AN EFFICIENT APPROACH TO VALUATION OF CREDIT BASKET PRODUCTS AND RATINGS TRIGGERED STEP-UP BONDS[*][†]

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Abstract: We propose an efficient method for valuation of various classes of credit derivatives. The method is simulation based and is shown to be very flexible and very well suited for pricing complex basket products such as CDOs, CDO2s, FTDS, as well as for pricing options on some special step-up corporate bonds. Our approach implements methodology and results of [2] and [3].

Keywords: Ratings migrations, Markovian model, Basket credit derivatives, Step-up bonds, Simulation.

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1 Introduction

When considering a collection of random phenomena, typically, an important issue to confront is the issue of (stochastic) dependence between these random phenomena.

1.1 Static vs Dynamic Approaches to Modeling Dependence

Suppose that τ_1 and τ_2 are non-negative random variables, defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P}),$ and modeling (random) times of occurrence of two different random events. For example, τ_i may represent tye time of default of an economic entity i. Now, for many different reasons, one may need to use the joint probability distribution of τ_1 and τ_2 (under the measure $\mathbb{P}$). If, ideally, this distribution is known, then the only problem at hand is to effectively apply the knowledge of the distribution. If however, as it is typically the case, this distribution is not known, then one faces the issue of, first, modeling the distribution, and, second, estimating/calibrating it. Unquestionably, there are many different ways in which modeling can be done. All depends on what input information one wants to use in the model, what aspects of the model are considered to be important, etc. .

We shall now briefly describe two possible approaches. The first one is the copula approach; we dub it a static approach. The second one is the intensity approach; we dub it a dynamic approach.

In the copula approach one uses as model inputs the marginal distributions of τ_1 and τ_2, say F_1 and F_2, as well as a copula function C. Then one constructs the joint distribution of τ_1 and τ_2, say F, as

$$F(t_1, t_2) = C(F_1(t_1), F_2(t_2)).$$

In other words,

$$\mathbb{P}(\tau_i \le t_i,\, i = 1, 2) = C(\mathbb{P}(\tau_1 \le t_1), \mathbb{P}(\tau_2 \le t_2)).$$

This of course is a very effective way of constructing joint distributions, especially when one uses tractable copula functions.

What happens though if one wants to create the joint distribution so that certain conditional probabilities are adequately modeled. For example, what if one is concerned with adequate modeling of

$$\mathbb{P}(\tau_1 \in \mathcal{T}_1, \tau_2 \in \mathcal{T}_2 | \tau_1 \in \bar{\mathcal{T}}_1, \tau_2 \in \bar{\mathcal{T}}_2), \tag{1}$$

for various $\mathcal{T}_1,\ \mathcal{T}_2,\ \bar{\mathcal{T}}_1$ and $\bar{\mathcal{T}}_2$. Or what if one wants to adequately model the intensities such as

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{P}(t < \tau_1 \le t + h | \tau_1 > 1, \tau_2 \le t). \tag{2}$$

It turns out that the copula approach can't, in general, be effectively used for this purpose.

What comes as a rescue is what we call an intensity, or dynamic approach. We like to think about this approach as a "bottom-up" approach, as opposed to the "top-down" copula approach. In the context of the toy-example considered here, the dynamic approach starts with the observation that one can associate to random times τ_1 and τ_2 a random process $X = (H^1, H^2),$ where $H^i_t := \mathbb{1}_{\{\tau_i \le t\}}$. The process X is obviously a Markov chain (in its own filtration), taking values in the state space $\mathcal{X} = \{(0,0), (0,1), (1,0), (1,1)\}$. Thus, its complete probabilistic characterization is provided by its intensity matrix (infinitesimal generator), say $\Lambda(t) = [\lambda_{xx'}(t)]_{x,x \in \mathcal{X}}$. Expressions in (1) and (2) can be easily represented in terms of function $\Lambda(\cdot)$. In particular, we have that

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{P}(t < \tau_1 \le t + h | \tau_1 > t, \tau_2 \le t) = \lambda_{(0,1)(1,1)}(t) + \lambda_{(0,1)(1,0)}(t).$$

Thus, our dynamic approach takes as an input to the model the process X.

Just as in the copula approach the art lies in choice of the copula function, in the dynamic approach the art lies in choice of the process X, and, in particular in choice of its infinitesimal characteristics. Various authors proposed various versions of the dynamic approach (cf. e.g. [1], [4], [7], [9], [10]). In this paper we take the version proposed in [2]. This approach is relevant for what we do here in Sections 2–6. In Section 7 we deal with a somewhat different problem of modeling dependence; there we exploit the dynamic approach developed in [3].

In the rest of this section we give some more reasons why we do not favor the copula approach over our dynamic approach. We focus on the factor copula approach; nevertheless, at least one of our remarks below (cf. (iii) below) applies to copula approach in general.

The factor copula approach, and in particular the one factor Li copula, has become a market standard for pricing CDOs and other basket credit products. The factor copula framework has certain attractive features, as it allows to specify separately marginal default probabilities and correlation structure, and leads to numerically tractable pricing algorithms, that are sometimes dubbed "quasi-analytical." We refer to [5] for discussion of copula models used in credit risk.

Inadequacy of the factor copula approach became apparent when the development of a liquid market of standardized CDO tranches gave rise to the concept of so called *correlation skews*. This "anomaly" reflected the inability of such copula models to fit market data, and spurred a vast literature on correlation skew modeling (cf. [6] and references therein). This literature, however, only attempts to extend the standard factor copula model, by incorporating some skew features. In particular, systemic event (e.g. "end of the World" scenario) and stochastic correlation models were introduced in an attempt to better fit market data by providing richer dynamics to the dependence structure. Basically, such models allow the correlation among default times to switch between a high probability regime of low correlation and a low probability regime of high correlation or even co-monotonicity (that is 100% correlation between default times (cf. [6])) . This behavior can be regarded as a static counterpart of the effect of default contagion, and considerably improves the fit to market data (cf. [6]). However, we believe that the copula framework still presents serious shortfalls, from both the economic and modeling perspectives. Below, we briefly discuss some of them:

(i) In the factor copula approach, default times are some transformation of mixtures of latent factors, on which an arbitrary correlation structure is imposed. This framework only allows to model the *effect* of default dependencies and does not attempt in any way to model the *causal relationship* among defaults events. The dependence among default times results from phenomena known as *default contagion* and *frailty*, which reflect the financial inter-dependence among the underlying (reference) entities, and their exposure to common market risk factors. In fact, we show next that contagion and frailty may not be captured by the one factor copula approach.

(ii) A straightforward computation shows that he conditional independence assumption, underlying factor copula models, may lead to counterintuitive assessments of conditional default probabilities. To justify this statement we first need to briefly describe the commonly used approach to construct one factor (Gaussian) copula. According to this approach the random default times are defined as follows:

$$\tau_i = F^{-1}(\Phi(\overline{V}_i)),$$

where

$$\overline{V}_i = \sqrt{1-\rho^2}V_i + \rho V, \quad i = 1, 2 \ldots, L.$$

In the above formulae V_i's are i.i.d. standard Normal random variables, V is a standard Normal latent factor, independent of V_i for all i, $\rho \in (-1, 1)$, and F is a prescribed (marginal) distribution of the random times τ_i. In particular, this construction makes it so that the random times τ_i are conditionally independent, given the common factor V.

Assume now that τ_i represents the default time of obligor i in the pool of L obligors. Next, define $N_t = \sum_{i=1}^L H_t^i$; that is, N_t is the total number of obligors in the pool of L obligors who defaulted by time t. It is easy to see that, conditionally on V, $N_t \sim Binomial(L, p_t(V))$, where

$$p_t(V) = \Phi\left(\frac{\Phi^{-1}(F(t)) - \rho V}{\sqrt{1-\rho^2}}\right),$$

where Φ is the the standard Normal cumulative distribution function.

Now, let us denote by $P^{|V}(A)$ the probability of A, conditional on the knowledge of the value assumed by the common factor V. Then, in view of the above construction, we obtain the following

$$
\begin{aligned}
P^{|V}(N_{t+h}=1, N_t=0) &= P^{|V}\left(\bigcup_{i=1}^{L}\left\{\{H^i_{t+h}=1\}\cap\{H^i_t=0\}\cap\bigcap_{j\neq i}\{H^j_{t+h}=0\}\right\}\right)\\
&= L(p_{t+h}(V)-p_t(V))(1-p_{t+h}(V))^{L-1},\\
P^{|V}(N_{t+h}=L, N_t=L-1) &= P^{|V}\left(\bigcup_{i=1}^{L}\left\{\{H^i_{t+h}=1\}\cap\{H^i_t=0\}\cap\bigcap_{j\neq i}\{H^j_t=1\}\right\}\right)\\
&= L(p_{t+h}(V)-p_t(V))p_t^{L-1}(V),\\
P^{|V}(N_t=0) = (1-p_t(V))^L, &\quad P^{|V}(N_t=L-1) = L(1-p_t(V))p_t^{L-1}(V).
\end{aligned}
$$

Consequently, we obtain that

$$
h(t,h,L,\rho,F) := \frac{P(N_{t+h}=1|N_t=0)}{P(N_{t+h}=L|N_t=L-1)}
$$
$$
= L\,\frac{\int_{-\infty}^{\infty}(p_{t+h}(v)-p_t(v))(1-p_{t+h}(v))^{L-1}d\Phi(v)}{\int_{-\infty}^{\infty}(p_{t+h}(v)-p_t(v))p_t(v))^{L-1}d\Phi(v)}\;\frac{\int_{-\infty}^{\infty}(1-p_t(v))p_t^{L-1}(v)d\Phi(v)}{\int_{-\infty}^{\infty}(1-p_t(v))^L d\Phi(v)}.
$$

