HEDGING OF CREDIT DEFAULT SWAPTIONS IN A HAZARD PROCESS MODEL

Tomasz R. Bielecki*
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA

Monique Jeanblanc†
Département de Mathématiques
Université d'Évry Val d'Essonne
91025 Évry Cedex, France
Institut Europlace de Finance

Marek Rutkowski‡
School of Mathematics and Statistics
University of New South Wales
Sydney, NSW 2052, Australia
and
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland

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Introduction

An important issue that should be addressed in the context of credit default swap (CDS) rates is a construction of an appropriate model in which a family of options written on credit default swap, referred to hereafter as credit default swaptions, can be valued and hedged. Some previous efforts in this directions were largely motivated by the market practice of using a suitable version of the Black swaption formula. For an option written on a single-name forward credit default swap, such pricing formula was derived by Schönbucher [30]-[31] and Jamshidian [22], who formally used the risk-neutral valuation formula in an intensity-based credit risk model, which was not fully specified. The derivations of a version of the Black formula for credit default swaptions presented in these papers are based on rather abstract approximation arguments for a positive martingale, as opposed to an explicit construction of a (lognormal) model for a family of CDS rates associated with a given tenor structure, in which the pricing of a credit default swaption could be supported by strict replication arguments. Such a construction was provided, albeit under some simplifying assumptions, in recent papers by Brigo [8]-[9]. To be more specific, Brigo [8]-[9] analyzes the joint dynamics of certain families of forward CDS rates under judiciously chosen martingale measures. He shows that in some cases (most notably, for a family of one-period forward CDS rates), it is possible to develop a change of a numéraire approach, which is analogous to arbitrage-free modeling of forward LIBOR rates. He also emphasizes the difficulties that arise in the context of modeling of a family of co-terminal forward CDS rates.

In an alternative approach, Ben-Ameur et al. [2] and Brigo and Cousot [12] deal with the valuation of European and Bermudan credit default swaptions within the framework of the intensity-based SSRD (Shifted Square-Root Diffusion) model, which was introduced previously by Brigo and Alfonsi [11]. In a recent work by Brigo and El-Bachir [13], a complete study of valuation of credit default swaptions in the SSRJD (Shifted Square-Root Jump-Diffusion) model is provided.

It is worth stressing that none of the above-mentioned papers addresses the issue of hedging of credit default swaptions. The only work, that we are aware of, that discusses hedging of CDS option is Zhang et al. [32]. The methodology used in this paper is driven by the idea of simultaneous derivation of the price of the swaption and of the hedging strategy by solving a sequence of some constrained optimization problems. It appears that the hedging strategy discussed in [32] is not necessarily a self-financing strategy, and, moreover, it is not quite clear what is the meaning of the calculated "price" of the swaption. In addition, the relationship (9), in [32], between the par CDS spread and the risk-neutral intensity of default, appears to be questionable. Such relationship is true in case of constant default intensity and constant recovery rate (in which case the par CDS spread is constant as well), but is not true in general. Consequently, there appears to be mismatch between postulated stochastic evolution of the par spread process and the stochastic evolution of the hazard rate process.

In our previous papers [4]-[6], we provided some general results regarding the issue of hedging of general defaultable claims in the copula-based and intensity-based frameworks. The goal of this work is to apply these results to the valuation and hedging of credit default swaptions in some specific hazard process models.

A strictly positive random variable τ, defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, is termed a *random time*. In view of its financial interpretation, we will refer to it as a *default time*. We define the default indicator process $H_t = \mathbb{1}_{\{\tau \le t\}}$ and we denote by $\mathbb{H}$ the filtration generated by this process. We assume that we are given, in addition, some auxiliary filtration $\mathbb{F}$ and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that we have $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as to the *full filtration*. It is clear that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time (but not necessarily an $\mathbb{F}$-stopping time). All processes are defined on the space $(\Omega, \mathbb{G}, \mathbb{Q}^*)$. In what follows, T is a finite horizon and unless otherwise stated, all processes considered are assumed to be $\mathbb{G}$-adapted and with càdlàg sample paths.

1 Dynamics of Asset Prices in a Hazard Process Model

We assume that the underlying market model is arbitrage-free, meaning that it admits a *spot martingale measure* $\mathbb{Q}^*$ (not necessarily unique) equivalent to $\mathbb{Q}$. A *spot martingale measure* is associated with the choice of the savings account B as a numéraire, in the sense that the price process of any tradeable security, which pays no coupons or dividends, is a $(\mathbb{Q}^*, \mathbb{G})$-martingale when it is discounted by the *savings account* B. As usual, B is given by

$$B_t = \exp\Big(\int_0^t r_u\,du\Big), \quad \forall\, t \in \mathbb{R}_+, \tag{1}$$

where the short-term r is assumed to follow an $\mathbb{F}$-progressively measurable stochastic process. The choice of a suitable term structure model is arbitrary and it is not discussed in the present work.

Let us denote by $G_t = \mathbb{Q}^*(\tau > t \,|\, \mathcal{F}_t)$ the *survival process* of τ with respect to a filtration $\mathbb{F}$. We postulate that $G_0 = 1$ and $G_t > 0$ for every $t \in \mathbb{R}_+$ (hence the case where τ is an $\mathbb{F}$-stopping time is excluded) so that the *hazard process* $\Gamma = -\ln G$ of τ with respect to the filtration $\mathbb{F}$ is well defined.

For any $\mathbb{Q}^*$-integrable and $\mathcal{F}_T$-measurable random variable Y, the following classic formula holds

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{T<\tau\}}Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\, G_t^{-1}\mathbb{E}_{\mathbb{Q}^*}(G_T Y \,|\, \mathcal{F}_t).$$

Clearly, the process G is a bounded $(\mathbb{Q}^*, \mathbb{G})$-supermartingale and thus it admits the unique Doob-Meyer decomposition $G = \mu - \nu$, where μ is a martingale part and ν is a predictable increasing process. Note that, if G is continuous, then the processes μ and ν are continuous as well.

In this section, we work under the following standing assumption.

Assumption 1.1 We postulate that G is a continuous process and the increasing process ν in its Doob-Meyer decomposition is absolutely continuous with respect to the Lebesgue measure, so that $d\nu_t = \upsilon_t\,dt$ for some $\mathbb{F}$-progressively measurable, non-negative process υ. We denote by λ the $\mathbb{F}$-progressively measurable process defined as $\lambda_t = G_t^{-1}\upsilon_t$. The process λ is called the $\mathbb{F}$-*intensity* of default time.

Assumption 1.1 implies that $dG_t = d\mu_t - \lambda_t G_t\,dt$, where the $(\mathbb{Q}^*, \mathbb{F})$-martingale μ is continuous. Moreover, continuity of G implies that $\mathbb{Q}^*(\tau = t) = 0$ for any $t \in \mathbb{R}_+$. Finally, it is known (see, e.g., [15] or [16]) that under Assumption 1.1 the process M, which is given by the formula

$$M_t = H_t - \int_0^{t\wedge\tau} \lambda_u\,du = H_t - \int_0^t (1 - H_u)\lambda_u\,du, \tag{2}$$

is a $(\mathbb{Q}^*, \mathbb{G})$-martingale.

We first recall some valuation results for defaultable claims, which are borrowed from [6].

Definition 1.1 By a *defaultable claim maturing at* T we mean a quadruple (X, A, Z, τ), where X is an $\mathcal{F}_T$-measurable random variable, $A = (A_t)_{t\in[0,T]}$ is an $\mathbb{F}$-adapted, continuous process of finite variation with $A_0 = 0$, $Z = (Z_t)_{t\in[0,T]}$ is an $\mathbb{F}$-predictable process and τ is a default time. The *dividend process* $D = (D_t)_{t\in\mathbb{R}_+}$ of a defaultable claim maturing at T equals, for every $t \in \mathbb{R}_+$,

$$D_t = X\mathbb{1}_{\{T<\tau\}}\mathbb{1}_{[T,\infty[}(t) + \int_{]0,t\wedge T]} (1 - H_u)\,dA_u + \int_{]0,t\wedge T]} Z_u\,dH_u.$$

The financial interpretation of this definition is as follows: X is the *promised payoff*, A represents the process of *promised dividends* and the process Z, termed the *recovery process*, specifies the recovery payoff at default.

Definition 1.2 The *ex-dividend price* process S associated with the dividend process D equals, for every $t \in [0,T]$,

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big).$$

It can be shown (see [6]) that the ex-dividend price of the defaultable claim (X, A, Z, τ) satisfies, for every $t \in [0,T]$,

$$S_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \, \mathbb{E}_{\mathbb{Q}^*}\bigg(B_T^{-1} G_T X \mathbb{1}_{\{t<T\}} + \int_t^T B_u^{-1} G_u Z_u \lambda_u \, du + \int_t^T B_u^{-1} G_u \, dA_u \, \bigg| \, \mathcal{F}_t \bigg). \tag{3}$$

This means, in particular, that for any $t \in [0,T]$ the ex-dividend price S equals $S_t = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t$ for the $\mathbb{F}$-adapted process $\widetilde{S}$, which is termed the *ex-dividend pre-default price* of a defaultable claim.

Definition 1.3 The *cumulative price* process S^c associated with the dividend process D is given by the following expression, for every $t \in [0,T]$,

$$S^c_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big) = S_t + B_t \int_{]0,t]} B_u^{-1} \, dD_u.$$

Under the assumption that all $(\mathbb{Q}^*, \mathbb{F})$-martingales are continuous (see Assumption 1.2 in [6]), the following result is valid (we refer to [6] for the proof).

