Technical article

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HJM with multiples

Developed in these pages last year, the Heath-Jarrow-Morton approach to hazard rate models allows arbitrage-free pricing of credit derivatives. By constructing a matrix of migration hazard rates, Tomasz Bielecki and Marek Rutkowski extend this approach to a multiple ratings environment

This article presents a new approach to modelling credit risk, evaluating defaultable debt and pricing credit derivatives. Our technique, based on Heath-Jarrow-Morton (HJM) methodology (1992), uses available information about credit spreads and recovery rates to model the intensity of credit migrations between various credit ratings classes. Our results complement previous work by Arvanitis, Gregory & Laurent (1999), Duffie & Singleton (1998), Jarrow, Lando & Turnbull (1997), Schönbucher (1998) and Thomas, Allen & Morkel-Kingsbury (1998), among others. We should also mention recent papers by Maksymiuk & Gatarek (1999) and Pugachevsky (1999), which deal with various generalisations of the HJM framework that cover credit risk (they do not, however, consider credit ratings).

Default-free and default-risky bonds

Let $B(t, T)$ and $D_{C_t}(t, T)$ denote time t prices of default-free and default-risky (or defaultable) zero-coupon bonds maturing at time T, respectively. The default-free bond pays \$1 at time T. The recovery payment for the default-risky bond needs to be modelled. The meaning of the subscript C_t in the notation $D_{C_t}(t, T)$ will be explained later. For simplicity of exposition, in this paper we focus on the recovery scheme in which the recovery payment is received by the holder of the defaultable bond at the maturity time of the bond (this is commonly referred to as the fractional recovery of treasury). Of course, if the defaultable bond does not default prior to or on the maturity date, then it pays \$1 at maturity.

We are concerned with modelling the dynamics for the price process $D_{C_t}(t, T)$, as well as with relating $B(t, T)$ and $D_{C_t}(t, T)$. This involves deriving a credit risk model that takes into account available data regarding credit spreads and recovery rates for various credit rating classes. By a credit risk model we mean a model for probabilities of migrations between various credit rating classes (including default). Using our credit risk model we then construct an arbitrage-free model of the defaultable term structure.

For a fixed horizon date $T^* > 0$, let (Ω, F, P) denote the underlying probability space, endowed with the filtration $F = (\mathcal{F}_t)_{t \in [0, T^*]}$. The process r_t represents the short-term interest rate and:

$$B_t = \exp\left(\int_0^t r_u du\right)$$

is the savings account, as usual. In addition, let the default-free instantaneous forward rate be $f(t, T)$, so that the price $B(t, T)$ of a unit default-free zero-coupon bond equals:

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right) \tag{1}$$

Suppose that there are K credit classes or states, the K^{th} state denoting the state of default. The risky bond can be in any of the states $i \in \mathcal{K} = \{1, \dots, K\}$, which represents its credit quality. For any $i < K$, we write $g_i(t, T)$, the conditional instantaneous forward rate for the risky bond that is in class i at time t. We assume the HJM-type dynamics for the instantaneous rates $f(t, T)$ and $g_i(t, T)$, $i = 1, \dots, K - 1$, under the real-world probability P, namely:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t \tag{2}$$

and:

$$dg_i(t,T) = \alpha_i(t,T)dt + \sigma_i(t,T)dW_t \tag{3}$$

where W is the Wiener process under P.

It should be stressed that the process:

$$D_i(t,T) = \exp\left(-\int_t^T g_i(t,u)du\right) \tag{4}$$

does not represent the price process of a tradeable security. In other words, "the risky bond that is in state i at time t" is not a tradeable asset. In the present framework, a particular defaultable bond is formally defined by its face value (by convention, equal to one), the maturity date T, the bond's recovery covenants and the bond's initial rating, which we denote by C_0^1. As a consequence of (2) and (3), we get the following dynamics of $B(t, T)$ and $D_i(t, T)$ under the equivalent risk-neutral probability P:

$$dB(t,T) = B(t,T)\left(r_t dt + b(t,T)dW_t^*\right) \tag{5}$$

and:

$$dD_i(t,T) = D_i(t,T)\left(\left(r_t + \mu_i(t)\right)dt + b_i(t,T)dW_t^*\right) \tag{6}$$

where $\mu_i(t)$ is an F-adapted stochastic process related to a Girsanov transformation, and $b_i(t, T) = -\int_t^T \sigma_i(t, u)du$ is the bond's (possibly deterministic) volatility. Finally, W^* denotes the Wiener process under P^*.

