Modelling of the Defaultable Term Structure: Conditionally Markov Approach*

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Abstract

The paper provides a detailed technical description of the Bielecki and Rutkowski (2000a,b) approach to the Heath-Jarrow-Morton type modelling of defaultable term structure of interest rates with multiple ratings. Special emphasis is put on the arbitrage-free feature of the model, as well as on the explicit construction of the conditionally Markov process of credit migrations.

Keywords: Defaultable term structure, HJM model, credit risk, credit migration, conditional Markov chain

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1 Introduction

Some of the basic elements of financial markets are *discount and coupon bonds*, which represent so called *fixed income instruments*. Coupon bonds can frequently be considered as portfolios of discount bonds with various maturities. That is why, in many respects, the main object of study with regard to fixed income instruments are discount bonds. A discount bond is an instrument that promises to pay a specified *notional amount* at a specified *maturity* date, say T. Thus, the value of a discount bond at any time $t \leq T$ is derived as a function of the notional amount, as well as some other factors. The dependence of the price of discount bonds on the physical time t and on the maturity dates T is known as the *term structure of interest rates*. It needs to be said though that the term structure of interest rates can be represented in many alternative ways, besides in terms of prices of discount bonds.

If a discount bond pays the promised notional amount at the maturity date T with certainty, then such a bond is called a default free bond. A good example of such bonds is provided by Treasury bonds. Frequently, a discount bond is not certain to pay the promised notional amount at the maturity date. If so, such a bond is considered to be prone to default and is known as *defaultable bond*. Most of the corporate bonds are defaultable bonds. The dependence of the price of defaultable discount bonds on the physical time t and on the maturity dates T is known as the *defaultable term structure of interest rates*.

This papers continues the line of research on *reduced-form* (or *intensity-based*) modeling of defaultable term structure of interest rates originated in Duffie (1994, 1998), Jarrow and Turnbull (1995), Jarrow et al. (1997), Duffie and Singleton (1998, 1999), Lando (1998), Schönbucher (1998), Thomas et al. (1998), Arvanitis et al. (1999), and Bielecki and Rutkowski (2000a,b). We do not make here any attempt to classify or scrutinize various models developed in previous works. For a detailed analysis of these approaches, the interested reader is referred to Chapters 12 and 13 in Bielecki and Rutkowski (2002).

From the mathematical perspective, the intensity-based modeling of random times hinges on the techniques of modeling random times developed in the reliability theory. The key concept in this

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methodology is the survival probability of a reference instrument or entity, or, more specifically, the hazard rate that represents the intensity of default. In the most simple version of the intensity-based approach, nothing is assumed about the factors generating this hazard rate. More sophisticated versions additionally include factor processes that possibly impact the dynamics of the credit spreads.

Important modeling aspects include: the choice of the underlying probability measure (*real-world* or *risk-neutral*), the goal of modeling (risk management or valuation of derivatives), and the source of intensities. In a typical reduced-form model, the value of the firm is not included in the model. The specification of intensities is based either on the model's calibration to market data or on the estimation based on historical observations.

Both in case of credit risk management and in case of valuation of credit derivatives, the possibility of migrations of underlying credit name between different rating grades is essential, as it reflects the fundamental feature of the real-life market of credit risk sensitive instruments (corporate bonds and loans). In practice, credit ratings are the natural attributes of credit names. Most authors (see, for instance, Jarrow et al. (1997) or Lando (1998)) were approaching the issue of modeling of the credit migrations from the Markovian perspective.

In Bielecki and Rutkowski (2000a) we presented a general outline of our Heath-Jarrow-Morton (HJM) type reduced-form approach to the modelling of defaultable term structures that correspond to multiple ratings of corporate bonds. The present paper complements in many ways the previous one, mainly by providing a complete presentation of mathematical aspects of our model.

The paper is organized as follows. First, in Section 2, we expand on the discussion of the case of one rating class – that is, the case when no credit migration other than default is allowed – thereby providing more insight into the model. Subsequently, we provide in Section 4 a detailed presentation of the multiple ratings version of our model. An explicit construction of the continuous-time migration process C is provided in Section 4. We do not show that C actually is a conditionally Markov chain under the (extended) spot martingale probability measure, however. For the proof of this property, see Lemma 11.3.1 in Section 11.3 of Bielecki and Rutkowski (2002), where other relevant properties of the credit migration process are also examined. Let us also mention that Section 13.2.8 of Bielecki and Rutkowski (2002) deals with an important issue of an equivalent change of the probability measure and the associated concepts of market prices for interest rate and credit risks. Section 5 concludes the paper with some pertinent remarks.

An important feature of our model is that it indeed is self-consistent; that is, it is an arbitrage-free model. In our previous paper, we have already indicated this arbitrage-free property of the model. In the present work, this important property receives a full justification, since we provide a detailed description of the enlarged probability space, which, when combined with the dynamics of the instantaneous forward rates and the dynamics of the migration process, underly the arbitrage-free feature of the model.

It should be acknowledged that our construction can be extended in several directions, and some of our assumptions can be weakened substantially. For instance, by combining our approach with results of Eberlein and Raible (1999), Eberlein and Õzkan (2001) have shown that the model can be extended to the case of a term structure model driven by a Lévy process (as opposed to the case of a standard Brownian motion examined here). The important issue of the model's calibration requires further studies.

2 HJM model with default

We first focus on a defaultable bond from a given rating class and we assume that it cannot migrate to another class before default. We assume that the dynamics of default-free and defaultable instantaneous forward rates are specified through the HJM approach. In other words, the coefficients of the real-world dynamics of instantaneous, continuously compounded, forward rates are taken as model's inputs. We assume, of course, that the model of default-free bond market is arbitrage-free. Our goal is to explain the dynamics of the defaultable instantaneous forward rate by introducing a judiciously chosen random time with a stochastic intensity, interpreted as the bond's default time. This random time will be defined as a totally inaccessible stopping time on an enlarged probability space. Unless explicitly stated otherwise, we assume here the fractional recovery of Treasury value scheme, with a constant recovery rate. Results of this section are close to these of Schönbucher (1998), the line of arguments is somewhat different, however.

In this section, we shall work under the following standing assumptions (HJM.1)–(HJM.3). For mild technical conditions under which expressions (2.1) and (2.2) are well defined, the reader is referred to Heath et al. (1992). We fix throughout a finite horizon date $T^* > 0$.

Condition (HJM.1) We are given a d-dimensional standard Brownian motion W, defined on the filtered probability space $(\tilde{\Omega}, \mathbb{F}, \mathbb{P})$, where $\mathbb{P}$ is interpreted as the real-world probability measure.

For the sake of convenience, we shall frequently postulate that the reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T^*]}$ is generated by the process $W_t,\ t \in [0, T^*]$.

Condition (HJM.2) For any fixed maturity $T \le T^*$, the *default-free instantaneous forward rate* $f(t,T)$ satisfies

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t, \tag{2.1}$$

where $\alpha(\cdot,T)$ and $\sigma(\cdot,T)$ are $\mathbb{F}$-adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

The next assumption specifies the dynamics for defaultable instantaneous forward rates.

Condition (HJM.3) For any fixed maturity $T \le T^*$, the *defaultable instantaneous forward rate* $g(t,T)$ satisfies

$$dg(t,T) = \tilde{\alpha}(t,T)\,dt + \tilde{\sigma}(t,T)\,dW_t \tag{2.2}$$

for some $\mathbb{F}$-adapted stochastic processes $\tilde{\alpha}(\cdot,T)$ and $\tilde{\sigma}(\cdot,T)$ with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

By definition, at time $t \le T$, the price of a default-free zero-coupon bond, with the maturity date T and unit face value, equals

$$B(t,T) = \exp\Big(-\int_t^T f(t,u)\,du\Big). \tag{2.3}$$

For any $t \le T$ we set

$$\tilde{D}(t,T) = \exp\Big(-\int_t^T g(t,u)\,du\Big). \tag{2.4}$$

Definition 2.1 *For any date* $t \le T \le T^*$, *the* instantaneous forward credit spread $s(t,T)$ *equals* $s(t,T) = g(t,T) - f(t,T)$.

It is clear that

$$\tilde{D}(t,T) = B(t,T)\exp\Big(-\int_t^T s(t,u)\,du\Big). \tag{2.5}$$

We shall interpret $\tilde{D}(t,T)$ as the *pre-default value* of a T-maturity zero-coupon corporate bond with fractional recovery of Treasury value. At the intuitive level, $\tilde{D}(t,T)$ represents the value of a zero-coupon corporate bond conditioned on the event that the bond has not yet defaulted by time t.

2.1 Default-free term structure

In the first step, we focus on a default-free term structure of interest rates. For the reader's convenience, we quote the following standard result, due to Heath et al. (1992).

Lemma 2.1 *The dynamics of the default-free bond price* $B(t,T)$ *are*

$$dB(t,T) = B(t,T)\big(a(t,T)\,dt + b(t,T)\,dW_t\big), \tag{2.6}$$

where

$$a(t,T) = f(t,t) - \alpha^*(t,T) + \tfrac{1}{2}\,|\sigma^*(t,T)|^2, \quad b(t,T) = -\sigma^*(t,T), \tag{2.7}$$

with $\alpha^*(t,T) = \int_t^T \alpha(t,u)\,du$ *and* $\sigma^*(t,T) = \int_t^T \sigma(t,u)\,du$.

We assume that it is also possible to invest in the default-free savings account $B_t = \exp(\int_0^t r_u\,du)$, corresponding to the *short-term interest rate* $r_t = f(t,t)$. The following condition is known to exclude arbitrage across default-free bonds for all maturities $T \le T^*$ and the savings account (see Heath et al. (1992) or Chapter 13 in Musiela and Rutkowski (1997)).

Condition (HJM.4) There exists an adapted $\mathbb{R}^d$-valued process β such that

$$\mathbb{E}_{\mathbb{P}}\Big\{ \exp\Big(\int_0^{T^*} \beta_u\,dW_u - \frac{1}{2}\int_0^{T^*} |\beta_u|^2\,du \Big) \Big\} = 1$$

and for any maturity $T \le T^*$ and any $t \in [0,T]$ we have

$$\tfrac{1}{2}|\sigma^*(t,T)|^2 - \alpha^*(t,T) = \sigma^*(t,T)\beta_t$$

or, equivalently, $\alpha(t,T) + \sigma(t,T)(\beta_t - \sigma^*(t,T)) = 0$.

