(\kappa) \tag{19}$$

where $S_t(\kappa)$ is the price of a CDS at time t, satisfies

$$dV_t(\phi, C) = \phi^1_t \big(dS_t(\kappa) + dD_t\big) - dC(t) = \phi^1_t \, d\widehat{S}_t(\kappa) - dC(t) \tag{20}$$

where the dividend process D of a CDS is in turn given by (4). Note that C represents both outflows and infusions of funds. It will be used to cover the running cashflows associated with a claim we wish to replicate.

Consider a defaultable claim (X, A, Z, τ) where X is a constant, A is a function of finite variation, and Z is some recovery function. In order to define replication of a defaultable claim (X, A, Z, τ), it suffices to consider trading strategies on the random interval $[0, \tau \wedge T]$.

Definition 2.4 We say that a trading strategy (ϕ, C) *replicates a defaultable claim* (X, A, Z, τ) if:
(i) the processes $\phi = (\phi^0, \phi^1)$ and $V(\phi, C)$ are stopped at $\tau \wedge T$,
(ii) $C(\tau \wedge t) = A(\tau \wedge t)$ for every $t \in [0, T]$,
(iii) the equality $V_{\tau \wedge T}(\phi, C) = Y$ holds, where the random variable Y equals

$$Y = X \mathbb{1}_{\{\tau > T\}} + Z(\tau) \mathbb{1}_{\{\tau \leq T\}}. \tag{21}$$

Remark. Alternatively, one may say that a self-financing trading strategy $\phi = (\phi, 0)$ (i.e., a trading strategy with $C = 0$) replicates a defaultable claim (X, A, Z, τ) if and only if $V_{\tau \wedge T}(\phi) = \widehat{Y}$, where we set

$$\widehat{Y} = X \mathbb{1}_{\{\tau > T\}} + A(\tau \wedge T) + Z(\tau) \mathbb{1}_{\{\tau \leq T\}}. \tag{22}$$

However, in the case of non-zero (possibly random) interest rates, it is more convenient to define replication of a defaultable claim via Definition 2.4, since the running payoffs specified by A are distributed over time and thus, in principle, they need to be discounted accordingly (this does not show in (22), since it is assumed here that $r = 0$).

Let us denote, for every $t \in [0, T]$,

$$\widetilde{Z}(t) = \frac{1}{G(t)} \left(X G(T) - \int_t^T Z(u) \, dG(u) \right) \tag{23}$$

and

$$\widetilde{A}(t) = \frac{1}{G(t)} \int_{]t,T]} G(u) \, dA(u). \tag{24}$$

Let π and $\widetilde{\pi}$ be the risk-neutral value and the pre-default risk-neutral value of a defaultable claim under $\mathbb{Q}^*$, so that $\pi_t = \mathbb{1}_{\{t < \tau\}} \widetilde{\pi}(t)$ for every $t \in [0, T]$. Also, let $\widehat{\pi}$ stand for its risk-neutral cumulative price. It is clear that $\widetilde{\pi}(0) = \pi(0) = \widehat{\pi}(0) = \mathbb{E}_{\mathbb{Q}^*}(\widehat{Y})$

Proposition 2.2 *The pre-default risk-neutral value of a defaultable claim* (X, A, Z, τ) *equals* $\widetilde{\pi}(t) = \widetilde{Z}(t) + \widetilde{A}(t)$ *and thus*

$$d\widetilde{\pi}(t) = \gamma(t)(\widetilde{\pi}(t) - Z(t)) \, dt - dA(t). \tag{25}$$

Moreover

$$d\pi_t = (Z(t) - \widetilde{\pi}(t-)) \, dM_t - dA(t \wedge \tau) \tag{26}$$

and

$$d\widehat{\pi}_t = (Z(t) - \widetilde{\pi}(t-)) \, dM_t. \tag{27}$$

Proof. The proof of equality $\widetilde{\pi}(t) = \widetilde{Z}(t) + \widetilde{A}(t)$ is similar to the derivation of formula (9). We have

$$\begin{aligned}\pi_t &= \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\}}Y + A(\tau\wedge T) - A(\tau\wedge t)\,\Big|\,\mathcal{H}_t\Big)\\ &= \mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\Big(XG(T) - \int_t^T Z(u)\,dG(u)\Big) + \mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\int_{]t,T]} G(u)\,dA(u)\\ &= \mathbb{1}_{\{t<\tau\}}(\widetilde{Z}(t) + \widetilde{A}(t)) = \mathbb{1}_{\{t<\tau\}}\widetilde{\pi}(t).\end{aligned}$$

By elementary computation, we obtain

$$d\widetilde{Z}(t) = \gamma(t)(\widetilde{Z}(t) - Z(t))\,dt, \quad d\widetilde{A}(t) = \gamma(t)\widetilde{A}(t)\,dt - dA(t),$$

and thus (25) holds. Finally, (26) follows easily from (25) and the integration by parts formula applied to the equality $\pi_t = (1 - H_t)\widetilde{\pi}(t)$ (see the proof of Lemma 2.2 for similar computations). The last formula is also clear. □

The next proposition shows that the risk-neutral value of a defaultable claim is also its replication price, that is, a defaultable claim derives its value from the price of the traded CDS.

Theorem 2.1 *Assume that the inequality $\widetilde{S}_t(\kappa) \neq \delta(t)$ holds for every $t \in [0,T]$. Let $\phi^1_t = \widetilde{\phi}_1(\tau\wedge t)$, where the function $\widetilde{\phi}_1 : [0,T] \to \mathbb{R}$ is given by the formula*

$$\widetilde{\phi}_1(t) = \frac{Z(t) - \widetilde{\pi}(t-)}{\delta(t) - \widetilde{S}_t(\kappa)}, \quad \forall\, t \in [0,T], \tag{28}$$

and let $\phi^0_t = V_t(\phi, A) - \phi^1_t S_t(\kappa)$, where the process $V(\phi, A)$ is given by the formula

$$V_t(\phi, A) = \widetilde{\pi}(0) + \int_{]0,\tau\wedge t]} \widetilde{\phi}_1(u)\,d\widehat{S}_u(\kappa) - A(t\wedge\tau). \tag{29}$$

Then the trading strategy (ϕ^0, ϕ^1, A) replicates a defaultable claim (X, A, Z, τ).

Proof. Assume first that a trading strategy $\phi = (\phi^0, \phi^1, C)$ is a replicating strategy for (X, A, Z, τ). By virtue of condition (i) in Definition 2.4 we have $C = A$ and thus, by combining (29) with (15), we obtain

$$dV_t(\phi, A) = \phi^1_t(\delta(t) - \widetilde{S}_t(\kappa))\,dM_t - dA(\tau\wedge t)$$

For ϕ^1 given by (28), we thus obtain

$$dV_t(\phi, A) = (Z(t) - \widetilde{\pi}(t-))\,dM_t - dA(\tau\wedge t).$$

It is thus clear that if we take $\phi^1_t = \widetilde{\phi}_1(\tau\wedge t)$ with $\widetilde{\phi}_1$ given by (28), and the initial condition $V_0(\phi, A) = \widetilde{\pi}(0) = \pi_0$, then we have that $V_t(\phi, A) = \pi(t)$ for every $t \in [0,T]$. It is now easily seen that all conditions of Definition 2.4 are satisfied since, in particular, $\pi_{\tau\wedge T} = Y$ with Y given by (21).
□

Remark. Of course, if we take as (X, A, Z, τ) a CDS with rate κ and recovery function δ, then we have $Z(t) = \delta(t)$ and $\widetilde{\pi}(t-) = \widetilde{\pi}(t) = \widetilde{S}_t(\kappa)$, so that $\phi^1_t = 1$ for every $t \in [0,T]$.

3 Dynamic Hedging of Basket Credit Derivatives

In this section, we shall examine hedging of first-to-default basket claims with single name credit default swaps on the underlying n credit names, denoted as $1, 2, \ldots, n$. Our standing assumption (A) is maintained throughout this section.

Let the random times $\tau_1, \tau_2, \ldots, \tau_n$ given on a common probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$ represent the default times of with n credit names. We denote by $\tau_{(1)} = \tau_1 \wedge \tau_2 \wedge \ldots \wedge \tau_n = \min(\tau_1, \tau_2, \ldots, \tau_n)$ the moment of the first default, so that no defaults are observed on the event $\{\tau_{(1)} > t\}$.

Let

$$F(t_1,t_2,\ldots,t_n)=\mathbb{Q}^*(\tau_1\le t_1,\tau_2\le t_2,\ldots,\tau_n\le t_n)$$

be the joint probability distribution function of default times. We assume that the probability distribution of default times is jointly continuous, and we write $f(t_1,t_2,\ldots,t_n)$ to denote the joint probability density function. Also, let

$$G(t_1,t_2,\ldots,t_n)=\mathbb{Q}^*(\tau_1> t_1,\tau_2> t_2,\ldots,\tau_n> t_n)$$

stand for the joint probability that the names $1,2,\ldots,n$ have survived up to times $t_1,t_2,\ldots,t_n$. In particular, the joint survival function equals

$$G(t,\ldots,t)=\mathbb{Q}^*(\tau_1>t,\tau_2>t,\ldots,\tau_n>t)=\mathbb{Q}^*(\tau_{(1)}>t)=G_{(1)}(t).$$

For each $i=1,2,\ldots,n$, we introduce the default indicator process $H^i_t=\mathbb{1}_{\{\tau_i\le t\}}$ and the corresponding filtration $\mathbb{H}^i=(\mathcal{H}^i_t)_{t\in\mathbb{R}_+}$ where $\mathcal{H}^i_t=\sigma(H^i_u:u\le t)$. We denote by $\mathbb{G}$ the joint filtration generated by default indicator processes $H^1,H^2,\ldots,H^n$, so that $\mathbb{G}=\mathbb{H}^1\vee\mathbb{H}^2\vee\ldots\vee\mathbb{H}^n$. It is clear that $\tau_{(1)}$ is a $\mathbb{G}$-stopping time as the infimum of $\mathbb{G}$-stopping times.

Finally, we write $H^{(1)}_t=\mathbb{1}_{\{\tau_{(1)}\le t\}}$ and $\mathbb{H}^{(1)}=(\mathcal{H}^{(1)}_t)_{t\in\mathbb{R}_+}$ where $\mathcal{H}^{(1)}_t=\sigma(H^{(1)}_u:u\le t)$.

Since we assume that $\mathbb{Q}^*(\tau_i=\tau_j)=0$ for any $i\neq j$, $i,j=1,2,\ldots,n$, we also have that

$$H^{(1)}_t=H^{(1)}_{t\wedge\tau_{(1)}}=\sum_{i=1}^n H^i_{t\wedge\tau_{(1)}}.$$

We make the standing assumption $\mathbb{Q}^*(\tau_{(1)}>T)=G_{(1)}(T)>0$.

For any $t\in[0,T]$, the event $\{\tau_{(1)}>t\}$ is an atom of the σ-field $\mathcal{G}_t$. Hence the following simple, but useful, result.