We remark that, letting $h \to 0$, the limit of $h(t,h,L)$ is the ratio between the jump intensity of N on the set $\{N_t = 0\}$ and the jump intensity of N on the set $\{N_t = L-1\}$. Now, fixing $t > 0$, $h > 0$ (small) and $L > 1$, we want to show that $h(t,h,L,\rho,F) > 1$ for some marginal distributions F. (Note that we always have $h(t,h,1,\rho,F) = 1$.) Now, assuming that the default times τ_i have exponential marginal laws with intensity λ, that is $F(t) = 1-\exp(-\lambda t)$, the quantity $h_t(t,h,L,\rho,\lambda)$ can be easily computed numerically. Below, we show a plot of $h(t,h,L,\lambda)$ for different values of λ.[1]

We conclude that the factor copula framework may fail to produce satisfactory default contagion effects; as the above example shows the jump intensity of the cumulative default process N may be much higher when $N_t = 0$ as compared to the the case when $N_t = L-1$. This shortfall appears to be a consequence of the fact that in the standard factor copula approach it is very hard to control the dynamics of the default intensities. The literature (cf. [13]) attempted to overcome this shortcoming by introducing copula based models, which allow the dynamics of the default intensities, and in particular the distribution of the default times, to be dynamically updated to default information. Still, such models seem not to provide adequate control over dynamics of the default intensities. In addition, these models are non-Markovian in the sense that the only conditioning information that can be effectively use is the information about the events of the form $\{\tau_i = t_i\}$, rather than information about the events of the form $\{\tau_i \le t_i\}$. In practice, within this framework, knowledge of the current state of the basket of reference entities is not sufficient to form conditional expectations. This obviously excludes the use of any of the Markovian technology, and in a Monte Carlo implementation, requires storing the entire history of the default processes.

(iii) The copula approach, as it stands now, appears to be useless when it comes to dealing with modeling dependence between random times of credit migrations, that is, dependence between random times when credit ratings of various obligors change, rather than just modeling dependence between random default times. In a credit migrations setting, one needs to model dependence structure between a possibly infinite collection of stopping times, which, to our knowledge, the existing copula technology cannot handle. New derivative products, such as some European corporate bonds, explicitly link their cash-flows to migrations in the ratings issued by one or more rating agencies. We believe that, in the near future, rating based products will become as popular as default driven derivatives and, for this reason, models that can handle such complex dynamics need to be studied.

[1] We plot the value of $h_t(L)$ for h=.01. By taking h smaller, the computation becomes unstable, however, from the numerical results $h(t,h,L,\lambda)$ is a decreasing function of h, and thus presumably the same conclusions are valid in the limit.

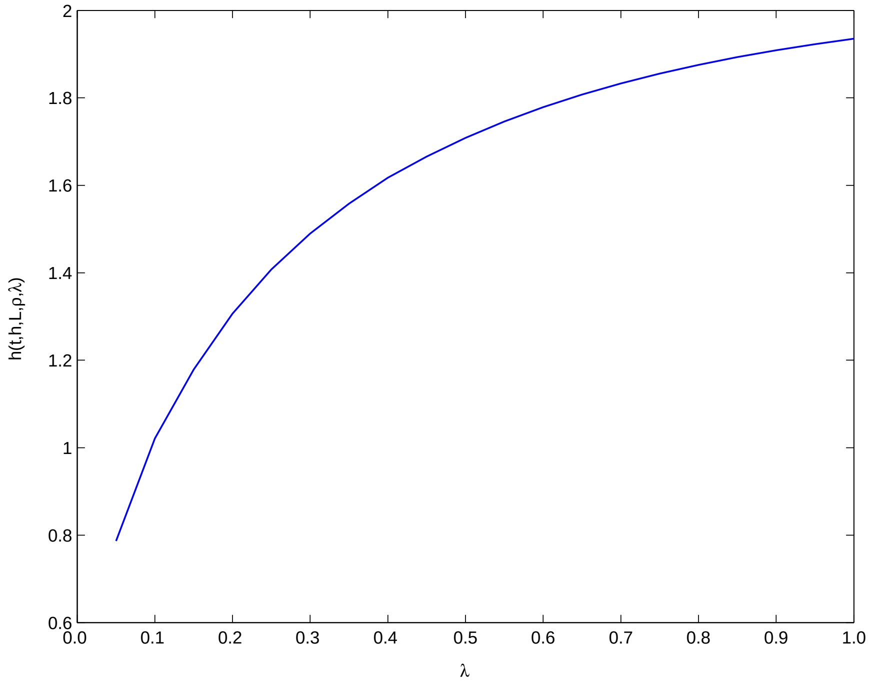

Figure 1: *Plot of* $h(t, h, L, \rho, \lambda)$ *with* $t = 1, h = .05, L = 10, \rho = .6$ *for different values of the intensity* λ*.*

1.2 What We Do in This Paper

The above reasons led us to abandon the standard copula framework and to attempt to make the step towards a *second generation* of basket credit models, which sacrifice some of the numerical tractability of factor copula approaches, to adopt more rigorous modeling assumptions and describe the causal relationships that drive the dependence among default events.

In Bielecki et al. [2] a fully Markovian model was presented so to provide a basis for systematic approach to valuation and hedging of basket credit derivatives. In this paper, we adopt the approach of [2] and adapt it to efficiently price some selected basket credit derivatives, such as CDOs, CDO2s, FSTDs, as well as credit quality triggered corporate step-up bonds.

The pricing models are calibrated to credit data provided by the GFI Group and the Citigroup. Bond data were provided by Bloomberg, through the direct feed available at the IIT's Stuart School of Business. The calibration and pricing results presented in Section 6 indicate extreme efficiency and robustness of our approach.[2]

In Bielecki et al. [3] the problem of dependence for some classes of random processes is studied. It turns out that some of the results derived in [3] are very well suited for application to pricing of ratings triggered corporate step-up bonds and we shall make use of them for this purpose.

In Section 2 we provide a formal description of the credit products that we discuss in this paper, representing relevant cash flows in terms of formulae that we find well suited for calibration and valuation applications. In Section 3 we summarize these aspects of the modeling approach presented in [2], which are relevant for practical implementations presented in this paper. A brief description of a simulation algorithm and of the calibration methodology we use, are given in Section 4 and in Section 5, respectively. Section 6 presents calibration and pricing results for selected credit products. Finally, Section 7 deals with step-up bonds. Valuation of related bond options will be dealt with in a future paper, as well as hedging issues will be addressed in a

[2]The pricing and calibration libraries are implemented in C++ interfaced through Excel spreadsheet.

follow-up paper.

2 Description of Relevant Credit Products

In this section, we describe the cash-flows associated to the main-stream basket credit products, focusing in particular on the recently developed standardized instruments like the Dow Jones Credit Default Swap indices (iTraxx and CDX), and the relative derivative contracts. In particular we will discuss Collateralized Debt Obligations (CDO), CDO squared and First to Default Swaps. We shall also describe credit quality triggered corporate step-up bonds.

2.1 CDS Indices

CDS indices[3] are static portfolios of equally weighted credit default swaps (CDSs) with standard maturities of five to ten years. Typically, the index matures few months before the underlying CDSs. The debt obligations underlying the CDSs in the pool are selected from among those with highest CDS trading volume in the respective industry sector. We will typically refer to the underlying debt obligations as to reference entities. CDS indices are typically issued by a pool of licensed financial institutions, known as the market makers. At the time of issuance, the market makers determine an annual rate known as *(index) spread*, to be paid out to investors on a periodic basis. By purchasing the index, an investor enters into a binding contract, whose main provisions are summarized below:

(i) The inception time of the contract is time[4] $t = 0$; the maturity time of the contract is T. At inception, the pool is composed of N reference credit names and its notional value[5] is N.

(ii) By purchasing the index, the investor sells protection to the market makers. Thus, the investor assumes the role of a protection seller and the market makers assume the role of protection buyers. In practice, the investors agrees to absorb all losses due to defaults in the reference portfolio, occurring between the time of inception and maturity. In case of default of a reference entity, the protection seller pays to the market makers the protection payment in the amount of $(1 - \delta)$, where $\delta \in [0, 1]$ is the agreed recovery rate (typically 40%). The notional on which the market maker pays the spread, henceforth referred to as *residual protection*, is then reduced by such amount. For instance, after the first default, the residual protection is updated as follows (recall that, at inception, the notional is N):

$$N \to N - (1 - \delta).$$

(iii) In exchange, the protection seller receives from the market maker a periodic fixed premium on the residual protection[6] at the annual rate of η. If, at inception, the market index spread is different from the issuance spread, the present value of the difference is settled through an upfront payment.

We denote by τ_i the random default time of the i^{th} name in the index and by H^i_t the right continuous process defined as $H^i_t = \mathbb{1}_{\{\tau_i \le t\}}$, $i = 1, 2, ..., N$. Also, let $\{t_j,\ j = 0, 1, ..., J\}$ with $0 = t_0$ and $t_J \le T$ denote the tenor of the premium leg payments dates. The discounted cumulative cash flows associated to a CDS index are as follows:

$$\textit{Premium Leg} = \eta \sum_{j=0}^{J} \beta_{t_j} \left(\sum_{i=1}^{N} 1 - H^i_{t_j}(1 - \delta) \right),$$

[3]We obtained description of rules governing CDS indices from www.iboxx.com. The rules described here apply to Series 3 iTraxx.

[4]Throughout the paper we shall set the inception dates of various products discussed here to $t = 0$. This is done so to simplify the notation; our discussion generalizes in a straightforward manner to any inception date $t \ge 0$. Note that with this convention the time of issuance of the index, say $t^{issuance}$, satisfies $t^{issuance} \le 0$. This of course may be unpleasing from the esthetics point of view, but has no bearing for both the qualitative and quantitative results of our paper.