Let β be some process satisfying the last condition. Then the probability measure $\mathbb{P}^*$, given by the formula

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\Big(\int_0^{T^*} \beta_u\,dW_u - \frac{1}{2}\int_0^{T^*} |\beta_u|^2\,du \Big), \quad \mathbb{P}\text{-a.s.},$$

is a *spot martingale measure* for the default-free term structure. Let us define a Brownian motion W^* under $\mathbb{P}^*$ by setting: $W^*_t = W_t - \int_0^t \beta_u\,du$ for $t \in [0,T^*]$. Then, for any fixed maturity $T \le T^*$, the discounted price of default-free zero coupon bond satisfies under $\mathbb{P}^*$

$$dZ(t,T) = Z(t,T)b(t,T)\,dW^*_t. \tag{2.8}$$

We assume from now on that the process β is uniquely determined; in other words, the default-free bonds market is complete (this assumption is not required for our further development, though). This means that any default-free contingent claim can be priced through the standard risk-neutral valuation formula. Finally, let us denote by $\mathbb{P}_T$ be the forward martingale probability measure for the date $T \le T^*$, so that

$$\frac{d\mathbb{P}_T}{d\mathbb{P}^*} = \frac{1}{B(0,T)B_T}, \quad \mathbb{P}^*\text{-a.s.}$$

2.2 Pre-default value of a corporate bond

The proof of the next result, which gives dynamics of the relative pre-default value of a corporate bond, is analogous to the proof of Lemma 2.1.

Lemma 2.2 *We have*

$$d\tilde{D}(t,T) = \tilde{D}(t,T)\big(\tilde{a}(t,T)\,dt + \tilde{b}(t,T)\,dW_t\big)$$

with

$$\tilde{a}(t,T) = g(t,t) - \tilde{\alpha}^*(t,T) + \tfrac{1}{2}\,|\tilde{\sigma}^*(t,T)|^2, \quad \tilde{b}(t,T) = -\tilde{\sigma}^*(t,T).$$

In view of Lemma 2.2, under $\mathbb{P}$ the process $\tilde{Z}(t,T) = B_t^{-1}\tilde{D}(t,T)$ satisfies

$$d\tilde{Z}(t,T) = \tilde{Z}(t,T)\big((\tilde{a}(t,T) - r_t)\,dt + \tilde{b}(t,T)\,dW_t\big). \tag{2.9}$$

Consequently, under the (unique) spot martingale probability measure $\mathbb{P}^*$, we have

$$d\tilde{Z}(t,T) = \tilde{Z}(t,T)\big(\lambda^*(t,T)\,dt + \tilde{b}(t,T)\,dW^*_t\big), \tag{2.10}$$

where for every $t \in [0,T]$ we set $\lambda^*(t,T) = \tilde{a}(t,T) - r_t + \tilde{b}(t,T)\beta_t$. It is useful to notice that

$$\lambda^*(t,T) = s_t - \tilde{\alpha}^*(t,T) + \tfrac{1}{2}\,|\tilde{\sigma}^*(t,T)|^2 - \tilde{\sigma}^*(t,T)\beta_t, \tag{2.11}$$

where $s_t = g(t,t) - f(t,t)$ is the *short-term credit spread.* As apparent from (2.11), the process $\lambda^*(t,T)$ may depend on the maturity date T, in general. However, we shall assume that this is not the case, so that the following assumption holds.

Condition (HJM.5) Processes $\lambda^*(t,T)$ given by (2.11) do not depend on T, namely, $\lambda^*(t,U) = \lambda^*(t,T)$ for every $t \in [0, U \wedge T]$ and every $U, T \le T^*$.

To emphasise that we work under assumption (HJM.5), we shall henceforth write λ^*_t, rather than $\lambda^*(t,T)$. In some instances, it is convenient to assume that the coefficients in (2.11) are chosen in such a way that the following property is valid (in the special case of zero recovery, Condition (HJM.6) implies that the risk-neutral intensity of default equals the short-term credit spread; see, formula (40) in Schönbucher (1998)).

Condition (HJM.6) We have $\lambda^*_t = s_t$ for every $t \in [0, T^*]$.

In view of (2.11), Condition (HJM.6) is satisfied whenever for every $0 \le t \le T \le T^*$ we have

$$\tfrac{1}{2}|\tilde{\sigma}^*(t,T)|^2 - \tilde{\alpha}^*(t,T) = \tilde{\sigma}^*(t,T)\beta_t$$

or, equivalently, $\tilde{\alpha}(t,T) + \tilde{\sigma}(t,T)(\beta_t - \tilde{\sigma}^*(t,T)) = 0$. If the coefficients $\alpha(t,T)$, $\sigma(t,T)$ and $\tilde{\sigma}(t,T)$ are given, the last equality uniquely specifies the drift coefficient $\tilde{\alpha}(t,T)$ in (2.2).

2.3 Default time of a corporate bond

We henceforth assume that conditions (HJM.1)–(HJM.5) are satisfied. Let $\delta \in [0,1)$ be a fixed number, referred to as the *recovery rate.* By virtue (2.10), we have

$$d\tilde{Z}(t,T) = \tilde{Z}(t,T)\big(\lambda^*_t\,dt + \tilde{b}(t,T)\,dW^*_t\big).$$

We introduce an auxiliary process $\lambda^*_{1,2}$, which satisfies, for every $t \in [0, T^*]$,

$$(\tilde{Z}(t,T) - \delta Z(t,T))\lambda^*_{1,2}(t) = \tilde{Z}(t,T)\lambda^*_t. \tag{2.12}$$

Notice that in the case of zero recovery (i.e., when $\delta = 0$), we simply have $\lambda^*_{1,2}(t) = \lambda^*_t$ for every $t \in [0,T]$. On the other hand, if we take $\delta > 0$ then the process $\lambda^*_{1,2}$ is strictly positive provided that $\tilde{D}(t,T) > \delta B(t,T)$ (recall that we have assumed that $\tilde{D}(t,T) < B(t,T)$).

Remarks. If the assumption $\tilde{D}(t,T) > \delta B(t,T)$ is relaxed, the process $\lambda^*_{1,2}$ is strictly positive provided that $\lambda^*_t(\tilde{Z}(t,T) - \delta Z(t,T)) > 0$ for every $t \in [0,T]$. Notice also that in general $\lambda^*_{1,2}$ depends both on the recovery rate δ and on maturity date T. We assume that the process $\lambda^*_{1,2}$ is strictly positive and that it is Lebesgue integrable on $[0,T^*]$, with probability 1.

We shall show that there exists a random time τ such that the process (we denote $H_t = \mathbb{1}_{\{\tau \le t\}}$)

$$M_t = H_t - \int_0^t \lambda^*_{1,2}(u)\mathbb{1}_{\{\tau > u\}}\,du, \quad \forall\, t \in [0,T], \tag{2.13}$$

is a martingale under a suitable extension $\mathbb{Q}^*$ of the spot martingale probability measure $\mathbb{P}^*$. In general, we cannot expect a stopping time τ with the desired properties to exist on the original probability space $(\tilde{\Omega}, \mathbb{F}, \mathbb{P}^*)$. For instance, if the underlying filtration is generated by a standard Brownian motion, which is the usual assumption imposed to ensure the uniqueness of the spot martingale probability measure $\mathbb{P}^*$, no stopping time with desired properties exists on the original space. The necessity of enlarging the underlying probability space is also closely related to the fact that it is not possible to replicate a defaultable bond using risk-free bonds; that is, the process $D^\delta(t,T)$ does not correspond to the wealth process of a self-financing portfolio of risk-free bonds.

Let us denote by $(\Omega, \mathcal{G}, \mathbb{Q}^*)$ the enlarged probability space. Our additional requirement is that W^* remains a standard Brownian motion when we substitute $\mathbb{P}^*$ with $\mathbb{Q}^*$. To satisfy all these requirements, it suffices to take a product space $(\tilde{\Omega} \times \hat{\Omega}, (\mathcal{F}_t \otimes \hat{\mathcal{F}})_{t \in [0,T^*]}, \mathbb{P}^* \otimes \hat{\mathbb{Q}})$, where the auxiliary probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{Q}})$ is large enough to support a unit exponential random variable, denoted by η in what follows. Then we set

$$\tau = \inf\Big\{t \in \mathbb{R}_+ : \int_0^t \lambda^*_{1,2}(u)\,du \ge \eta\Big\}.$$

We extend W^* (and all other relevant processes) to the enlarged space by setting $W_t^*(\tilde{\omega},\hat{\omega}) = W_t^*(\tilde{\omega})$, etc. We preserve the notation $\mathbb{F}$ for the trivial extension of $\mathbb{F}$ to the enlarged probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, and we introduce the filtration $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T^*]}$ generated by the random time τ: $\mathcal{H}_t = \sigma(H_u : u \le t)$, where $H_u = 1\!\!1_{\{\tau\le u\}}$ is the jump process associated with the default time τ. Finally, we set $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in [0, T^*]$. Then, it is clearly seen that the desired properties hold under $\mathbb{Q}^* = \mathbb{P}^* \otimes \hat{\mathbb{Q}}$. In particular, the process M given by (2.13) is a $\mathbb{G}$-local martingale under $\mathbb{Q}^*$ and W^* is a $\mathbb{G}$-Brownian motion under $\mathbb{Q}^*$. Notice that for obvious reasons the independence of τ and W^* does not hold. The proof of the following auxiliary result is left to the reader.