Lemma 3.1 *Let X be a $\mathbb{Q}^*$-integrable stochastic process. Then*

$$\mathbb{E}_{\mathbb{Q}^*}(X_t\,|\,\mathcal{G}_t)\mathbb{1}_{\{\tau_{(1)}>t\}}=\widetilde{X}(t)\mathbb{1}_{\{\tau_{(1)}>t\}}$$

where the function $\widetilde{X}:[0,T]\to\mathbb{R}$ is given by the formula

$$\widetilde{X}(t)=\frac{\mathbb{E}_{\mathbb{Q}^*}\big(X_t\mathbb{1}_{\{\tau_{(1)}>t\}}\big)}{G_{(1)}(t)},\quad \forall\,t\in[0,T].$$

If X is a $\mathbb{G}$-adapted, $\mathbb{Q}^*$-integrable stochastic process then

$$X_t=X_t\mathbb{1}_{\{\tau_{(1)}\le t\}}+\widetilde{X}(t)\mathbb{1}_{\{\tau_{(1)}>t\}},\quad \forall\,t\in[0,T].$$

By convention, the function $\widetilde{X}:[0,T]\to\mathbb{R}$ is called the *pre-default value* of X.

3.1 First-to-Default Intensities

In this section, we introduce the so-called *first-to-default intensities.* This natural concept will prove useful in the valuation and hedging of the first-to-default basket claims.

Definition 3.1 The function $\widetilde{\lambda}_i:\mathbb{R}_+\to\mathbb{R}_+$ given by

$$\widetilde{\lambda}_i(t)=\lim_{h\downarrow 0}\frac{1}{h}\,\mathbb{Q}^*(t<\tau_i\le t+h\,|\,\tau_{(1)}>t) \tag{30}$$

is called the *ith first-to-default intensity.* The function $\widetilde{\lambda}:\mathbb{R}_+\to\mathbb{R}_+$ given by

$$\widetilde{\lambda}(t)=\lim_{h\downarrow 0}\frac{1}{h}\,\mathbb{Q}^*(t<\tau_{(1)}\le t+h\,|\,\tau_{(1)}>t) \tag{31}$$

is called the *first-to-default intensity.*

Let us denote

$$\partial_i G(t,\dots,t) = \frac{\partial G(t_1,t_2,\dots,t_n)}{\partial t_i}_{\big| t_1=t_2=\ldots=t_n=t}.$$

Then we have the following elementary lemma summarizing the properties of the first-to-default intensity.

Lemma 3.2 *The ith first-to-default intensity $\widetilde{\lambda}_i$ satisfies, for $i=1,2,\dots,n$,*

$$\begin{aligned}\widetilde{\lambda}_i(t) &= \frac{\int_t^\infty \ldots \int_t^\infty f(u_1,\dots,u_{i-1},t,u_{i+1},\dots,u_n)\,du_1\ldots du_{i-1}du_{i+1}\ldots du_n}{G(t,\dots,t)}\\ &= \frac{\int_t^\infty \ldots \int_t^\infty F(du_1,\dots,du_{i-1},t,du_{i+1},\dots,du_n)}{G_{(1)}(t)} = -\frac{\partial_i G(t,\dots,t)}{G_{(1)}(t)}.\end{aligned}$$

The first-to-default intensity $\widetilde{\lambda}$ satisfies

$$\widetilde{\lambda}(t) = -\frac{1}{G_{(1)}(t)}\frac{dG_{(1)}(t)}{dt} = \frac{f_{(1)}(t)}{G_{(1)}(t)} \tag{32}$$

where $f_{(1)}(t)$ is the probability density function of $\tau_{(1)}$. The equality $\widetilde{\lambda}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)$ holds.

Proof. Clearly

$$\widetilde{\lambda}_i(t) = \lim_{h\downarrow 0}\frac{1}{h}\,\frac{\int_t^\infty \ldots \int_t^{t+h} \ldots \int_t^\infty f(u_1,\dots,u_i,\dots,u_n)\,du_1\ldots du_i\ldots du_n}{G(t,\dots,t)}$$

and thus the first asserted equality follows. The second equality follows directly from (31) and the definition of $G_{(1)}$. Finally, equality $\widetilde{\lambda}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)$ is equivalent to the equality

$$\lim_{h\downarrow 0}\frac{1}{h}\sum_{i=1}^n \mathbb{Q}^*(t<\tau_i\le t+h \,|\, \tau_{(1)}>t) = \lim_{h\downarrow 0}\frac{1}{h}\,\mathbb{Q}^*(t<\tau_{(1)}\le t+h \,|\, \tau_{(1)}>t),$$

which in turn is easy to establish. □

Remarks. The ith first-to-default intensity $\widetilde{\lambda}_i$ should not be confused with the (marginal) intensity function λ_i of τ_i, which is defined as

$$\lambda_i(t) = \frac{f_i(t)}{G_i(t)}, \quad \forall\, t\in\mathbb{R}_+,$$

where f_i is the (marginal) probability density function of τ_i, that is,

$$f_i(t) = \int_0^\infty \ldots \int_0^\infty f(u_1,\dots,u_{i-1},t,u_{i+1},\dots,u_n)\,du_1\ldots du_{i-1}du_{i+1}\ldots du_n,$$

and $G_i(t) = 1-F_i(t) = \int_t^\infty f_i(u)\,du$. Indeed, we have that $\widetilde{\lambda}_i \ne \lambda_i$, in general. However, if $\tau_1,\dots,\tau_n$ are mutually independent under $\mathbb{Q}^*$ then $\widetilde{\lambda}_i = \lambda_i$, that is, the first-to-default and marginal default intensities coincide.

It is also rather clear that the first-to-default intensity $\widetilde{\lambda}$ is not equal to the sum of marginal default intensities, that is, we have that $\widetilde{\lambda}(t) \ne \sum_{i=1}^n \lambda_i(t)$, in general.

3.2 First-to-Default Martingale Representation Theorem

We now state an integral representation theorem for a $\mathbb{G}$-martingale stopped at $\tau_{(1)}$ with respect to some basic processes. To this end, we define, for $i=1,2,\dots n$,

$$\widehat{M}^i_t = H^i_{t\wedge\tau_{(1)}} - \int_0^{t\wedge\tau_{(1)}} \widetilde{\lambda}_i(u)\,du, \quad \forall\, t\in\mathbb{R}_+. \tag{33}$$

Then we have the following *first-to-default martingale representation theorem.*

Proposition 3.1 *Consider the $\mathbb{G}$-martingale $\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y \,|\, \mathcal{G}_t)$, $t \in [0,T]$, where Y is a $\mathbb{Q}^*$-integrable random variable given by the expression*

$$Y = \sum_{i=1}^{n} Z_i(\tau_i) \mathbb{1}_{\{\tau_i \le T,\, \tau_i = \tau_{(1)}\}} + X \mathbb{1}_{\{\tau_{(1)} > T\}} \tag{34}$$

for some functions $Z_i : [0,T] \to \mathbb{R}$, $i = 1,2,\dots,n$ and some constant X. Then $\widehat{M}$ admits the following representation

$$\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Y) + \sum_{i=1}^{n} \int_{]0,t]} h_i(u)\, d\widehat{M}^i_u \tag{35}$$

where the functions h_i, $i = 1,2,\dots,n$ are given by

$$h_i(t) = Z_i(t) - \widehat{M}_{t-} = Z_i(t) - \widetilde{M}(t-), \quad \forall\, t \in [0,T], \tag{36}$$

where $\widetilde{M}$ is the unique function such that $\widehat{M}_t \mathbb{1}_{\{\tau_{(1)} > t\}} = \widetilde{M}(t) \mathbb{1}_{\{\tau_{(1)} > t\}}$ for every $t \in [0,T]$. The function $\widetilde{M}$ satisfies $\widetilde{M}_0 = \mathbb{E}_{\mathbb{Q}^*}(Y)$ and

$$d\widetilde{M}(t) = \sum_{i=1}^{n} \widetilde{\lambda}_i(t) \big(\widetilde{M}(t) - Z_i(t)\big)\, dt. \tag{37}$$

More explicitly

$$\widetilde{M}(t) = \mathbb{E}_{\mathbb{Q}^*}(Y) \exp\left\{ \int_0^t \widetilde{\lambda}(s)\, ds \right\} - \int_0^t \sum_{i=1}^{n} \widetilde{\lambda}_i(s) Z_i(s) \exp\left\{ \int_s^t \widetilde{\lambda}(u)\, du \right\} ds.$$

Proof. To alleviate notation, we provide the proof of this result in a bivariate setting only. In that case, $\tau_{(1)} = \tau_1 \wedge \tau_2$ and $\mathcal{G}_t = \mathcal{H}^1_t \vee \mathcal{H}^2_t$. We start by noting that

$$\widehat{M}_t = \mathbb{E}_{\mathbb{Q}^*}(Z_1(\tau_1) \mathbb{1}_{\{\tau_1 \le T,\, \tau_2 > \tau_1\}} \,|\, \mathcal{G}_t) + \mathbb{E}_{\mathbb{Q}^*}(Z_2(\tau_2) \mathbb{1}_{\{\tau_2 \le T,\, \tau_1 > \tau_2\}} \,|\, \mathcal{G}_t) + \mathbb{E}_{\mathbb{Q}^*}(X \mathbb{1}_{\{\tau_{(1)} > T\}} \,|\, \mathcal{G}_t),$$

and thus (see Lemma 3.1)

$$\mathbb{1}_{\{\tau_{(1)} > t\}} \widehat{M}_t = \mathbb{1}_{\{\tau_{(1)} > t\}} \widetilde{M}(t) = \mathbb{1}_{\{\tau_{(1)} > t\}} \sum_{i=1}^{3} \widetilde{Y}^i(t)$$

where the auxiliary functions $\widetilde{Y}^i : [0,T] \to \mathbb{R}$, $i = 1,2,3$, are given by

$$\widetilde{Y}^1(t) = \frac{\int_t^T du Z_1(u) \int_u^\infty dv f(u,v)}{G_{(1)}(t)}, \ \widetilde{Y}^2(t) = \frac{\int_t^T dv Z_2(v) \int_v^\infty du f(u,v)}{G_{(1)}(t)}, \ \widetilde{Y}^3(t) = \frac{X G_{(1)}(T)}{G_{(1)}(t)}.$$

By elementary calculations and using Lemma 3.2, we obtain

$$\begin{aligned} \frac{d\widetilde{Y}^1(t)}{dt} &= -\frac{Z_1(t) \int_t^\infty dv f(t,v)}{G_{(1)}(t)} - \frac{\int_t^T du Z_1(u) \int_u^\infty dv f(u,v)}{G^2_{(1)}(t)} \frac{dG_{(1)}(t)}{dt} \\ &= -Z_1(t) \frac{\int_t^\infty dv f(t,v)}{G_{(1)}(t)} - \frac{\widetilde{Y}^1(t)}{G_{(1)}(t)} \frac{dG_{(1)}(t)}{dt} \\ &= -Z_1(t) \widetilde{\lambda}_1(t) + \widetilde{Y}^1(t)(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)), \end{aligned} \tag{38}$$

and thus, by symmetry,

$$\frac{d\widetilde{Y}^2(t)}{dt} = -Z_2(t) \widetilde{\lambda}_2(t) + \widetilde{Y}^2(t)(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)). \tag{39}$$

Moreover

$$\frac{d\widetilde{Y}^3(t)}{dt} = -\frac{XG_{(1)}(T)}{G_{(1)}^2(t)}\,\frac{dG_{(1)}(t)}{dt} = \widetilde{Y}^3(t)(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)). \tag{40}$$

Hence, recalling that $\widetilde{M}(t) = \sum_{i=1}^3 \widetilde{Y}^i(t)$, we obtain from (38)-(40)

$$d\widetilde{M}(t) = -\widetilde{\lambda}_1(t)\big(Z_1(t) - \widetilde{M}(t)\big)\,dt - \widetilde{\lambda}_2(t)\big(Z_2(t) - \widetilde{M}(t)\big)\,dt \tag{41}$$