[5]We henceforth assume that the face value of each reference entity is one. Thus the total notional of the index is N.

[6]Whenever a reference entity defaults, its weight in the index is set to zero. By purchasing one unit of index the protection seller owes protection only on those names that have not yet defaulted at time of inception.

$$\textit{Protection Leg} = \sum_{i=1}^{N} \beta_{\tau_i}(1-\delta)H_T^i,$$

where $\beta_t := \exp(-\int_0^t r_s ds)$ is the discount factor.

2.2 Collateralized Debt Obligations

Collateralized Debt Obligations (CDO) are credit derivatives backed by portfolios of assets. If the underlying portfolio is made up of bonds, loans or other securitized receivables, such products are known as *cash* CDOs. Alternatively, the underlying portfolio may consist of credit derivatives referencing a pool of debt obligations. In the latter case, CDOs are said to be *synthetic*. Because of their recently acquired popularity, we focus our discussion on standardized (synthetic) CDO contracts backed by CDS indices. We begin with an overview of the product:

(i) The time of inception of the contract is $t = 0$, the maturity is T. The notional of the CDO contract is the residual protection (as defined above) of the underlying CDS index at the time of inception. We shall assume that, at inception, the CDO notional is N.

(ii) The credit risk (the potential loss due to credit events) borne by the reference pool is layered into different risk levels. The range in between two adjacent risk levels is called a *tranche*. The lower bound of a tranche is usually referred to as *attachment point* and the upper bound as *detachment point*. The credit risk is sold in these tranches to protection sellers. For instance, in a typical CDO contract on iTraxx, the credit risk is split into equity, mezzanine, and senior tranches, corresponding to $0-3\%, 3-6\%, 6-9\%, 9-12\%$, and $12-22\%$ of the losses, respectively. At inception, the notional value of each tranche is the CDO notional weighted by the respective *tranche width*.

(iii) The tranche buyer sells partial protection to the pool owner, by agreeing to absorb the pool's losses comprised in between the tranche attachment and detachment point. This is better understood by an example: Assume that, at inception, the protection seller purchases one currency unit worth of the $6-9\%$ tranche. One year later, as a consequence of a series of default events, the cumulative loss breaks through the tranche attachment point, reaching 8%. The protection seller fulfills his/her obligation by paying out two thirds ($= \frac{8\%-6\%}{9\%-6\%}$) of a currency unit to the market maker. The tranche notional is then reduced to one third of its pre-default event value. We refer to the remaining tranche notional as to *residual tranche protection*.

(iv) In exchange, up until maturity, the CDO issuer (protection buyer) makes periodic spread payments to the tranche buyer on the residual tranche protection. Returning to our example, after the loss reaches 8%, premium payments are made on one third ($= \frac{9\%-8\%}{9\%-6\%}$) of the tranche notional, until the next credit event occurs or the contract matures.

We denote by L_l and U_l the lower and upper attachment points of the l^{th} tranche and by κ_l its market spread. It is convenient to introduce the cumulative loss process,

$$\Gamma_t = \sum_{i=1}^{N} H_t^i(1-\delta). \tag{3}$$

Also, let $A_l = N\,L_l$, $B_l = N\,U_l$ and $C_l = B_l - A_l$ denote, respectively, the monetary values of the attachment point, detachment point and width of the l^{th} tranche. The cumulative default payments process on the l^{th} tranche of the CDO is then:

$$M_t^l = (\Gamma_t - A_l)\mathbb{1}_{[A_l,B_l]}(\Gamma_t) + C_l\mathbb{1}_{(B_l,N]}(\Gamma_t).$$

Purchasing one unit of the l^{th} tranche generates the following discounted cash flows:

$$\textit{Premium Leg} = \kappa_l \sum_{j=0}^{J} \beta_{t_j}(C_l - M_{t_j}^l), \ \textit{Protection Leg} = \int_0^T \beta_t dM_t^l$$

We remark here, that the equity tranche of the CDO on iTraxx or CDX is quoted as an upfront rate, say κ_0, on the total tranche notional, in addition to 500 basis points (5% rate) paid annually on the residual tranche protection. The premium leg payment, in this case, is as follows:

$$\kappa_0 C_0 + \sum_{j=0}^{J} \beta_{t_j}(.05)(C_0 - M_{t_j}^0).$$

Note that, in the above, we assume that premium payments start immediately at inception of the CDO contract, and no accrual convention is in force. This may not be always the case, and the cash flow formulas should be adjusted accordingly.

Finally, we remark that, although in the above we discussed standardized CDOs on CDS indexes, in section 6.2.1 , we will also compute the fair spreads of bespoke tranches (tranches of CDO's on customized pools of CDS contract, and/or tranches with customized attachment and detachment points).

2.3 CDO Squared

Squared CDOs (also denoted as CDO2, CDO-2 or CDO^2) have gained considerable popularity in the last twelve to eighteen months. A prototypical synthetic CDO-2 is backed by a portfolio ("outer" CDO) consisting of other synthetic CDO tranches ("inner" CDOs). The outer CDO may refer to up to 1000 (not necessarily distinct) names, although, in general the number of referenced obligors ranges between 250 and 400. Due to the limited number of liquid CDS in the market, there might be a considerable amount of overlapping among the inner CDOs. This means that an underlying reference name might be present in more than one of the inner CDO contracts. As a consequence, a default event might simultaneously affect more than one of the inner CDO tranches, and this leads to the necessity of keeping track of the identity of the defaulted entities. In what follows, we provide a brief description of this very exotic product:

(i) The time of inception of the contract is $t = 0$, the maturity is T. Clearly, the outer CDO matures at or before the maturity dates[7] of the inner CDOs. The notional of the outer CDO is the sum of the notionals of the inner CDO tranches (as defined in the previous section).

(ii) The notional of the outer CDO is, again, layered into credit levels, or tranches. We shall call the tranches of the outer and inner CDOs outer and inner tranches, respectively. Each outer tranche is responsible for a portion of the losses suffered by the outer CDO notional, which arise as a consequence of the losses incurred by the inner tranches.

(iii) The buyer of a tranche in the outer CDO sells partial protection, by agreeing to absorb the losses comprised in between the outer tranche attachment and detachment points. This is better understood by a simplistic example: Consider a CDO squared backed by the mezzanine tranches of three CDO contracts. The protection seller purchases the equity outer tranche (having, for example, attachment points $0 - 5\%$). Assume that credit name XYZ is referred to by all of the inner CDOs. Assume, in addition, that at the default time of XYZ, say τ_{XYZ}, the cumulative loss in two out of three inner CDOs breaks through the attachment point of the respective mezzanine tranche. Then, assuming a recovery rate of δ, at τ_{XYZ} the protection seller pays $2(1-\delta)$ and the residual protection of the outer equity tranche is reduced by the same amount.

(iv) In exchange, the protection seller makes periodic spread payments on the residual notional of the outer tranche.

We shall need the following notation. Let the outer CDO be backed by $m = 1, \ldots, M$ inner CDO tranches, with respective attachment points $L_{l(m)}, U_{l(m)}$. Let N_m denote the size (and notional) of the m^{th} inner CDO, $A_{l(m)} = N_m\, L_{l(m)}$, $B_{l(m)} = N_m\, U_{l(m)}$ and $C_{l(m)} = B_{l(m)} - A_{l(m)}$.

[7] The inner CDOs may mature at different dates.

The cumulative loss in the m^{th} pool is defined as (cf. (3)):

$$\Gamma_t^m = \sum_{i=1}^{N_m} H_t^i(1-\delta).$$

and the relative default payments process is

$$M_t^{l(m)} = (\Gamma_t^m - A_{l(m)})\mathbb{1}_{[A_{l(m)},B_{l(m)}]}(\Gamma_t^m) + C_{l(m)}\mathbb{1}_{(B_{l(m)},N_m]}(\Gamma_t^m).$$

In addition, we define the cumulative loss in the outer CDO as

$$\Gamma_t^{sq} = \sum_{m=1}^{M} M_t^{l(m)}$$

and the relative "outer" default payment process:

$$M_t^{sq} = (\Gamma_t^{sq} - A_p)\mathbb{1}_{[A_p,B_p]}(\Gamma_t^{sq}) + C_p\mathbb{1}_{(B_p,N^{sq}]}(\Gamma_t^{sq})$$

where, $N^{sq} = \sum_{m=1}^{M} C_{l(m)}$ is the notional of the outer CDO, and, analogously to above $A_p = N^{sq}L_p$, $B_p = N^{sq}U_p$ and $C_p = B_p - A_p$ are the (monetary values of) upper, lower attachments, and width of the p^{th} outer tranche.

Purchasing one unit of the p^{th} outer tranche (paying periodic premium υ_p) generates the following discounted cash flows:

$$\textit{Premium Leg} = \upsilon^p \sum_{j=0}^{J} \beta_{t_j}(C_p - M_{t_j}^{sq}),\ \textit{Protection Leg} = \int_0^T \beta_t dM_t^{sq}.$$

2.4 N$^{\text{th}}$-to-default Swaps

N$^{\text{th}}$-to-default swaps (NTDS) are basket credit instruments backed by portfolios of single name CDSs. Since the growth in popularity of CDS indices and their associated derivatives, NTDS have become rather illiquid. Currently, such products are typically customized bank to client contracts, and hence relatively bespoke to the client's credit portfolio. For this reason, we focus our attention on First to Default Swap contracts issued on the iTraxx index, which are the only ones with a certain degree of liquidity[8]. Standardized FTDS are now issued on each of the iTraxx sector sub-indices. Each FTDS is backed by an equally weighted portfolio of five single name CDSs in the relative sub-index, chosen according to some liquidity criteria. The main provisions contained in a FTDS contract are the following:

(i) The time of inception of the contract is $t = 0$, the maturity is T.