Proposition 1.1 *Let μ be the $(\mathbb{Q}^*, \mathbb{F})$-martingale part of the Doob-Meyer decomposition of G and let m be the $(\mathbb{Q}^*, \mathbb{F})$-martingale given by the formula*

$$m_t = \mathbb{E}_{\mathbb{Q}^*}\bigg(B_T^{-1} G_T X + \int_0^T B_u^{-1} G_u Z_u \lambda_u \, du + \int_0^T B_u^{-1} G_u \, dA_u \, \bigg| \, \mathcal{F}_t \bigg).$$

(i) The dynamics of the ex-dividend price S on $[0,T]$ are

$$\begin{aligned} dS_t &= -S_{t-} \, dM_t + (1 - H_t)\big((r_t S_t - \lambda_t Z_t) \, dt + dA_t\big) \\ &\quad + (1 - H_t) G_t^{-1} \big(B_t \, dm_t - S_t \, d\mu_t\big) + (1 - H_t) G_t^{-2} \big(S_t \, d\langle \mu \rangle_t - B_t \, d\langle \mu, m \rangle_t\big) \, . \end{aligned}$$

(ii) The dynamics of the predefault price $\widetilde{S}$ on $[0,T]$ are

$$\begin{aligned} d\widetilde{S}_t &= \big((\lambda_t + r_t)\widetilde{S}_t - \lambda_t Z_t\big) \, dt + dA_t + G_t^{-1} \big(B_t \, dm_t - \widetilde{S}_t \, d\mu_t\big) \\ &\quad + G_t^{-2} \big(\widetilde{S}_t \, d\langle \mu \rangle_t - B_t \, d\langle \mu, m \rangle_t\big). \end{aligned}$$

(iii) The dynamics of the cumulative price S^c on $[0,T]$ are

$$\begin{aligned} dS^c_t &= r_t S^c_t \, dt + (Z_t - S_{t-}) \, dM_t \\ &\quad + (1 - H_t) G_t^{-1} \big(B_t \, dm_t - S_t \, d\mu_t\big) + (1 - H_t) G_t^{-2} \big(S_t \, d\langle \mu \rangle_t - B_t \, d\langle \mu, m \rangle_t\big). \end{aligned}$$

2 Credit Default Swaptions

We are in a position to analyze credit default swaps and related options in a hazard process model introduced in Section 1. In this section, we maintain the standing Assumption 1.1.

2.1 Forward CDS

A forward CDS can be initiated at any time $s \in [0, U]$ and it gives default protection over the future time interval $[U, T]$. If the reference entity defaults prior to the start date U the contract is terminated and no payments are made. In what follows, the recovery process δ and the time period $[U, T]$ are fixed.

Definition 2.1 A *forward CDS* with start date U, maturity T, a constant rate κ and recovery at default is a defaultable claim $(0, A, Z, \tau)$ where $A_t = -\kappa \mathbb{1}_{[U,T]}(t)\, dL_t$ and $Z_t = \delta_t \mathbb{1}_{[U,T]}(t)$ for every $t \in [0, T]$. An $\mathbb{F}$-predictable process $\delta : [0, T] \to \mathbb{R}$ represents the *default protection* and a constant κ is the *CDS rate.*

An increasing process L represents the tenor structure of fee payments (see, for instance, [8]-[9] or [29]). For a stylized forward CDS, we may set $L_t = t$ (cf. [5]-[6]). This convention is not necessary, however, for the further developments.

Since a forward CDS does not pays any dividends prior to the start date U, its price $S_t(\kappa)$ for any $t \in [s, U]$ can be considered as either the cum-dividend price or the ex-dividend price. Therefore, the price of a forward CDS at any date $t \in [s, U]$ equals

$$S_t(\kappa) = S^c_t(\kappa) = B_t\, \mathbb{E}_{\mathbb{Q}^*}\bigg(\mathbb{1}_{\{U \le \tau \le T\}} B^{-1}_\tau \delta_\tau - \kappa \int_{]\tau\wedge U, \tau\wedge T]} B^{-1}_u \, dL_u \,\Big|\, \mathcal{G}_t \bigg) \tag{4}$$

and formula (3) becomes

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t}\, \mathbb{E}_{\mathbb{Q}^*}\bigg(-\int_U^T B^{-1}_u \delta_u \, dG_u - \kappa \int_{]U,T]} B^{-1}_u G_u \, dL_u \,\Big|\, \mathcal{F}_t \bigg) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa). \tag{5}$$

We also write

$$\widetilde{S}_t(\kappa) = \widetilde{P}(t, U, T) - \kappa \widetilde{A}(t, U, T), \tag{6}$$

where $\widetilde{P}(t, U, T)$ represents the pre-default value at time t of the protection leg and the *CDS annuity* $\widetilde{A}(t, U, T)$ represents the pre-default value at time t of the survival annuity stream per unit of the rate κ, which is usually expressed in basis points. It is worth noting that $\widetilde{A}$ is a strictly positive process. Without loss of generality, we may and do assume that $s = 0$.

By a *forward market CDS* at time $t \in [0, U]$ we mean a forward CDS in which the rate is chosen in such a way that the contract is valueless at time t. The corresponding (pre-default) *forward CDS rate* is an $\mathcal{F}_t$-measurable random variable $\kappa(t, U, T)$ that solves the equation $\widetilde{S}_t(\kappa(t, U, T)) = 0$. It is thus clear that, for every $t \in [0, U]$,

$$\kappa(t, U, T) = \frac{\widetilde{P}(t, U, T)}{\widetilde{A}(t, U, T)} = -\frac{\mathbb{E}_{\mathbb{Q}^*}\Big(\int_U^T B^{-1}_u \delta_u \, dG_u \,\Big|\, \mathcal{F}_t \Big)}{\mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]U,T]} B^{-1}_u G_u \, dL_u \,\Big|\, \mathcal{F}_t \Big)} = \frac{p_t}{a_t}, \tag{7}$$

where the $(\mathbb{Q}^*, \mathbb{F})$-martingales $(p_t,\, t \in [0, U])$ and $(a_t,\, t \in [0, U])$ are given by the following expressions

$$p_t = -\mathbb{E}_{\mathbb{Q}^*}\Big(\int_U^T B^{-1}_u \delta_u \, dG_u \,\Big|\, \mathcal{F}_t \Big), \quad a_t = \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]U,T]} B^{-1}_u G_u \, dL_u \,\Big|\, \mathcal{F}_t \Big). \tag{8}$$

Lemma 2.1 *For a forward CDS with a constant rate κ we have, for every $t \in [0, U]$,*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t, U, T)(\kappa(t, U, T) - \kappa). \tag{9}$$

Proof. It suffices to observe that

$$\begin{aligned} S_t(\kappa) &= S_t(\kappa) - S_t(\kappa(t, U, T)) \\ &= \mathbb{1}_{\{t<\tau\}} \big(\widetilde{P}(t, U, T) - \kappa \widetilde{A}(t, U, T) - \widetilde{P}(t, U, T) + \kappa(t, U, T) \widetilde{A}(t, U, T) \big), \end{aligned}$$

as required. □

2.2 Credit Default Swaption

Let us fix some expiry date $0 < R \leq U$ and let us consider a call option with zero strike on the value of a forward start CDS. The swaption's payoff at its expiry date R thus equals $C_R = (S_R(\kappa))^+$. Using Lemma 2.1, we obtain

$$C_R = \mathbb{1}_{\{R<\tau\}} \widetilde{A}(R,U,T)(\kappa(R,U,T) - \kappa)^+.$$

The formula above shows that a call option with zero strike on the value of a forward CDS with rate κ is formally equivalent to a call option on a forward CDS rate.

We are thus interested in the value C_t of this claim at time $t \in [0,R]$. It is worth stressing that a credit default swaption is considered here as a derivative asset in a market model in which CDSs with maturities R, U and T are assumed to be traded assets. The corresponding forward CDSs can be easily synthetized by static positions in traded CDSs.

Lemma 2.2 *Assume that the claim C_T is attainable. Then the price at time $t \in [0,R]$ of a credit default swaption equals*

$$C_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}^*} \left(\frac{G_R}{B_R} \widetilde{A}(R,U,T) \big(\kappa(R,U,T) - \kappa \big)^+ \,\Big|\, \mathcal{F}_t \right). \tag{10}$$

Proof. It suffices to use the risk-neutral valuation formula. □

In order to simplify formula (10), we shall now define an equivalent probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_R)$. Towards this end, we recall that the process $(a_t,\, t \in [0,T])$ is a strictly positive $(\mathbb{Q}^*, \mathbb{F})$-martingale. Then, we take the Radon-Nikodým density of $\widehat{\mathbb{Q}}$ with respect to $\mathbb{Q}^*$ to be given as

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}^*} = \frac{a_R}{a_0}, \quad \mathbb{Q}^*\text{-a.s.} \tag{11}$$

Therefore, for every $t \in [0,R]$,

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}^*} \Big| \mathcal{F}_t = \frac{a_t}{a_0}, \quad \mathbb{Q}^*\text{-a.s.}$$

It is advantageous to work under $\widehat{\mathbb{Q}}$ since the process $(p_t = a_t \kappa(t,U,T),\, t \in [0,R])$ is a $(\mathbb{Q}^*, \mathbb{F})$-martingale, so that the forward CDS rate $(\kappa(t,U,T),\, t \in [0,R])$ is a $(\widehat{\mathbb{Q}}, \mathbb{F})$-martingale. In addition, the pricing formula for a credit default swaption takes a simpler form under $\widehat{\mathbb{Q}}$, as seen from the following result (cf. [14] or [29]).

Proposition 2.1 *The price of a credit default swaption is given by the following expression, for every $t \in [0,R]$,*

$$C_t = \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t,U,T)\, \mathbb{E}_{\widehat{\mathbb{Q}}} \big((\kappa(R,U,T) - \kappa)^+ \,\big|\, \mathcal{F}_t \big) = \mathbb{1}_{\{t<\tau\}} \widetilde{C}_t, \tag{12}$$

where $\widetilde{C}_t$ is the pre-default value of a credit default swaption at time t.

Proof. Using (10), we obtain

$$\begin{aligned}
C_t &= \mathbb{1}_{\{t<\tau\}} B_t G_t^{-1}\, \mathbb{E}_{\mathbb{Q}^*} \left(G_R B_R^{-1} \widetilde{A}(R,U,T)(\kappa(R,U,T) - \kappa)^+ \,\Big|\, \mathcal{F}_t \right) \\
&= \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t,U,T) a_t^{-1}\, \mathbb{E}_{\mathbb{Q}^*} \left(a_R (\kappa(R,U,T) - \kappa)^+ \,\Big|\, \mathcal{F}_t \right) \\
&= \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t,U,T)\, \mathbb{E}_{\widehat{\mathbb{Q}}} \big((\kappa(R,U,T) - \kappa)^+ \,\big|\, \mathcal{F}_t \big),
\end{aligned}$$

where the last equality follows from the Bayes formula. □

To proceed further, we assume that the filtration $\mathbb{F}$ is generated by a (possibly multidimensional) Brownian motion W under $\mathbb{Q}^*$. Since p and a are strictly positive $(\mathbb{Q}^*, \mathbb{F})$-martingales, we deduce from the predictable representation theorem for a Brownian filtration that

$$dp_t = p_t \sigma_t^p \, dW_t, \quad da_t = a_t \sigma_t^a \, dW_t,$$

for some $\mathbb{F}$-predictable processes σ^p and σ^a.