Lemma 2.3 *For a fixed $t \in [0,T^*]$, let ξ, η be the two $\mathcal{F}_t$-measurable random variables such that $\xi = \eta$ on $\{\tau > t\}$. Then we have $\xi = \eta$, $\mathbb{Q}^*$-a.s.*

We are in a position to specify the price process of a T-maturity defaultable bond with fractional recovery of Treasury value. We first introduce an auxiliary process $\hat{Z}(t,T)$ by postulating that $\hat{Z}(t,T)$ solves the following SDE

$$\begin{aligned} d\hat{Z}(t,T) &= \hat{Z}(t,T)\big(\tilde{b}(t,T)1\!\!1_{\{\tau>t\}} + b(t,T)1\!\!1_{\{\tau\le t\}}\big)\,dW_t^* \\ &\quad + (\delta Z(t,T) - \hat{Z}(t-,T))\,dM_t \end{aligned} \tag{2.14}$$

with the initial condition $\hat{Z}(0,T) = \tilde{Z}(0,T)$. For obvious reasons, the process $\hat{Z}(t,T)$, if well defined, is a local martingale under $\mathbb{Q}^*$. Combining (2.14) with (2.13), we obtain

$$\begin{aligned} d\hat{Z}(t,T) &= \hat{Z}(t,T)\big(\tilde{b}(t,T)1\!\!1_{\{\tau>t\}} + b(t,T)1\!\!1_{\{\tau\le t\}}\big)\,dW_t^* \\ &\quad + (\hat{Z}(t,T) - \delta Z(t,T))\lambda_{1,2}^*(t)1\!\!1_{\{\tau>t\}}\,dt + (\delta Z(t,T) - \hat{Z}(t-,T))\,dH_t. \end{aligned}$$

On the other hand, inserting (2.10) into (2.12), we find that $\tilde{Z}(t,T)$ obeys

$$d\tilde{Z}(t,T) = (\tilde{Z}(t,T) - \delta Z(t,T))\lambda_{1,2}^*(t)\,dt + \tilde{Z}(t,T)\tilde{b}(t,T)\,dW_t^*. \tag{2.15}$$

It is thus easy to see that $\hat{Z}(t,T) = \tilde{Z}(t,T)$ on $[0,\tau[$, and thus $\hat{Z}(t,T)$ also satisfies the following SDE:

$$\begin{aligned} d\hat{Z}(t,T) &= \hat{Z}(t,T)\big(\tilde{b}(t,T)1\!\!1_{\{\tau>t\}} + b(t,T)1\!\!1_{\{\tau\le t\}}\big)\,dW_t^* \\ &\quad + \hat{Z}(t,T)\lambda_t^*1\!\!1_{\{\tau>t\}}\,dt + (\delta Z(t,T) - \hat{Z}(t-,T))\,dH_t. \end{aligned}$$

Next, from (2.8), for any $t \in [0,T]$ we obtain

$$\hat{Z}(t,T) = 1\!\!1_{\{\tau>t\}}\tilde{Z}(t,T) + \delta 1\!\!1_{\{\tau\le t\}}Z(t,T). \tag{2.16}$$

To check (2.16), it is enough to solve the SDE in question first on the random interval $[0,\tau[$ and subsequently on $[\tau, T]$. In view of the last equality, we may represent the Itô differential of $\hat{Z}(t,T)$ in another way, namely,

$$\begin{aligned} d\hat{Z}(t,T) &= \big(\tilde{Z}(t,T)\tilde{b}(t,T)1\!\!1_{\{\tau>t\}} + \delta Z(t,T)b(t,T)1\!\!1_{\{\tau\le t\}}\big)\,dW_t^* \\ &\quad + \tilde{Z}(t,T)\lambda_t^*1\!\!1_{\{\tau>t\}}\,dt + (\delta Z(t,T) - \tilde{Z}(t-,T))\,dH_t. \end{aligned}$$

In the next step, we introduce the price process $D^\delta(t,T)$ of a T-maturity defaultable bond. For any $t \in [0,T]$, the process $D^\delta(t,T)$ is defined through the formula

$$D^\delta(t,T) = B_t\hat{Z}(t,T) = 1\!\!1_{\{\tau>t\}}\tilde{D}(t,T) + \delta 1\!\!1_{\{\tau\le t\}}B(t,T), \tag{2.17}$$

where the second equality is an immediate consequence of (2.16). In the case when $\delta = 0$, the process $\hat{Z}(t,T)$ vanishes on the stochastic interval $[\tau, T]$, and we have

$$d\hat{Z}(t,T) = \hat{Z}(t,T)\big(\lambda_t^*\,dt + \tilde{b}(t,T)\,dW_t^*\big) - \hat{Z}(t-,T)\,dH_t. \tag{2.18}$$

It is interesting to notice that $\hat{Z}(t,T)$ also satisfies

$$\begin{aligned} d\hat{Z}(t,T) &= \big(\tilde{Z}(t,T)\tilde{b}(t,T)1\!\!1_{\{\tau>t\}} + \delta Z(t,T)b(t,T)1\!\!1_{\{\tau\le t\}}\big)\,dW_t^* \\ &\quad + (\tilde{Z}(t,T) - \delta Z(t,T))\lambda_{1,2}^*(t)1\!\!1_{\{\tau>t\}}\,dt \\ &\quad + (\delta Z(t,T) - \tilde{Z}(t,T))\,dH_t. \end{aligned}$$

This means that the process $\hat{Z}(t,T)$ can alternatively be introduced through the expression

$$\begin{aligned} d\hat{Z}(t,T) &= \big(\tilde{Z}(t,T)\tilde{b}(t,T)1\!\!1_{\{\tau>t\}} + \delta Z(t,T)b(t,T)1\!\!1_{\{\tau\le t\}}\big)\,dW_t^* \\ &\quad + (\delta Z(t,T) - \tilde{Z}(t,T))dM_t \end{aligned} \tag{2.19}$$

with $\hat{Z}(0,T) = \tilde{Z}(0,T)$. We shall use an analogous approach in the next section. To simplify the exposition, we shall make throughout the following technical assumption that will also be in force in Section 3.1 (although the process $\hat{Z}(t,T)$ is defined differently in the next section). We stress that the following condition is a technical one; it will be used in demonstrating the martingale property of the discounted price of our defaultable bond.

Condition (HJM.7) The process $\hat{Z}(t,T)$, given by the stochastic differential equation (2.14) (or, equivalently, by expression (2.19)), is a $\mathbb{G}$-martingale (as opposed to a local martingale) under $\mathbb{Q}^*$.

Let us now focus on the migration process. In the present setting, the two-dimensional migration process $\tilde{C} = (C, \hat{C})$ (see Section 4.1) lives on four states, since we have $K = 2$. We may and do assume that $\tilde{C}_0 = (C_0, \hat{C}_0) = (1,1)$. We also assume that $\hat{C}_t = 1$ for every t. Thus, the only relevant states for the process $\tilde{C}$ are $(1,1)$ and $(2,1)$. The state $(1,1)$ is the *pre-default state,* and the state $(2,1)$ is the absorbing *default state.* Since the component $\hat{C}$ is uniquely determined by the past of the first component, C, it is clear that we only need to specify the dynamics for C.

We postulate that the $\mathbb{F}$-conditional intensity matrix for C equals

$$\Lambda_t^* = \begin{pmatrix} -\lambda_{1,2}^*(t) & \lambda_{1,2}^*(t) \\ 0 & 0 \end{pmatrix}.$$

The default time τ is given by the formula

$$\tau = \inf\{t \in \mathbb{R}_+ : C_t = 2\} = \inf\{t \in \mathbb{R}_+ : (C_t, \hat{C}_t) = (2,1)\}. \tag{2.20}$$

Using (2.17), for every $t \in [0,T]$ we obtain

$$\begin{aligned} D_C(t,T) &:= 1\!\!1_{\{C_t=1\}}\,\tilde{D}(t,T) + \delta 1\!\!1_{\{C_t=2\}}\,B(t,T) \\ &= 1\!\!1_{\{\tau>t\}}\,\tilde{D}(t,T) + \delta 1\!\!1_{\{\tau\le t\}}\,B(t,T) = D^\delta(t,T) \end{aligned}$$

as expected. The component $\hat{C}$ plays no essential role in the present setting. Its relevance will show up in case of multiple credit ratings, though.

In the remaining part of this section, we shall frequently use the notation $1\!\!1_{\{\tau>t\}}$ and $1\!\!1_{\{\tau\le t\}}$, rather than $1\!\!1_{\{C_t=1\}}$ and $1\!\!1_{\{C_t=2\}}$, respectively.

2.4 Case of zero recovery rate

Conditions (HJM.1)–(HJM.5) and (HJM.7) are assumed below. We shall now examine in detail the case of zero recovery rate. We already know that for $\delta = 0$, the matrix Λ^* takes the following form (cf. (2.12)):

$$\Lambda_t^* = \begin{pmatrix} -\lambda_t^* & \lambda_t^* \\ 0 & 0 \end{pmatrix}.$$

This means that the $\mathbb{F}$-intensity of the default time equals λ^*. In particular, it coincides with the short-term credit spread $s(t,t)$ if (HJM.6) is valid. Let $D^0(t,T)$ be given by (2.17) with $\delta = 0$, i.e., $D^0(t,T) = 1\!\!1_{\{\tau>t\}}\tilde{D}(t,T)$, and let $\mathbb{Q}_T$ be the forward martingale probability measure associated with $\mathbb{Q}^*$ through the formula

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}^*} = \frac{1}{B(0,T)B_T}, \quad \mathbb{Q}^*\text{-a.s.} \tag{2.21}$$

It is apparent that we have $\mathbb{Q}_T = \mathbb{P}_T$ on $(\Omega, \mathcal{F}_T)$.

Proposition 2.1 (i) *Under the spot martingale probability measure* $\mathbb{Q}^*$, *we have*

$$dD^0(t,T) = D^0(t,T)\Big(\big(\tilde{a}(t,T) + \tilde{b}(t,T)\beta_t\big)\,dt + \tilde{b}(t,T)\,dW^*_t\Big) - D^0(t-,T)\,dH_t.$$

(ii) *The following risk-neutral valuation formulae are valid*

$$D^0(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1}\mathbb{1}_{\{\tau>T\}}\,|\,\mathcal{G}_t) = B(t,T)\,\mathbb{Q}_T\{\tau > T\,|\,\mathcal{G}_t\}. \tag{2.22}$$

(iii) *The pre-default value process satisfies*

$$\tilde{D}(t,T) = B(t,T)\,\frac{\mathbb{Q}_T\{\tau > T\,|\,\mathcal{F}_t\}}{\mathbb{Q}_T\{\tau > t\,|\,\mathcal{F}_t\}}.$$

Proof. The first statement is an immediate consequence of (2.17), combined with (2.9), (2.16) and (2.18). From (2.10), we obtain

$$d\tilde{D}(t,T) = \tilde{D}(t,T)\big((r_t + \lambda^*_t)\,dt + \tilde{b}(t,T)\,dW^*_t\big),$$

and thus (recall that $\tilde{D}(T,T) = 1$)

$$\tilde{D}(t,T) = \tilde{B}_t\,\mathbb{E}_{\mathbb{P}^*}(\tilde{B}_T^{-1}\,|\,\mathcal{F}_t) = \tilde{B}_t\,\mathbb{E}_{\mathbb{Q}^*}(\tilde{B}_T^{-1}\,|\,\mathcal{G}_t),$$

where we denote: $\tilde{B}_t = \exp\big(\int_0^t (r_u + \lambda^*_u)\,du\big)$. If we now define the process $V_t = \tilde{D}(t,T)$, then this process is just like the process V introduced in Proposition 8.3.2 in Bielecki and Rutkowski (2002), with $Z = 0$ and $X = 1$. Since $\Delta V_\tau = 0$ (this holds since we know that the process $\tilde{D}(t,T)$ is continuous), using Corollary 8.3.1 in Bielecki and Rutkowski (2002) we obtain the first equality in (2.22):

$$D^0(t,T) = \mathbb{1}_{\{\tau>t\}}\tilde{D}(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1}\mathbb{1}_{\{\tau>T\}}\,|\,\mathcal{G}_t).$$

The second equality in (2.22) follows from the Bayes rule and (2.21). Part (iii) follows from part (ii) and Lemma 2.3. □

2.5 Case of a non-zero recovery rate

We shall work under assumptions (HJM.1)–(HJM.5) and (HJM.7). The next result deals with the fractional recovery of Treasury value scheme with an arbitrary recovery rate $\delta \in [0,1)$. Since Proposition 2.2 covers the case of zero recovery, (2.22) can also be seen as a special case of formula (2.24).