(ii) By investing in a FTDS, the protection seller agrees to absorb the loss produced by the first default in the reference portfolio

(iii) In exchange, the protection seller is paid a periodic premium, known as FTDS spread, computed on the residual protection. The premium is paid through the first default time. We denote the FTDS spread by φ.

Recall that $\{t_j, j = 0, 1, ..., J\}$ with $0 = t_0$ and $t_J \leq T$ denotes the tenor of the premium leg payments dates. Also, denote by $\tau^{(1)}$ the (random) time of the first default in the pool. The discounted cumulative cash flows associated to a FTDS on an iTraxx sub-index containing N names are as follows (again we assume that each name in the basket has notional equal to one):

$$\textit{Premium Leg} = \varphi \sum_{j=0}^{J} \beta_{t_j}\mathbb{1}_{\{\tau^{(1)} \geq t_j\}}, \quad \textit{Protection Leg} = \beta_{\tau_i}(1-\delta)\mathbb{1}_{\{\tau^{(1)} \leq T\}}.$$

[8]Thanks to Matt Woodhams from GFI Group for his valuable comments in this regard.

2.5 Ratings Triggered Corporate Step-Up Bonds

These bonds were issued by some European telecom companies in the recent 5-6 years. As of now, to our knowledge, these products are not traded in baskets , however they are of interest because they offer protection against credit events other than defaults. In particular, ratings triggered corporate step-up bonds (step-up bonds for short) are corporate coupon issues for which the coupon payment depends on the issuer's credit quality: in principle, the coupon payment increases when the credit quality of the issuer declines. In practice, for such bonds, credit quality is reflected in credit ratings assigned to the issuer by at least one credit ratings agency (Moody's-KMV or Standard&Poor's). The provisions linking the cash flows of the step-up bonds to the credit rating of the issuer have different step amounts and different rating event triggers. In some cases, a step-up of the coupon requires a downgrade to the trigger level by both rating agencies. In other cases, there are step-up triggers for actions of each rating agency. Here, a downgrade by one agency will trigger an increase in the coupon regardless of the rating from the other agency. Provisions also vary with respect to step-down features which, as the name suggests, trigger a lowering of the coupon if the company regains its original rating after a downgrade. In general, there is no step-down below the initial coupon for ratings exceeding the initial rating. Next, we give a brief summary of the most common provisions characterizing the payoff of a step-up bond (typically, a step-up bond is subject to a selection of the provisions listed below):

(i) Step-up: The coupon increases if the rating decreases and hits the rating-trigger.

(ii) Step-down: The coupon decreases if the rating increases over the rating-trigger after the trigger level was previously hit.

(iii) One-off: The coupon increases only once, even if the rating falls further below the rating-trigger; for bonds that are not one-off, each further decrease in the rating, causes a further increase in the coupon.

(iv) And/or: Determines whether the coupon is adjusted if both Moody's and S&P ratings hit the trigger, or whether the adjustment occurs if either Moody's or S&P ratings hit the trigger level.

(v) Accrual: the coupon increases may be enforced either starting from the next coupon payment or immediately following a rating action.

Let X_t stand for some indicator of credit quality at time t (note that in this case, the process X may be composed of two distinct rating processes). Assume that $t_i,\ i = 1, 2, \ldots, n$ are coupon payment dates. In this paper we assume the convention that coupon paid at date t_n depends only on the rating history through date t_{n-1}, that is: $c_n = c(X_t,\ t \leq t_{n-1})$ are the coupon payments. In other words, we assume that no accrual convention is in force.

Assuming that the bond's notional amount is $1,$ the cumulative discounted cash flow of the step-up bond is (as usual we assume that the current time is 0):

$$(1 - H_T)\beta_T + \int_{(0,T]} (1 - H_u)\beta_u\, dC_u + \beta_\tau Z_\tau H_T, \tag{4}$$

where $C_t = \sum_{t_i \leq t} c_i$, τ is the bond's default time, $H_t = \mathbb{1}_{\tau \leq t},$ and where Z_t is a (predictable) recovery process.

3 Markovian Market Model

In this section, we give a brief description of the Markovian market model we implement for valuation and hedging of basket credit instruments. This framework is a special case of the more general model introduced in Bielecki et al.[2], which allows to incorporate information relative to the dynamic evolution of credit ratings in the pricing of basket instruments. We begin with some notation.

Let the underlying probability space be denoted by $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, where $\mathbb{Q}$ is a risk neutral measure inferred from the market (we shall discuss this in further detail when addressing the issue of model calibration),

$\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ is a filtration containing all information available to market agents. The filtration $\mathbb{H}$ carries information about the evolution of credit events, such as changes in credit ratings or defaults of respective credit names. The filtration $\mathbb{F}$ is a reference filtration containing information pertaining to the evolution of relevant macroeconomic variables.

We consider N obligors (or credit names) and we assume that the current credit quality of each reference entity can be classified into $\mathcal{K} := \{1, 2, \dots, K\}$ rating categories. By convention, the category K corresponds to default. Let X^l, $l = 1, 2, \dots, N$ be processes on $(\Omega, \mathcal{G}, \mathbb{Q})$ taking values in the finite state space $\mathcal{K}$. The process X^l represents the evolution of credit ratings of the l^{th} reference entity. We define the *default time* τ_l of the l^{th} reference entity by setting

$$\tau_l = \inf\{\, t > 0 \,:\, X^l_t = K\} \tag{5}$$

We assume that the default state K is absorbing, so that for each name the default event can only occur once.

We denote by $X = (X^1, X^2, \dots, X^N)$ the joint credit rating process of the portfolio of N credit names. The state space of X is $\mathcal{X} := \mathcal{K}^N$ and the elements of $\mathcal{X}$ will be denoted by $x = \{x^1, \dots, x^N\}$. We postulate that the filtration $\mathbb{H}$ is the natural filtration of the process X and that the filtration $\mathbb{F}$ is generated by a $\mathbb{R}^n$ valued factor process, Y, representing the evolution of relevant economic variables, like short rate or equity price processes.

We assume that the process $M = (X, Y)$ is jointly Markov under $\mathbb{Q}$, so that we have, for every $0 \le t \le s$, $x \in \mathcal{X}$, and any set $\mathcal{Y}$ from the state space of Y,

$$\mathbb{Q}(X_s = x, Y_s \in \mathcal{Y} \,|\, \mathcal{H}_t \vee \mathcal{F}^Y_t) = \mathbb{Q}(X_s = x, Y_s \in \mathcal{Y} \,|\, X_t, Y_t). \tag{6}$$

The process M is constructed as a Markov chain modulated by a Lévy process. We shall refer to X (Y) as the *Markov chain* (*Lévy*) component of M. Given $X_t = x$ and $Y_t = y$, the intensity matrix of the Markov chain component is given by $\Lambda_t = [\lambda(x, x'; y)]_{x' \in \mathcal{X}}$. The Lévy component satisfies the SDE:

$$dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dW_t + \int_{\mathbb{R}^n} g(Y_{t-}, y')\,N(dy', dt),$$

where, for a fixed $y \in \mathbb{R}^n$, $N(dy', dt)$ is a counting process with Lévy measure $\nu(dy')$ and $\sigma(y)$ satisfies $\sigma(y)\sigma(y)^{\mathsf{T}} = a(y)$. We provide the following structure to the generator of the process M.

$$\begin{aligned}\mathbf{A}f(x, y) = (1/2) \sum_{i,j=1}^{n} a_{ij}(y)\partial_i\partial_j f(x, y) + \sum_{i=1}^{n} b_i(y)\partial_i f(x, y)\\ + \int_{\mathbb{R}^n} \big(f(x, y + g(y, y')) - f(x, y)\big)\nu(dy')\\ + \sum_{l=1}^{N}\sum_{\xi\in\mathcal{K}} \lambda^l(x, x^\xi_l; y) f(x^\xi_l, y),\end{aligned} \tag{7}$$

where we write $x^\xi_l = (x^1, x^2, \dots, x^{l-1}, \xi, x^{l+1}, \dots, x^N)$.

Note that the model specified by (7) does not allow for simultaneous jumps of the components X^l and $X^{l'}$ for $l \neq l'$. In other words, the ratings of different credit names may not change simultaneously. Nevertheless, this is not a serious lack of generality, as the ratings of both credit names may still change in an arbitrarily small time interval. The advantage is that, for the purpose of simulation of paths of process X, rather than dealing with $\mathcal{X} \times \mathcal{X}$ intensity matrix $[\lambda(x, x'; y)]$, we shall deal with N intensity matrices $[\lambda^l(x, x^\xi_l; y)]$, each of dimension $\mathcal{K} \times \mathcal{K}$ (for any fixed y). We stress that, within the present set-up, the current credit rating of the credit name l directly impacts the intensity of transition of the rating of the credit name l', and vice versa. This property, known as *frailty*, may contribute to default contagion.