Lemma 2.3 *Let the filtration $\mathbb{F}$ be generated by a Brownian motion W under $\mathbb{Q}^*$. Then the forward CDS rate $(\kappa(t,U,T),\, t \in [0,R])$ is $(\widehat{\mathbb{Q}}, \mathbb{F})$-martingale and*

$$d\kappa(t,U,T) = \kappa(t,U,T)\sigma_t^\kappa \, d\widehat{W}_t \tag{13}$$

where the $\mathbb{F}$-predictable process σ^κ satisfies $\sigma^\kappa = \sigma^p - \sigma^a$ and the process $\widehat{W}$, which is given by the formula

$$\widehat{W}_t = W_t - \int_0^t \sigma_u^a \, du, \quad \forall\, t \in [0,R],$$

is a $(\widehat{\mathbb{Q}}, \mathbb{F})$-Brownian motion.

Proof. It suffices to apply the Itô formula to (7) and use the Girsanov theorem. □

To a large extent, the above result is model independent. In particular, if the process $\sigma^p - \sigma^a$ is deterministic so that Black formula can be applied for pricing and hedging. Such an assumption is quite constraining, as explained, for example, in Brigo and El-Bachir [13]. In Section 4, we shall present pricing and hedging results for the model in which $\sigma^p - \sigma^a$ is not deterministic, but in which hedging and pricing results are still numerically feasible due to special structure of the hazard process.

2.3 Hedging with a Forward-Start CDS and the Swap Portfolio

In order to get the simplest form of a hedging strategy for a credit default swaption, let us assume that a forward CDS with a fixed rate κ is traded. As a second traded instrument, we take the bond portfolio corresponding to the CDS annuity. Note that $\widetilde{A}(t,U,T)$ can be seen as the pre-default value at time $t \in [0,U]$ of a particular portfolio of defaultable bonds with zero recovery, referred to hereafter as the *swap portfolio*. If default occurs at some date $t \in [0,U]$, the wealth of this portfolio will necessarily fall to zero. Of course, the same property holds for the forward CDS as well as for the credit default swaption. This means that in what follows we will only need to focus on the dynamics of the pre-default value of the swaption and the pre-default wealth of a hedging portfolio.

Let $A(t,U,T)$ be the price process of the swap portfolio at time $t \in [0,U]$. Formally, we set $A(t,U,T) = \mathbb{1}_{\{t<\tau\}}\widetilde{A}(t,U,T)$. Recall also that $S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}\widetilde{S}_t(\kappa)$.

Let $\varphi = (\varphi^1, \varphi^2)$ be a *trading strategy*, where φ^1 and φ^2 are $\mathbb{G}$-predictable processes. The wealth of φ equals, for any $t \in [0,R]$,

$$V_t(\varphi) = \varphi_t^1 S_t(\widehat{\kappa}) + \varphi_t^2 A(t,U,T)$$

and thus the pre-default wealth satisfies, for any $t \in [0,T]$,

$$\widetilde{V}_t(\varphi) = \varphi_t^1 \widetilde{S}_t(\widehat{\kappa}) + \varphi_t^2 \widetilde{A}(t,U,T).$$

Of course, the equality $V_t(\varphi) = \mathbb{1}_{\{t<\tau\}}\widetilde{V}_t(\varphi)$ holds for any $t \in [0,R]$ and thus it suffices to examine a replicating strategy on the interval $[0, \tau \wedge R]$. Therefore, it is enough to search for $\mathbb{F}$-predictable processes $\widetilde{\varphi}^i$, $i = 1,2$ such that for every $t \in [0,R]$ we have that $\mathbb{1}_{\{t<\tau\}}\varphi_t^i = \widetilde{\varphi}_t^i$ for $i = 1,2$. We then say that φ replicates a credit default swaption if $\widetilde{V}_t(\widetilde{\varphi}) = \widetilde{C}_t$ for every $t \in [0,R]$ or, equivalently, if $V_t(\varphi) = C_t$ for every $t \in [0,R]$.

A replicating strategy φ is required to be *self-financing*, in the sense that

$$d\widetilde{V}_t(\varphi) = d\widetilde{V}_t(\widetilde{\varphi}) = \widetilde{\varphi}^1_t \, d\widetilde{S}_t(\widehat{\kappa}) + \widetilde{\varphi}^2_t \, d\widetilde{A}(t,U,T).$$

It can be easily shown by Itô's formula that the relative pre-default wealth satisfies

$$d(\widetilde{V}_t(\widetilde{\varphi})/\widetilde{A}(t,U,T)) = \widetilde{\varphi}^1_t \, d(\widetilde{S}_t(\widehat{\kappa})/\widetilde{A}(t,U,T)). \tag{14}$$

Proposition 2.2 *Assume that the Brownian motion W is one-dimensional. Then the replicating strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \widetilde{\varphi}^2)$ for the credit default swaption is given by, for any $t \in [0, \tau \wedge R]$,*

$$\widetilde{\varphi}^1_t = \frac{\widetilde{C}_t \sigma^C_t}{\kappa_t \sigma^\kappa_t}, \quad \widetilde{\varphi}^2_t = \frac{\widetilde{C}_t - \widetilde{\varphi}^1_t \widetilde{S}_t(\kappa)}{\widetilde{A}(t,U,T)}. \tag{15}$$

Proof. On the one hand, the predictable representation theorem for a Brownian filtration yields

$$d(\widetilde{C}_t/\widetilde{A}(t,U,T)) = \widetilde{C}_t \sigma^C_t \, d\widehat{W}_t.$$

On the other hand, in view of (9), (13) and (14), we obtain

$$d(\widetilde{V}_t(\widetilde{\varphi})/\widetilde{A}(t,U,T)) = \widetilde{\varphi}^1_t \, d(\widetilde{S}_t(\widehat{\kappa})/\widetilde{A}(t,U,T)) = \widetilde{\varphi}^1_t \, d\kappa(t,U,T) = \widetilde{\varphi}^1_t \kappa(t,U,T) \sigma^\kappa_t \, d\widehat{W}_t. \tag{16}$$

A comparison of the formulae above yields the expression for the hedge ratio $\widetilde{\varphi}^1$. Standard arguments show that the strategy $\widetilde{\varphi}$ given by (15) is self-financing and its pre-default wealth satisfies $\widetilde{V}_t(\widetilde{\varphi}) = \widetilde{C}_t$ for every $t \in [0, R]$. As already mentioned above, it default occurs prior to or at expiration date R of a swaption then the wealth of the portfolio φ falls to zero, and the same property is satisfied by the price process of the credit default swaption. □

In view of the last result, the problem of searching for a replicating strategy for a credit default swaption is reduced to explicit computations of processes $\widetilde{C}$ and κ appearing in formula (15) and the respective volatilities σ^C and σ^κ. Of course, such computations far from trivial, in general, and they rarely lead to closed-form analytical results. In the next subsection, we will consider the classic case where they are fairly standard.

2.4 Black Formula for Credit Default Swaptions

The goal of this section is to examine briefly the case of a deterministic volatility of a forward CDS rate. For a more detailed discussion of the use of the Black formula in valuing credit default (index) swaptions, the interested reader is referred to Brigo and Morini [14], Morini and Brigo [27], and Rutkowski and Armstrong [29]. Let us only mention here that although an arbitrage-free model of a family of forward CDS rates underpinning Proposition 2.3 is rather difficult to construct, the pricing formula (17) corresponds to the market convention for valuing credit default swaptions, and thus it is natural to consider it as a benchmark.

Proposition 2.3 *Assume that the Brownian motion W is one-dimensional and the volatility σ^κ of the forward CDS rate $(\kappa(t,U,T),\, t \in [0,R])$ is deterministic. Then the pre-default value of a credit default swaption maturing at R equals, for every $t \in [0,R]$,*

$$\widetilde{C}_t = \widetilde{A}(t,U,T) \Big(\kappa_t N\big(d_+(\kappa_t,t,R)\big) - N\big(d_-(\kappa_t,t,R)\big) \Big), \tag{17}$$

where we write $\kappa_t = \kappa(t,U,T)$ and where

$$d_\pm(\kappa_t,t,R) = \frac{\ln(\kappa_t/\kappa) \pm \frac{1}{2}\int_t^R (\sigma^\kappa(u))^2 \, du}{\sqrt{\int_t^R (\sigma^\kappa(u))^2 \, du}}.$$

Equivalently,

$$\widetilde{C}_t = \widetilde{P}(t,U,T) \, N\big(d_+(\kappa_t,t,R)\big) - \widetilde{A}(t,U,T) \, N\big(d_-(\kappa_t,t,R)\big). \tag{18}$$

The replicating strategy $\widetilde{\varphi}$ is given by (15) with $\widetilde{\varphi}^1_t = N\big(d_+(\kappa(t,U,T),t,R)\big)$.

Proof. Derivation of the pricing formula (17) relies on standard arguments and thus it is omitted. To find the hedging strategy, it suffices to observe that, by a straightforward application of the Itô formula to expression (17), the volatility of the process $\widetilde{C}$ equals, for every $t \in [0,T]$,

$$\sigma_t^C = (\widetilde{C}_t)^{-1} \kappa_t \sigma_t^\kappa \, N\big(d_+(\kappa(t,U,T),t,R)\big).$$

The results now follows by Proposition 2.2. □

In practice, hedging should rather be done by taking positions at any date t in the *market CDS*, that is, the just-issued CDS with the fixed spread $\kappa(t,U,T)$. An explicit representation for this strategy in continuous-time is more cumbersome, however, since one needs to deal with an uncountable family of traded assets (see Section 1.1.3 in [6]).

3 Modeling of Default Time

The goal of this section is to analyze a method of modeling the default time, which, under some assumptions, will be shown to be consistent with a hazard process model of Section 1. In particular, we will establish some formulae for the volatilities of $\mathbb{F}$-martingales p and a introduced in Section 2. To this end, we need to introduce suitable definitions. Let then τ be a random time defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$.

Definition 3.1 For any fixed $u \in \mathbb{R}_+$, we define the $\mathbb{F}$-martingale $G_t^u = \mathbb{Q}^*(\tau > u \,|\, \mathcal{F}_t)$ for $t \in [0,T]$.

For conciseness, we shall frequently write G_t instead of G_t^t (of course, this convention is consistent with notation introduced in Section 1). Recall that the process $(G_t,\, t \in [0,T])$ is an $\mathbb{F}$-supermartingale. We also assume that G is a strictly positive process.

We will work throughout under the following standing assumption, which also underpins the papers by El Karoui et al. [16] and Jeanblanc and Le Cam [23].