Consequently, since the function $\widetilde{M}$ is continuous, we have, on the event $\{\tau_{(1)} > t\}$,

$$d\widehat{M}_t = -\widetilde{\lambda}_1(t)\big(Z_1(t) - \widehat{M}_{t-}\big)\,dt - \widetilde{\lambda}_2(t)\big(Z_2(t) - \widehat{M}_{t-}\big)\,dt.$$

We shall now check that both sides of equality (35) coincide at time $\tau_{(1)}$ on the event $\{\tau_{(1)} \le T\}$. To this end, we observe that we have, on the event $\{\tau_{(1)} \le T\}$,

$$\widehat{M}_{\tau_{(1)}} = Z_1(\tau_1)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + Z_2(\tau_2)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}},$$

whereas the right-hand side in (35) is equal to

$$\begin{aligned}
&\widehat{M}_0 + \int_{]0,\tau_{(1)}[} h_1(u)\,d\widehat{M}_u^1 + \int_{]0,\tau_{(1)}[} h_2(u)\,d\widehat{M}_u^2 \\
&\quad + \mathbb{1}_{\{\tau_{(1)}=\tau_1\}} \int_{[\tau_{(1)}]} h_1(u)\,dH_u^1 + \mathbb{1}_{\{\tau_{(1)}=\tau_2\}} \int_{[\tau_{(1)}]} h_2(u)\,dH_u^2 \\
&\quad = \widetilde{M}(\tau_{(1)}-) + \big(Z_1(\tau_1) - \widetilde{M}(\tau_{(1)}-)\big)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + \big(Z_2(\tau_2) - \widetilde{M}(\tau_{(1)}-)\big)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}} \\
&\quad = Z_1(\tau_1)\mathbb{1}_{\{\tau_{(1)}=\tau_1\}} + Z_2(\tau_2)\mathbb{1}_{\{\tau_{(1)}=\tau_2\}}
\end{aligned}$$

as $\widetilde{M}(\tau_{(1)}-) = \widehat{M}_{\tau_{(1)}-}$. Since the processes on both sides of equality (35) are stopped at $\tau_{(1)}$, we conclude that equality (35) is valid for every $t \in [0,T]$. Formula (37) was also established in the proof (see formula (41)). □

The next result shows that the basic processes $\widehat{M}^i$ are in fact $\mathbb{G}$-martingales. They will be referred to as the *basic first-to-default martingales.*

Corollary 3.1 *For each $i = 1,2,\dots,n$, the process $\widehat{M}^i$ given by the formula* (33) *is a $\mathbb{G}$-martingale stopped at $\tau_{(1)}$.*

Proof. Let us fix $k \in \{1,2,\dots,n\}$. It is clear that the process $\widehat{M}^k$ is stopped at $\tau_{(1)}$. Let $\widetilde{M}^k(t) = \int_0^t \widetilde{\lambda}_i(u)\,du$ be the unique function such that

$$\mathbb{1}_{\{\tau_{(1)}>t\}}\widehat{M}_t^i = \mathbb{1}_{\{\tau_{(1)}>t\}}\widetilde{M}^k(t), \quad \forall\, t \in [0,T].$$

Let us take $h_k(t) = 1$ and $h_i(t) = 0$ for any $i \ne k$ in formula (35), or equivalently, let us set

$$Z_k(t) = 1 + \widetilde{M}^k(t), \quad Z_i(t) = \widetilde{M}^k(t),\ i \ne k,$$

in the definition (34) of the random variable Y. Finally, the constant X in (34) is chosen in such a way that the random variable Y satisfies $\mathbb{E}_{\mathbb{Q}^*}(Y) = \widehat{M}_0^k$. Then we may deduce from (35) that $\widehat{M}^k = \widehat{M}$, and thus $\widehat{M}^k$ is manifestly a $\mathbb{G}$-martingale. □

3.3 Price Dynamics of the ith CDS

As traded assets in our model, we take the constant savings account and a family of single-name CDSs with default protections δ_i and rates κ_i. For convenience, we assume that the CDSs have the same maturity T, but this assumption can be easily relaxed. The ith traded CDS is formally defined by its dividend process

$$D_t^i = \int_{(0,t]} \delta_i(u)\, dH_u^i - \kappa_i(t \wedge \tau_i), \quad \forall\, t \in [0,T].$$

Consequently, the price at time t of the ith CDS equals

$$S_t^i(\kappa_i) = \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t<\tau_i\le T\}}\delta_i(\tau_i)\,|\,\mathcal{G}_t\big) - \kappa_i\,\mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t<\tau_i\}}\big((\tau_i\wedge T)-t\big)\,\big|\,\mathcal{G}_t\big). \tag{42}$$

To replicate a first-to-default claim, we only need to examine the dynamics of each CDS on the interval $[0, \tau_{(1)} \wedge T]$. The following lemma will prove useful in this regard.

Lemma 3.3 *We have, on the event $\{\tau_{(1)} > t\}$,*

$$S_t^i(\kappa_i) = \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}}\delta_i(\tau_{(1)}) + \sum_{j\neq i}\mathbb{1}_{\{t<\tau_{(1)}=\tau_j\le T\}}S^i_{\tau_{(1)}}(\kappa_i) - \kappa_i\mathbb{1}_{\{t<\tau_{(1)}\}}(\tau_{(1)}\wedge T - t)\,\Big|\,\mathcal{G}_t\Big).$$

Proof. We first note that the price $S_t^i(\kappa_i)$ can be represented as follows, on the event $\{\tau_{(1)} > t\}$,

$$\begin{aligned} S_t^i(\kappa_i) \;=\; &\mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}}\delta_i(\tau_{(1)}) + \sum_{j\neq i}\mathbb{1}_{\{t<\tau_{(1)}=\tau_j\le T\}}\big(\mathbb{1}_{\{\tau_{(1)}<\tau_i\le T\}}\delta_i(\tau_i\wedge T) \\ &- \kappa_i\mathbb{1}_{\{\tau_{(1)}<\tau_i\}}(\tau_i-\tau_{(1)})\big)\,\Big|\,\mathcal{G}_t\Big) - \kappa_i\,\mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t<\tau_{(1)}\}}(\tau_{(1)}\wedge T - t)\,\big|\,\mathcal{G}_t\big). \end{aligned}$$

By conditioning first on the σ-field $\mathcal{G}_{\tau_{(1)}}$, we obtain the claimed formula. □

Representation established in Lemma 3.3 is by no means surprising; it merely shows that in order to compute the price of a CDS prior to the first default, we can either do the computations in a single step, by considering the cash flows occurring on $]t, \tau_i \wedge T]$, or we can compute first the price of the contract at time $\tau_{(1)} \wedge T$, and subsequently value all cash flows occurring on $]t, \tau_{(1)} \wedge T]$. However, it also shows that we can consider from now on not the original ith CDS but the associated CDS contract with random maturity $\tau_i \wedge T$.

Similarly as in Section 2.3, we write $S_t^i(\kappa_i) = \mathbb{1}_{\{t<\tau_{(1)}\}}\widetilde{S}_t^i(\kappa_i)$ where the pre-default price $\widetilde{S}_t^i(\kappa_i)$ satisfies

$$\widetilde{S}_t^i(\kappa_i) = \widetilde{\delta}^i(t,T) - \kappa_i\widetilde{A}^i(t,T) \tag{43}$$

where $\widetilde{\delta}^i(t,T)$ and $\kappa\widetilde{A}^i(t,T)$ are pre-default values of the protection leg and the fee leg respectively.

For any $j \neq i$, we define a function $S^i_{t|j}(\kappa_i) : [0,T] \to \mathbb{R}$, which represents the price of the ith CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$. Formally, this quantity is defined as the unique function satisfying

$$\mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}}S^i_{\tau_{(1)}|j}(\kappa_i) = \mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}}S^i_{\tau_{(1)}}(\kappa_i)$$

so that

$$\mathbb{1}_{\{\tau_{(1)}\le T\}}S^i_{\tau_{(1)}}(\kappa_i) = \sum_{j\neq i}\mathbb{1}_{\{\tau_{(1)}=\tau_j\le T\}}S^i_{\tau_{(1)}|j}(\kappa_i).$$

Let us examine the case of two names. Then the function $S^1_{t|2}(\kappa_1)$, $t \in [0,T]$, represents the price of the first CDS at time t on the event $\{\tau_{(1)} = \tau_2 = t\}$.

Lemma 3.4 *The function $S^1_{v|2}(\kappa_1)$, $v \in [0,T]$, equals*

$$S^1_{v|2}(\kappa_1) = \frac{\int_v^T \delta_1(u) f(u,v)du}{\int_v^\infty f(u,v)\,du} - \kappa_1\frac{\int_v^T du\int_u^\infty dz f(z,v)}{\int_v^\infty f(u,v)\,du}. \tag{44}$$

Proof. Note that the conditional c.d.f. of τ_1 given that $\tau_1 > \tau_2 = v$ equals

$$\mathbb{Q}^*(\tau_1 \le u \,|\, \tau_1 > \tau_2 = v) = F_{\tau_1|\tau_1>\tau_2=v}(u) = \frac{\int_v^u f(z,v)\,dz}{\int_v^\infty f(z,v)\,dz}, \quad \forall\, u \in [v,\infty],$$

so that the conditional tail equals

$$G_{\tau_1|\tau_1>\tau_2=v}(u) = 1 - F_{\tau_1|\tau_1>\tau_2=v}(u) = \frac{\int_u^\infty f(z,v)\,dz}{\int_v^\infty f(z,v)\,dz}, \quad \forall\, u \in [v,\infty]. \tag{45}$$

Let J be the right-hand side of (44). It is clear that

$$J = -\int_v^T \delta_1(u)\,dG_{\tau_1|\tau_1>\tau_2=v}(u) - \kappa_1 \int_v^T G_{\tau_1|\tau_1>\tau_2=v}(u)\,du.$$

Combining Lemma 2.1 with the fact that $S^1_{\tau_{(1)}}(\kappa_i)$ is equal to the conditional expectation with respect to σ-field $\mathcal{G}_{\tau_{(1)}}$ of the cash flows of the ith CDS on $]\tau_{(1)} \vee \tau_i, \tau_i \wedge T]$, we conclude that J coincides with $S^1_{v|2}(\kappa_1)$, the price of the first CDS on the event $\{\tau_{(1)} = \tau_2 = v\}$. □

The following result extends Lemma 2.2.