3.1 Valuation of Basket Credit Derivatives in the Markovian Framework

We now discuss the pricing of the basket instruments introduced in section two of the paper. In particular, computing the fair spreads of such products involves evaluating the conditional expectation, under the risk neutral measure $\mathbb{Q}$, of some quantities related to the cash flows associated to each instrument. In the case of CDS indices, CDOs, CDO2s and FTDS, the fair spread is such that, at inception, the value of the contract is exactly zero, i.e the risk neutral expectations of the fixed leg and protection leg payments are identical. The following expressions are derived from the discounted cumulative cash flows given in Section 2 (refer to this section for the precise definitions of the quantities appearing in the formulae below). They represent initial (at time $t = 0$) values of spreads and prices, given the state of the market at inception, $(X_0, Y_0) = (x, y)$:

- the fair spread of a single name CDS is:

$$\eta^l = \frac{\mathbf{E}^{x,y}_{\mathbb{Q}}\Big(\beta_{\tau_l} H^l_T\Big)(1-\delta)}{\mathbf{E}^{x,y}_{\mathbb{Q}}\Big(\sum_{j=0}^{J} \beta_{t_j}(1 - H^l_{t_j})\Big)}$$

- the fair spread of a CDS index is:

$$\eta = \frac{\mathbf{E}^{x,y}_{\mathbb{Q}} \sum_{i=1}^{L} \beta_{\tau_i}(1-\delta) H^i_T}{\mathbf{E}^{x,y}_{\mathbb{Q}} \sum_{j=0}^{J} \beta_{t_j} \Big(\sum_{i=1}^{N} 1 - H^i_{t_j}(1-\delta)\Big)}$$

- the fair spread of the CDO equity tranche is:

$$\kappa_0 = \frac{1}{C_0}\mathbf{E}^{x,y}_{\mathbb{Q}}\Big(\int_0^T \beta_t dM^0_t - \sum_{j=0}^{J} \beta_{t_j} .05(C_0 - M^0_{t_j})\Big)$$

- the fair spread of the l^{th} CDO tranche is:

$$\kappa_l = \frac{\mathbf{E}^{x,y}_{\mathbb{Q}}\Big(\int_0^T \beta_t dM^l_t\Big)}{\mathbf{E}^{x,y}_{\mathbb{Q}}\Big(\sum_{j=0}^{J} \beta_{t_j}(C_l - M^l_{t_j})\Big)}$$

- the fair spread of the p^{th} tranche of the CDO squared is:

$$\upsilon_p = \frac{\mathbf{E}^{x,y}_{\mathbb{Q}}\Big(\int_0^T \beta_t dM^{sq}_t\Big)}{\mathbf{E}^{x,y}_{\mathbb{Q}}\Big(\sum_{j=0}^{J} \beta_{t_j}(C_p - M^{sq}_{t_j})\Big)}$$

- the fair spread of a First To Default Swap is:

$$\varphi = \frac{\mathbf{E}^{x,y}_{\mathbb{Q}}\Big(\beta_{\tau_i}(1-\delta)\mathbb{1}_{\{\tau^{(1)} \le T\}}\Big)}{\mathbf{E}^{x,y}_{\mathbb{Q}}\Big(\sum_{j=0}^{J} \beta_{t_j} \mathbb{1}_{\{\tau^{(1)} \ge t_j\}}\Big)}$$

- fair value of the step-up bond is:

$$D = \mathbf{E}^{x,y}_{\mathbb{Q}}\Big((1 - H_T)\beta_T + \int_{(0,T]} (1 - H_u)\beta_u \, dC_u + \beta_\tau Z_\tau H_T\Big)$$

Depending on the dimensionality of the problem, the above conditional expectations will be evaluated either by means of Monte Carlo simulation, or by means of some other numerical method and, in the low dimensional cases, even analytically . In the next sections we address the practical issues of implementing the proposed theoretical framework.

4 Model Implementation

In this section, we discuss the practical implementation of our model. In particular we provide further structure to the generator of the Markov chain component of the joint process (X, Y) and specify a general functional form for its transition intensities. We then briefly describe a recursive procedure for simulating the evolution of the process X.

4.1 Specification of Credit Ratings Transition Intensities

Because we need to simulate the joint process (X, Y), it is important to specify its form in such a way to avoid unnecessary computational complexity. As noted earlier, the structure of the generator $\mathbf{A}$ that we postulate makes it so that simulation of the evolution of process X reduces to recursive simulation of the evolution of processes X^l, whose state spaces are only of size K each. In order to facilitate simulations even further, we also postulate that each migration process X^l behaves like a birth-and-death process with absorption at default, and with possible jumps to default from every intermediate state. In addition, we shall assume that the factor process, Y, is independent of X. Conditional upon $(X_t, Y_t) = (x, y)$, the infinitesimal generator governing the evolution of the credit ratings of the l^{th} name is the sub-stochastic matrix:

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ \\ \vdots \\ K-1 \\ K
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & \cdots & K-1 & K \end{array} \\
\begin{pmatrix}
\lambda^l_{1,1} & \lambda^l_{1,2} & 0 & \cdots & 0 & \lambda^l_{1,K} \\
\lambda^l_{2,1} & \lambda^l_{2,2} & \lambda^l_{2,3} & \cdots & 0 & \lambda^l_{2,K} \\
0 & \lambda^l_{3,2} & \lambda^l_{3,3} & \cdots & 0 & \lambda^l_{3,K} \\
 & & & & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda^l_{K-1,K-1} & \lambda^l_{K-1,K} \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\end{array},
$$

where, with a slight change of notation, $\lambda^l_{x^l,\xi} = \lambda^l_{x^l,\xi}(x, y) = \lambda^l(x, x^\xi_l; y)$.
The functional form of the transition intensities should reflect the specific characteristics of the instruments we need to price and should be chosen to obtain the best possible fit in the calibration phase.

4.2 Simulation Algorithm

In general, a simulation of the evolution of the process X entails high computational costs, as the the cardinality of the state space of X is equal to K^N. Thus, for example, in case of $K = 18$ rating categories, as in Moody's ratings, and in case of a portfolio of $N = 100$ credit ratings, the state space has 18^{100} elements. However, the specific assumptions on the structure of the generator allow to simulate the process in a recursive fashion, which has a relatively low computational complexity. We consider here simulations of sample paths over a generic time interval, $[t_1, t_2]$, where $0 \leq t_1 < t_2$, and assume that the time t_1 state of the process (X, Y) is (x, y). Generating one sample path will, in general, involve the following steps:

Step 1: in Step 1, a sample path of the process Y is simulated. Recall that the dynamics of the factor process are described by the SDE

$$
\begin{aligned}
dY_t &= b(Y_t)\,dt + \sigma(Y_t)\,dW_t + \int_{\mathbb{R}^n} g(Y_{t-}, y')\,N(dy', dt) \\
Y_{t_1} &= y
\end{aligned}
$$

Any standard procedure can be used to simulate a sample path of Y (the reader is referred, for example, to Kloeden and Platen [11]). We denote by $\widehat{Y}$ the simulated sample path of Y.

Step 2: generate a sample path of X on the interval $[t_1, t_2]$.

Step 2.1: simulate the first jump time of the process X in the time interval $[t_1, t_2]$. Towards this end, draw from a unit exponential distribution. We denote by $\widehat{\eta}_1$ the value of the first draw. The simulated value of the first jump time, τ, is then given by:

$$\tau = \inf\left\{ t > t_1 \ : \ \int_{t_1}^{t} \lambda(x, \widehat{Y}_u)\, du \geq \widehat{\eta}_1 \right\},$$

where

$$\lambda(x, \widehat{Y}_t) := -\sum_{i=1}^{N} \lambda^i_{x^i,x^i}(x, \widehat{Y}_t)$$

and

$$\lambda^l_{x^l,x^l}(x, \widehat{Y}_t) = -\lambda^l_{x^l,x^l-1}(x, \widehat{Y}_t) - \lambda^l_{x^l,x^l+1}(x, \widehat{Y}_t) - \lambda^l_{x^l,K}(x, \widehat{Y}_t).$$

If $\tau > t_2$ return to step 1, otherwise go to Step 2.2.

Step 2.2: simulate which component of the vector process X jumps at time τ, by drawing from the conditional distribution:

$$Q(X^l_\tau \neq X^l_{\tau-}) = -\frac{\lambda^l_{x^l,x^l}(x, \widehat{Y}_\tau)}{\lambda(x, \widehat{Y}_\tau)}$$

Recall that $\lambda^l_{x^l,x^l}(x, \widehat{Y}_t) = 0$ if $x^l = K$, since K is an absorbing state.

Step 2.3: assume the i^{th} obligor jumps at τ. Simulate the direction of the jump by drawing from the conditional distribution

$$Q^i(X^i_\tau = \xi) = -\frac{\lambda^i_{x^i,\xi}(x, \widehat{Y}_\tau)}{\lambda^i_{x^i,x^i}(x, \widehat{Y}_\tau)}$$

where

$$\xi = \{x^i - 1; x^i + 1; K\}$$

Step 2.4: update the state of X and set $t_1 = \tau$. Repeat Steps 2.1-2.3 on $[t_1, t_2]$ until $\tau > t_2$

Step 3: calculate the simulated value of a relevant functional. For instance, assume that Y represents the short rate process, and is used as a discount factor, i.e $\int_0^t Y_t = -\ln B_t$. In order to compute the protection leg of a CDS index, one would evaluate

$$\sum_{i=1}^{L} \frac{B_{\tau_i}}{B_t}(1 - \delta)(H^i_T - H^i_t)(\omega)$$

at each run ω, and obtain the Monte Carlo estimate by averaging over all sample paths.

5 Model Calibration

In the previous sections we assumed a risk neutral pricing measure as given. Arbitrage free pricing, in fact, requires the existence of a risk neutral measure, under which the price processes in the underlying market are martingales. In our market model, relevant assets are the single name CDSs contained in the indices, the indices themselves, and the relative derivative products. It is a standing assumption that financial markets actually are arbitrage free, and a risk neutral measure can thus be inferred from the prevailing market prices. Choosing a risk-neutral probability measure such as to reproduce the prices of traded derivative prices is known as model calibration.