Assumption 3.1 There exists a family of $\mathbb{F}$-adapted processes $(f_t^x,\, t \in [0,T])$, where $x \in \mathbb{R}_+$, such that, for any $u \in \mathbb{R}_+$,

$$G_t^u = \int_u^\infty f_t^x \, dx, \quad \forall\, t \in [0,T].$$

Assumption 3.1 implies, in particular, that the probability distribution of the random variable τ has the probability density function $f_0^{\cdot}$ with respect to Lebesgue measure, so that $\mathbb{Q}^*(\tau \in dx) = f_0^x \, dx$. More generally, for any $t \in [0,T]$, the random variable $f_t^{\cdot}$ represents the conditional density of τ with respect to the σ-field $\mathcal{F}_t$, that is, $f_t^x \, dx = \mathbb{Q}^*(\tau \in dx \,|\, \mathcal{F}_t)$. To alleviate notation, we shall write $f_t^t = f_t$ and we denote $\widehat{\lambda}_t = G_t^{-1} f_t$.

Note that Assumption 3.1 implies that any $(\mathbb{Q}^*, \mathbb{F})$-martingale is a $(\mathbb{Q}^*, \mathbb{G})$-semimartingale (see Jacod [19] or Jeanblanc and Le Cam [23]). Moreover, we have the following well known result.

Lemma 3.1 *Under Assumption 3.1, the process $(\widehat{M}_t,\, t \in [0,T])$, given by the formula*

$$\widehat{M}_t = H_t - \int_0^{t\wedge\tau} \widehat{\lambda}_u \, du,$$

is a $(\mathbb{Q}^*, \mathbb{G})$-martingale.

Proof. For the sake of completeness, we provide the proof of the lemma (it can be found in [16] and [23]). Let us fix $0 \le s < t \le T$. We have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^*}(\widehat{M}_t - \widehat{M}_s \,|\, \mathcal{G}_s) &= \mathbb{1}_{\{s<\tau\}} \, \mathbb{E}_{\mathbb{Q}^*}\bigg(\mathbb{1}_{\{s<\tau\le t\}} - \int_s^{t\wedge\tau} \widehat{\lambda}_u \, du \,\Big|\, \mathcal{G}_s \bigg) \\
&= \mathbb{1}_{\{s<\tau\}} \, \mathbb{E}_{\mathbb{Q}^*}\bigg(\mathbb{1}_{\{s<\tau\le t\}} - \mathbb{1}_{\{t<\tau\}} \int_s^t \widehat{\lambda}_u \, du - \mathbb{1}_{\{\tau\le t\}} \int_s^t \mathbb{1}_{\{u<\tau\le t\}} \widehat{\lambda}_u \, du \,\Big|\, \mathcal{G}_s \bigg).
\end{aligned}$$

For any $0 \le s < t \le T$, the following equalities hold, on the event $\{s < \tau\}$,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{s<\tau\le t\}} \,|\, \mathcal{G}_s\big) &= \frac{1}{G_s}(G_s - G_s^t),\\
\mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\}} \int_s^t \widehat{\lambda}_u \, du \,\Big|\, \mathcal{G}_s\Big) &= \frac{1}{G_s}\, \mathbb{E}_{\mathbb{Q}^*}\Big(G_t \int_s^t \widehat{\lambda}_u \, du \,\Big|\, \mathcal{F}_s\Big),\\
\mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{\tau\le t\}} \int_s^t \mathbb{1}_{\{u<\tau\le t\}} \widehat{\lambda}_u \, du \,\Big|\, \mathcal{G}_s\Big) &= \frac{1}{G_s}\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^t \widehat{\lambda}_u (G_t^u - G_t)\, du \,\Big|\, \mathcal{F}_s\Big),
\end{aligned}
$$

where two last equalities are obtained by conditioning with respect to $\mathcal{F}_t$. Consequently,

$$
\mathbb{E}_{\mathbb{Q}^*}(\widehat{M}_t - \widehat{M}_s \,|\, \mathcal{G}_s) = \mathbb{1}_{\{s<\tau\}} \frac{1}{G_s} \left(G_s - G_s^t - \mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^t \widehat{\lambda}_u G_t^u \, du \,\Big|\, \mathcal{F}_s\Big)\right).
$$

To conclude that $\mathbb{E}_{\mathbb{Q}^*}(\widehat{M}_t - \widehat{M}_s \,|\, \mathcal{G}_s) = 0$ for $s \le t$, it suffices to note that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^t \widehat{\lambda}_u G_t^u \, du \,\Big|\, \mathcal{F}_s\Big) &= \int_s^t \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{E}_{\mathbb{Q}^*}\big(\widehat{\lambda}_u G_t^u \,|\, \mathcal{F}_u\big) \,\Big|\, \mathcal{F}_s\Big)\, du\\
&= \int_s^t \mathbb{E}_{\mathbb{Q}^*}\Big(\widehat{\lambda}_u \, \mathbb{E}_{\mathbb{Q}^*}\big(G_t^u \,|\, \mathcal{F}_u\big) \,\Big|\, \mathcal{F}_s\Big)\, du = \int_s^t \mathbb{E}_{\mathbb{Q}^*}(\widehat{\lambda}_u G_u \,|\, \mathcal{F}_s)\, du\\
&= \mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^t f_u \, du \,\Big|\, \mathcal{F}_s\Big) = G_s - G_s^t,
\end{aligned}
$$

since $\mathbb{E}_{\mathbb{Q}^*}(G_t^u \,|\, \mathcal{F}_u) = \mathbb{Q}^*(\tau > u \,|\, \mathcal{F}_u) = G_u$ for $u \le t$. □

The next standing assumption will allow us to make use of the predictable representation property of the Brownian filtration.

Assumption 3.2 The filtration $\mathbb{F}$ is generated by a one-dimensional Brownian motion $(W_t,\, t \in [0,T])$ under $\mathbb{Q}^*$.

Under Assumption 3.2, for any fixed $u \in \mathbb{R}_+$, the $(\mathbb{Q}^*, \mathbb{F})$-martingale G^u admits the following integral representation, for $t \in [0,T]$,

$$
G_t^u = G_0^u + \int_0^t g_s^u \, dW_s \tag{19}
$$

for some $\mathbb{F}$-predictable, real-valued process $(g_t^u,\, t \in [0,T])$. Similarly, for any fixed $x \in \mathbb{R}_+$, the process $(f_t^x,\, t \in [0,T])$ is a non-negative $(\mathbb{Q}^*, \mathbb{F})$-martingale and thus there exists an $\mathbb{F}$-predictable process $(\sigma_t^x,\, t \in [0,T])$ such that, for $t \in [0,T]$,

$$
f_t^x = f_0^x + \int_0^t \sigma_s^x \, dW_s. \tag{20}
$$

Since the stochastic Fubini's theorem yields

$$
G_t^u = \int_u^\infty f_t^x \, dx = \int_u^\infty \Big(f_0^x + \int_0^t \sigma_s^x \, dW_s\Big) dx = G_0^u + \int_0^t dW_s \int_u^\infty \sigma_s^x \, dx,
$$

we conclude that the following relationship is valid, for every $u \in \mathbb{R}_+$ and $t \in [0,T]$,

$$
g_t^u = \int_u^\infty \sigma_t^x \, dx.
$$

By applying the Itô-Wentzell-Kunita formula (see Theorem 3.3.1 in Kunita [25] or Section 5 below), we obtain the following auxiliary result, where we denote $g_s^s = g_s$ and $f_s^s = f_s$.

Lemma 3.2 *Under Assumptions 3.1-3.2, the Doob-Meyer decomposition of the survival process G reads, for every $t \in [0,T]$,*

$$G_t = G_0 + \int_0^t g_u \, dW_u - \int_0^t f_u \, du. \tag{21}$$

In particular, G is a continuous process.

The following result shows that under Assumptions 3.1-3.2 we can make use of all results of Section 1.

Lemma 3.3 *Assumptions 3.1-3.2 imply that Assumption 1.1 is satisfied. Moreover, the equality $\widehat{\lambda} = \lambda$ is valid and thus the process $\widehat{M} = M$ is a $(\mathbb{Q}^*, \mathbb{G})$-martingale.*

Proof. In view of Lemma 3.2, for the process ν in the Doob-Meyer decomposition of G we obtain $d\nu_t = f_t \, dt = \widehat{\lambda}_t G_t \, dt$. Therefore, Assumption 1.1 is satisfied and the equality $\widehat{\lambda} = \lambda$ holds. This in turn implies that, for every $t \in [0,T]$,

$$\widehat{M}_t = H_t - \int_0^{t\wedge\tau} \widehat{\lambda}_u \, du = H_t - \int_0^{t\wedge\tau} \lambda_u \, du = M_t$$

and thus $\widehat{M}$ is a $(\mathbb{Q}^*, \mathbb{G})$-martingale since, under Assumption 1.1, the process M given by (2) is known to be a $(\mathbb{Q}^*, \mathbb{G})$-martingale. □

It is shown in [23] that, under Assumptions 3.1-3.2, the process

$$\widetilde{W}_t = W_t - \int_0^{t\wedge\tau} \frac{g_u}{G_u} \, du + \int_{t\wedge\tau}^t \frac{\sigma_u^\tau}{f_u^\tau} \, du \tag{22}$$

is a $(\mathbb{Q}^*, \mathbb{G})$-Brownian motion. We take this result for granted and we refer to [23] for the proof. Let us note that on the event $\{t < \tau\}$ the process $\widetilde{W}$ satisfies

$$d\widetilde{W}_t = dW_t - g_t G_t^{-1} \, dt.$$

3.1 Price Dynamics of a Defaultable Claim

The following assumption will allow us to establish more explicit representations for (pre-default) prices.

Assumption 3.3 The quantities Z, X, A and B are deterministic. To emphasize this feature, we will write $Z(t), A(t), B(t)$ and $\beta(t) = B^{-1}(t)$, rather than Z_t, A_t, B_t and $\beta_t = B_t^{-1}$.

The main goal of the next result is to derive a more explicit representation for the volatility term appearing in the price dynamics.