Lemma 3.5 *The dynamics of the pre-default price $\widetilde{S}^i_t(\kappa_i)$ are*

$$d\widetilde{S}^i_t(\kappa_i) = \widetilde{\lambda}(t)\widetilde{S}^i_t(\kappa_i)\,dt + \Big(\kappa_i - \delta_i(t)\widetilde{\lambda}_i(t) - \sum_{j\neq i}^n S^i_{t|j}(\kappa_i)\widetilde{\lambda}_i(t)\Big)dt \tag{46}$$

where $\widetilde{\lambda}(t) = \sum_{i=1}^n \widetilde{\lambda}_i(t)$, or equivalently,

$$d\widetilde{S}^i_t(\kappa_i) = \widetilde{\lambda}_i(t)\big(\widetilde{S}^i_t(\kappa_i) - \delta_i(t)\big)\,dt + \sum_{j\neq i} \widetilde{\lambda}_j(t)\big(\widetilde{S}^i_t(\kappa_i) - S^i_{t|j}(\kappa_i)\big)\,dt + \kappa_i dt. \tag{47}$$

The cumulative price of the ith CDS stopped at $\tau_{(1)}$ satisfies

$$\widehat{S}^i_t(\kappa_i) = S^i_t(\kappa_i) + \int_0^t \delta_i(u)\,dH^i_{u\wedge\tau_{(1)}} + \sum_{j\neq i}\int_0^t S^i_{u|j}(\kappa_i)\,dH^j_{u\wedge\tau_{(1)}} - \kappa_i(\tau_{(1)}\wedge t), \tag{48}$$

and thus

$$d\widehat{S}^i_t(\kappa_i) = \big(\delta_i(t) - \widetilde{S}^i_{t-}(\kappa_i)\big)\,d\widehat{M}^i_t + \sum_{j\neq i}\big(S^i_{t|j}(\kappa_i) - \widetilde{S}^i_{t-}(\kappa_i)\big)\,d\widehat{M}^j_t. \tag{49}$$

Proof. We shall consider the case $n = 2$. Using the formula derived in Lemma 3.3, we obtain

$$\widetilde{\delta}^1(t,T) = \frac{\int_t^T du\,\delta_1(u)\int_u^\infty dv f(u,v)}{G_{(1)}(t)} + \frac{\int_t^T dv\,S^1_{v|2}(\kappa_1)\int_v^\infty du f(u,v)}{G_{(1)}(t)}. \tag{50}$$

By adapting equality (38), we get

$$d\widetilde{\delta}^1(t,T) = \Big((\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t))\widetilde{g}_1(t) - \widetilde{\lambda}_1(t)\delta_1(t) - \widetilde{\lambda}_2(t)S^1_{t|2}(\kappa_1)\Big)dt. \tag{51}$$

To establish (46)-(47), we need also to examine the fee leg. Its price equals

$$\mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau_{(1)}\}}\kappa_1\big((\tau_{(1)}\wedge T) - t\big)\,\Big|\,\mathcal{G}_t\Big) = \mathbb{1}_{\{t<\tau_{(1)}\}}\kappa_1\widetilde{A}^i(t,T),$$

To evaluate the conditional expectation above, it suffices to use the c.d.f. $F_{(1)}$ of the random time $\tau_{(1)}$. As in Section 2.2 (see the proof of Lemma 2.1), we obtain

$$\widetilde{A}^i(t,T) = \frac{1}{G_{(1)}(t)}\int_t^T G_{(1)}(u)\,du, \tag{52}$$

and thus

$$d\widetilde{A}^i(t,T) = \big(1 + (\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t))\widetilde{A}^i(t,T)\big)\,dt.$$

Since $\widetilde{S}^1_t(\kappa_1) = \widetilde{\delta}^i(t,T) - \kappa_i\widetilde{A}^i(t,T)$, the formulae (46)-(47) follow. Formula (48) is rather clear. Finally, dynamics (49) can be easily deduced from (47) and (48) □

3.4 Risk-Neutral Valuation of a First-to-Default Claim

In this section, we shall analyze the risk-neutral valuation of first-to-default claims on a basket of n credit names.

Definition 3.2 A *first-to-default claim* (*FTDC*) with maturity T is a defaultable claim $(X, A, Z, \tau_{(1)})$ where X is a constant amount payable at maturity if no default occurs, $A : [0, T] \to \mathbb{R}$ with $A_0 = 0$ is a function of bounded variation representing the dividend stream up to $\tau_{(1)}$, and $Z = (Z_1, Z_2, \ldots, Z_n)$ is the vector of functions $Z_i : [0, T] \to \mathbb{R}$ where $Z_i(\tau_{(1)})$ specifies the recovery received at time $\tau_{(1)}$ if the ith name is the first defaulted name, that is, on the event $\{\tau_i = \tau_{(1)} \le T\}$.

We define the *risk-neutral value* π of an FTDC by setting

$$\pi_t = \sum_{i=1}^n \mathbb{E}_{\mathbb{Q}^*}\Big(Z_i(\tau_i)\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}} + \mathbb{1}_{\{t<\tau_{(1)}\}}\int_t^T (1 - H^{(1)}_u)\, dA(u) + X\mathbb{1}_{\{\tau_{(1)}>T\}}\,\Big|\, \mathcal{G}_t\Big),$$

and the *risk-neutral cumulative value* $\widehat{\pi}$ of an FTDC by the formula

$$\begin{aligned}\widehat{\pi}_t \;&=\; \sum_{i=1}^n \mathbb{E}_{\mathbb{Q}^*}\Big(Z_i(\tau_i)\mathbb{1}_{\{t<\tau_{(1)}=\tau_i\le T\}} + \mathbb{1}_{\{t<\tau_{(1)}\}}\int_t^T (1 - H^{(1)}_u)\, dA(u)\,\Big|\, \mathcal{G}_t\Big)\\ &\quad + \mathbb{E}_{\mathbb{Q}^*}(X\mathbb{1}_{\{\tau_{(1)}>T\}}|\mathcal{G}_t) + \sum_{i=1}^n \int_0^t Z_i(u)\, dH^i_{u\wedge\tau_{(1)}} + \int_0^t (1 - H^{(1)}_u)\, dA(u)\end{aligned}$$

where the last two terms represent the past dividends. Let us stress that the risk-neutral valuation of an FTDC will be later supported by replication arguments (see Theorem 3.1), and thus risk-neutral value π of an FTDC will be shown to be its replication price.

By the *pre-default risk-neutral value* associated with a $\mathbb{G}$-adapted process π, we mean the function $\widetilde{\pi}$ such that $\pi_t\mathbb{1}_{\{\tau_{(1)}>t\}} = \widetilde{\pi}(t)\mathbb{1}_{\{\tau_{(1)}>t\}}$ for every $t \in [0, T]$. Direct calculations lead to the following result, which can also be deduced from Proposition 3.1.

Lemma 3.6 *The pre-default risk-neutral value of an FTDC equals*

$$\widetilde{\pi}(t) = \sum_{i=1}^n \frac{\Psi_i(t)}{G_{(1)}(t)} + \frac{1}{G_{(1)}(t)}\int_t^T G_{(1)}(u)\, dA(u) + X\frac{G_{(1)}(T)}{G_{(1)}(t)} \tag{53}$$

where

$$\Psi_i(t) = \int_{u_i=t}^T \int_{u_1=u_i}^\infty \ldots \int_{u_{i-1}=u_i}^\infty \int_{u_{i+1}=u_i}^\infty \ldots \int_{u_n=u_i}^\infty Z_i(u_i)\, F(du_1, \ldots, du_{i-1}, du_i, du_{i+1}, \ldots, du_n).$$

The next result extends Proposition 2.2 to the multi-name set-up. Its proof is similar to the proof of Lemma 3.5, and thus it is omitted.

Proposition 3.2 *The pre-default risk-neutral value of an FTDC satisfies*

$$d\widetilde{\pi}(t) = \sum_{i=1} \widetilde{\lambda}_i(t)\big(\widetilde{\pi}(t) - Z_i(t)\big)\, dt - dA(t). \tag{54}$$

Moreover, the risk-neutral value of an FTDC satisfies

$$d\pi_t = \sum_{i=1}^n \,(Z_i(t) - \widetilde{\pi}(t-))\; d\widehat{M}^i_u - dA(\tau_{(1)} \wedge t), \tag{55}$$

and the risk-neutral cumulative value $\widehat{\pi}$ *of an FTDC satisfies*

$$d\widehat{\pi}_t = \sum_{i=1}^n \,(Z_i(t) - \widetilde{\pi}(t-))\; d\widehat{M}^i_u. \tag{56}$$

3.5 Dynamic Replication of a First-to-Default Claim

Let $B = 1$ and single-name CDSs with prices $S^1(\kappa_1), \dots, S^n(\kappa_n)$ be traded assets. We say that a $\mathbb{G}$-predictable process $\phi = (\phi^0, \phi^1, \dots, \phi^n)$ and a function C of finite variation with $C(0) = 0$ define a *self-financing strategy with dividend stream* C if the wealth process $V(\phi, C)$, defined as

$$V_t(\phi, C) = \phi_t^0 + \sum_{i=1}^n \phi_t^i S_t^i(\kappa_i), \tag{57}$$

satisfies

$$dV_t(\phi, C) = \sum_{i=1}^n \phi_t^i \big(dS_t^i(\kappa_i) + dD_t^i\big) - dC(t) = \sum_{i=1}^n \phi_t^i \, d\widehat{S}_t^i(\kappa_i) - dC(t) \tag{58}$$

where $S^i(\kappa_i)$ ($\widehat{S}^i(\kappa_i)$, respectively) is the price (cumulative price, respectively) of the ith CDS.

Definition 3.3 We say that a trading strategy (ϕ, C) *replicates an FTDC* $(X, A, Z, \tau_{(1)})$ if:
(i) the processes $\phi = (\phi^0, \phi^1, \dots, \phi^n)$ and $V(\phi, C)$ are stopped at $\tau_{(1)} \wedge T$,
(ii) $C(\tau_{(1)} \wedge t) = A(\tau_{(1)} \wedge t)$ for every $t \in [0, T]$,
(iii) the equality $V_{\tau_{(1)} \wedge T}(\phi, C) = Y$ holds, where the random variable Y equals

$$Y = X \mathbb{1}_{\{\tau_{(1)} > T\}} + \sum_{i=1}^n Z_i(\tau_{(1)}) \mathbb{1}_{\{\tau_i = \tau_{(1)} \le T\}}. \tag{59}$$

We are now in a position to extend Theorem 2.1 to the case of a first-to-default claim on a basket of n credit names.

Theorem 3.1 *Assume that* $\det N(t) \neq 0$ *for every* $t \in [0, T]$*, where*

$$N(t) = \begin{bmatrix} \delta_1(t) - \widetilde{S}_t^1(\kappa_1) & S_{t|1}^2(\kappa_2) - \widetilde{S}_t^2(\kappa_2) & . & S_{t|1}^n(\kappa_n) - \widetilde{S}_t^n(\kappa_n) \\ S_{t|2}^1(\kappa_1) - \widetilde{S}_t^1(\kappa_1) & \delta_2(t) - \widetilde{S}_t^2(\kappa_2) & . & S_{t|2}^n(\kappa_n) - \widetilde{S}_t^n(\kappa_n) \\ \dots & . & . & . \\ S_{t|n}^1(\kappa_1) - \widetilde{S}_t^1(\kappa_1) & S_{t|n}^2(\kappa_1) - \widetilde{S}_t^2(\kappa_1) & . & \delta_n(t) - \widetilde{S}_t^n(\kappa_n) \end{bmatrix}$$

Let $\widetilde{\phi}(t) = (\widetilde{\phi}_1(t), \widetilde{\phi}_2(t), \dots, \widetilde{\phi}_n(t))$ *be the unique solution to the equation* $N(t)\widetilde{\phi}(t) = h(t)$ *where* $h(t) = (h_1(t), h_2(t), \dots, h_n(t))$ *with* $h_i(t) = Z_i(t) - \widetilde{\pi}(t-)$ *and where* $\widetilde{\pi}$ *is given by Lemma 3.6. More explicitly, the functions* $\widetilde{\phi}_1, \widetilde{\phi}_2, \dots, \widetilde{\phi}_n$ *satisfy, for* $t \in [0, T]$ *and* $i = 1, 2, \dots, n$*,*

$$\widetilde{\phi}_i(t)\big(\delta_i(t) - \widetilde{S}_t^i(\kappa_i)\big) + \sum_{j \neq i} \widetilde{\phi}_j(t)\big(S_{t|i}^j(\kappa_j) - \widetilde{S}_t^j(\kappa_j)\big) = Z_i(t) - \widetilde{\pi}(t-). \tag{60}$$