Proposition 2.2 *The price process* $D^\delta(t,T)$ *of a corporate bond equals*

$$D^\delta(t,T) = \mathbb{1}_{\{C_t=1\}}\exp\Big(-\int_t^T g(t,u)\,du\Big) + \delta\mathbb{1}_{\{C_t=2\}}\exp\Big(-\int_t^T f(t,u)\,du\Big)$$

or, equivalently,

$$D^\delta(t,T) = \mathbb{1}_{\{C_t=1\}}\tilde{D}(t,T) + \delta\mathbb{1}_{\{C_t=2\}}B(t,T). \tag{2.23}$$

Moreover, the risk-neutral valuation formula holds

$$D^\delta(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}^*}\big(\delta B_T^{-1}\mathbb{1}_{\{\tau\le T\}} + B_T^{-1}\mathbb{1}_{\{\tau>T\}}\,\big|\,\mathcal{G}_t\big). \tag{2.24}$$

Furthermore,

$$D^\delta(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(\delta\mathbb{1}_{\{\tau\le T\}} + \mathbb{1}_{\{\tau>T\}}\,|\,\mathcal{G}_t\big),$$

where $\mathbb{Q}_T$ *is the forward martingale probability measure for the date* T, *associated with the spot martingale probability measure* $\mathbb{Q}^*$ *through* (2.21).

Proof. The first two formulae follow from (2.3) and (2.4), combined with (2.17) and (2.20). In view of (2.17), it is also clear that $D^\delta(T,T) = \delta\mathbb{1}_{\{\tau\le T\}} + \mathbb{1}_{\{\tau>T\}}$. To establish (2.24), it is thus enough to show that the discounted process $B_t^{-1}D^\delta(t,T)$ is a martingale under $\mathbb{Q}^*$. This is obvious, however, since by virtue of (2.17) we have $B_t^{-1}D^\delta(t,T) = \hat{Z}(t,T)$. In view of (2.24), the last equality is an immediate consequence of the Bayes rule and the definition of the probability measure $\mathbb{Q}_T$. □

Let us notice that equations (2.23) and (2.24) are quite natural and expected outcomes. The first equation reflects the dynamical behavior of the defaultable bond consistent with the financial interpretation of the instantaneous forward rates $f(t,T)$, $g(t,T)$, and the recovery rate δ. The second equation is a special case of the standard risk-neutral valuation result. Let us notice that in most valuation models for defaultable bonds, such a formula is postulated, rather than established.

3 HJM model with credit migrations

We shall now construct an arbitrage-free model of a defaultable term structure of interest rates with migrations of the credit rankings of a corporate bond between several rating classes. In this section, we start with:

- a pre-specified default-free term structure, given in terms of the corresponding instantaneous forward rates,
- pre-specified term structures corresponding to a given finite collection of credit classes, formally represented by a finite family of instantaneous forward rates.

We shall maintain here Conditions (HJM.1), (HJM.2) and (HJM.4); however, we shall rename them as (BR.1), (BR.2) and (BR.4), respectively. Similarly as in Section 2, we first assume the fractional recovery of Treasury value scheme. Alternative recovery schemes are studied in Section 3.4.

3.1 Model's assumption

We now assume that the set of credit rating classes is $\mathcal{K} = \{1, \ldots, K\}$, where the class K corresponds to the default event. For any $i = 1, \ldots, K-1$, we write $\delta_i \in [0, 1)$ to denote the corresponding (deterministic) recovery rate. We shall first focus on the fractional recovery of Treasury value scheme, so that δ_i can be seen as the fraction of par paid at bond's maturity, if the bond, which is currently in the i^{th} rating class, defaults. For the sake of brevity, we shall denote $\delta = (\delta_1, \ldots, \delta_{K-1})$.

We shall combine the risk-free term structure of Section 2.1 with $K-1$ different term structures corresponding to the $K-1$ pre-default credit rating classes (the discussion in the previous section regarded the case of $K = 2$). Within the present framework, Condition (HJM.3) takes the following form.

Condition (BR.3) For any $T \le T^*$, the instantaneous forward rate $g_i(t,T)$, corresponding to the rating class $i = 1, \ldots, K$ satisfies under $\mathbb{P}$

$$dg_i(t,T) = \alpha_i(t,T)\,dt + \sigma_i(t,T)\,dW_t,$$

where $\alpha_i(t,T)$, $\sigma_i(t,T)$, $t \in [0,T]$, are adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

Remarks. It will be sometimes convenient to assume in addition that

$$g_{K-1}(t,t) > g_{K-2}(t,t) > \cdots > g_1(t,t) > f(t,t). \tag{3.25}$$

This assumption is not necessary, however.

As before, the price of a T-maturity default-free discount bond is denoted by $B(t,T)$, so that

$$B(t,T) = \exp\Big(-\int_t^T f(t,u)\,du\Big), \tag{3.26}$$

and we denote $Z(t,T) = B_t^{-1}B(t,T)$. For any $i = 1, \ldots, K-1$ we set

$$D_i(t,T) = \exp\Big(-\int_t^T g_i(t,u)\,du\Big). \tag{3.27}$$

Formulae analogous to (2.7) hold for processes $B(t,T)$ and $D_i(t,T)$, $i = 1, \ldots, K-1$, after a suitable change of notation. In particular, we now denote

$$a_i(t,T) = g_i(t,t) - \alpha_i^*(t,T) + \tfrac{1}{2}\,|\sigma_i^*(t,T)|^2, \quad b_i(t,T) = -\sigma_i^*(t,T),$$

where

$$\alpha_i^*(t,T) = \int_t^T \alpha_i(t,u)\,du, \quad \sigma_i^*(t,T) = \int_t^T \sigma_i(t,u)\,du.$$

Recall that Condition (HJM.4), which is assumed throughout this section and is now called Condition (BR.4), defines the process β. Given the process β, Condition (HJM.5) takes the following form.

Condition (BR.5) For $i = 1, \ldots, K-1$, the process $\lambda_i^*(\cdot, T)$, which is given by the formula

$$\lambda_i^*(t,T) = a_i(t,T) - f(t,t) + b_i(t,T)\beta_t, \quad \forall\, t \in [0,T],$$

does not depend on the maturity T.

Remarks. If we also assume that

$$a_i(t,T) + b_i(t,T)\beta_t = g_i(t,T),$$

then $\lambda_i^*(t) = g_i(t,t) - f(t,t)$, so that obviously $\lambda_i^*(t) > 0$ for $i = 1, \ldots, K$. More generally, arguing along the same lines as in the preceding section, one can show that processes λ_i^* are strictly positive. It is worth stressing, however, that neither the strict positivity of λ_i^*s, nor their independence of maturity T, are necessary requirements for our further developments.

We make the standing assumptions (BR.1)–(BR.5). Proceeding as in Section 2, we construct the spot martingale probability measure $\mathbb{P}^*$ for the risk-free term structure of interest rates. Under $\mathbb{P}^*$, the process $Z(t,T) = B_t^{-1} B(t,T)$ satisfies

$$dZ(t,T) = Z(t,T) b(t,T)\, dW_t^*.$$

Likewise, if we define processes $Z_i(t,T) = B_t^{-1} D_i(t,T)$ for $i = 1, \ldots, K-1$, we obtain the following dynamics for $Z_i(t,T)$ under $\mathbb{P}^*$ (cf. (2.10))

$$dZ_i(t,T) = Z_i(t,T)\big(\lambda_i^*(t)\, dt + b_i(t,T)\, dW_t^*\big).$$

3.2 Migration process

The next step is to introduce a conditionally Markov chain C on the state space $\mathcal{K} = \{1, \ldots, K\}$. To construct C, one needs to enlarge the underlying probability space. Suitable extensions of $\mathcal{F}_t$ and $\mathbb{P}^*$ will be denoted by $\mathcal{G}_t$ and $\mathbb{Q}^*$, respectively (for the construction of C, see Section 4). The $\mathbb{F}$-conditional infinitesimal generator of C equals

$$\Lambda_t^* = \begin{pmatrix} \lambda_{1,1}^*(t) & \ldots & \lambda_{1,K}^*(t) \\ \cdot & \ldots & \cdot \\ \lambda_{K-1,1}^*(t) & \ldots & \lambda_{K-1,K}^*(t) \\ 0 & \ldots & 0 \end{pmatrix},$$

where $\lambda_{ii}^*(t) = -\sum_{j \neq i} \lambda_{ij}^*(t)$ for $i = 1, \ldots, K-1$ and λ_{ij}^* are $\mathbb{F}$-adapted, strictly positive processes. To ensure that the model is arbitrage free, the processes λ_{ij}^* will also be assumed to satisfy condition (BR.6).

The fact that K is the absorbing state corresponds to our assumption that the defaulted firm actually goes bankrupt. Although a firm's reorganization or takeover can also be envisaged, from the viewpoint of bondholders these possible extensions are not relevant.

Remarks. Lando (1998) also uses a conditionally Markov process to model credit ratings. In contrast to our endogenous HJM-type approach, he exogenously specifies the credit migration process by introducing an auxiliary process of state-variables that drives the intensities of migrations. In addition, the construction of the migration process given in Section 4 of Lando (1998) seems to be incorrect.