Proposition 3.1 *Under Assumptions 3.1-3.3, the dynamics of the pre-default price $\widetilde{S}$ are*

$$d\widetilde{S}_t = \big((r(t) + \lambda_t)\widetilde{S}_t - \lambda_t Z(t)\big) \, dt + dA(t) + \zeta_t \big(dW_t - g_t G_t^{-1} \, dt\big)$$

and the dynamics of the cumulative price S^c are

$$dS_t^c = r(t) S_t^c \, dt + (Z(t) - \widetilde{S}_t) \, dM_t + (1 - H_t)\zeta_t \, d\widetilde{W}_t,$$

where the $(\mathbb{Q}^*, \mathbb{G})$-Brownian motion $\widetilde{W}$ is given by (22) and the process ζ equals, for every $t \in [0,T]$,

$$\zeta_t = G_t^{-1}\big(B(t)\nu_t - \widetilde{S}_t g_t\big) \tag{23}$$

with

$$\nu_t = \beta(T) X G_t^T + \int_t^T \beta(u) Z(u) \sigma_t^u \, du + \int_t^T \beta(u) g_t^u \, dA(u). \tag{24}$$

Proof. Under Assumptions 3.1-3.3, using the martingale properties of the processes G^u and f^u, we obtain from (3)

$$S_t = \mathbb{1}_{\{t<\tau\}} \frac{B(t)}{G_t}\Big(\beta(T)XG_t^T + \int_t^T \beta(u)\big(Z(u)f_t^u\,du + G_t^u\,dA(u)\big)\Big) \tag{25}$$

and

$$\begin{aligned} m_t &= \beta(T)XG_t^T + \int_0^t \beta(u)Z(u)f_u\,du + \int_t^T \beta(u)Z(u)f_t^u\,du \\ &\quad + \int_0^t \beta(u)G_u\,du + \int_t^T \beta(u)G_t^u\,du. \end{aligned}$$

It follows that

$$dm_t = \Big(\beta(T)XG_t^T + \int_t^T \beta(u)Z(u)\sigma_t^u\,du + \int_t^T \beta(u)g_t^u\,dA(u)\Big)dW_t = \nu_t\,dW_t,$$

where the process ν is given by (24). Hence, using the equality $\mu_t = \int_0^t g_s\,dW_s$, we obtain

$$d\langle\mu\rangle_t = g_t^2\,dt, \quad d\langle\mu, m\rangle_t = g_t\nu_t\,dt$$

In view of (23), the asserted formulae follow directly from Proposition 1.1. □

3.2 Price Dynamics of a Forward CDS

In what follows, we shall work under Assumptions 3.1-3.3. Let $S(\kappa)$ be the price of a forward CDS with the protection payment $\delta(\tau)$ at time τ on the event $\{U \le \tau \le T\}$, where δ is some function, κ is the constant spread and L is an increasing function. By the risk-neutral valuation formula (4), the price $S_t(\kappa)$ of a forward CDS equals, for every $t \in [0, U]$,

$$S_t(\kappa) = B(t)\,\mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U\le\tau\le T\}}\beta(\tau)\delta(\tau) - \kappa \int_{]\tau\wedge U, \tau\wedge T]} \beta(u)\,dL(u)\,\Big|\,\mathcal{G}_t\Big)$$

for a deterministic increasing function L specifying the tenor structure of fee payments.

The following result is a rather straightforward consequence of Proposition 3.1.

Corollary 3.1 *Under Assumptions 3.1-3.3, the price $S(\kappa)$ satisfies, for every $t \in [0, U]$,*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B(t)}{G_t}\Big(\int_U^T \beta(u)\delta(u)f_t^u\,du - \kappa\int_{]U,T]} \beta(u)G_t^u\,dL(u)\Big). \tag{26}$$

The dynamics of the process $\widetilde{S}(\kappa)$ are, for $t \in [0, U]$,

$$d\widetilde{S}_t(\kappa) = (r(t) + \lambda_t)\widetilde{S}_t(\kappa)\,dt + \zeta_t\Big(dW_t - \frac{g_t}{G_t}\,dt\Big),$$

where

$$\zeta_t = \frac{B(t)}{G_t}\Big(\int_U^T \beta(u)\delta(u)\sigma_t^u\,du - \kappa\int_{]U,T]} \beta(u)g_t^u\,dL(u)\Big) - \widetilde{S}_t(\kappa)\frac{g_t}{G_t}.$$

Consequently, the ex-dividend prices satisfies, for $t \in [0, U]$,

$$dS_t(\kappa) = r(t)S_t(\kappa)\,dt - S_{t-}(\kappa)\,dM_t + (1 - H_t)\zeta_t\Big(dW_t - \frac{g_t}{G_t}\,dt\Big)$$

and the cumulative price satisfies, for $t \in [0, U]$,

$$dS_t^c(\kappa) = r_tS_t^c(\kappa)\,dt + \big(\delta(t) - S_{t-}(\kappa)\big)\,dM_t - (1 - H_t)\zeta_t\Big(dW_t - \frac{g_t}{G_t}\,dt\Big).$$

The final step is the computation of the volatilities appearing in dynamics (13) of the forward CDS rate. Of course, the usefulness of Lemma 3.4 depends on the possibility of explicit computations of these volatilities. In Section 4, we will argue that such computations can be performed in the CIR intensity model. Recall that the positive $(\mathbb{Q}^*, \mathbb{F})$-martingales p and a are defined by (8).

Lemma 3.4 *Under Assumptions 3.1-3.3, the volatilities σ^p and σ^a of positive $(\mathbb{Q}^*, \mathbb{F})$-martingales p and a are given by the following expressions*

$$\begin{aligned} \sigma_t^p &= \Big(\int_U^T \beta(u)\delta(u)\sigma_t^u \, du\Big)\Big(\int_U^T \beta(u)\delta(u) f_t^u \, du\Big)^{-1}, \\ \sigma_t^a &= \Big(\int_{]U,T]} \beta(u) g_t^u \, dL(u)\Big)\Big(\int_{]U,T]} \beta(u) G_t^u \, dL(u)\Big)^{-1}. \end{aligned}$$

Proof. Using the martingale properties of processes G^u and f^u, we obtain

$$\begin{aligned} p_t &= -\mathbb{E}_{\mathbb{Q}^*}\Big(\int_U^T \beta(u)\delta(u)\, dG_u \,\Big|\, \mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{Q}^*}\Big(\int_U^T \beta(u)\delta(u) f_u \, du \,\Big|\, \mathcal{F}_t\Big) \\ &= \int_U^T \beta(u)\delta(u)\, \mathbb{E}_{\mathbb{Q}^*}(f_u|\mathcal{F}_t)\, du = \int_U^T \beta(u)\delta(u) f_t^u \, du, \\ a_t &= \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]U,T]} \beta(u) G_u \, dL(u) \,\Big|\, \mathcal{F}_t\Big) = \int_{]U,T]} \beta(u)\mathbb{E}_{\mathbb{Q}^*}(G_u|\mathcal{F}_t)\, dL(u) = \int_{]U,T]} \beta(u) G_t^u \, dL(u). \end{aligned}$$

Therefore,

$$\begin{aligned} dp_t &= \int_U^T \beta(u)\delta(u)\, d_t f_t^u du = \Big(\int_U^T \beta(u)\delta(u)\sigma_t^u \, du\Big)\, dW_t, \\ da_t &= \int_{]U,T]} \beta(u)\, d_t G_t^u dL(u) = \Big(\int_{]U,T]} \beta(u) g_t^u \, dL(u)\Big)\, dW_t. \end{aligned}$$

We conclude that

$$dp_t = p_t \sigma_t^p \, dW_t, \quad da_t = a_t \sigma_t^a \, dW_t$$

with the volatilities σ^p and σ^a given in the statement of the lemma. □

3.3 Immersion Property

It is not uncommon to construct a default time τ in such a way that the filtration $\mathbb{F}$ is *immersed* in $\mathbb{G}$. Recall that a filtration $\mathbb{F}$ is said to be *immersed* in a filtration $\mathbb{G}$ under $\mathbb{Q}^*$, where $\mathbb{F} \subset \mathbb{G}$, if any $(\mathbb{Q}^*, \mathbb{F})$-martingale is a $(\mathbb{Q}^*, \mathbb{G})$-martingale; this condition is also frequently referred to as the (H) hypothesis.

Assumption 3.4 The filtration $\mathbb{F}$ is immersed in the filtration $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$.

In our setting, Assumption 3.4 implies that G is an increasing process and, for any $u \in [0, T]$, the $\mathbb{F}$-martingale G^u is stopped at time u. The following lemma is thus easy to establish.

Lemma 3.5 *Under Assumptions 3.1-3.4, we have that $g_t = 0$ for every $t \in [0, T]$. Moreover, $f_t^u = f_u^u = f_u$ for every $0 \le u < t \le T$ and thus $\sigma_t^u = 0$ for every $0 \le u < t \le T$. Consequently, the equality $W = \widetilde{W}$ holds and thus an $(\mathbb{Q}^*, \mathbb{F})$-Brownian motion W is also a $(\mathbb{Q}^*, \mathbb{G})$-Brownian motion.*

By combining Proposition 3.1 with Lemma 3.2, we obtain the following result. Of course, this result can also be applied to a forward CDS.

Corollary 3.2 *Under Assumptions 3.1-3.4, we have that, for every $t \in [0,T]$,*

$$d\widetilde{S}_t = \Big((r(t) + \lambda_t)\widetilde{S}_t - \lambda_t Z(t) \Big)\, dt + dA(t) + \zeta_t \, dW_t$$

and the dynamics of the cumulative price are

$$dS^c_t = r(t) S^c_t \, dt + (Z(t) - \widetilde{S}_t)\, dM_t + (1 - H_t)\zeta_t \, dW_t$$

with $\zeta_t = G_t^{-1} B(t) \nu_t^T$.

3.4 Modeling of f_t^x

We are looking for a family of non-negative martingales f^x such that $\int_0^\infty f_t^x \, dx = 1$, so that they can be considered as probability densities in x.

3.4.1 A backward methodology

Assume that, for some $T \leq \infty$, a family f_T^x of $\mathcal{F}_T$-measurable non negative random variables is given, so that

$$\int_0^\infty f_T^x \, dx = 1.$$

Then, setting

$$f_t^x := \mathbb{E}(f_T^x | \mathcal{F}_t) \tag{27}$$

will provide a family of densities. A family of f_T^x can be constructed in different ways. We shall present two possible constructions.

A first example is to start with a family $\varphi(x, \alpha)$ of densities in $x \in \mathbb{R}_+$, depending on a parameter $\alpha \in A \subset \mathbb{R}^d$, so that:

$$\int_{\mathbb{R}_+} \varphi(x, \alpha) dx = 1.$$

Then, take X an $\mathcal{F}_T$-measurable random variable, taking value in A, and $f_T^x := \varphi(x, X)$. In particular we can take $\varphi(x, \alpha) = \alpha e^{-\alpha x}$. In that case, $f_t^x = \mathbb{E}(X e^{-xX} | \mathcal{F}_t)$ can be computed as $\partial_x \mathbb{E}(e^{-xX} | \mathcal{F}_t)$.