Let us set $\phi_t^i = \widetilde{\phi}_i(\tau_{(1)} \wedge t)$ *for* $i = 1, 2, \dots, n$ *and let*

$$\phi_t^0 = V_t(\phi, A) - \sum_{i=1}^n \phi_t^i S_t^i(\kappa_i), \quad \forall\, t \in [0, T], \tag{61}$$

where the process $V(\phi, A)$ *is given by the formula*

$$V_t(\phi, A) = \widetilde{\pi}(0) + \sum_{i=1}^n \int_{]0, \tau_{(1)} \wedge t]} \widetilde{\phi}_i(u) \, d\widehat{S}_u^i(\kappa_i) - A(\tau_{(1)} \wedge t). \tag{62}$$

Then the trading strategy (ϕ, A) *replicates an FTDC* $(X, A, Z, \tau_{(1)})$*.*

Proof. The proof is based on similar arguments as the proof of Theorem 2.1. It suffices to check that under the assumption of the theorem, for a trading strategy (ϕ, A) stopped at $\tau_{(1)}$, we obtain from (58) and (49) that

$$dV_t(\phi, A) = \sum_{i=1}^n \phi_t^i \Big(\big(\delta_i(t) - \widetilde{S}_{t-}^i(\kappa_i)\big) \, d\widehat{M}_t^i + \sum_{j \neq i} \big(S_{t|j}^i(\kappa_i) - \widetilde{S}_{t-}^i(\kappa_i)\big) \, d\widehat{M}_t^j \Big) - dA(\tau_{(1)} \wedge t).$$

For $\phi^i_t = \widetilde{\phi}_i(\tau_{(1)} \wedge t)$, where the functions $\widetilde{\phi}_1, \widetilde{\phi}_2, \ldots, \widetilde{\phi}_n$ solve (60), we thus obtain

$$dV_t(\phi, A) = \sum_{i=1}^{n} (Z_i(t) - \widetilde{\pi}(t-))\, d\widehat{M}^i_t - dA(\tau_{(1)} \wedge t).$$

By comparing the last formula with (55), we conclude that if, in addition, $V_0(\phi, A) = \pi_0 = \widetilde{\pi}_0$ and ϕ^0 is given by (61), then the strategy (ϕ, A) replicates an FTDC $(X, A, Z, \tau_{(1)})$. □

3.6 Conditional Default Distributions

In the case of first-to-default claims, it was enough to consider the unconditional distribution of default times. As expected, in order to deal with a general basket defaultable claim, we need to analyze conditional distributions of default times. It is possible to follow the approach presented in preceding sections, and to explicitly derive the dynamics of all processes of interest on the time interval $[0, T]$. However, since we deal here with a simple model of joint defaults, it suffices to make a non-restrictive assumption that we work on the canonical space $\Omega = \mathbb{R}^n$, and to use simple arguments based on conditioning with respect to past defaults.

Suppose that k names out of a total of n names have already defaulted. To introduce a convenient notation, we adopt the convention that the $n - k$ non-defaulted names are in their original order $j_1 < \ldots < j_{n-k}$, and the k defaulted names $i_1, \ldots, i_k$ are ordered in such a way that $u_1 < \ldots < u_k$. For the sake of brevity, we write $D_k = \{\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\}$ to denote the *information structure* of the past k defaults.

Definition 3.4 The *joint conditional distribution function* of default times $\tau_{j_1}, \ldots, \tau_{j_{n-k}}$ equals, for every $t_1, \ldots, t_{n-k} > u_k$,

$$F(t_1, \ldots, t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k) = \mathbb{Q}^*\big(\tau_{j_1} \le t_1, \ldots, \tau_{j_{n-k}} \le t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\big).$$

The *joint conditional survival function* of default times $\tau_{j_1}, \ldots, \tau_{j_{n-k}}$ is given by the expression

$$G(t_1, \ldots, t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k) = \mathbb{Q}^*\big(\tau_{j_1} > t_1, \ldots, \tau_{j_{n-k}} > t_{n-k} \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\big)$$

for every $t_1, \ldots, t_{n-k} > u_k$.

As expected, the conditional first-to-default intensities are defined using the joint conditional distributions, instead of the joint unconditional distribution. We write $G_{(1)}(t \,|D_k) = G(t, \ldots, t \,|D_k)$.

Definition 3.5 Given the event D_k, for any $j_l \in \{j_1, \ldots, j_{n-k}\}$, the *conditional first-to-default intensity* of a surviving name j_l is denoted by $\widetilde{\lambda}_{j_l}(t \,|D_k) = \widetilde{\lambda}_{j_l}(t \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k)$, and is given by the formula

$$\widetilde{\lambda}_{j_l}(t \,|D_k) = \frac{\int_t^\infty \int_t^\infty \ldots \int_t^\infty dF(t_1, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_{n-k} | D_k)}{G_{(1)}(t \,|D_k)}, \quad \forall\, t \in [u_k, T].$$

In Section 3.3, we introduced the processes $S^i_{t|j}(\kappa_j)$ representing the value of the ith CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$. According to the notation introduced above, we thus dealt with the conditional value of the ith CDS with respect to $D_1 = \{\tau_j = t\}$. It is clear that to value a CDS for each surviving name we can proceed as prior to the first default, except that now we should use the conditional distribution

$$F(t_1, \ldots, t_{n-1} \,|\, D_1) = F(t_1, \ldots, t_{n-1} \,|\, \tau_j = j), \quad \forall\, t_1, \ldots, t_{n-1} \in [t, T],$$

rather than the unconditional distribution $F(t_1, \ldots, t_n)$ employed in Proposition 3.6. The same argument can be applied to any default event D_k. The corresponding conditional version of Proposition 3.6 is rather easy to formulate and prove, and thus we feel there is no need to provide an explicit conditional pricing formula here.

The conditional first-to-default intensities introduced in Definition 3.5 will allow us to construct the conditional first-to-default martingales in a similar way as we defined the first-to-default martingales M^i associated with the first-to-default intensities $\underset{\sim}{\lambda}_i$. However, since any name can default at any time, we need to introduce an entire family of conditional martingales, whose compensators are based on intensities conditioned on the information structure of past defaults.

Definition 3.6 Given the default event $D_k = \{\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\}$, for each surviving name $j_l \in \{j_1, \ldots, j_{n-k}\}$, we define the *basic conditional first-to-default martingale* $\widehat{M}^{j_l}_{t|D_k}$ by setting

$$\widehat{M}^{j_l}_{t|D_k} = H^{j_l}_{t\wedge\tau_{(k+1)}} - \int_{u_k}^{t} \mathbb{1}_{\{u<\tau_{(k+1)}\}} \widetilde{\lambda}_{j_l}(u\,|D_k)\, du, \quad \forall\, t \in [u_k, T]. \tag{63}$$

The process $\widehat{M}^{j_l}_{t|D_k}$, $t \in [u_k, T]$, is a martingale under the *conditioned probability measure* $\mathbb{Q}^*|D_k$, that is, the probability measure $\mathbb{Q}^*$ conditioned on the event D_k, and with respect to the filtration generated by default processes of the surviving names, that is, the filtration $\mathcal{G}^{D_k}_t := \mathcal{H}^{j_1}_t \vee \ldots \vee \mathcal{H}^{j_{n-k}}_t$ for $t \in [u_k, T]$.

Since we condition on the event D_k, we have $\tau_{(k+1)} = \tau_{j_1} \wedge \tau_{j_2} \wedge \ldots \wedge \tau_{j_{n-k}}$, so that $\tau_{(k+1)}$ is the first default for all surviving names. Formula (63) is thus a rather straightforward generalization of formula (33). In particular, for $k = 0$ we obtain $\widehat{M}^i_{t|D_0} = \widehat{M}^i_t$, $t \in [0, T]$, for any $i = 1, 2, \ldots, n$. The martingale property of the process $\widehat{M}^{j_l}_{t|D_k}$, stated in Definition 3.6, follows from Proposition 3.3 (it can also be seen as a conditional version of Corollary 3.1).

We are in the position to state the conditional version of the first-to-default martingale representation theorem of Section 3.2. Formally, this result is nothing else than a restatement of the martingale representation formula of Proposition 3.1 in terms of conditional first-to-default intensities and conditional first-to-default martingales.

Let us fix an event D_k write $\mathbb{G}^{D_k} = \mathbb{H}^{j_1} \vee \ldots \vee \mathbb{H}^{j_{n-k}}$.

Proposition 3.3 *Let Y be a random variable given by the formula*

$$Y = \sum_{l=1}^{n-k} Z_{j_l|D_k}(\tau_{j_l}) \mathbb{1}_{\{\tau_{j_l} \le T,\, \tau_{j_l} = \tau_{(k+1)}\}} + X \mathbb{1}_{\{\tau_{(k+1)} > T\}} \tag{64}$$

for some functions $Z_{j_l|D_k} : [u_k, T] \to \mathbb{R}$, $l = 1, 2, \ldots, n-k$, and some constant X (possibly dependent on D_k). Let us define

$$\widehat{M}_{t|D_k} = \mathbb{E}_{\mathbb{Q}^*|D_k}(Y \,|\, \mathcal{G}^{D_k}_t), \quad \forall\, t \in [u_k, T]. \tag{65}$$

Then $\widehat{M}_{t|D_k}$, $t \in [u_k, T]$, is a $\mathbb{G}^{D_k}$-martingale with respect to the conditioned probability measure $\mathbb{Q}^*|D_k$ and it admits the following representation, for $t \in [u_k, T]$,

$$\widehat{M}_{t|D_k} = \widehat{M}_{0|D_k} + \sum_{l=1}^{n-k} \int_{]u_k, t]} h_{j_l}(u|D_k)\, d\widehat{M}^{j_l}_{u|D_k}$$

where the processes h_{j_l} are given by

$$h_{j_l}(t\,|D_k) = Z_{j_l|D_k}(t) - \widehat{M}_{t-|D_k}, \quad \forall\, t \in [u_k, T].$$

Proof. The proof relies on a direct extension of arguments used in the proof of Proposition 3.1 to the context of conditional default distributions. Therefore, it is left to the reader. □

3.7 Recursive Valuation of a Basket Claim

We are ready extend the results developed in the context of first-to-default claims to value and hedge general basket claims. A generic basket claim is any contingent claim that pays a specified amount on each default from a basket of n credit names and a constant amount at maturity T if no defaults have occurred prior to or at T.