It will be apparent from Condition (BR.6) that the intensities λ_{ij}^* of credit migrations may depend on the maturity T and on the vector of recovery rates δ. Thus, to each maturity T and to every recovery vector δ there may correspond a different migration process. Nevertheless, this feature of the model will not interfere with the property of absence of arbitrage between defaultable bonds of various maturities (and possibly various recovery profiles). This is because the enlarged probability space $(\Omega, (\mathcal{G}_t)_{t \in [0,T^*]}, \mathbb{Q}^*)$ does not depend on neither T nor δ, and the processes $D^\delta(\cdot, T)$, introduced later in this section, are martingales on $(\Omega, (\mathcal{G}_t)_{t \in [0,T^*]}, \mathbb{Q}^*)$, regardless of the particular values of $T \leq T^*$ and $\delta \in [0,1)^{K-1}$. As usual, we shall write $H_t^i = \mathbb{1}_{\{C_t = i\}}$ for $i = 1, \ldots, K$. Let us define

$$M_t^{ij} = H_t^{ij} - \int_0^t \lambda_{ij}^*(s) H_s^i\, ds, \quad \forall\, t \in [0,T],$$

for $i = 1, \dots, K-1$, $j = 1, \dots, K$, and $j \neq i$, where, as before, H_t^{ij} represents the number of transitions from i to j by C over the time interval $(0, t]$. It can be shown (see Proposition 11.3.1 in Bielecki and Rutkowski (2002)) that M_t^{ij} is a local martingale on the enlarged probability space $(\Omega, (\mathcal{G}_t)_{t \in [0,T^*]}, \mathbb{Q}^*)$. In Section 4.1, we have also defined the 'pre-jump' component $\hat{C}$ of the two-dimensional conditional Markov chain $\tilde{C} = (C, \hat{C})$. Observe that $\hat{C}_t = C_{u(t)-}$, where $u(t) = \sup\{u \leq t : C_u \neq C_t\}$. By convention, $\sup \emptyset = 0$, therefore $\hat{C}_t = C_t$ if $C_u = C_0$ for every $u \in [0, t]$. In other words, $u(t)$ is the time of the last jump of C before (and including) time t, and $\hat{C}_t$ represents the last state of C before the jump to the current state.

3.3 General case

We are in a position to examine the general case. For any $K \geq 3$, we define an auxiliary process $\hat{Z}(t,T)$ by postulating that it satisfies:

$$
\begin{aligned}
d\hat{Z}(t,T) &= \sum_{i=1}^{K-1} H_t^i Z_i(t,T) b_i(t,T)\, dW_t^* + \sum_{i=1}^{K-1} \delta_i H_t^{iK} Z(t,T) b(t,T)\, dW_t^* \\
&\quad + \sum_{i,j=1,\, i\neq j}^{K-1} \big(Z_j(t,T) - Z_i(t,T)\big)\, dM_t^{ij} + \sum_{i=1}^{K-1} \big(\delta_i Z(t,T) - Z_i(t,T)\big)\, dM_t^{iK}
\end{aligned}
$$

with the initial condition:

$$
\hat{Z}(0,T) = \sum_{i=1}^{K-1} H_0^i Z_i(0,T).
$$

We shall now introduce the 'consistency' condition. Its role is to ensure that the process $\hat{Z}(\cdot, T)$ is a local martingale under $\mathbb{Q}^*$. For the sake of brevity, we shall write $Z_i(t)$ rather than $Z_i(t,T)$.

Condition (BR.6) For any $i = 1, \dots, K-1$ and every $t \in [0,T]$ the following equalities are satisfied:

$$
\sum_{j=1}^{K-1} \lambda_{ij}^*(t)\big(Z_j(t) - Z_i(t)\big) + \lambda_{iK}^*(t)\big(\delta_i Z(t) - Z_i(t)\big) + \lambda_i^*(t) Z_i(t) = 0.
$$

Under the assumption above, the process $\hat{Z}(t,T)$ is governed by the following expression:

$$
\begin{aligned}
d\hat{Z}(t,T) = &\sum_{i,j=1,\, i\neq j}^{K-1} \big(Z_j(t,T) - Z_i(t,T)\big)\, dH_t^{ij} \\
&+ \sum_{i=1}^{K-1} \big(\delta_i Z(t,T) - Z_i(t,T)\big)\, dH_t^{iK} \\
&+ \sum_{i=1}^{K-1} H_t^i\, dZ_i(t,T) + \sum_{i=1}^{K-1} \delta_i H_t^{iK}\, dZ(t,T).
\end{aligned}
$$

The following result furnishes more convenient representations for the auxiliary process $\hat{Z}(t,T)$.

Lemma 3.4 *Under assumption* (BR.6), *the process* $\hat{Z}(t,T)$ *satisfies*

$$
\hat{Z}(t,T) = \sum_{i=1}^{K-1} \big(H_t^i Z_i(t,T) + \delta_i H_t^{iK} Z(t,T)\big)
$$

or, equivalently,

$$
\hat{Z}(t,T) = \mathbb{1}_{\{C_t \neq K\}}\, Z_{C_t}(t,T) + \delta_{\hat{C}_t} \mathbb{1}_{\{C_t = K\}}\, Z(t,T). \tag{3.28}
$$

Moreover, $\hat{Z}(t,T)$ is the unique solution to the SDE

$$
\begin{aligned}
d\hat{Z}(t,T) = & \sum_{i,j=1,\, i\neq j}^{K-1} \big(Z_j(t,T) - H_t^i \hat{Z}(t-,T)\big)\, dM_t^{ij} \\
& + \sum_{i=1}^{K-1} \big(\delta_i Z(t,T) - H_t^i \hat{Z}(t-,T)\big)\, dM_t^{iK} \\
& + \sum_{i=1}^{K-1} H_t^i \hat{Z}(t,T) b_i(t,T)\, dW_t^* + H_t^K \hat{Z}(t,T) b(t,T)\, dW_t^*
\end{aligned}
$$

with the initial condition $\hat{Z}(0,T) = \sum_{i=1}^{K-1} H_0^i Z_i(0,T)$.

Proof. The lemma can be proved by reasoning as in the case of $K = 3$, and thus the proof is left to the reader. □

As expected, we define the value process of a T-maturity zero-coupon corporate bond by setting

$$
D^\delta(t,T) = B_t \hat{Z}(t,T) = 1\!\!1_{\{C_t \neq K\}} D_{C_t}(t,T) + \delta_{\hat{C}_t} 1\!\!1_{\{C_t = K\}} B(t,T). \tag{3.29}
$$

The next result immediately follows from the properties of $\hat{Z}(t,T)$.

Proposition 3.3 *Under the risk-neutral probability $\mathbb{Q}^*$, the dynamics of the price process $D^\delta(t,T)$ are*

$$
\begin{aligned}
dD^\delta(t,T) = & \sum_{i,j=1,\, i\neq j}^{K-1} \big(D_j(t,T) - D_i(t,T)\big)\, dH_t^{ij} \\
& + \sum_{i=1}^{K-1} \big(\delta_i B(t,T) - D_i(t,T)\big)\, dH_t^{iK} + \sum_{i=1}^{K-1} H_t^i\, dD_i(t,T) \\
& + \sum_{i=1}^{K-1} \delta_i H_t^{iK}\, dB(t,T) + r_t D^\delta(t,T)\, dt,
\end{aligned}
$$

where

$$
dB(t,T) = B(t,T)\big(r_t\, dt + b(t,T)\, dW_t^*\big)
$$

and

$$
dD_i(t,T) = D_i(t,T)\big((r_t + \lambda_i^*(t))\, dt + b_i(t,T)\, dW_t^*\big).
$$

The next theorem shows that the process $D^\delta(t,T)$, formally introduced through (3.29), can be given an intuitive interpretation in terms of the default time and recovery rates. To this end, we make the following technical assumption (cf. Condition (HJM.7) of Section 2.3).

Condition (BR.7) The process $\hat{Z}(t,T)$, given by expression (3.28), is a $\mathbb{G}$-martingale (as opposed to a local martingale) under $\mathbb{Q}^*$.

The main result of this section holds under assumptions (BR.1)–(BR.7).

Theorem 3.1 *For any $i = 1, \ldots, K-1$, let $\delta_i \in [0,1)$ be the recovery rate for a defaultable bond from the i^{th} rating class. The price process $D^\delta(t,T)$ of a T-maturity defaultable bond equals*

$$
D^\delta(t,T) = 1\!\!1_{\{C_t \neq K\}} e^{-\int_t^T g_{C_t}(t,u)\, du} + \delta_{\hat{C}_t} 1\!\!1_{\{C_t = K\}} e^{-\int_t^T f(t,u)\, du},
$$

that is,

$$
D^\delta(t,T) = 1\!\!1_{\{C_t \neq K\}} D_{C_t}(t,T) + \delta_{\hat{C}_t} 1\!\!1_{\{C_t = K\}} B(t,T),
$$

Equivalently,

$$
D^\delta(t,T) = B(t,T)\Big(1\!\!1_{\{C_t \neq K\}} e^{-\int_t^T s_{C_t}(t,u)\, du} + \delta_{\hat{C}_t} 1\!\!1_{\{C_t = K\}}\Big),
$$

where $s_i(t,u) = g_i(t,u) - f(t,u)$ represents the i^{th} instantaneous forward credit spread. Moreover, $D^\delta(t,T)$ is given by the risk-neutral valuation formula, namely,

$$D^\delta(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}^*}\big(\delta_{\hat{C}_T} B_T^{-1} 1\!\!1_{\{\tau \le T\}} + B_T^{-1} 1\!\!1_{\{\tau > T\}} \,|\, \mathcal{G}_t\big), \tag{3.30}$$

where τ is the default time, i.e., $\tau = \inf\{t \in \mathbb{R}_+ : C_t = K\}$. The last formula can also be rewritten as follows:

$$D^\delta(t,T) = B(t,T) \, \mathbb{E}_{\mathbb{Q}_T}\big(\delta_{\hat{C}_T} 1\!\!1_{\{\tau \le T\}} + 1\!\!1_{\{\tau > T\}} \,|\, \mathcal{G}_t\big),$$

where $\mathbb{Q}_T$ is the forward martingale probability measure for the date T, associated with $\mathbb{Q}^*$ through (2.21).

Proof. The first formula is an immediate consequence of (3.29) combined with (3.26) and (3.27). For the second, notice first that in view of the second equality in (3.29) and the definition of τ, the process $D^\delta(t,T)$ satisfies the terminal condition

$$D^\delta(T,T) = \delta_{\hat{C}_T} 1\!\!1_{\{\tau \le T\}} + 1\!\!1_{\{\tau > T\}}.$$

Furthermore, using the first equality in (3.29), we deduce the discounted process $B_t^{-1} D^\delta(t,T)$ coincides with $\hat{Z}(t,T)$, and thus it is a $\mathbb{Q}^*$-martingale. Equality (3.30) is thus obvious. □

3.4 Alternative recovery schemes

Our next goal is to show that our approach can be easily extended to cover other recovery rules. To this end, we introduce the following condition.

Condition (BR.8) The (terminal) *recovery profile* is given in terms of a $(K-1)$-dimensional process $\delta(t) = \big(\delta_1(t), \ldots, \delta_{K-1}(t)\big)$. Each process $\delta_i(t)$, $i = 1, \ldots, K-1$ is an $\mathbb{F}$-adapted and (locally) bounded process on the original probability space $(\tilde{\Omega}, \mathbb{F}, \mathbb{P})$.