Credit migrations

Let $C_t = (C_t^1, C_t^2)$ denote a two-dimensional conditional Markov process taking values in $K \times K$. In financial interpretation, process C models migrations between credit grades. More specifically, C_t^1 is the current rating of a bond and C_t^2 represents its previous rating grade. It is thus natural to assume that the states (K, i), $i \in \mathcal{K}$ are absorbing. This idea of a "twin state" conditional Markov chain is similar to that suggested in Arvanitis, Gregory & Laurent (1999).

We wish to model the price process of a defaultable bond for a given initial credit state C_0 at time 0. We may therefore assume that $(C_0^1, C_0^2) = (i, i)$ for some $i \neq K$). We need to take into account not only the fluctuations of the price due to the presence of the Wiener noise (interest rate risk), but also the sudden jumps that are due to rating upgrades or downgrades (credit risk).

Let $\delta_i \in [0, 1)$, $i = 1, \dots, K - 1$, denote the recovery rates.This means that if the T-maturity unit bond defaults before or at time T, its owner is entitled to the payout δ_i at maturity date T, provided that the bond belonged to class i just before default occurred. Such a recovery scheme is commonly referred to as the fractional recovery of treasury value in financial literature.

To construct the arbitrage-free defaultable term structure, it is important to specify appropriately the infinitesimal generator of C^1 at time t, given the σ-field $\mathcal{F}_t$, that is, the K-dimensional matrix:

$$\Lambda_t = \begin{pmatrix} \lambda_{1,1}(t) & \dots & \lambda_{1,K}(t) \\ \cdot & \dots & \cdot \\ \lambda_{K-1,1}(t) & \dots & \lambda_{K-1,K}(t) \\ 0 & \dots & 0 \end{pmatrix} \quad (7)$$

where $\lambda_{i,i}(t) = -\Sigma_{j\neq i}\lambda_{i,j}(t)$ for $i = 1, \dots, K-1$, and where $\lambda_{i,j}$ are F-adapted processes. To this end, we need to postulate that the processes $\lambda_{i,j}$ satisfy the following consistency condition: for each $i = 1, \dots, K-1$ and for every $t \in [0, T]$:

$$\sum_{j=1, j\neq 1}^{K-1}\lambda_{i,j}(t)\big(D_j(t,T) - D_i(t,T)\big) + \lambda_{i,K}(t)\big(\delta_i Z(t,T) - D_i(t,T)\big) + \mu_i(t)D_i(t,T) = 0 \quad (8)$$

where we set $Z(t, T) = B(t, T)/Bt$, so that:

$$dZ(t,T) = Z(t,T)b(t,T)dW_t^*$$

Let us stress that the entries of the matrix Λ should be chosen in such a way that $\lambda_{i,j}$, $i \neq j$ follow non-negative processes. In a special case of zero recovery (ie, when $\delta_i = 0$ for $i = 1, \dots, K-1$) we may take, for instance $\lambda_{iK}(t) = \mu_i(t)$ for $i = 1, \dots, K-1$ and $\lambda_{ij} = 0$ for each i when $j \leq K-1$.

To produce a process C with desired properties we need to enlarge the underlying probability space $(\tilde{\Omega}, \tilde{F}, Q^*)$, where Q^* is the extended risk-neutral probability. The filtration $\tilde{F} = (\tilde{F}_t)_{t\in[0,T^*]}$ is an enlargement of Wiener filtration, and also accounts for random shocks leading to credit migrations.

Let us set:

$$M_{i,j}(t) := H_{i,j}(t) - \int_0^t \lambda_{i,j}(s)H_i(s)ds, \qquad \forall t \in [0,T]$$

where $H_i(t) = I_{\{C_t^1 = i\}}$, and $H_{i,j}(t)$ represents the number of transitions from i to j by C^1 over the time interval $(0, t]$. It can be shown that the processes $M_{i,j}$ are $\tilde{F}$-martingales under the extended risk-neutral probability Q^*.