Another example is to assume that

$$f_T^x = \lambda_T^x \exp\left(- \int_0^x \lambda_T^u \, du \right),$$

where $\lambda_T^u, u \geq 0$, is a family of non-negative $\mathcal{F}_T$-measurable random variables such that $\int_0^\infty \lambda_T^u du = \infty$.

3.4.2 A forward construction

Here we consider a random field f_t^x, $x, t \geq 0$, where

$$f_t^x = \lambda_t^x \exp\Big(- \int_0^x \lambda_t^u \, du \Big)$$

for a family of non-negative processes $\lambda^u, \ u \geq 0$, that satisfy

$$d\lambda_t^u = \lambda_t^u \big(a_t^u \, dt + \sigma_t^u \, dW_t \big). \tag{28}$$

Our aim is to find conditions on the coefficients a_t^u, σ_t^u, so that for each $x \geq 0$ the process f^x is a non-negative martingale.

Lemma 3.6 *The martingale property of f^x is equivalent to the following condition*

$$a_t^x - \int_0^x a_t^u \lambda_t^u \, du - \sigma_t^x \int_0^x \lambda_t^u \sigma_t^u \, du + \frac{1}{2}\left(\int_0^x \sigma_t^u \lambda_t^u du\right)^2 = 0. \tag{29}$$

Proof. Application Itô's lemma yields

$$\begin{aligned} df_t^x &= \exp\Big(-\int_0^x \lambda_t^u \, du\Big)\left(d_t\lambda_t^x - \lambda_t^x \left(\int_0^x d_t\lambda_t^u \, du + \frac{1}{2}(\int_0^x \lambda_t^u \sigma_t^u du)^2\right) dt - \sigma_t^x \left(\lambda_t^x \int_0^x \lambda_t^u \sigma_t^u du\right) dt\right) \\ &= \exp\Big(-\int_0^x \lambda_t^u \, du\Big)\lambda_t^x \left(\mu_t^x dt + \Sigma_t^x \, dW_t\right) \end{aligned}$$

with $\mu_t^x = a_t^x - \int_0^x a_t^u \lambda_t^u \, du - \sigma_t^x \int_0^x \lambda_t^u \sigma_t^u \, du + \frac{1}{2}\left(\int_0^x \sigma_t^u \lambda_t^u du\right)^2$. □

Condition (29) is satisfied, for example, if $a_t^x = \sigma_t^x \int_0^x \lambda_t^u \sigma_t^u \, du$, in which case equation (28) takes the form

$$d\lambda_t^x = \lambda_t^x \sigma_t^x \left(\int_0^t \lambda_t^u \sigma_t^u \, du\right) dt + \lambda_t^x \sigma_t^x dW_t. \tag{30}$$

Please note the obvious analogy of the above equation with HJM model for the instantaneous forward interest rates. Also, please note that it is not guaranteed that the solution of the equation (30) is non-negative.

We shall provide a general template for models yielding non-negative families $\lambda^u, \ u \geq 0$ of solutions to (30). Towards this end we shall first rewrite the equation as

$$\lambda_t^x = \lambda_0^x + \int_0^t \psi_s^x \Psi_s^x \, ds + \int_0^t \psi_s^x dW_s, \tag{31}$$

where $\psi_t^x = \lambda_t^x \sigma_t^x$ and $\Psi_t^x = \int_0^x \psi_t^u \, du$. Non-negativity of λ^x is satisfied if

- process ψ^x is non-negative, and hence the process $\psi^x \Psi^x$ is non-negative as well,
- the process $Z_t^x := \lambda_0^x + \int_0^t \psi_s^x dW_s$ is a Doléans-Dade exponential with initial value $\lambda_0^x > 0$, that is, Z^x satisfies $Z_t^x = \lambda_0^x + \int_0^t Z_s^x b_s^x dW_s$, for some process b^x. A sufficient for this to happen is that ψ^x is such that for every $t \geq 0$ we have

  $$\int_0^t \psi_s^x dW_s = \int_0^t b_s^x Z_s^x dW_s.$$

  This means that

  $$\psi_t^x = b_t^x Z_t = b_t^x \lambda_0^x \exp\left(\int_0^t b_s^x dW_s - \frac{1}{2}\int_0^t (b_s^x)^2 ds\right). \tag{32}$$

We thus obtain the following

Proposition 3.2 *Let $\lambda_0^x > 0$ and b^x be a non-negative $\mathbb{F}$-adapted process, where $x \geq 0$. Define ψ_t^x by*

$$\psi_t^x = b_t^x \, \lambda_0^x \, \exp\left(\int_0^t b_s^x \, dW_s - \frac{1}{2}\int_0^t (b_s^x)^2 \, ds\right), \tag{33}$$

and let

$$f_t^x = \lambda_t^x \exp\left(-\int_0^x \lambda_t^v \, dv\right),$$

where

$$\lambda_t^x = \lambda_0^x + \int_0^x \psi_s^x \, dW_s + \int_0^t \psi_s^x \Psi_s^x \, ds.$$

If the family $\lambda^x, \ x \geq 0$, satisfies the property that $\int_0^\infty \lambda_t^x \, dx = \infty$, then the family $(f^x, \ x \geq 0)$ satisfies the required assumptions specified in the beginning of this section.

3.4.3 A Cox process approach

We give another method motivated by construction of the first jump of a Cox processes.

Assume that a non-negative $\mathbb{F}$-adapted process λ is given and set $\Lambda_t = \int_0^t \lambda_s ds$. Let Θ be a random variable independent of $\mathcal{F}_\infty$ with unit exponential law, and let V be an $\mathcal{F}_\infty$-measurable non-negative random variable. We define:

$$\tau = \inf\{t : \Lambda_t \geq \Theta/V\}.$$

For any $x \geq 0$ and $t \geq 0$ we have

$$\mathbb{P}(\tau > x|\mathcal{F}_t) \quad = \quad \mathbb{E}\left(\mathbb{P}\left(\Lambda_x V < \Theta|\, \mathcal{F}_\infty\right)|\mathcal{F}_t\right) = \mathbb{E}\left(\exp -(V\Lambda_x)|\mathcal{F}_t\right) = \int_x^\infty f_t^u du,$$

with

$$f_t^x = -\frac{d}{dx}\mathbb{E}\left(\exp -(V\Lambda_x)|\, \mathcal{F}_t\right) = \mathbb{E}\left(V\lambda_x \, \exp\left(-V\Lambda_x\right)|\mathcal{F}_t\right).$$

It is straightforward to see that the family $(f^x,\ x \geq 0)$ satisfies the required assumptions specified in the beginning of this section.

3.4.4 A functional approach

We assume that f_t^x is a strictly positive random field. Then, we have that for all $x \geq 0$

$$f_t^x = f_0^x + \int_0^t f_u^x \, \Sigma_u^x \, dW_u,$$

for some predictable process Σ^x. Observe that the normalization condition $\int_0^\infty f_t^x dx = 1$, which needs to be satisfied for all $t \geq 0$, implies that

$$\int_0^\infty f_t^x \, \Sigma_t^x \, dx = 0$$

for all $t \geq 0$. A sufficient condition for the random field Σ_t^x to satisfy the above condition is that

$$\Sigma_t^x = \Psi_t^x - \int_0^\infty f_t^y \, \Psi_t^y \, dy,$$

for some family of predictable processes Ψ_t^y. Thus, f_t^x satisfies

$$f_t^x = f_0^x + \int_0^t f_u^x \left(\Psi_u^x - \int_0^\infty f_u^y \, \Psi_u^y \, dy \right) dW_t. \tag{34}$$

This indicates a possible method for modeling of f_t^x in terms of f_0^x and Ψ_u^x starting from the above equation.

Any family f^x satisfying the above equation is obviously a family of local martingales fulfilling the normalization condition. However, non-negativity and martingale properties of such a family are not obvious. At the moment we were unable to construct any example of family f^x satisfying the above equation and fulfilling the required assumptions specified in the beginning of this section. We note that equation of this type has been recently studied by Macrina et. al [17].

4 Intensity-Based Modeling

The goal of this section is to provide an example of a model in which our standing assumptions are satisfied and to show that this model is amenable for quasi-explicit computations of the price of a credit default swaption and its replicating strategy in term of the underlying forward CDS and the corresponding swap portfolio process.

It is not uncommon to start modeling by specifying the dynamics of the default intensity process. Following this approach, let us postulate that we are given a non-negative and $\mathbb{F}$-predictable process λ defined on some probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, which is endowed with a filtration $\mathbb{F}$. The default time is defined by the formula

$$\tau = \inf \Big\{ t \in \mathbb{R}_+ : \int_0^t \lambda_u \, du \geq x \Big\}, \tag{35}$$

where x is a random variable with the unit exponential distribution, independent of the filtration $\mathbb{F}$. Note that τ can be seen as the moment of the first jump of a Cox process with the intensity process λ. Let us denote $\Lambda_t = \int_0^t \lambda_u \, du$. It is easily seen that the $(\mathbb{Q}^*, \mathbb{F})$-martingale $(G_t^u, \, t \in [0,T])$ satisfies

$$G_t^u = \mathbb{E}_{\mathbb{Q}^*}\big(e^{-\Lambda_u} \,\big|\, \mathcal{F}_t\big) \ \text{ for } t \in [0,u[, \quad G_t^u = e^{-\Lambda_u} \ \text{ for } t \in [u,T],$$

so that the process G equals $G_t = e^{-\Lambda_t}$ for every $t \in [0,T]$. Moreover, the $(\mathbb{Q}^*, \mathbb{F})$-martingale $(f_t^x, \, t \in [0,T])$ satisfies

$$f_t^x = \mathbb{E}_{\mathbb{Q}^*}\big(\lambda_x e^{-\Lambda_x} \,\big|\, \mathcal{F}_t\big) \ \text{ for } t \in [0,x[, \quad f_t^x = \lambda_x e^{-\Lambda_x} \ \text{ for } t \in [x,T].$$

It is well known that the immersion property holds between $\mathbb{F}$ and $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. If, in addition, $\mathbb{F}$ is the Brownian filtration then Assumptions 3.1-3.4 are satisfied.