Definition 3.7 A *basket claim* associated with a family of n credit names is given as $(X, A, \bar{Z}, \bar{\tau})$ where X is a constant amount payable at maturity only if no default occurs prior to or at T, the vector $\bar{\tau} = (\tau_1, \dots, \tau_n)$ represents default times, and the time-dependent matrix $\bar{Z}$ represents the payoffs at defaults, specifically,

$$\bar{Z} = \begin{bmatrix} Z_1(t\,|D_0) & Z_2(t\,|D_0) & . & Z_n(t\,|D_0) \\ Z_1(t\,|D_1) & Z_2(t\,|D_1) & . & Z_n(t\,|D_1) \\ . & . & . & . \\ Z_1(t\,|D_{n-1}) & Z_2(t\,|D_{n-1}) & . & Z_n(t\,|D_{n-1}) \end{bmatrix}.$$

Note that the above matrix $\bar{Z}$ is presented in the shorthand notation. In fact, in each row we need to specify, for an arbitrary choice of the event $D_k = \{\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k\}$ and any name $j_l \notin \{i_1, \dots, i_k\}$, the *conditional payoff function* at the moment of the $(k+1)$th default:

$$Z_{j_l}(t\,|D_k) = Z_{j_l}(t\,|\,\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k), \quad \forall\, t \in [u_k, T].$$

If the financial interpretation, the function $Z_{j_l}(t\,|D_k)$ determines the recovery payment at the default of the name j_l, conditional on the event D_k, on the event $\{\tau_{j_l} = \tau_{(k+1)} = t\}$, that is, assuming that the name j_l is the first defaulting name among all surviving names. In particular, $Z_i(t\,|D_0) := Z_i(t)$ represents the recovery payment at the default of the ith name at time $t \in [0, T]$, given that no defaults have occurred prior to t, that is, at the moment of the first default (note that the symbol D_0 means merely that we consider a situation of no defaults prior to t).

Example 3.1 Let us consider the kth-to-default claim for some fixed $k \in \{1, 2, \dots, n\}$. Assume that the payoff at the kth default depends only on the moment of the kth default and the identity of the kth-to-default name. Then all rows of the matrix $\bar{Z}$ are equal to zero, except for the kth row, which is $[Z_1(t\,|\,k-1), Z_2(t\,|\,k-1), \dots, Z_n(t\,|\,k-1)]$ for $t \in [0, T]$. We write here $k-1$, rather than D_{k-1}, in order to emphasize that the knowledge of timings and identities of the k defaulted names is not relevant under the present assumptions.

More generally, for a generic basket claim in which the payoff at the ith default depends on the time of the ith default and identity of the ith defaulting name only, the recovery matrix $\bar{Z}$ reads

$$\bar{Z} = \begin{bmatrix} Z_1(t) & Z_2(t) & . & Z_n(t) \\ Z_1(t\,|1) & Z_2(t\,|1) & . & Z_n(t\,|1) \\ . & . & . & . \\ Z_1(t\,|n-1) & Z_2(t\,|n-1) & . & Z_n(t\,|n-1) \end{bmatrix}$$

where $Z_j(t\,|k-1)$ represents the payoff at the moment $\tau_{(k)} = t$ of the kth default if j is the kth defaulting name, that is, on the event $\{\tau_j = \tau_{(k)} = t\}$. This shows that in several practically important examples of basket credit derivatives, the matrix $\bar{Z}$ will have a simple structure.

It is clear that any basket claim can be represented as a static portfolio of kth-to-default claims for $k = 1, 2, \dots, n$. However, this decomposition does not seem to be advantageous for our purposes. In what follows, we prefer to represent a basket claim as a sequence of *conditional first-to-default claims*, with the same value between any two defaults as our basket claim. In that way, we will be able to directly apply results developed for the case of first-to-default claims and thus to produce a simple iterative algorithm for the valuation and hedging of a basket claim.

Instead of stating a formal result, using a rather heavy notation, we prefer to first focus on the computational procedure for valuation and hedging of a basket claim. The important concept in this procedure is the *conditional pre-default price*

$$\widetilde{Z}(t\,|D_k) = \widetilde{Z}(t\,|\,\tau_{i_1} = u_1, \dots, \tau_{i_k} = u_k), \quad \forall\, t \in [u_k, T],$$

of a "conditional first-to-default claim". The function $\widetilde{Z}(t\,|D_k)$, $t \in [u_k, T]$, is defined as the risk-neutral value of a conditional FTDC on $n-k$ surviving names, with the following recovery payoffs upon the first default at any date $t \in [u_k, T]$

$$\widehat{Z}_{j_l}(t\,|\,D_k) = Z_{j_l}(t\,|\,D_k) + \widetilde{Z}(t\,|\,D_k, \tau_{j_l} = t). \tag{66}$$

Assume for the moment that for any name $j_m \notin \{i_1, \ldots, i_k, j_l\}$ the conditional recovery payoff $\widehat{Z}_{j_m}(t \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k, \tau_{j_l} = u_{k+1})$ upon the first default after date u_{k+1} is known. Then we can compute the function

$$\widetilde{Z}(t \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k, \tau_{j_l} = u_{k+1}), \quad \forall\, t \in [u_{k+1}, T],$$

as in Lemma 3.6, but using conditional default distribution. The assumption that the conditional payoffs are known is in fact not restrictive, since the functions appearing in right-hand side of (66) are known from the previous step in the following recursive pricing algorithm.

- *First step.* We first derive the value of a basket claim assuming that all but one defaults have already occurred. Let $D_{n-1} = \{\tau_{i_1} = u_1, \ldots, \tau_{i_{n-1}} = u_{n-1}\}$. For any $t \in [u_{n-1}, T]$, we deal with the payoffs

$$\widehat{Z}_{j_1}(t \,|D_{n-1}) = Z_{j_1}(t \,|D_{n-1}) = Z_{j_1}(t \,|\, \tau_{i_1} = u_1, \ldots, \tau_{i_{n-1}} = u_{n-1}),$$

  for $j_1 \notin \{i_1, \ldots, i_{n-1}\}$ where the recovery payment function $Z_{j_1}(t \,|D_{n-1})$, $t \in [u_{n-1}, T]$, is given by the specification of the basket claim. Hence we can evaluate the pre-default value $\widetilde{Z}(t \,|D_{n-1})$ at any time $t \in [u_{n-1}, T]$, as a value of a conditional first-to-default claim with the said payoff, using the conditional distribution under $\mathbb{Q}^*|D_{n-1}$ of the random time $\tau_{j_1} = \tau_{i_n}$ on the interval $[u_{n-1}, T]$.

- *Second step.* In this step, we assume that all but two names have already defaulted. Let $D_{n-2} = \{\tau_{i_1} = u_1, \ldots, \tau_{i_{n-2}} = u_{n-2}\}$. For each surviving name $j_1, j_2 \notin \{i_1, \ldots, i_{n-2}\}$, the payoff $\widehat{Z}_{j_l}(t \,|D_{n-2})$, $t \in [u_{n-2}, T]$, of a basket claim at the moment of the next default formally comprises the recovery payoff from the defaulted name j_l which is $Z_{j_l}(t \,|D_{n-2})$ and the pre-default value $\widetilde{Z}(t \,|\, D_{n-2}, \tau_{j_l} = t)$, $t \in [u_{n-2}, T]$, which was computed in the first step. Therefore, we have

$$\widehat{Z}_{j_l}(t \,|\, D_{n-2}) = Z_{j_l}(t \,|\, D_{n-2}) + \widetilde{Z}(t \,|\, D_{n-2}, \tau_{j_l} = t), \quad \forall\, t \in [u_{n-2}, T].$$

  To find the value of a basket claim between the $(n-2)$th and $(n-1)$th default, it suffices to compute the pre-default value of the conditional FTDC associated with the two surviving names, $j_1, j_2 \notin \{i_1, \ldots, i_{n-2}\}$. Since the conditional payoffs $\widehat{Z}_{j_1}(t \,|D_{n-2})$ and $\widehat{Z}_{j_2}(t \,|D_{n-2})$ are known, we may compute the expectation under the conditional probability measure $\mathbb{Q}^*|D_{n-2}$ in order to find the pre-default value of this conditional FTDC for any $t \in [u_{n-2}, T]$.

- *General induction step.* We now assume that exactly k default have occurred, that is, we assume that the event $D_k = \{\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k\}$ is given. From the preceding step, we know the function $\widetilde{Z}(t \,|D_{k+1})$ where $D_k = \{\tau_{i_1} = u_1, \ldots, \tau_{i_k} = u_k, \tau_{j_l} = u_{k+1}\}$. In order to compute $\widetilde{Z}(t \,|D_k)$, we set

$$\widehat{Z}_{j_l}(t \,|\, D_k) = Z_{j_l}(t \,|\, D_k) + \widetilde{Z}(t \,|\, D_k, \tau_{j_l} = t), \quad \forall\, t \in [u_k, T], \tag{67}$$

  for any $j_1, \ldots, j_{n-k} \notin \{i_1, \ldots, i_k\}$, and we compute $\widetilde{Z}(t \,|D_k), t \in [u_k, T]$, as the risk-neutral value under $\mathbb{Q}^*|D_k$ at time of the conditional FTDC with the payoffs given by (67).

We are in the position state the valuation result for a basket claim, which can be formally proved using the reasoning presented above.

Proposition 3.4 *The risk-neutral value at time $t \in [0, T]$ of a basket claim $(X, A, \bar{Z}, \bar{\tau})$ equals*

$$\pi_t = \sum_{k=0}^{n-1} \widetilde{Z}(t \,|D_k) \mathbb{1}_{[\tau_{(k)} \wedge T, \tau_{(k+1)} \wedge T[}(t), \quad \forall\, t \in [0, T],$$

where $D_k = D_k(\omega) = \{\tau_{i_1}(\omega) = u_1, \ldots, \tau_{i_k}(\omega) = u_k\}$ for $k = 1, 2, \ldots, n$, and D_0 means that no defaults have yet occurred.

3.8 Recursive Replication of a Basket Claim

From the discussion of the preceding section, it is clear that a basket claim can be conveniently interpreted as a specific sequence of conditional first-to-default claims. Hence it is easy to guess that the replication of a basket claim should refer to hedging of the underlying sequence of conditional first-to-default claims. In the next result, we denote $\tau_{(0)} = 0$.

Theorem 3.2 *For any $k = 0, 1, \ldots, n$, the replicating strategy ϕ for a basket claim $(X, A, \bar{Z}, \bar{\tau})$ on the time interval $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ coincides with the replicating strategy for the conditional FTDC with payoffs $\widehat{Z}(t \,|D_k)$ given by (67). The replicating strategy $\phi = (\phi^0, \phi^{j1}, \ldots, \phi^{j_{n-k}}, A)$, corresponding to the units of savings account and units of CDS on each surviving name at time t, has the wealth process*

$$V_t(\phi, A) = \phi^0_t + \sum_{l=1}^{n-i} \phi^{j_l}_t S^{j_l}_t(\kappa_{j_l})$$

where processes ϕ^{j_l}, $l = 1, 2, \ldots, n-k$ can be computed by the conditional version of Theorem 3.1.

Proof. We know that the basket claim can be decomposed into a series of conditional first-to-default claims. So, at any given moment of time $t \in [0, T]$, assuming that k defaults have already occurred, our basket claim is equivalent to the conditional FTDC with payoffs $\widehat{Z}(t \,|\, D_k)$ and the pre-default value $\widetilde{Z}(t \,|D_k)$. This conditional FTDC is alive up to the next default $\tau_{(k+1)}$ or maturity T, whichever comes first. Hence it is clear that the replicating strategy of a basket claim over the random interval $[\tau_k \wedge T, \tau_{k+1} \wedge T]$ need to coincide with the replicating strategy for this conditional first-to-default claim, and thus it can be found along the same lines as in Theorem 3.1, using the conditional distribution under $\mathbb{Q}^*|D_k$ of defaults for surviving names. □

4 Applications to Copula-Based Credit Risk Models

In this section, we will apply our previous results to some specific models, in which some common copulas are used to model dependence between default times (see, for instance, Cherubini et al. [7], Embrechts et al. [8], Frey et al. [9], Laurent and Gregory [12], Li [13] or McNeil et al. [14]). It is fair to admit that copula-based credit risk models are not fully suitable for a dynamical approach to credit risk, since the future behavior of credit spreads can be predicted with certainty, up to the observations of default times. Hence they are unsuitable for hedging of option-like contracts on credit spreads. On the other hand, however, these models are of a common use in practical valuation credit derivatives and thus we decided to present them here. Of course, our results are more general, so that they can be applied to an arbitrary joint distribution of default times (i.e., not necessarily given by some copula function). Also, in the follow-up work [2] we extend the results of this work to a fully dynamical set-up.