Before we complete the description of the general recovery structure, we define an auxiliary process $\hat{Z}(t,T)$ by setting

$$\hat{Z}(t,T) = \sum_{i=1}^{K-1} (H_t^i Z_i(t,T) + \delta_i(t) H_t^{iK} Z(t,T))$$

or, equivalently,

$$\hat{Z}(t,T) = 1\!\!1_{\{C_t \ne K\}} Z_{C_t}(t,T) + \delta_{\hat{C}_t}(t) 1\!\!1_{\{C_t = K\}} Z(t,T).$$

Condition (BR.6) of Section 3.3, which serves to specify the transition intensities of the migration process C, now takes the following form.

Condition (BR.9) For any $i = 1, \ldots, K-1$ and every $t \in [0,T]$, the following equalities are satisfied:

$$\sum_{j=1}^{K-1} \lambda_{ij}^*(t)\big(Z_j(t) - Z_i(t)\big) + \lambda_{iK}^*(t)\big(\delta_i(t) Z(t) - Z_i(t)\big) + \lambda_i^*(t) Z_i(t) = 0.$$

We may now complete the description of the financial interpretation of the general recovery structure. Given the migration process C specified via Condition (BR.9), it is postulated that the recovery payment of $\delta_{\hat{C}_\tau}(\tau)$ occurs at the maturity date T (provided that $\tau \le T$). Specifically, if $\tau \le T$ and if the rating class immediately preceding the default is the i^{th} class (i.e., $\hat{C}_T(\tau) = i$), then the recovery payment received by the bondholders at time T equals $\delta_i(\tau)$. Observe now that under (BR.9) the auxiliary process $\hat{Z}(t,T)$ is a $\mathbb{G}$-local martingale under $\mathbb{Q}^*$. As before, we postulate that this process is a martingale, and we define the price process $\hat{D}^\delta(t,T)$ of a T-maturity defaultable bond by the following expression

$$\hat{D}^\delta(t,T) = B_t \hat{Z}(t,T) = 1\!\!1_{\{\tau > t\}} D_{C_t}(t,T) + 1\!\!1_{\{\tau \le t\}} \delta_{\hat{C}_\tau}(\tau) B(t,T).$$

Consequently, we obtain

$$\hat{D}^\delta(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}^*}\big(\delta^* B_T^{-1} 1\!\!1_{\{\tau \le T\}} + B_T^{-1} 1\!\!1_{\{\tau > T\}} \,\big|\, \mathcal{G}_t\big),$$

where $\delta^* = \delta_{\hat{C}_\tau}(\tau)$. We shall specify the above results to the two particular recovery schemes that were also considered in the previous section.

Fractional recovery of par value. Suppose first that $\delta_i(t) = \delta_i B^{-1}(t,T)$ for $i = 1, \ldots, K-1$. Then

$$\hat{D}^\delta(t,T) = \mathbb{1}_{\{\tau > t\}} D_{C_t}(t,T) + \mathbb{1}_{\{\tau \le t\}} \delta_{\hat{C}_\tau} B^{-1}(\tau,T) B(t,T).$$

This corresponds to the random payoff $\delta^* = \delta_{\hat{C}_\tau} B^{-1}(\tau,T)$ at time T. Consequently, we obtain the following expression for the price process of a T-maturity defaultable bond:

$$\hat{D}^\delta(t,T) = \mathbb{1}_{\{\tau > t\}} D_{C_t}(t,T) + \mathbb{1}_{\{\tau \le t\}} \delta^* B(t,T).$$

The last equality leads to the following risk-neutral representation:

$$\hat{D}^\delta(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}^*}\big(\delta_{\hat{C}_\tau} B^{-1}(\tau,T) B_T^{-1} \mathbb{1}_{\{\tau \le T\}} + B_T^{-1} \mathbb{1}_{\{\tau > T\}} \,\big|\, \mathcal{G}_t\big).$$

Fractional recovery of market value. Assume that for $i = 1, \ldots, K-1$ the recovery processes are given as $\delta_i(t) = \delta_i D_i(t,T) B^{-1}(t,T)$. Then

$$\hat{D}^\delta(t,T) = \mathbb{1}_{\{\tau > t\}} D_{C_t}(t,T) + \mathbb{1}_{\{\tau \le t\}} \delta_{\hat{C}_\tau} D_{\hat{C}_\tau}(\tau,T) B^{-1}(\tau,T) B(t,T).$$

Consequently, the arbitrage price of a zero coupon bond has now the following risk-neutral representation:

$$\hat{D}^\delta(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}^*}\big(\delta_{\hat{C}_\tau} D_{\hat{C}_\tau}(\tau,T) B^{-1}(\tau,T) B_T^{-1} \mathbb{1}_{\{\tau \le T\}} + B_T^{-1} \mathbb{1}_{\{\tau > T\}} \,\big|\, \mathcal{G}_t\big).$$

4 Continuous-time conditionally Markov chains

We examine the case of continuous-time processes, and thus the time parameter t is assumed to take values in $\mathbb{R}_+$. For the sake of the reader's convenience, we shall work under the risk-neutral probability. For this reason, the intensities of transitions are denoted by $\lambda^*_{ij}(t)$ rather than $\lambda_{ij}(t)$. Let us observe that the construction given below also provides a method for constructing an ordinary $\mathbb{G}$-Markov chain; in this case, the transition intensities are deterministic functions of time.

We consider a probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$ endowed with some filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$, such that $\mathbb{F} \subseteq \mathbb{G}$. Let C be a $\mathcal{K}$-valued stochastic process defined on this probability space, where $\mathcal{K} = \{1, \ldots, K\}$. As usual, $\mathbb{F}^C$ denotes the filtration generated by the process C. It is natural to assume that C is a $\mathbb{G}$-adapted process, so that $\mathbb{F}^C \subseteq \mathbb{G}$.

Definition 4.1 *A process C is called a* conditionally $\mathbb{G}$-Markov chain relative to $\mathbb{F}$ *and under $\mathbb{Q}^*$ if for every $0 \le t \le s$ and any function $h : \mathcal{K} \to \mathbb{R}$ we have*

$$\mathbb{E}_{\mathbb{Q}^*}(h(C_s) \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}^*}\big(h(C_s) \,|\, \mathcal{F}_t \vee \sigma(C_t)\big).$$

We shall also say that C is an $\mathbb{F}$-*conditional* $\mathbb{G}$-*Markov chain* under $\mathbb{Q}^*$ if C satisfies the above definition.

Remarks. (i) Observe that if the (reference) filtration $\mathbb{F}$ is trivial (i.e. $\mathcal{F}_t = \{\emptyset, \Omega\}$ for all t) then the above definition reduces to the ordinary definition of a Markov chain. (ii) A Cox process (cf. Lando (1998)) provides a good example of a conditional Markov chain.

Let $\Lambda^*_t = [\lambda^*_{ij}(t)]_{1\le i,j\le K}$, $t \in \mathbb{R}_+$, denote an $\mathbb{F}$-progressively measurable, bounded, matrix-valued process (the boundedness is postulated for the sake of simplicity). For every $i \in \mathcal{K}$, $t \in \mathbb{R}_+$, and any function $h : \mathcal{K} \to \mathbb{R}$, we denote $\Lambda^*_t h(i) = \sum_{j=1}^K \lambda^*_{ij}(t) h(j)$.

Definition 4.2 *An $\mathbb{F}$-progressively measurable, bounded, matrix-valued process Λ^* is called an* $\mathbb{F}$-conditional infinitesimal generator *for a $\mathcal{K}$-valued $\mathbb{F}$-conditional $\mathbb{G}$-Markov chain C under $\mathbb{Q}^*$ if for any function $h : \mathcal{K} \to \mathbb{R}$ the process M^h, given as*

$$M^h_t = h(C_t) - h(C_0) - \int_0^t \Lambda^*_u h(C_u)\, du, \quad \forall\, t \in \mathbb{R}_+,$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}^*$.

In view of the natural interpretation of the process $\lambda^*_{ij}(t)$, $t \in \mathbb{R}_+$, as the $\mathbb{F}$-conditional intensity of transition from the state i to the state j, the $\mathbb{F}$-conditional infinitesimal generator Λ^* is also commonly referred to as the *matrix of stochastic intensities* for C under $\mathbb{Q}^*$. We return to the case of $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^C$. Under an assumption that an $\mathbb{F}$-conditional infinitesimal generator process Λ^* is given a priori, we make take the filtration $\mathbb{F}^{\Lambda^*}$ generated by Λ^* as the reference filtration $\mathbb{F}$, and we obtain the following condition:

$$\mathbb{E}_{\mathbb{Q}^*}\big(h(C_s)\,|\,\mathcal{F}^{\Lambda^*}_t \vee \mathcal{F}^C_t\big) = \mathbb{E}_{\mathbb{Q}^*}\big(h(C_s)\,|\,\mathcal{F}^{\Lambda^*}_t \vee \sigma(C_t)\big).$$

4.1 Construction of a conditionally Markov chain

We shall now provide a formal construction of an $\mathbb{F}$-conditional $\mathbb{G}$-Markov chain C associated with a given infinitesimal generator. The construction given below is inspired, in particular, by Chapter 2 in Davis (1993), Section 7.3 in Last and Brandt (1995), and Sections 2.3-2.4 in Yin and Zhang (1997).

We fix the underlying probability space $(\tilde{\Omega}, \mathcal{F}, \mathbb{P}^*)$, and we assume that it is endowed with the two filtrations, $\mathbb{F}$ and $\tilde{\mathbb{G}}$, satisfying the 'usual conditions' and such that $\mathbb{F} \subseteq \tilde{\mathbb{G}}$. We consider a $K \times K$ matrix Λ^* of bounded, $\mathbb{F}$-progressively measurable stochastic processes

$$\Lambda^*_t = \begin{pmatrix} \lambda^*_{1,1}(t) & \dots & \lambda^*_{1,K-1}(t) & \lambda^*_{1,K}(t) \\ \cdot & \dots & \cdot & \cdot \\ \lambda^*_{K-1,1}(t) & \dots & \lambda^*_{K-1,K-1}(t) & \lambda^*_{K-1,K}(t) \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

The matrix Λ^* will play the role of the matrix of stochastic intensities. We assume that processes λ^*_{ij}, $i \neq j$, are non-negative and

$$\lambda^*_{ii}(t) = -\sum_{j \neq i} \lambda^*_{ij}(t), \quad \forall\, t \in \mathbb{R}_+.$$

Since the last row of the matrix Λ^*_t is zero, the state K will be an absorbing state for C under probability $\mathbb{Q}^*$ that is defined below.