This "inverse" problem is seldom feasible thus, typically, calibrating reduces to achieving the best approximation to market prices within a given model class. The quality of such approximation is a good indicator of the ability of the model to reproduce and explain the workings of the underlying markets. The main-stream copula models are known not to provide a good fit to market data, giving rise to the concept of correlation skews and other market inconsistencies, which really reflect the inability of these models to capture the market risk factors. Calibration provides an important test for our proposed framework.

5.1 Calibration Procedure

Calibration of the risk neutral parameters of the model, that is, the parameters corresponding to the risk neutral measure, can be split into two separate problems: calibration of the dynamics of the factor process[9] Y, and calibration of the transition intensities of the process X. Since, in general, the dynamics of the rating process need to be simulated, calibration becomes a delicate phase. We denote the output of the simulation (which may include individual CDS spreads, index spread, the CDO tranche spreads, etc.) by[10]

$$E^{\Theta}\left(f\left(\widehat{X}_t, \widehat{Y}_t;\ {}_{t\leq T}\right)\right),$$

where $\Theta \in \mathbf{\Theta} \subset \mathbb{R}^n$ is a vector of parameters to be calibrated. Calibrating the model is done by means of solving the following minimization problem:

$$\inf_{\Theta\in\mathbf{\Theta}} \left|\left|E^{\Theta}\left(f\left(\widehat{X}_t, \widehat{Y}_t;\ {}_{t\leq T}\right)\right) - \mathbf{M}\right|\right|$$

where $\mathbf{M} \in \mathbb{R}^n$ is a vector of market data corresponding to the simulation output and $\|\cdot\|$ is a norm in $\mathbb{R}^n$.

Since the degree of smoothness of the map $\Theta \to E^{\Theta}\left(f\left(\widehat{X}_t, \widehat{Y}_t;\ {}_{t\leq T}\right)\right)$ is unknown and function evaluations are typically done by Monte Carlo simulation, it is best to use algorithms that do not require computation of gradients. In particular we suggest using the downhill simplex method (also known as the Nelder-Mead algorithm) or Powell Golden Search method to perform the minimization. The need for computational speed becomes evident in this phase, since each function evaluation requires simulating a large number of sample paths.

6 Applications of the Markovian Set Up to Pricing Basket Credit Derivatives via Simulation

In the following, we specialize the general Markovian framework to pricing selected credit derivatives. First, in Section 6.1 we shall calibrate a Markovian model to market quotes on individual CDSs, CDS indices, and synthetic CDOs derived from CDS indices. Then, in Sections 6.2.2 and 6.2.3 we shall apply this framework and the calibration results for pricing FTDSs, customized CDOs, and CDO2. In Section 7 we shall calibrate another version of our model to market quotes relative to one step-up bond and price different (step-up bond) issues.

6.1 Calibration of a Two State Markov Model for Pricing Basket Credit Derivatives

In this section, we consider a special version of the general Markov model described above and we calibrate its parameters to market data.

Recall that $X_t = (X_t^1, X_t^2, \ldots, X_t^N)$ denotes the joint credit ratings process of the portfolio of N credit names. We assume that the current credit quality of each name in the pool can be classified into two rating categories, i.e. $\mathcal{K} := \{0, K\}$, where, 0 is a pre-default state[11], and as usual, K denotes the default state. Under the term default, we encompass all credit events that warrant a protection payment. In this case, these include actual bankruptcy of the obligor, as well as situations of extreme financial distress (i.e. the obligor files for chapter 11).

Note that we have decided to reduce the state space of each ratings process X^i to two states only. This modeling decision rests upon empirical reasons, and appears to be adequate with regard to (basket) credit instruments whose cash flows are not explicitly tied to changes in credit quality of underlying credit names,

[9] Calibration of Levy models is discussed in [8].

[10] Function f represents the simulation output that is relevant to the given application, $\widehat{X}$, $\widehat{Y}$ denote simulated paths of processes X and Y and E^{Θ} is the MonteCarlo estimate of the expected value of f under the "measure" Θ.

[11] Thus, the "state" 0 represents all the ratings $1, 2, \ldots, K-1$.

but rather depend only on occurrence of default. It needs to be stressed though that credit quality of an obligor, as reflected in the value of the corresponding CDS spread, provides a useful implicit quantification of credit ratings of the obligor, and as such is in some way used in our specialized model.[12]

We postulate that, under the risk neutral measure $\mathbb{Q}$, the jump intensities (default intensities, in this case) of the Markov chain component X are as follows:

$$\lambda^l_{0,K}(t) = h(\eta^l, X_t; \Theta), \tag{8}$$

where h is a judiciously chosen function, Θ is a vector of model parameters and η^l denotes the spread of the CDS corresponding to the l^{th} reference name, at inception. Recall that state K is absorbing, so that $\lambda^l_{K,0}(t) = 0$. The discount factor β_t is obtained by interpolation of the term structure of T-Bonds at inception.

In order to price consistently the underlying CDSs and the CDO tranches, we calibrate the intensity parameter vector Θ to univariate and multivariate default information, provided by the single name CDS spreads and by the CDO tranche spreads, respectively.[13] The model fits both single name CDS spreads and CDO tranche spreads. The simulation scheme converges rather quickly, with the (relative) standard error of the estimate ranging between 1% and 4%, after 100000 simulation runs. As for computational speed, 100000 paths can be generated in few seconds on a 1.5 mhz computer. In addition, the calibration of the model takes only few minutes of computing time, provided that the optimization algorithm starts from a sensible initial guess. A comparison of market and model generated spreads are illustrated in Table 1, Table 2 and Table 3. As usual, the equity tranche is quoted as an up-front premium, in addition to the contractual 500 bps.

Tranche	Model Spreads	Market Spreads
0-3%	23.79 %	24 %
3-6%	83.83	83
6-9%	24.55	27
9-12%	14.58	14
12-22%	8.43	9

Table 1: *Fit of two state Markov model to iTraxx market data on 31-August-2005. All values are quoted in bps*

Tranche	Model Spreads	Market Spreads
0-3%	24.52 %	24.9 %
3-6%	71.75	71.5
6-9%	22.04	24
9-12%	12.68	11.5
12-22%	6.41	6.65

Table 2: *Fit of two state Markov model to iTraxx market data on 05-November-2005.*

Tranche	Model Spreads	Market Spreads
0-3%	42.46 %	43.5 %
3-7%	123.15	123
7-10%	27.07	30
10-15%	14.04	13
15-30%	6.28	6.5

Table 3: *Fit of two state Markov model to DJ CDX market data on 10-November-2005.*

[12] In order to deal with instruments whose cash flows explicitly depend on changes in credit ratings, such as credit quality triggered step-up bonds (cf. Section 7), explicit quantification of relevant credit ratings will be needed.

[13] The market data is relative to CDO on iTraxx as quoted on August 31 2005 and November 5 2005 and on DJ CDX as quoted on November 10 2005. The data was courteously provided by GFI

Once the risk neutral parameters are calibrated, one can price other credit instruments referring to some or all of the names in the iTraxx index. In the next section, we use the calibrated model to price a CDO on iTraxx with customized tranche attachments, a CDO squared and an FTDS.

6.1.1 Correlation Skew

Due to space limitation we do not provide here results regarding fitting to market data of the individual CDS spreads generated by the model. As we said, we obtain a very good fit, which is illustrated by the following figure of implied correlation skew,

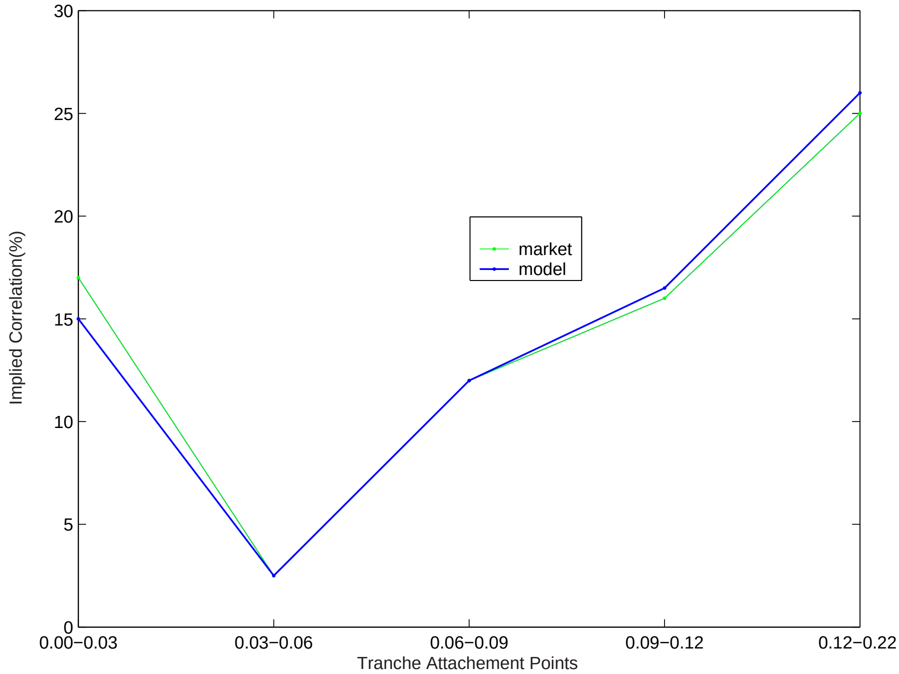

Figure 2: Correlation skew

6.2 Pricing of Selected Basket Credit Derivatives via Simulation

6.2.1 Pricing of Bespoke CDOs: Bespoke Pools and Tranchelets

Bespoke Pools

Our methodology is easily applicable to pricing bespoke CDOs where the customization is done with respect to names in the pool. The catch is, however, that we can deal with such bespoke CDOs given the customization is done relative to the pools of names that were used for calibration purposes. That means, for example, that we can deal with a bespoke CDO referring to a subset of names in any standard market index (such as iTraxx-..., CDX, or a combination of those). Sample pricing results are presented in Table 4

Tranchelets: Bespoke Attachment–Detachment Points

Using relevant calibrated data, we price a bespoke CDO on the iTraxx S3 index with customized attachment points. The pricing results are shown in Table 5.