4.1 CIR Intensity Model

For the sake of concreteness, we examine a special case of the Cox process model in which the default intensity process λ is governed by the CIR dynamics

$$d\lambda_t = \mu(\lambda_t)\, dt + \nu(\lambda_t)\, dW_t, \quad \lambda_0 > 0,$$

where $\mu(\lambda) = a - b\lambda$, $\nu(\lambda) = c\sqrt{\lambda}$ and W is a one-dimensional Brownian motion, which generates the filtration $\mathbb{F}$. It is well known that under the assumption that $2a > c$ the unique solution to this SDE is strictly positive. We postulate that the default time τ is given by formula (35).

Let us denote, for arbitrary $0 \leq t \leq u \leq T$,

$$H_t^u = \mathbb{E}_{\mathbb{Q}^*}\big(e^{-(\Lambda_u - \Lambda_t)} \,\big|\, \mathcal{F}_t\big) = \frac{G_t^u}{G_t}. \tag{36}$$

It is known (see, e.g., [28], page 357, or [24], Section 6.3.4) that

$$H_t^u = e^{m(t,u) - n(t,u)\lambda_t} = \widehat{H}(\lambda_t, t, u), \tag{37}$$

where we write

$$\widehat{H}(y,t,u) = e^{m(t,u) - n(t,u)y} \tag{38}$$

and where the functions m and n are given by explicit formulae, specifically,

$$m(t,T) = \frac{2a}{c^2} \ln \left\{ \frac{\gamma e^{b(T-t)/2}}{\gamma \cosh \gamma(T-t) + \frac{1}{2} b \sinh \gamma(T-t)} \right\}$$

and

$$n(t,T) = \frac{\sinh \gamma(T-t)}{\gamma \cosh \gamma(T-t) + \frac{1}{2} b \sinh \gamma(T-t)},$$

where in turn $2\gamma = (b^2 + 2c^2)^{1/2}$. It is important to notice that, for any fixed $t \in \mathbb{R}_+$, the function $n(t,T)$, $T \geq t$, is strictly increasing. Moreover, the function n is strictly positive and thus, for any fixed u and t, the auxiliary function $\widehat{H}(y,t,u)$ is decreasing and continuous in $y \in \mathbb{R}_+$.

Now, let $D^0(t,T)$ be the price at time t of a unit defaultable zero-coupon bond with zero recovery maturing at $T \geq t$, and let $B(t,T)$ be the price at time t of a unit discount bond maturing at $T \geq t$. It is well known that if interest rates are independent of the intensity process, then $D^0(t,T)$ is given by the following formula

$$D^0(t,T) = \mathbb{1}_{\{t<\tau\}} B(t,T) H_t^T. \tag{39}$$

4.2 Volatility of a Forward CDS Rate

Our first goal in the CIR intensity model is to analyze the volatility of the forward CDS rate. To this end, we need to find the integral representations of $\mathbb{F}$-martingales G^u and f^x.

Lemma 4.1 *For any fixed* $0 < t \leq T$ *and* $t < u$, *we have* $dG_t^u = g_t^u \, dW_t$, *where*

$$g_t^u = -e^{-\Lambda_t} H_t^u \nu(\lambda_t) n(t,u). \tag{40}$$

For any fixed $0 < x \leq T$ *and every* $t \leq x$, *we have that*

$$f_t^x = -e^{-\Lambda_t} \partial_x H_t^x = e^{-\Lambda_t} \alpha_t^x H_t^x, \tag{41}$$

where

$$\alpha_t^x = \lambda_t \partial_x n(t,x) - \partial_x m(t,x).$$

Moreover, for $t < x$, *the equality* $d_t f_t^x = \sigma_t^x \, dW_t$ *holds with*

$$\sigma_t^x = e^{-\Lambda_t} H_t^x \nu(\lambda_t) \big(n_x(t,x) - \alpha_t^x n(t,x) \big). \tag{42}$$

Proof. Let us first establish (41). To this end, we note that

$$f_t^x = \mathbb{E}_{\mathbb{Q}^*}(\lambda_x e^{-\Lambda_x} \,|\, \mathcal{F}_t) = e^{-\Lambda_t} \, \mathbb{E}_{\mathbb{Q}^*}(\lambda_x e^{-(\Lambda_x - \Lambda_t)} \,|\, \mathcal{F}_t) = -e^{-\Lambda_t} \partial_x H_t^x.$$

Using (38), we obtain

$$\partial_x H_t^x = (\partial_x m(t,x) - \lambda_t \partial_x n(t,x)) H_t^x = -\alpha_t^x H_t^x,$$

so that equality (41) is valid. Since G^u and f^x are $\mathbb{F}$-martingales, to derive (40) and (42), it suffices to focus on martingale terms in their differentials. Noting that $G_t^u = e^{\Lambda_t} H_t^u$ and applying the Itô formula to (36), we obtain (40). Similarly, to establish (42), it suffices to apply the Itô integration by parts formula to $\alpha_t^x H_t^x$. □

For simplicity of presentation, in the next result we reduce our attention to the case of null interest rate, that is, we set $B \equiv 1$.

Proposition 4.1 *Assume that* $\delta > 0$ *is a positive constant and* $B \equiv 1$. *Then*

$$\sigma_t^p = \nu(\lambda_t) \frac{H_t^T n(t,T) - H_t^U n(t,U)}{H_t^T - H_t^U} \tag{43}$$

and

$$\sigma_t^a = \nu(\lambda_t) \frac{\int_{]U,T]} H_t^u n(t,u) \, dL(u)}{\int_{]U,T]} H_t^u \, dL(u)}. \tag{44}$$

Proof. Under the present assumptions, we obtain from Lemma 3.4

$$\sigma_t^p = \Big(\int_U^T \sigma_t^u \, du \Big) \Big(\int_U^T f_t^u \, du \Big)^{-1}$$

and

$$\sigma_t^a = \Big(\int_{]U,T]} g_t^u \, dL(u) \Big) \Big(\int_{]U,T]} G_t^u \, dL(u) \Big)^{-1}.$$

Using (41) and (42), we get

$$\sigma_t^p = \frac{e^{-\Lambda_t} \nu(\lambda_t) \int_U^T H_t^u \big(\alpha_t^u n(t,u) + n_s(t,u) \big) \, du}{e^{-\Lambda_t} \int_U^T \alpha_t^u H_t^u \, du}.$$

Since

$$\int_U^T H_t^u \big(\alpha_t^u n(t,u) + n_s(t,u) \big) \, du = \int_U^T \partial_s (H_t^u n(t,u)) \, du = H_t^T n(t,T) - H_t^U n(t,U)$$

and

$$\int_U^T \alpha_t^u H_t^u \, du = H_t^T - H_t^U,$$

we conclude that (43) holds. For σ^a, using (36) and (40), we obtain

$$\sigma_t^a = \frac{-e^{-\Lambda_t} \nu(\lambda_t) \int_{]U,T]} H_t^u n(t,u) \, dL(u)}{e^{-\Lambda_t} \int_{]U,T]} H_t^u \, dL(u)}.$$

This completes the proof. □

Remark. In the general case of non-zero interest rates the formulae change accordingly. For example

$$\sigma_t^p = \nu(\lambda_t) \frac{H_t^T n(t,T) \beta(T) - H_t^U n(t,U) \beta(U) + \int_U^T r(u) \beta(u) H_t^u n(t,u) du}{H_t^T \beta(T) - H_t^U \beta(U) + \int_U^T r(u) \beta(u) H_t^u du}. \tag{45}$$

In view of Proposition 4.1, it is natural to conjecture that the volatility σ^κ of the forward CDS rate is not deterministic, thereby precluding possibility of justifying the use of Black formula in the CIR stochastic intensity model.

4.3 Credit Default Swaption

We shall use here the setup of Section 3.2, and we shall build upon the ideas borrowed from [13] (see also [10] and [20]). Throughout this section, it is assumed that $\delta(t) = \delta$, where δ is a positive constant. Moreover, we assume that the spot rate is a non-negative deterministic function of time (we refer to [13] for a discussion of this assumption in the context of valuation of default swaptions).

Recall that a credit default swaption is formally equivalent to a defaultable claim $(C_R, 0, 0, \tau)$, where $C_R = \big(S_R(\kappa)\big)^+$. Observe that under present assumptions the price at time t of a unit discount bond maturing at time u satisfies $B(t,u) = B(t)B^{-1}(u) = B(t)\beta(u)$. Let us write $\lambda_t^u = f_t^u G_t^{-1}$ so that $\lambda_t^u = \hat{h}(\lambda_t, t, u)$, where

$$\hat{h}(y,t,u) = -\partial_u \widehat{H}(y,t,u),$$

and where function $\widehat{H}$ is given in (38). Hence, in view of (26), we see that

$$C_R = \mathbb{1}_{\{R<\tau\}} \left(\delta \int_U^T B(R,u) \lambda_R^u \, du - \kappa \int_{]U,T]} B(R,u) H_R^u \, dL(u) \right)^+.$$

A straightforward computation leads to the following representation

$$C_R = \mathbb{1}_{\{R<\tau\}} \left(\delta B(R,U) H_R^U - \int_{]U,T]} H_R^u \, d\chi(u) \right)^+, \tag{46}$$

where the function $\chi : \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$d\chi(u) = -\delta \frac{\partial B(R,u)}{\partial u}\, du + \kappa B(R,u)\, dL(u) + \delta B(R,u)\, d\mathbb{1}_{[T,\infty[}(u).$$

Let us define auxiliary functions $\zeta,\ \psi : \mathbb{R}_+ \to \mathbb{R}_+$ by setting

$$\psi(y) = \int_{]U,T]} \widehat{H}(y,R,u)\, d\chi(u), \quad \zeta(x) = \delta B(R,U)\widehat{H}(x,R,U).$$

We note that χ generates a non-negative measure μ_χ on $(]U,T], \mathcal{B}(]U,T]))$ and $\mu_\chi(]U,T]) > 0$. Therefore, the strictly positive function ψ is strictly decreasing. In addition, ψ is continuous, and thus its inverse is continuous too. Moreover, the swaption's payoff admits the following representation

$$C_R = \mathbb{1}_{\{R<\tau\}} \Big(\delta B(R,U)\widehat{H}(\lambda_R,R,U) - \psi(\lambda_R)\Big)^+. \tag{47}$$

Our goal now is to examine the representation (47) in more detail. Towards this end we shall analyze existence of solution to the following equation

$$\zeta(\lambda_R) = \psi(y),$$

or, more explicitly

$$\delta B(R,U)\widehat{H}(\lambda_R,R,U) = \int_{]U,T]} \widehat{H}(y,R,u)\, d\chi(u), \tag{48}$$

which needs to be understood as equation in variable $y \geq 0$.