To construct an associated conditionally Markov chain, we need to enlarge the underlying probability space. To this end, we introduce two sequences, $U_{1,k}$, $U_{2,k}$, $k = 1, 2, \dots$, of mutually independent random variables uniformly distributed on $[0,1]$. We may and do assume that they are defined on a Hilbert cube $(\Omega^U, \mathcal{F}^U, \mathbb{P}^U)$ (see Section 23 in Davis (1992)). The generic elements of $\tilde{\Omega}$, Ω^U and of the set $\mathcal{K}$ are denoted by $\tilde{\omega}$, $\omega^U = (\omega^U_{1,1}, \omega^U_{2,1}, \omega^U_{1,2}, \omega^U_{2,2}, \omega^U_{1,3}, \dots)$ and i, respectively. Assume that the initial law μ belongs to $\mu(\mathcal{K})$, where $\mu(\mathcal{K})$ stands for the set of all probability distributions on the space $\bar{\Omega} := \mathcal{K}$. Let $C_0 : \bar{\Omega} \to \mathcal{K}$ be a random variable distributed according to μ. We may and do assume that $C_0(i) = i$ (since the generic element of $\bar{\Omega}$ is denoted by $\bar{\omega}$, we shall also write $C_0(\bar{\omega}) = \bar{\omega}$).

The following notation will be used throughout for the survival functions of the jump times of the process C that we are going to construct:

$$G(t, i, \tilde{\omega}) := e^{\int_0^t \lambda^*_{ii}(v,\tilde{\omega})\,dv} = e^{-\int_0^t \lambda^*_i(v,\tilde{\omega})\,dv},$$

where we denote $\lambda^*_i(v, \tilde{\omega}) = -\lambda^*_{ii}(v, \tilde{\omega})$ for $i = 1, \dots, K$. We define an auxiliary mapping $\mathbf{T} : \mathcal{K} \times [0,\infty) \times [0,1] \times \tilde{\Omega} \to [0,\infty]$ by setting

$$\mathbf{T}(i, s, u, \tilde{\omega}) = \inf\Big\{\, t \geq 0 \, : \, \frac{G(t+s, i, \tilde{\omega})}{G(s, i, \tilde{\omega})} \leq u \Big\},$$

where by convention, $\inf \emptyset = \infty$. Equivalently,

$$\mathbf{T}(i, s, u, \tilde{\omega}) = \inf\Big\{\, t \geq 0 \, : e^{-\int_s^{t+s} \lambda^*_i(v,\tilde{\omega})\,dv} \leq u \Big\}.$$

Let $\mathbf{C} : [0,1] \times \mathcal{K} \times [0,\infty) \times \tilde{\Omega} \to \mathcal{K}$ be any mapping such that, for every $i, j \in \mathcal{K}$, $j \neq i$,

$$\ell\big(\{u \in [0,1] : \mathbf{C}(u, i, t, \tilde{\omega}) = j\}\big) = \begin{cases} \frac{\lambda^*_{ij}(t,\tilde{\omega})}{\lambda^*_i(t,\tilde{\omega})}, & \lambda^*_i(t,\tilde{\omega}) > 0, \\ 0, & \lambda^*_i(t,\tilde{\omega}) = 0, \end{cases}$$

where $\ell(A)$ stands for the Lebesgue measure of the set A. Finally, we define the enlarged probability space by setting:

$$(\Omega, \mathcal{G}, \mathbb{Q}^*) = (\tilde{\Omega} \times \Omega^U \times \bar{\Omega}, \tilde{\mathcal{G}}_\infty \otimes \mathcal{F}^U \otimes 2^{\mathcal{K}}, \mathbb{P}^* \otimes \mathbb{P}^U \otimes \mu).$$

Filtrations defined on the component subspaces are extended to the enlarged space in an obvious way and their denotation is preserved. So, for example, the filtration $\mathbb{F}$ defined on $(\tilde{\Omega}, \mathcal{F}, \mathbb{P}^*)$ is extended to $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, and is still denoted as $\mathbb{F}$.

Step 1: Construction of the 1^{st} *jump time.* Let $\tau_0 := 0$. We define (for brevity, we shall frequently write simply ω instead of $(\tilde{\omega}, \omega^U, \bar{\omega})$)

$$\eta_1(\omega) = \eta_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}) := \mathbf{T}(\bar{\omega}, 0, U_{1,1}(\omega^U_{1,1}), \tilde{\omega})$$

or, more explicitly,

$$\eta_1(\omega) = \inf\Big\{\, t \geq 0 : e^{-\int_0^t \lambda^*_{C_0}(v,\tilde{\omega})\,dv} \leq U_{1,1} \Big\}.$$

Put another way,

$$\eta_1(\omega) = \inf\Big\{\, t \geq 0 : \int_0^t \lambda^*_{C_0}(v,\tilde{\omega})\,dv \geq \tilde{e}_{1,1} \Big\},$$

where $\tilde{e}_{1,1} := -\log U_{1,1}$ is a unit exponential random variable. We define the first jump time τ_1 by setting $\tau_1 = \tau_0 + \eta_1$ so that $\tau_1 = \tau_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega})$. It is thus clear that τ_1 is a random variable on $(\Omega, \mathcal{G}, \mathbb{Q}^*)$.

In fact, τ_1 only depends on the following variables: $\tilde{\omega}$, $\omega^U_{1,1}$ and $\bar{\omega}$. It is also clearly seen that, for every $t > 0$,

$$\mathbb{Q}^*\left\{\tau_1 > t \,\middle|\, \mathcal{F}_t \vee \sigma(C_0)\right\}(\omega) = \frac{G(t, C_0, \tilde{\omega})}{G(0, C_0, \tilde{\omega})} = e^{-\int_0^t \lambda^*_{C_0}(v,\tilde{\omega})\,dv}.$$

Consequently,

$$\mathbb{Q}^*\{\tau_1 > t\} = \mathbb{E}_{\mathbb{Q}^*}\big(G(t, C_0, \tilde{\omega})\big) = \mathbb{E}_{\mathbb{Q}^*}\Big(e^{-\int_0^t \lambda^*_{C_0}(v,\tilde{\omega})\,dv}\Big).$$

Let us observe that the uniform boundedness of the processes λ^*_{ij} implies that $\mathbb{Q}^*\{\tau_1 = 0\} = 0$. Finally, since by assumption $\int_0^\infty \lambda^*_i(t)\,dt = \infty$ for any $i = 1, \ldots, K-1$ we have $\mathbb{Q}^*\{\tau_1 < \infty\} = 1$.

To make sure that the conditional jump distribution can be defined as in Step 2 below, we now check that $\mathbb{Q}^*\{\lambda^*_{C_0}(\tau_1) = 0\} = 0$ or, equivalently, that the equality

$$\mathbb{Q}^*\{\lambda^*_i(\tau_1) = 0,\, C_0 = i\} = 0$$

is valid for every $i = 1, \ldots, K-1$. From the construction of the jump time τ_1, it can be easily deduced that for any bounded, $\mathbb{F}$-adapted stochastic process Z we have

$$\mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{C_0=i\}} Z_{\tau_1(\omega)}(\tilde{\omega})\big) = \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{C_0=i\}} \int_0^\infty Z_t \lambda^*_i(t) e^{-\int_0^t \lambda^*_i(s)ds} dt\Big).$$

By applying the last formula to the bounded, $\mathbb{F}$-adapted process $Z_t = \mathbb{1}_{\tilde{B}}(t)$, where $\tilde{B} = \{(t,\tilde{\omega}) : \lambda^*_i(t,\tilde{\omega}) = 0\}$, we obtain

$$\mathbb{Q}^*\{\lambda^*_i(\tau_1) = 0,\, C_0 = i\} = \mathbb{Q}^*\{(\tau_1(\omega), \tilde{\omega}) \in \tilde{B},\, C_0 = i\} = 0.$$

Step 2: Construction of the 1^{st} *jump.* For any $\omega = (\tilde{\omega}, \omega^U, \bar{\omega})$, we define $\bar{C}_1(\omega)$ by setting

$$\bar{C}_1(\omega) = \bar{C}_1(\tilde{\omega}, \omega^U_{1,1}, \omega^U_{2,1}, \bar{\omega}) := \mathbf{C}(U_{2,1}(\omega^U_{2,1}), C_0(\bar{\omega}), \tau_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}), \tilde{\omega}).$$

It is clear that $\bar{C}_1$ is a random variable on $(\Omega, \mathcal{G}, \mathbb{Q}^*)$. Also, it is apparent that $\bar{C}_1$ depends on $\tilde{\omega}$, $\omega^U_{1,1}$, $\omega^U_{2,1}$ and $\bar{\omega}$ only. Moreover, we have

$$\mathbb{Q}^*\big\{\bar{C}_1 = j \,\big|\, \mathcal{G}^{1,0}_{\tau_1}\big\}(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}) = \frac{\lambda^*_{C_0,j}(\tau_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}), \tilde{\omega})}{\lambda^*_{C_0}(\tau_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}), \tilde{\omega})},$$

where we set $\mathcal{G}^{1,0}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \sigma(C_0)$, where $\mathcal{H}^1_t = \sigma\left(\mathbb{1}_{\{\tau_1 \leq s\}} : 0 \leq s \leq t\right)$ (notice that $\sigma(\tau_1) \subset \mathcal{H}^1_{\tau_1}$).