Tranche	Model Spreads	Composition
3-7%	172.26	consumers+telecom+auto
3-7%	108.42	consumers+industrial+telecom+auto
5-10%	433.27	hi-vol

Table 4: *Mezzanine and Senior tranche spreads of bespoke CDO on sub-pools of iTraxx .*

Tranche	Model Spreads
0-3%	24.52 %
3-9%	40.75
9-16%	9.35
16-21%	4.15
21-35%	1.35

Table 5: *Pricing of customized CDO tranches on iTraxx index S3, November 5 2005.*

6.2.2 Pricing of FTDSs on CDS indices

Next, we price an FTDS on a portfolio of credit names referring to the iTraxx S3 index and listed in Table 6., along with market data and pricing results[14]:

Sector	Entity	Bid/Ask Market Spread	Model Spreads
Autos	VOLKSWAGEN	44 / 46	45.39
Energy	SUEZ	26 / 27	25.01
Financials	Bayerische Hypo	18 / 20	18.75
Industrials	Bayer	24 / 26	25.23
TMT	FRANCE TELECOM	42 / 44	43.87
Consumer	MARKS AND SPENCER	63 / 65	65.82
FTDS Spread		78%/86%	85%

Table 6: *November 5, 2005: Diversified FTDS composition and market quote v. model output. The FTDS spread is quoted as a percentage of the sum of the underlying CDS spreads.*

The table shows that the version of our model used here prices consistently FTDS when calibrated to CDO and iTraxx data, and it suggests that the model is able to capture the dynamics of the credit market in a realistic fashion.

6.2.3 Pricing of CDO2s

In this section, we price the $5-20\%$ (outer) tranche of a CDO squared comprising the $3-6\%$ (inner) tranche of the standard iTraxx CDO and the $5-10\%$ (inner) tranche of a synthetic CDO referring to the iTraxx Consumers sub-index. The total number of names referencing the outer CDO is $125+30$, thus the total notional of the outer CDO is $125(.06-.03)+30(.1-.05)$. The average spread of the Consumers sub-index and of the iTraxx index are 51.3333 and 39.562 bps., respectively. The pricing results are summarized in table 7.

7 Pricing Ratings Triggered Step-Up Bonds via Simulation

Some of the results developed in Bielecki et al. [3] appear to be ideally suited for applications to pricing and hedging of ratings triggered step-up bonds. Here, using results of [3], we shall apply a simulation approach

[14] The market data is relative to diversified FTDS contract as quoted on November 5 2005. The data was courteously provided by Citigroup

Tranche	Model Spreads
(Inner) iTraxx 3-6%	71.85
(Inner) iTraxx-Cons 5-10%	123.66
(Outer) iTraxx + iTraxx-Cons 5-25%	447.625

Table 7: *Pricing of* $5-25\%$ *CDO squared tranche on iTraxx* $3-6\%$ *and iTraxx-Cons* $5-10\%$*, November 5 2005.*

to pricing ratings triggered step-up bonds. Let us consider a ratings triggered step-up bond (cf. Section 2.5) issued by an obligor XYZ. Recall that, typically, cash-flows associated with a step-up bond depend on ratings assigned to XYZ by both Moody‘s Investors Service (Moody’s in what follows) and Standard & Poor’s (S&P in what follows). Thus, a straightforward way to model joint credit migrations would be to consider a credit migration process X such that $X_t = (M_t, SP_t),$ where M_t and SP_t denote the time t credit rating assigned to XYZ by Moody’s and SP_t, respectively. We assume that process M is a time-homogeneous Markov chain w.r.t. its natural filtration, under the statistical probability $\mathbb{P},$ and that its state space is $\mathcal{K} = \{1, 2, \ldots, K\}$. Likewise, we assume that process SP is a time-homogeneous Markov chain w.r.t. its natural filtration, under the statistical probability $\mathbb{P},$ and that its state space is $\mathcal{K} = \{1, 2, \ldots, K\}$.

Typically, we are only provided with individual statistical characteristics of each of the processes M and SP. Thus, in a sense, we know the marginal distributions of the joint process X under the measure $\mathbb{P}$ (where M and SP are considered as the "univariate" margins). The crucial issue is thus the appropriate modeling of dependence between processes M and SP. In particular, we want to model dependence, under $\mathbb{P},$ between M and SP so that the joint process X is a time-homogeneous Markov chain, and so that the components M and SP are time-homogeneous Markov chains with given $\mathbb{P}$-generators, say A^M and A^{SP}, respectively. Thus, essentially, we need to model a $\mathbb{P}$-generator matrix, say A^X, so that process X is a time-homogeneous Markov chain with $\mathbb{P}$-generator A^X and that processes M and SP are time-homogeneous Markov chains with $\mathbb{P}$-generators A^M and A^{SP}. An approach to deal with this problem has recently been developed in Bielecki et al. [3] and it is summarized below.

7.1 Multivariate Markov Chains with Given Marginals, and Markov Copulae

We fix an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. On this space we consider a bivariate time-homogenous Markov chain, say $X = (X_1, X_2)$, whose components take values in finite state spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ with cardinalities K_1 and K_2, respectively. The joint process will therefore live in a state space $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$ of cardinality $K = K_1 \times K_2$. Let us denote by $A^X = [a^X_{ih,jk}]_{i,j\in\mathcal{K}_1,\, h,k\in\mathcal{K}_2}$ the infinitesimal generator matrix for X, and let us introduce the following structural assumption,

Assumption (S)

$$\sum_{k\in\mathcal{K}_2} a^X_{ih,jk} =: a^1_{ij},\ \forall i,j\in\mathcal{K}_1, i\neq j,\ \forall h\in\mathcal{K}_2, \tag{9}$$

$$\sum_{j\in\mathcal{K}_1} a^X_{ih,jk} =: a^2_{hk},\ \forall h,k,\in\mathcal{K}_2, h\neq k,\ \forall i\in\mathcal{K}_1. \tag{10}$$

Let us additionally define

$$a^1_{ii} := -\sum_{j\in\mathcal{K}_1, j\neq i} a^1_{ij}, \quad \forall i\in\mathcal{K}_1, \tag{11}$$

$$a^2_{hh} := -\sum_{k\in\mathcal{K}_2, k\neq h} a^2_{hk}, \quad \forall h\in\mathcal{K}_2. \tag{12}$$

The following result comes from Bielecki et al. [3],

Lemma 7.1 *Suppose Assumption (S) is satisfied. Then, each marginal process* $X^n,\ n = 1, 2,$ *is a time-homogeneous Markov chain w.r.t. to its own filtration. Moreover, the infinitesimal generator matrix for* X^n *is* $A^n = [a^n_{ij}]_{i,j\in\mathcal{K}_n},$ *with* $a^n_{ij},\ n = 1, 2$ *defined trough (9) –(12).*

Now, suppose we are given infinitesimal $\mathbb{P}$-generators $A^M = [a^M_{ij}]$ and $A^{SP} = [a^{SP}_{hk}]$ for univariate chains M and SP. Thus, letting $a^1_{ij} = a^M_{ij}$ and $a^2_{ij} = a^{SP}_{ij}$, we are given the right hand sides of equations in the system (9) –(10). If we can now solve this system of equations for $a^X_{ih,jk}$'s, so that the resulting matrix[15] $A^X = [a^X_{ih,jk}]_{i,j\in\mathcal{K}_1,\, h,k\in\mathcal{K}_2}$ satisfies conditions for a generator matrix of a Markov chain. Then, using Lemma 7.1 we may conclude that X is a bivariate Markov chain under $\mathbb{P}$, whose marginals are Markov chains with the same distributions as M and SP under under $\mathbb{P}$. Thus, indeed, the system (9) –(10) essentially serves as a "Markov copula" between the Markovian margins M, SP and the bivariate Markov chain X.

Note that, typically, the system (9)–(10) contains many more variables than equations. Thus, one can create several bivariate Markov chains X with the given margins M and SP. In financial applications this feature leaves a lot of room for various modeling options and for calibration of the model. For example, as observed by Lando and Mortensen [12] although the ratings assigned by S&P and Moody's to the same company do not necessarily coincide, split ratings are rare and are usually only observed in short time intervals. This feature can easily be modelled using the Markovian copula system (9) –(10) via imposing side constraints for the unknowns $a^X_{ih,jk}$'s. In order to model such observed behavior of the joint rating process, we thus impose additional constraints on the variables in the system (9) –(10). Specifically, we postulate that

$$a^X_{ih,jk} = \begin{cases} 0, & \text{if } i \neq j \text{ and } h \neq k \text{ and } j \neq k, \\ \alpha \min(a^M_{ij}, a^{SP}_{hk}), & \text{if } i \neq j \text{ and } h \neq k \text{ and } j = k, \end{cases} \tag{13}$$

where $\alpha \in [0,1]$ is a modelling parameter. Using constraint (13) we can easily solve system (9) –(10) (in this case the system actually becomes fully decoupled) and we can obtain the generator of the joint process. The interpretation of constraint (13) is the following: The components M and SP of the process X migrate according to their marginal laws, but they tend to join, that is, they tend to both take the same values. The strength of such tendency is measured by the parameter α. When $\alpha = 0$ then, in fact, the two components are independent processes; when $\alpha = 1$ the intensity of both components migrating simultaneously to the same rating category is maximum (given the specified functional form for the intensities of common jumps).