For simplicity of exposition and in order to get more explicit formulae, we only consider the bivariate situation and we make the following standing assumptions.

Assumptions (B). We assume from now on that:
(i) we are given an FTDC $(X, A, Z, \tau_{(1)})$ where $Z = (Z_1, Z_2)$ for some constants Z_1, Z_2 and X,
(ii) the default times τ_1 and τ_2 have exponential marginal distributions with parameters λ_1 and λ_2,
(ii) the recovery δ_i of the ith CDS is constant and $\kappa_i = \lambda_i \delta_i$ for $i = 1, 2$ (see Example 2.1).

Before proceeding to computations, let us note that

$$\int_{u=t}^{T} \int_{v=u}^{\infty} G(du, dv) = -\int_t^T G(du, u)$$

and thus, assuming that the pair (τ_1, τ_2) has the joint probability density function $f(u, v)$,

$$\int_t^T du \int_u^{\infty} dv f(u, v) = -\int_t^T \partial_1 G(u, u)\, du$$

and

$$dv \int_a^b f(u,v)\,du = G(a,dv) - G(b,dv) = dv\big(\partial_2 G(b,v) - \partial_2 G(a,v)\big)$$

$$\int_v^T du \int_u^\infty dz f(z,v) = -\int_v^T \partial_2 G(u,v)\,du.$$

4.1 Independent Default Times

Let us first consider the case where the default times τ_1 and τ_2 are independent (this corresponds to the product copula $C(u,v) = uv$). In view of independence, the marginal intensities and the first-to-default intensities can be easily shown to coincide. We have, for $i = 1,2$

$$G_i(u) = \mathbb{Q}^*(\tau_i > u) = e^{-\lambda_i u}$$

and thus the joint survival function equals

$$G(u,v) = G_1(u)G_2(v) = e^{-\lambda_1 u}e^{-\lambda_2 v}.$$

Consequently

$$F(du,dv) = G(du,dv) = \lambda_1\lambda_2 e^{-\lambda_1 u}e^{-\lambda_2 v}\,dudv = f(u,v)\,dudv$$

and $G(du,u) = -\lambda_1 e^{-(\lambda_1+\lambda_2)u}\,du$.

Proposition 4.1 *Assume that the default times τ_1 and τ_2 are independent. Then the replicating strategy for an FTDC $(X,0,Z,\tau_{(1)})$ is given as*

$$\widetilde{\phi}^1(t) = \frac{Z_1 - \widetilde{\pi}(t)}{\delta_1}, \quad \widetilde{\phi}^2(t) = \frac{Z_2 - \widetilde{\pi}(t)}{\delta_2}$$

where

$$\widetilde{\pi}(t) = \frac{(Z_1\lambda_1 + Z_2\lambda_2)}{\lambda_1+\lambda_2}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + Xe^{-(\lambda_1+\lambda_2)(T-t)}.$$

Proof. From the previous remarks, we obtain

$$\begin{aligned}
\widetilde{\pi}(t) &= \frac{Z_1\int_t^T\int_u^\infty dF(u,v)}{G(t,t)} + \frac{Z_2\int_t^T\int_v^\infty dF(u,v)}{G(t,t)} + X\frac{G(T,T)}{G(t,t)} \\
&= \frac{Z_1\lambda_1\int_t^T e^{-(\lambda_1+\lambda_2)u}du}{e^{-(\lambda_1+\lambda_2)t}} + \frac{Z_2\lambda_2\int_t^T e^{-(\lambda_1+\lambda_2)v}dv}{e^{-(\lambda_1+\lambda_2)t}} + X\frac{G(T,T)}{G(t,t)} \\
&= \frac{Z_1\lambda_1}{(\lambda_1+\lambda_2)}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + \frac{Z_2\lambda_2}{(\lambda_1+\lambda_2)}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + X\frac{G(T,T)}{G(t,t)} \\
&= \frac{(Z_1\lambda_1 + Z_2\lambda_2)}{\lambda_1+\lambda_2}(1 - e^{-(\lambda_1+\lambda_2)(T-t)}) + Xe^{-(\lambda_1+\lambda_2)(T-t)}.
\end{aligned}$$

Under the assumption of independence of default times, we also have that $S^i_{t|j}(\kappa_i) = \widetilde{S}^i_t(\kappa_i)$ for $i,j = 1,2$ and $i \neq j$. Furthermore from Example 2.1, we have that $\widetilde{S}^i_t(\kappa_i) = 0$ for $t \in [0,T]$ and thus the matrix $N(t)$ in Theorem 3.1 reduces to

$$N(t) = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}.$$

The replicating strategy can be found easily by solving the linear equation $N(t)\widetilde{\phi}(t) = h(t)$ where $h(t) = (h_1(t), h_2(t))$ with $h_i(t) = Z_i - \widetilde{\pi}(t-) = Z_i - \widetilde{\pi}(t)$ for $i = 1,2$. □

As another important case of a first-to-default claim, we take a first-to-default swap (FTDS). For a stylized FTDS we have $X = 0$, $A(t) = -\kappa_{(1)}t$ where $\kappa_{(1)}$ is the *swap spread*, and $Z_i(t) = \delta_i \in [0,1)$ for some constants δ_i, $i = 1,2$. Hence an FTDS is formally given as an FTDC $(0, -\kappa_{(1)}t, (\delta_1,\delta_2), \tau_{(1)})$.

Under the present assumptions, we easily obtain

$$\pi_0 = \widetilde{\pi}(0) = \frac{1 - e^{\lambda T}}{\lambda} \Big((\lambda_1 \delta_1 + \lambda_2 \delta_2) - \kappa_{(1)} \Big)$$

where $\lambda = \lambda_1 + \lambda_2$. The *FTDS market spread* is the level of $\kappa_{(1)}$ that makes the FTDS valueless at initiation. Hence in this elementary example this spread equals $\lambda_1 \delta_1 + \lambda_2 \delta_2$. In addition, it can be shown that under the present assumptions we have that $\widetilde{\pi}(t) = 0$ for every $t \in [0, T]$.

Suppose that we wish to hedge the short position in the FTDS using two CDSs, say CDS^i, $i = 1, 2$, with respective default times τ_i, protection payments δ_i and spreads $\kappa_i = \lambda_i \delta_i$. Recall that in the present set-up we have that, for $t \in [0, T]$,

$$S^i_{t|j}(\kappa_i) = \widetilde{S}^i_t(\kappa_i) = 0, \quad i, j = 1, 2, \ i \neq j. \tag{68}$$

Consequently, we have here that $h_i(t) = -Z_i(t) = -\delta_i$ for every $t \in [0, T]$. It then follows from equation $N(t)\widetilde{\phi}(t) = h(t)$ that $\widetilde{\phi}_1(t) = \widetilde{\phi}_2(t) = 1$ for every $t \in [0, T]$ and thus $\phi^0_t = 0$ for every $t \in [0, T]$. This result is by no means surprising: we hedge a short position in the FTDS by holding a static portfolio of two single-name CDSs since, under the present assumptions, the FTDS is equivalent to such a portfolio of corresponding single name CDSs. Of course, one would not expect that this feature will still hold in a general case of dependent default times.

The first equality in (68) is due to the standing assumption of independence of default times τ_1 and τ_2 and thus it will no longer be true for other copulas (see foregoing subsections). The second equality follows from the postulate that the risk-neutral marginal distributions of default times are exponential. In practice, the risk-neutral marginal distributions of default times will be obtained by fitting the model to market data (i.e., market prices of single name CDSs) and thus typically they will not be exponential. To conclude, both equalities in (68) are unlikely to hold in any real-life implementation. Hence this example show be seen as the simplest illustration of general results and we do not pretend that it has any practical merits. Nevertheless, we believe that it might be useful to give a few more comments on the hedging strategy considered in this example.

Suppose that a dealer sells one FTDS and hedges his short position by holding a portfolio composed of one CDS^1 contract and one CDS^2 contract. Let us consider the event $\{\tau_{(1)} = \tau_1 < T\}$. The cumulative premium the dealer collects on the time interval $[0, t], \ t \leq \tau_{(1)}$, for selling the FTDS equals $(\lambda_1 \delta_1 + \lambda_2 \delta_2)t$. The protection coverage that the dealer has to pay at time $\tau_{(1)}$ equals δ_1 and the FTDS is terminated at τ_1. Now, the cumulative premium the dealer pays on the time interval $[0, t], \ t \leq \tau_{(1)}$, for holding the portfolio of one CDS^1 contract and one CDS^2 contract is $(\lambda_1 \delta_1 + \lambda_2 \delta_2)t$. At time τ_1, the dealer receives the protection payment of δ_1. The CDS^1 is terminated at time τ_1; the dealer still holds the CDS^2 contract, however. Recall, though, that the ex-dividend price (i.e., the market price) of this contract is zero. Hence the dealer may unwind the contract at time $\tau_{(1)}$ at no cost (again, this only holds under the assumption of independence and exponential marginals). Consequently the dealer's P/L is flat (zero) over the lifetime of the FTDS and the dealer has no positions in the remaining CDS at the first default time. Though we consider here the simplest set-up, it is plausible that a similar interpretation of a hedging strategy will also apply in a more general framework.

4.2 Archimedean Copulas

We now proceed to the case of exponentially distributed, but dependent, default times. Their interdependence will be specified by a choice of some *Archimedean copula*. Such examples were studied by Hua [11]; we present and simplify here some of his computations. Recall that a bivariate Archimedean copula is defined as

$$C(u, v) = \varphi^{-1}(\varphi(u), \varphi(v))$$

where φ is called the *generator* of a copula.