To explain the conditional Markov feature of C^1, let us denote by F_t^C the σ-field generated by the observation of credit migration process C up to time t. Then for arbitrary $s > t$ and $i,j \in K$ we have:

$$Q^*\left\{C_s = (i,j)\middle| F_t \vee F_t^C\right\} = Q^*\left\{C_s = (i,j)\middle| F_t \vee \left\{C_t = \left(C_t^1, C_t^2\right)\right\}\right\}$$

At the intuitive level, the past and current behaviour of bond prices determines the intensity of jump to another rating grade. The formula above provides the risk-neutral probability that the bond is in the credit grade i at time $s > t$, and the immediately preceding bond's class as j, given the bond was in the credit class C_t^1 at time t, which was immediately preceded by class C_t^2. (Note that the event $\{C_t = (i, i)\}$ indicates that the bond has never left the credit class i prior to time t.) Our credit risk model will generate such risk-neutral probabilities after it is calibrated to market data. It should be stressed that historically observed migration probabilities, reported by rating agencies, are not directly exploited in our approach. The link between the risk-neutral probabilities and the real-world ones is given, as expected, by the market prices of credit risk.

Defaultable bond price

We specify the dynamics under the risk-neutral probability Q^* of the price process $D_{C_t}(t, T)$ of a defaultable bond by setting:

$$dD_{C_t}(t,T) = \sum_{i,j=1, i\neq j}^{K-1}\big(D_j(t,T) - D_i(t,T)\big)dM_{i,j}(t) + \sum_{i=1}^{K-1}\big(\delta_i B(t,T) - D_i(t,T)\big)dM_{i,K}(t) + \sum_{i=1}^{K-1}H_i(t)D_i(t,T)b_i(t,T)dW_t^* + \sum_{i=1}^{K-1}\delta_i H_{i,K}(t)B(t,T)b(t,T)dW_t^*$$

Notice that the process $D_{C_t}(t, T)$ follows a (local) martingale under Q^*. Furthermore, condition (8) implies that:

$$dD_{C_t}(t,T) = \sum_{i,j=1,i\neq j}^{K-1}\left(D_j(t,T) - D_i(t,T)\right)dH_{i,j}(t) + \sum_{i=1}^{K-1}\left(\delta_i B(t,T) - D_i(t,T)\right)dH_{i,K}(t)$$
$$+\sum_{i=1}^{K-1}H_i(t)dD_i(t,T) + \sum_{i=1}^{K-1}\delta_i H_{i,K}(t)dB(t,T) + r_t D_{C_t}(t,T)dt$$

where the differentials $dB(t, T)$ and $dD_i(t, T)$ are given by (5) and (6).

It appears that the price process of a defaultable bond, for any initial condition C_0, is given by the following intuitively clear expression (notice that both components of the process C enter the formula below):

$$D_{C_t}(t,T) = I_{\{C_t^1\neq K\}}\exp\left(-\int_t^T g_{C_t^1}(t,u)du\right) + \delta_{C_t^2} I_{\{C_t^1=K\}}\exp\left(-\int_t^T f(t,u)du\right) \qquad (9)$$

for every $t \in [0, T]$. Put another way:

$$D_{C_t}(t,T) = I_{\{C_t^1\neq K\}}D_{C_t^1}(t,T) + \delta_{C_t^2} I_{\{C_t^1=K\}}\exp\left(-\int_t^T f(t,u)du\right) \qquad (10)$$

Therefore, for any initial condition C_0, at any time t we have $D_{C_t}(t, T) = D_i(t, T)$ on the set $\{C_t^1 = i\}$ for every $i < K$. Furthermore, $D_{C_t}(t, T) = \delta_i B(t, T)$ on the set $\{(C_t^1, C_t^2) = (K, i)\}$. We thus see that $D_i(t, T)$ does indeed represent the price at time t of a T-maturity defaultable bond, provided that the bond is currently in the i^{th} credit class. Due to the conditional Markovian structure of the model, the value $D_i(t, T)$ does not depend on the history of a particular defaultable bond, so we have a unique price for all defaultable bonds that are currently in a given credit class. For each $i \in \mathcal{K}$, we define the i^{th} credit spread $\gamma_i(t, u)$ by setting $\gamma_i(t, u) = g_i(t, u) - f(t, u)$.