7.2 Markovian Changes of Measure

For valuation purposes the statistical probability measure needs to be changed to an equivalent pricing measure. Typically, the Radon-Nikodym density is chosen in such a way that the resulting (risk-neutral) default probabilities are consistent with the term structure of CDS spreads (default data). In addition, we require that the process X, which is Markovian under the statistical measure, is also Markovian (w.r.t. its natural filtration) under the pricing measure. As a consequence, such change of measure must be chosen with some care. We refer the reader to [15] and [2] for a detailed discussion about Markovian changes of measure and their relation to changes of numeraire. We remark that, although the choice of the new probability measure is done so to preserve Markov property of the joint process X, the two components X^M and X^{SP} may not be Markov (w.r.t. their natural filtration) under the new probability measure.

Recall that $A^X = [a^X_{ih,jk}]$ denotes the generator of X under the statistical probability measure $\mathbb{P}$. Given any vector $h = [h_{11}, \cdots, h_{KK}] \in \mathbb{R}^{K^2}$ satisfying very mild conditions[16], we can change statistical measure $\mathbb{P}$ to an equivalent "risk-neutral" measure $\mathbb{Q}$ in such a way that process X is a time-homogeneous Markov chain under $\mathbb{Q}$, and its $\mathbb{Q}$-infinitesimal generator is given by

$$\widetilde{A}^X = [\widetilde{a}_{ih,jk}],$$

where $\widetilde{a}_{ih,jk} = a_{ih,jk}\frac{h_{jk}}{h_{ih}}$ for $ih \neq jk$ and $\widetilde{a}_{ih,jk} = -\sum_{jk\neq ih} a_{ih,jk}\frac{h_{jk}}{h_{ih}}$ for $ih = jk$. An arbitrary choice of vector h may lead to a heavy parametrization of the pricing model. We suggest that the vector h_{ij} be chosen as follows:

$$h_{ij} = \exp(\alpha_1 i + \alpha_2 j), \quad \forall i,j \in \mathcal{K},$$

where α_1 and α_2 are parameters to be calibrated. Calibration and pricing results indicate that this is a good choice.

[15] System (9) –(10) does not include diagonal elements of A^X. These elements are obtained as $a^X_{ih,ih} = -\sum_{(j,k)\in\mathcal{K}} a^X_{ih,jk}$.

[16] It suffices that each component be positive and bounded

7.3 Model Calibration

With the above choice of dependence structure and measure change, the model is fully specified by three parameters, namely $\alpha,\ \alpha_1,\ \alpha_2$. In order to obtain risk neutral prices for rating triggered step–up bonds, these parameters must be calibrated to market data.

Let us consider a vanilla bond, which is equivalent to the given step–up bond. By equivalent, we mean a coupon bearing bond, backed by the same company, whose all provisions, other than the step-up provision, are identical to those of the given step–up bond. That is, maturity and coupon dates are the same, and the coupons of the equivalent bond are equal to the fixed coupons of the step–up bond. In addition, credit risk is the same and liquidity risk is comparable. The term vanilla means that the step-up provision is not present.

One would presume, then, that the price of a step–up bond is equal to the price of the equivalent vanilla bond plus the (positive) value of the step–up provision. If such an equivalent vanilla bond were traded on the market, such presumption could be tested by comparing the market price of the step-up bond to the market price of the equivalent vanilla bond, in expectation that the former is not less than the latter.

Unfortunately, in general, equivalent vanilla bonds are not traded on the market. However, their price can be synthesized by applying a standard bootstrapping-interpolation procedure to the market prices of traded vanilla bonds. The value of the step–up provision, given by the difference between the market price of the step–up bond and the synthetic price of the corresponding equivalent bond is, surprisingly, negative. Such behavior was already noted by some recent empirical literature (cf. eg. [12]), which provides strong evidence that the market typically "underprices" step-up bonds, and often the (implied) market value of the step-up provision is negligible and even negative. Consequently, it appears that the presumption that the price of a step–up bond is equal to the price of the equivalent vanilla bond plus the (positive) value of the step–up provision is incorrect. Such findings suggest that step–up bond investors are, actually, more risk averse than vanilla bond investors. In particular, on the theoretical level, this means that the implied pricing kernel used to price step–up bonds should be different from the pricing kernel used when pricing vanilla bonds. For our purposes, the above implies that the model parameters, or at least those relative to credit migrations, should not be calibrated to vanilla bond data. Nevertheless, these data provide useful information and should not be ignored. In particular, under the assumptions given below, vanilla bond prices can be used to compute a term structure of firm-specific, liquidity adjusted, discount factors (risk–free rate + liquidity spread).

Our first assumption is that the vanilla bond market assesses likelihood of the default event in the same way as the CDS (Credit Default Swap) market[17]. Our second assumption is that liquidity risk is priced identically by the step–up and vanilla bond markets.

Given the above, we can apply a standard bootstrapping-interpolation procedure to a pool of reference bonds[18] to obtain a term-structure of firm specific, liquidity adjusted, zero-coupons. The straightforward procedure is briefly described below. We are given a set of J reference bonds with associated cash-flows $CF^j_{t^j_i}$, $j = 1, \dots, J$, and coupon dates $t^j_0 = 0, \dots, t^j_N = T^j$ such that $T^1 < T^2 < \cdots < T^J$. The cash-flows are then adjusted by the default probability implied by the CDS spreads. Let τ denote the default time of the relevant obligor, the default adjusted cash-flows are $\widetilde{CF}^j_{t^j_i} = CF^j_{t^j_i}\mathbb{Q}(\tau > t^j_i)$. The interpolation-bootstrapping procedure is now applied to the reference bonds with default-risk adjusted cash flows, so that the resulting discount factors account only for the firm specific liquidity spread[19]. At this point, the price of an arbitrary step–up bond can be computed by simulating the evolution of the joint rating process and the relative discounted cash-flows[20]. The simulation and calibration procedures are analogous to those described in section 4.

In view of the observation that step–up bond investors do not use the pricing rule prevailing in the vanilla bond market, the model parameters, $\alpha,\ \alpha_1,\ \alpha_2$ are calibrated to step-up bond data only.

[17]This is not necessary since default risk can be inferred from yield spreads in the bond market, but the high liquidity of the CDS market makes it a preferable choice.

[18]We adopt here terminology from [12] to denote vanilla bonds of several maturities which have comparable liquidity and are issued by the same company as the relevant step-up bond.

[19]Plus market risk spreads other than credit spread.

[20]Simulation seems to be the only feasible computation technique, because of certain path dependencies in the payoff structure, induced by the step–down provision present in most step–up issues. Such path dependency is well explained in [12].

7.3.1 Calibration Results

In this section we present calibration results. The bond data, obtained from Bloomberg's Corporate Bonds section, is relative to mid market quotes on April 5, 2006.

We calibrated the model parameters to a DT (Deutsche Telecom) step-up issue described in the table below:

ISIN	$XS0132407957$
Maturity	$07/11/11$
Coupon	$6\frac{5}{8}$ Annual
Step provision	$\begin{cases} +50 \text{ bps}, & \text{if both downgraded below single Aaa3/A-;} \\ -50 \text{ bps}, & \text{if both subsequently upgraded above Baa1/BBB+.} \end{cases}$

Table 8: *DT step-up issue on April 5, 2006.*

Given the default probability implied by the 5-y CDS spread of DT (46 bps), the liquidity adjusted discount rates are obtained using the above mentioned bootstrapping-interpolation procedure from the following pool of reference bonds:

ISIN	Maturity	Coupon	Mid-Price
XS0141544691	01/22/07	$5\frac{1}{4}$	1.015698
DE0002317807	05/20/08	$5\frac{1}{4}$	1.031821
XS0242840345	02/02/09	3	0.979798
XS0217817112	04/22/09	3	0.978352
XS0210319090	01/19/10	$3\frac{1}{4}$	0.976716
XS0210318795	01/19/15	4	0.960349

Table 9: *Reference bonds pool on April 5, 2006.*

The calibration results are given in the following table:

	Model Price	Market Price
Bond Price	1.11705	1.11705
Step-up provision	.00574	-

Table 10: *Calibration results*

We remark that, since our calibration problem is overdetermined (three parameters are calibrated to one piece of data), the value of the step-up provision is not uniquely defined. This problem can be easily overcome by calibrating the model to more step-up issues of different maturities and/or provisions.

7.4 Valuation of Step-up Bonds

Using the calibrated model, we price selected issues of DT step-up bonds; we refer to Tables 11 and 12 for the description of the bonds.

Table 13 presents the pricing results as well as the corresponding market quotes. The results are satisfactory, indicating that the model is robust and prices consistently across maturities and step–up provisions.

ISIN	$XS0113709264$
Maturity	$07/06/10$
Coupon	$6\frac{5}{8}$ Annual
Step provision	$\begin{cases} +50 \text{ bps}, & \text{if both downgraded below single Aaa3/A-;} \\ -50 \text{ bps}, & \text{if both subsequently upgraded above Baa1/BBB+.} \end{cases}$

Table 11: *DT step-up issue $XS0113709264$ on April 5, 2006.*

ISIN	$XS0155788150$
Maturity	$10/07/09$
Coupon	$6\frac{1}{2}$ Annual
Step provision	$\begin{cases} +50 \text{ bps}, & \text{if both downgraded below Baa1/BBB+;} \\ -50 \text{ bps}, & \text{if both subsequently upgraded above Baa2/BBB.} \end{cases}$

Table 12: *DT step-up issue $XS0155788150$ on April 5, 2006.*

ISIN	XS0113709264	XS0155788150
	Mkt Price/Model Price	Mkt Price/Model Price
Bond Price	1.10105/1.103546	1.08435/1.08685
Step-up provision	- /.003752	- /.00215

Table 13: *Pricing results using calibrated model*

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