4.2.1 Clayton Copula

Recall that the generator of the *Clayton copula* is given as

$$\varphi(s) = s^{-\theta} - 1, \quad s \in \mathbb{R}_+,$$

for some strictly positive parameter θ. Hence the bivariate Clayton copula can be represented as follows

$$C(u,v) = C_\theta^{Clayton}(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}.$$

Under the present assumptions, the corresponding joint survival function $G(u,v)$ equals

$$G(u,v) = C(G_1(u), G_2(v)) = (e^{\lambda_1 u\theta} + e^{\lambda_2 v\theta} - 1)^{-\frac{1}{\theta}}$$

so that

$$\frac{G(u,dv)}{dv} = -\lambda_2 e^{\lambda_2 v\theta}(e^{\lambda_1 u\theta} + e^{\lambda_2 v\theta} - 1)^{-\frac{1}{\theta}-1}$$

and

$$f(u,v) = \frac{G(du,dv)}{dudv} = (\theta+1)\lambda_1\lambda_2 e^{\lambda_1 u\theta + \lambda_2 v\theta}(e^{\lambda_1 u\theta} + e^{\lambda_2 v\theta} - 1)^{-\frac{1}{\theta}-2}.$$

Proposition 4.2 *Let the joint distribution of (τ_1, τ_2) be given by the Clayton copula with $\theta > 0$. Then the replicating strategy for an FTDC $(X, 0, Z, \tau_{(1)})$ is given by the expressions*

$$\widetilde{\phi}_1(t) = \frac{\delta_2(Z_1 - \widetilde{\pi}(t)) + S^2_{t|1}(\kappa_2)(Z_2 - \widetilde{\pi}(t))}{\delta_1\delta_2 - S^1_{t|2}(\kappa_1)S^2_{t|1}(\kappa_2)}, \tag{69}$$

$$\widetilde{\phi}_2(t) = \frac{\delta_1(Z_2 - \widetilde{\pi}(t)) + S^1_{t|2}(\kappa_1)(Z_1 - \widetilde{\pi}(t))}{\delta_1\delta_2 - S^1_{t|2}(\kappa_1)S^2_{t|1}(\kappa_2)}, \tag{70}$$

where

$$\begin{aligned}
\widetilde{\pi}(t) \;=\;& Z_1 \frac{\int_{e^{\lambda_1\theta t}}^{e^{\lambda_1\theta T}} (s + s^{\frac{\lambda_2}{\lambda_1}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}} + Z_2 \frac{\int_{e^{\lambda_2\theta t}}^{e^{\lambda_2\theta T}} (s + s^{\frac{\lambda_1}{\lambda_2}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}} \\
&+ X \frac{(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}}}{(e^{\lambda_1\theta t} + e^{\lambda_2\theta t} - 1)^{-\frac{1}{\theta}}},
\end{aligned}$$

$$\begin{aligned}
S^1_{v|2}(\kappa_1) \;=\;& \delta_1 \frac{[(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}-1} - (e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}]}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}} \\
&- \kappa_1 \frac{\int_v^T (e^{\lambda_1\theta u} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1} du}{(e^{\lambda_1\theta v} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1}},
\end{aligned}$$

and

$$\begin{aligned}
S^2_{u|1}(\kappa_2) \;=\;& \delta_2 \frac{[(e^{\lambda_1\theta T} + e^{\lambda_2\theta T} - 1)^{-\frac{1}{\theta}-1} - (e^{\lambda_1\theta u} + e^{\lambda_2\theta u} - 1)^{-\frac{1}{\theta}-1}]}{(e^{\lambda_1\theta u} + e^{\lambda_2\theta u} - 1)^{-1/\theta-1}} \\
&- \kappa_2 \frac{\int_u^T (e^{\lambda_1\theta u} + e^{\lambda_2\theta v} - 1)^{-\frac{1}{\theta}-1} dv}{(e^{\lambda_1\theta u} + e^{\lambda_2\theta u} - 1)^{-\frac{1}{\theta}-1}}.
\end{aligned}$$

Proof. Using the observation that

$$\begin{aligned}
\int_t^T du \int_u^\infty f(u,v)dv \;&=\; \int_t^T \lambda_1 e^{\lambda_1 u\theta}(e^{\lambda_1 u\theta} + e^{\lambda_2 u\theta} - 1)^{-\frac{1}{\theta}-1}\, du \\
&=\; \frac{1}{\theta}\int_{e^{\lambda_1\theta t}}^{e^{\lambda_1\theta T}} (s + s^{\frac{\lambda_2}{\lambda_1}} - 1)^{-\frac{1}{\theta}-1}\, ds
\end{aligned}$$

and thus by symmetry

$$\int_t^T dv \int_v^\infty f(u,v)du = \frac{1}{\theta}\int_{e^{\lambda_2\theta t}}^{e^{\lambda_2\theta T}} (s + s^{\frac{\lambda_1}{\lambda_2}} - 1)^{-\frac{1}{\theta}-1}\, ds.$$

We thus obtain

$$\begin{aligned}\widetilde{\pi}(t) &= \frac{Z_1\int_t^T\int_u^\infty dG(u,v)}{G(t,t)} + \frac{Z_2\int_t^T\int_v^\infty dG(u,v)}{G(t,t)} + X\frac{G(T,T)}{G(t,t)}\\ &= Z_1\frac{\int_{e^{\lambda_1\theta t}}^{e^{\lambda_1\theta T}}(s + s^{\frac{\lambda_2}{\lambda_1}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t}+e^{\lambda_2\theta t}-1)^{-\frac{1}{\theta}}} + Z_2\frac{\int_{e^{\lambda_2\theta t}}^{e^{\lambda_2\theta T}}(s + s^{\frac{\lambda_1}{\lambda_2}} - 1)^{-\frac{1}{\theta}-1}\, ds}{\theta(e^{\lambda_1\theta t}+e^{\lambda_2\theta t}-1)^{-\frac{1}{\theta}}}\\ &\quad + X\frac{(e^{\lambda_1\theta T}+e^{\lambda_2\theta T}-1)^{-\frac{1}{\theta}}}{(e^{\lambda_1\theta t}+e^{\lambda_2\theta t}-1)^{-\frac{1}{\theta}}}.\end{aligned}$$

We are in a position to determine the replicating strategy. Under the standing assumption that $\kappa_i = \lambda_i\delta_i$ for $i=1,2$ we still have that $\widetilde{S}^i_t(\kappa_i) = 0$ for $i=1,2$ and for $t\in[0,T]$. Hence the matrix $N(t)$ reduces to

$$N(t) = \begin{bmatrix} \delta_1 & -S^2_{t|1}(\kappa_2)\\ -S^1_{t|2}(\kappa_1) & \delta_2\end{bmatrix}$$

where

$$\begin{aligned}S^1_{v|2}(\kappa_1) &= \delta_1\frac{\int_v^T f(u,v)\,du}{\int_v^\infty f(u,v)\,du} - \kappa_1\frac{\int_v^T\int_u^\infty f(z,v)\,dzdu}{\int_v^\infty f(u,v)\,du}\\ &= \delta_1\frac{G(T,dv)-G(v,dv)}{G(v,dv)} + \kappa_1\frac{\int_t^T G(u,dv)}{G(v,dv)}\\ &= \delta_1\frac{[(e^{\lambda_1\theta T}+e^{\lambda_2\theta T}-1)^{-\frac{1}{\theta}-1} - (e^{\lambda_1\theta v}+e^{\lambda_2\theta v}-1)^{-\frac{1}{\theta}-1}]}{(e^{\lambda_1\theta v}+e^{\lambda_2\theta v}-1)^{-\frac{1}{\theta}-1}}\\ &\quad - \kappa_1\frac{\int_v^T(e^{\lambda_1\theta u}+e^{\lambda_2\theta v}-1)^{-\frac{1}{\theta}-1}\,du}{(e^{\lambda_1\theta v}+e^{\lambda_2\theta v}-1)^{-\frac{1}{\theta}-1}}.\end{aligned}$$

The expression for $S^2_{u|1}(\kappa_2)$ can be found by analogous computations. By solving the equation $N(t)\widetilde{\phi}(t) = h(t)$, we obtain the desired expressions (69)-(70). □

Similar computations can be done for the valuation and hedging of a first-to-default swap.

4.2.2 Gumbel Copula

As another of an Archimedean copula, we consider the *Gumbel copula* with the generator

$$\varphi(s) = (-\ln s)^\theta, \quad s\in\mathbb{R}_+,$$

for some $\theta \geq 1$. The bivariate Gumbel copula can thus be written as

$$C(u,v) = C_\theta^{Gumbel}(u,v) = e^{-[(-\ln u)^\theta + (-\ln v)^\theta]^{\frac{1}{\theta}}}.$$

Under our standing assumptions, the corresponding joint survival function $G(u,v)$ equals

$$G(u,v) = C(G_1(u),G_2(v)) = e^{-(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}.$$

Consequently

$$\frac{dG(u,v)}{dv} = -G(u,v)\lambda_2^\theta v^{\theta-1}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}$$

and

$$\frac{dG(u,v)}{dudv} = G(u,v)(\lambda_1\lambda_2)^\theta(uv)^{\theta-1}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-2}\big((\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}} + \theta - 1\big).$$

Proposition 4.3 *Let the joint distribution of (τ_1, τ_2) be given by the Gumbel copula with $\theta \geq 1$. Then the replicating strategy for an FTDC $(X, 0, Z, \tau_{(1)})$ is given by (69)-(70) with*

$$\widetilde{\pi}(t) = (Z_1\lambda_1^\theta + Z_2\lambda_2^\theta)\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) + Xe^{-\lambda(T-t)},$$

$$\begin{aligned} S^1_{v|2}(\kappa_1) &= \delta_1 \frac{e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1} - e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}} \\ &\quad - \kappa_1 \frac{\int_v^T e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}\,du}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}, \end{aligned}$$

$$\begin{aligned} S^2_{u|1}(\kappa_2) &= \delta_2 \frac{e^{-(\lambda_1^\theta u^\theta + \lambda_2^\theta T^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta u^\theta + \lambda_2^\theta T^\theta)^{\frac{1}{\theta}-1} - e^{-\lambda v}\lambda^{1-\theta}u^{1-\theta}}{e^{-\lambda v}\lambda^{1-\theta}u^{1-\theta}} \\ &\quad - \kappa_2 \frac{\int_u^T e^{-(\lambda_1^\theta u^\theta + \lambda_2^\theta T^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}\,dv}{e^{-\lambda v}\lambda^{1-\theta}u^{1-\theta}}. \end{aligned}$$

Proof. We have

$$\begin{aligned} \int_t^T \int_u^\infty dG(u,v) &= \int_t^T \lambda_1^\theta(\lambda_1^\theta + \lambda_2^\theta)^{\frac{1}{\theta}-1} e^{-(\lambda_1^\theta + \lambda_2^\theta)^{\frac{1}{\theta}} u}\,du \\ &= (-\lambda_1^\theta \lambda^{-\theta} e^{-\lambda u})|_{u=t}^{u=T} = \lambda_1^\theta\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) \end{aligned}$$

where $\lambda = (\lambda_1^\theta + \lambda_2^\theta)^{\frac{1}{\theta}}$. Similarly

$$\int_t^T \int_v^\infty dG(u,v) = \lambda_2^\theta\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}).$$

Furthermore $G(T,T) = e^{-\lambda T}$ and $G(t,t) = e^{-\lambda t}$. Hence

$$\begin{aligned} \widetilde{\pi}(t) &= Z_1 \frac{\int_t^T \int_u^\infty dG(u,v)}{G(t,t)} + Z_2 \frac{\int_t^T \int_v^\infty dG(u,v)}{G(t,t)} + X\frac{G(T,T)}{G(t,t)} \\ &= Z_1\lambda_1^\theta\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) + Z_2\lambda_2^\theta\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) + Xe^{-\lambda(T-t)} \\ &= (Z_1\lambda_1^\theta + \delta_2 Z_2^\theta)\lambda^{-\theta}(e^{-\lambda t} - e^{-\lambda T}) + Xe^{-\lambda(T-t)}. \end{aligned}$$

In order to find the replicating strategy, we proceed as in the proof of Proposition 4.2. Under the present assumptions, we have

$$\begin{aligned} S^1_{v|2}(\kappa_1) &= \delta_1 \frac{\int_v^T f(u,v)du}{\int_v^\infty f(u,v)du} - \kappa_1 \frac{\int_v^T \int_u^\infty f(z,v)dzdu}{\int_v^\infty f(u,v)du} \\ &= \delta_1 \frac{e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1} - e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}} \\ &\quad - \kappa_1 \frac{\int_v^T e^{-(\lambda_1^\theta T^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}}}(\lambda_1^\theta u^\theta + \lambda_2^\theta v^\theta)^{\frac{1}{\theta}-1}du}{e^{-\lambda v}\lambda^{1-\theta}v^{1-\theta}}. \end{aligned}$$

This completes the proof. □

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