Since $\lim_{y\to\infty} \widehat{H}(y,R,u) = 0$ and thus $\lim_{y\to\infty} \psi(y) = 0$, it is clear that if $\zeta(\lambda_R) < \psi(0)$ a.s. then there exists a unique positive random variable λ_R^* that solves equation (48) almost surely. If $\zeta(\lambda_R) \geq \psi(0)$ a.s., then $\zeta(\lambda_R) > \psi(y)$ a.s. for any $y > 0$ and therefore, almost surely, there is no positive solution to equation (48).[1]

We now have the following result,

Proposition 4.2 *Assume that the inequality $\zeta(\lambda_R) < \psi(0)$ is valid almost surely. Then*

$$C_R = \mathbb{1}_{\{R<\tau\}} \int_{]U,T]} \big(\widehat{H}(\lambda_R^*,R,u) - \widehat{H}(\lambda_R,R,u)\big)^+ \, d\chi(u). \tag{49}$$

If the inequality $\zeta(\lambda_R) \geq \psi(0)$ is satisfied almost surely, then

$$C_R = \mathbb{1}_{\{R<\tau\}} \bigg(\delta B(R,U)\widehat{H}(\lambda_R,R,U) - \int_{]U,T]} \widehat{H}(\lambda_R,R,u)\, d\chi(u)\bigg). \tag{50}$$

Proof. If $\zeta(\lambda_R) < \psi(0)$ a.s. then, in view of (46) and of our discussion following equation (48), we have that

$$\begin{aligned} C_R &= \mathbb{1}_{\{R<\tau\}} \bigg(\int_{]U,T]} \big(\widehat{H}(\lambda_R^*,R,u) - \widehat{H}(\lambda_R,R,u)\big)\, d\chi(u)\bigg)^+ \\ &= \mathbb{1}_{\{R<\tau\}} \bigg(\int_{]U,T]} \big(\widehat{H}(\lambda_R^*,R,u) - \widehat{H}(\lambda_R,R,u)\big)\, \mu_\chi(du)\bigg)^+. \end{aligned}$$

To prove (49), it suffices to recall that μ_χ is a non-negative measure and the sign of the expression $\widehat{H}(y^*,U,u) - \widehat{H}(\lambda_U,U,u)$ is constant with respect to u. The validity of (50) is obvious. □

Recalling (39) we thus obtain the following

[1] The "intermediate" case, when $0 < \mathbb{Q}^*(\zeta(\lambda_R) < \psi(0)) < 1$ can be analyzed accordingly.

Corollary 4.1 *Assume that the inequality $\zeta(\lambda_R) < \psi(0)$ is valid almost surely. Then the payoff of a credit default swaption with expiration date U equals*

$$C_R = \int_{]U,T]} \big(K(u)D^0(R,R) - D^0(R,u)\big)^+ \, d\widehat{\chi}(u), \tag{51}$$

where $d\widehat{\chi}(u) = B^{-1}(R,u)\, d\chi(u)$ and $K(u) = \widehat{H}(\lambda_R^*, R, u)B(R,u)$.

If the inequality $\zeta(\lambda_R) \geq \psi(0)$ is satisfied almost surely, we have that

$$C_R = \int_{]U,T]} \big(K(u)D^0(R,R) - D^0(R,u)\big)\, d\widehat{\chi}(u). \tag{52}$$

Let us focus on the case $\zeta(\lambda_R) < \psi(0)$ a.s.. From Corollary 4.1, we conclude that the credit default swaption is formally equivalent in the present setup to a weighted portfolio of *survival claims* $(C_R^u, 0, 0, \tau)$ maturing at R and indexed by $u \in [U,T]$, where C_R^u equals

$$C_R^u = \big(K(u)D^0(R,R) - D^0(R,u)\big)^+ = \mathbb{1}_{\{R<\tau\}}\big(K(u) - \widetilde{D}^0(R,u)\big)^+$$

The problem of hedging of a credit default swaption is thus reduced to a problem of hedging of options on zero-coupon defaultable bonds with zero recovery. For this purpose, we can employ the techniques developed in Section 4.2.2 of [4]. Specifically, we take as the hedging instruments two defaultable zero coupon bonds, say $Y^1(t) = D^0(t,T_1) =: \mathbb{1}_{\tau>t}\widetilde{Y}_t^1$ and $Y^2(t) = D^0(t,T_2) =: \mathbb{1}_{\tau>t}\widetilde{Y}_t^2$, with maturities $T_1, T_2 > T$, and sensitive to the same default time τ. Now, as shown in [4] if we can find a constant $x(u)$ and a predictable process $(\varphi_t^2(u),\ t \in [0,R])$ such that

$$x(u) + \int_0^R \varphi_s^2(u)\, d\widetilde{Y}_s^{2,1} = (K(u) - \widetilde{D}^0(R,u))^+,$$

where $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2/\widetilde{Y}_t^1$, then we can find a predictable process $(\varphi_t^1(u),\ t \in [0,R])$ such that the pair $\varphi(u) := (\varphi^1(u), \varphi^2(u))$ is a self-financing portfolio replicating the claim $\mathbb{1}_{\{R<\tau\}}\big(K(u) - \widetilde{D}^0(R,u)\big)^+$. Assuming that $x(u)$ and $\varphi(u)$ are sufficiently regular functions of u, we can apply a stochastic Fubini theorem (cf. e.g. [24]) to conclude that $\varphi = (\varphi^1, \varphi^2)$, where $\varphi_t^i = \int_{]U,T]} \varphi^i(u)\, d\hat{\chi}(u)$, is a self-financing portfolio replicating C_R. Finally, we observe that one can synthesize a zero coupon corporate bond with a portfolio of coupon paying corporate bonds and a bank account. So, in conclusion, we see that, at least in principle, the payoff C_R can be replicated by a self-financing portfolio of corporate coupon bonds and a bank account.

In the particular case where $R = U$, one has $\widehat{H}(y,U,U) = 1$ and $B(U,U) = 1$, therefore $\zeta(x) = \delta$ for any $x \in \mathbb{R}^+$. Then, (48) reduces to

$$\delta = \int_{]U,T]} \widehat{H}(y,R,u)\, d\chi(u),$$

and the solution λ_R^* is deterministic. In this case, we may use equality (51) to derive first an explicit pricing formula for a swaption in terms of the intensity process λ and subsequently to use this formula for analytical computations of the volatility σ^C of the swaption's pre-default price. In principle, this would allow us to compute the replicating strategy in terms of the underlying forward CDS and the corresponding swap portfolio, as derived in Proposition 2.2. In fact, the pricing formula for a put option written on a defaultable bond corresponds here to the well known bond option formula in the CIR interest rate model and thus it is readily available. Therefore, the computation of the volatility process σ^C is also feasible, at least in principle, by an application of the Itô formula to the pricing formula for C_t, $t \in [0,U]$.

In case of $R < U$ it seems rather difficult to employ (49) for analytical computations of the volatility of the swaption's price. That is why, one may want to consider an alternative approach involving another possible representation of C_R. To obtain this representation, let us recall that $\zeta(x) = \delta B(R,U)\widehat{H}(x,R,U)$, and observe that $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly positive and strictly decreasing. Let $x^* > 0$ denote a solution (if it exists) to the deterministic equation

$$\zeta(x) = \psi(x). \tag{53}$$

Lemma 4.2 *If the inequality $\zeta(0) < \psi(0)$ holds then there exists a unique solution $x^* > 0$ to equation (53) and $\zeta(x) > \psi(x)$ for $x > x^*$. If $\zeta(0) \geq \psi(0)$ then $\zeta(x) \geq \psi(x)$ for every $x \in \mathbb{R}_+$.*

Proof. Recall that the inequality $n(R,U) < n(R,u)$ holds for every $u \in]U,T]$. It is easy to deduce that $\zeta'(x) = -n(R,U)\zeta(x)$ whereas $\psi'(x) < -n(R,U)\psi(x)$ for every $x > 0$. This implies that a unique solution $x^* > 0$ to equation (53) exists if $\zeta(0) < \psi(0)$. It is also clear that there is no strictly positive solution to this equation if $\zeta(0) \geq \psi(0)$. □

Assume first that $\zeta(0) \geq \psi(0)$. Then (47) becomes

$$C_R = \mathbb{1}_{\{R<\tau\}} \Big(\delta B(R,U)\widehat{H}(\lambda_R,R,U) - \psi(\lambda_R)\Big) = \mathbb{1}_{\{R<\tau\}} \big(\zeta(\lambda_R) - \psi(\lambda_R)\big). \tag{54}$$

If $\zeta(0) < \psi(0)$ then (47) can be represented as follows

$$C_R = \mathbb{1}_{\{R<\tau\}}\mathbb{1}_{\{\lambda_R>x^*\}} \Big(\delta B(R,U)\widehat{H}(\lambda_R,R,U) - \psi(\lambda_R)\Big) = \mathbb{1}_{\{R<\tau\}}\mathbb{1}_{\{\lambda_R>x^*\}} \big(\zeta(\lambda_R) - \psi(\lambda_R)\big). \tag{55}$$

5 Appendix: Itô-Kunita-Wentzell Formula

For the reader's convenience, we recall here the Itô-Kunita-Wentzell formula. Let $F_t(x)$ be a family of stochastic processes, continuous in $(t,x) \in (\mathbb{R}_+ \times \mathbb{R}^d)$ a.s., and satisfying the following conditions:
(i) for each $t > 0$, $x \to F_t(x)$ is C^2 from $\mathbb{R}^d$ to $\mathbb{R}$,
(ii) for each x, $(F_t(x), t \geq 0)$ is a continuous semimartingale

$$dF_t(x) = \sum_{j=1}^n f_t^j(x)\, dM_t^j,$$

where M^j are continuous semimartingales, and $f^j(x)$ are stochastic processes continuous in (t,x), such that for every $s > 0$, the map $x \to f_s^j(x)$ is C^1, and for every x, $f^j(x)$ is an adapted process. Let $X = (X^1, \cdots, X^d)$ be a continuous semimartingale. Then

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \sum_{j=1}^n \int_0^t f_s^j(X_s)\, dM_s^j + \sum_{i=1}^d \int_0^t \frac{\partial F_s}{\partial x_i}(X_s)\, dX_s^i \\ &+ \sum_{i=1}^d \sum_{j=1}^n \int_0^t \frac{\partial f_s}{\partial x_i}(X_s)\, d\langle M^j, X^i\rangle_s + \frac{1}{2}\sum_{i,k=1}^d \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_k}\, d\langle X^k, X^i\rangle_s. \end{aligned}$$

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