Combining (1) with (4), we get:

$$D_i(t,T) = B(t,T)\exp\left(-\int_t^T \gamma_i(t,u)du\right)$$

Also:

$$D_{C_t}(t,T) = B(t,T)\left\{I_{\{C_t^1\neq K\}}\exp\left(-\int_t^T \gamma_{C_t^1}(t,u)du\right) + \delta_{C_t^2} I_{\{C_t^1=K\}}\right\} \qquad (11)$$

To simplify formulas (9) and (11), it is convenient to denote $f(t, T) = g_K(t, T)$, so that $\gamma_K(t, u) = 0$. Then (9) and (11) become:

$$D_{C_t}(t,T) = X_t\exp\left(-\int_t^T g_{C_t^1}(t,u)du\right)$$

and:

$$D_{C_t}(t,T) = B(t,T)X_t\exp\left(-\int_t^T \gamma_{C_t^1}(t,u)du\right)$$

respectively,where X_t is the promised payout from the defaultable bond, as at time t:

$$X_t = I_{\{C_t^1\neq K\}} + \delta_{C_t^2} I_{\{C_t^1=K\}}$$

Finally, let us introduce the default time by setting:

$$\tau = \inf\left\{t \in R_+ : C_t^1 = K\right\}$$

Thus X_t can be represented as follows:

$$X_t = I_{\{t<\tau\}} + \delta_{C_t^2} I_{\{t\geq\tau\}}$$

In the case of zero recovery, ie, when $\delta_i = 0$ for every $i < K$, we obtain (cf (10)):

$$D_{C_t}(t,T) = I_{\{t<\tau\}}\exp\left(-\int_t^T g_{C_t^1}(t,u)du\right) = I_{\{t<\tau\}}B(t,T)\exp\left(-\int_t^T \gamma_{C_t^1}(t,u)du\right)$$

Suppose, on the contrary, that $\delta_i = 1$ for every $i < K$. We then expect to have $D_{C_t}(t, T) = B(t, T)$ for every t. For this to hold, it is enough to assume, quite reasonably, that for any i the "conditional" defaultable rate $g_i(t, T)$ coincides with the risk-free rate $f(t, T)$.

We also obtain other interesting characterisations of the defaultable bond process. One of them is the following version of the classic risk-neutral valuation formula:

$$D_{C_t}(t,T) = B_t E_{Q^*}\left(\delta_{C_T^2} B_T^{-1} I_{\{T\geq\tau\}} + B_T^{-1} I_{\{T<\tau\}} \middle| \tilde{F}_t\right) \qquad (12)$$

The last formula makes it apparent that our approach is consistent with the standard risk-neutral valuation approach to defaultable claims.

Conclusion

We have constructed an arbitrage-free model for the prices of default-free and defaultable debt in the case of multiple credit rating classes. The model could be quite adequately – though somewhat lengthily – referred to as the credit-spreads-based HJM-type arbitrage-free term structure model with multiple ratings. Let us summarise its most relevant features:

☐ As the primitive objects (or inputs) in our approach, we have chosen the default-free and defaultable instantaneous forward rates, as well as the credit recovery rates. Our arbitrage-free pricing model for defaultable bonds is constructed so that it supports these data.

☐ Our fundamental pricing equation (9) represents the price process of the defaultable debt $D_{C_t}(t, T)$ in terms of: (a) the price process $B(t, T)$ of default-free debt; (b) the credit recovery rates δ_i's; (c) the credit migration process C_t; and (d) the credit spread processes $\gamma_i(t, u)$'s. This equation is intuitively clear.

☐ Other useful representations of the price process $D_{C_t}(t, T)$ have also been derived. Most notably, equation (12) provides a version of the risk-neutral valuation formula. The process $D_{C_t}(t, T)$ is represented here in terms of: (a) the price process $B(t, T)$ of default-free debt; (b) the credit recovery rates δ_i's; (c) the credit migration process C_t; and (d) the equivalent risk-neutral probability Q^* and the enlarged filtration $\tilde{F}$. It is worth emphasising that Q^* encompasses both the market prices for interest rate risk and market prices for the credit risk.

☐ The arbitrage-free property of the model is provided by our consistency condition (8). The credit migration intensities $\lambda_{i,j}$'s are not uniquely determined by this condition, in general. Calibration of the model to market data is required to pick up the right solution. ■

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