Step 3: Construction of the 2^{nd} *jump time.* In order to define the second jump time, we first set

$$\begin{aligned}\eta_2(\omega) &= \eta_2(\tilde{\omega}, \omega^U_{1,1}, \omega^U_{2,1}, \omega^U_{1,2}, \bar{\omega}) \\ &= \mathbf{T}(\bar{C}_1(\tilde{\omega}, \omega^U_{1,1}, \omega^U_{2,1}, \bar{\omega}), \tau_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}), U_{1,2}(\omega^U_{1,2}), \tilde{\omega}).\end{aligned}$$

More explicitly,

$$\eta_2 = \inf\Big\{ t \geq 0 \, : e^{-\int_{\tau_1}^{\tau_1+t} \lambda^*_{\bar{C}_1}(v)dv} \leq U_{1,2} \Big\}$$

or, equivalently,

$$\eta_2 = \inf\Big\{ t \geq 0 \, : \, \int_0^t \lambda^*_{\bar{C}_1}(v)\, dv \geq \tilde{e}_{1,2} \Big\},$$

where $\tilde{e}_{1,2} := -\log U_{1,2}$. As expected, we define the time of the second jump by setting $\tau_2 = \tau_1 + \eta_2$. The random variable τ_2, defined on the probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, depends only on the following variables: $\tilde{\omega}$, $\omega^U_{1,1}$, $\omega^U_{2,1}$, $\omega^U_{1,2}$, and $\bar{\omega}$. Again, it can be easily verified that

$$\mathbb{Q}^*\{\eta_2 > t \,|\, \mathcal{F}_{t+\tau_1} \vee \mathcal{H}^1_{\tau_1} \vee \sigma(\bar{C}_1)\}(\omega) = \frac{G(t + \tau_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}), \bar{C}_1, \tilde{\omega})}{G(\tau_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}), \bar{C}_1, \tilde{\omega})},$$

where $\bar{C}_1 = \bar{C}_1(\tilde{\omega}, \omega^U_{1,1}, \omega^U_{2,1}, \bar{\omega})$, and so

$$\mathbb{Q}^*\{\eta_2 > t\} = \mathbb{E}_{\mathbb{Q}^*}\left(\frac{G(t + \tau_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}), \bar{C}_1, \tilde{\omega})}{G(\tau_1(\tilde{\omega}, \omega^U_{1,1}, \bar{\omega}), \bar{C}_1, \tilde{\omega})} \right).$$

Arguing along similar lines as in Step 1, one may easily check that the following equalities hold true: $\mathbb{Q}^*\{\eta_2 = 0\} = 0$, $\mathbb{Q}^*\{\eta_2 < \infty\} = 1$, and $\mathbb{Q}^*\{\lambda^*_{\bar{C}_1}(\tau_2) = 0\} = 0$.

Step 4: Construction of the 2^{nd} *jump.* The random variable $\bar{C}_2$ is defined through the following expression:

$$\bar{C}_2(\omega) := \mathbf{C}(U_{2,2}(\omega^U_{2,2}), \bar{C}_1(\tilde{\omega}, \omega^U_{1,1}, \omega^U_{2,1}, \bar{\omega}), \eta_2(\tilde{\omega}, \omega^U_{1,1}, \omega^U_{2,1}, \omega^U_{1,2}, \bar{\omega}), \tilde{\omega}).$$

As in Step 2, it can be checked that

$$\mathbb{Q}^*\{\bar{C}_2 = j \,|\, \mathcal{G}^{2,1}_{\tau_2}\} = \frac{\lambda^*_{\bar{C}_1,j}(\tau_2)}{\lambda^*_{\bar{C}_1}(\tau_2)},$$

where we set $\mathcal{G}^{2,1}_t = \mathcal{F}_t \vee \mathcal{H}^2_t \vee \sigma(\bar{C}_1)$, and $\mathcal{H}^2_t = \sigma\left(\mathbb{1}_{\{\tau_2 \leq s\}} : 0 \leq s \leq t\right)$ (notice that $\sigma(\tau_2) \subset \mathcal{H}^2_{\tau_1}$).

Step 5: Construction of the k^{th} *jump time and the* k^{th} *jump.* In a similar way as in previous steps, we may construct the k^{th} jump time $\tau_k = \tau_{k-1} + \eta_k$ as well as the k^{th} jump $\bar{C}_k$ for the process C. More specifically, for every $t > 0$ we have

$$\mathbb{Q}^*\big\{\eta_k > t \,\big|\, \mathcal{F}_{t+\tau_{k-1}} \vee \mathcal{H}^{k-1}_{\tau_{k-1}} \vee \sigma(\bar{C}_{k-1})\big\} = \frac{G(t + \tau_{k-1}, \bar{C}_{k-1})}{G(\tau_{k-1}, \bar{C}_{k-1})}$$

and

$$\mathbb{Q}^*\big\{\bar{C}_k = j \,\big|\, \mathcal{G}^{k,k-1}_{\tau_k}\big\} = \frac{\lambda^*_{\bar{C}_{k-1},j}(\tau_k)}{\lambda^*_{\bar{C}_{k-1}}(\tau_k)},$$

where $\mathcal{G}^{k,k-1}_t = \mathcal{F}_t \vee \mathcal{H}^k_t \vee \sigma(\bar{C}_{k-1})$ and $\mathcal{H}^k_t = \sigma\left(\mathbb{1}_{\{\tau_k \leq s\}} : 0 \leq s \leq t\right)$. Observe that, in view of the assumed uniform boundedness of processes λ^*_{ij}, we have: $\tau_k \to \infty$ with probability 1 as k tends to ∞.

Step 6: Construction of C. To obtain a conditionally Markov chain C with values in the state space $\mathcal{K}$, it suffices to set $C_t = \bar{C}_{k-1}$ for $t \in [\tau_{k-1}, \tau_k)$ and any $k \geq 1$. This achieves the canonical construction of an $\mathbb{F}$-conditional $\mathbb{G}$-Markov chain associated with a given $\mathbb{F}$-adapted, matrix-valued, absolutely continuous stochastic process Λ^*_t, $t \in \mathbb{R}_+$.

Remarks. In that follows, we shall also use a two-dimensional migration process, which will be specified as follows: the first component of the two-dimensional migration process is the process C defined above. By definition, the second component of the migration process, denoted by $\hat{C}$, equals

$$\hat{C}_t = \begin{cases} C_0, & \text{if } t \in [0, \tau_2) \\ \bar{C}_{k-1}, & \text{if } t = [\tau_k, \tau_{k+1}),\ k \geq 2. \end{cases}$$

In this way, we obtain a two-dimensional migration process $\tilde{C}_t = (C_t, \hat{C}_t)$ with the finite state space $\mathcal{K} \times \mathcal{K}$. It should be made clear that the second component, $\bar{C}$, is introduced for the sake of convenience only (it will be used to specify the payoff in case of default). As shown in Section 11.3 of Bielecki and Rutkowski (2002), both the process C and the process $\hat{C} = (C, \bar{C})$ are $\mathbb{F}$-conditional Markov chains.

4.2 Forward Kolmogorov equation

For any function $h : \mathcal{K} \to \mathbb{R}$ we denote, for every $i \in \mathcal{K}$ and $t \in \mathbb{R}_+$,

$$\Lambda_t^* h(i) = \sum_{j=1}^{K} \lambda_{ij}^*(t) h(j).$$

For the proof of the next auxiliary result we refer to Bielecki and Rutkowski (2002) (see Proposition 11.3.1 therein).

Proposition 4.4 *For any function $h : \mathcal{K} \to \mathbb{R}$, the process M^h, given by the formula*

$$M_t^h = h(C_t) - \int_0^t \Lambda_u^* h(C_u)\, du,$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}^*$.

Let $\mathcal{P}^*(t, s)$ be the $\mathbb{F}$-conditional transition probability matrix for the process C under the probability measure $\mathbb{Q}^*$, specifically,

$$\mathcal{P}^*(t, s) := \big[p_{ij}^*(t, s)\big]_{1 \leq i,j \leq K},$$

where, for every $i, j = 1, \ldots, K$,

$$p_{ij}^*(t, s) := \mathbb{Q}^*\big\{C_t = j \,\big|\, \mathcal{F}_t \vee \{C_t = i\}\big\}.$$

It is useful to observe that the martingale property with respect to the filtration $\mathbb{G}$ of the process M^h implies immediately that that for $s \leq t$

$$\mathbb{E}_{\mathbb{Q}^*}\Big(h(C_s) - \int_t^s \Lambda_u^* h(C_u)\, du \,\Big|\, \mathcal{G}_t\Big) = h(C_t).$$

In particular, by applying the last formula to the function $h = \mathbb{1}_{\{j\}}$, we obtain the following equality

$$\mathbb{Q}^*\big\{C_s = j \,\big|\, \mathcal{G}_t\big\} = H_t^j - \int_t^s \mathbb{E}_{\mathbb{Q}^*}\big(\lambda_{C_u, j}^*(u) \,\big|\, \mathcal{G}_t\big)\, du.$$

We conclude that for any $i \in \mathcal{K}$ we have

$$p_{ij}^*(t, s) = \mathbb{1}_{\{j\}}(i) - \int_t^s \sum_{k=1}^{K} \mathbb{E}_{\mathbb{Q}^*}\Big(\lambda_{kj}^*(u) p_{ik}^*(t, u) \,\Big|\, \mathcal{F}_t \vee \{C_t = i\}\Big)\, du.$$

In view of the last equality, we are in a position to state the following result that clarifies the conditional Markov property of C. The proof of Corollary 1 is omitted (see Corollary 11.3.3 in Bielecki and Rutkowski (2002)).

Corollary 4.1 *For any fixed $t \in \mathbb{R}_+$, the transition probability matrix $\mathcal{P}^*(t,s)$ satisfies the forward Kolmogorov equation*

$$\frac{d\mathcal{P}^*(t,s)}{ds} = \mathcal{P}^*(t,s)\Lambda^*(t,s), \quad s \geq t,$$

with the initial condition $\mathcal{P}^*(t,t) = \mathrm{Id}$, where

$$\Lambda^*(t,s) = \mathbb{E}_{\mathbb{Q}^*}\big(\Lambda^*_s \,\big|\, \mathcal{F}_t \vee \sigma(C_t)\big)$$

and Id *is the K-dimensional identity matrix.*

5 Final remarks

Observe that in general even if all four pieces of data – namely: the maturity date, the transition intensities, the recovery scheme and the initial rating – are identical for the two zero-coupon bonds, the bonds themselves may not be identical. In fact, if they are issued by two different entities, the associated migration processes C and C' are also distinct, in general. More specifically, if we consider the joint migration process (C, C'), then the marginal finite-dimensional distributions for C and C' are identical, but in general $C \not\equiv C'$. If $C \not\equiv C'$, the credit migration processes C and C' may be either (conditionally) independent or dependent. In case of independent migration processes C and C', no statistical dependence between credit migrations of the two bonds appears. In case of mutually dependent migration processes, one needs to calibrate the dependence structure (or, more crudely, the correlation structure) between C and C'.

The foregoing remarks are valid if one considers an application of the general methodology presented in this paper to the valuation and hedging of individual defaultable bonds – that is, corporate bonds issued by particular institutions, as well as to the valuation and hedging of related credit derivatives. As an alternative, let us mention that the methodology presented in this paper may be applied to a totality of alike defaultable bonds – that is, to the totality of bonds for which all four features listed above coincide.

In the latter approach, we identify all such bonds and we substitute them with a *representative bond* with an associated representative migration process. This application of our methodology aims at valuation and hedging of credit derivatives that are tied to the average market value of corporate bonds of a given credit quality. Thus, the correlation structure between individual bonds is deliberately disregarded. All that really matters in this interpretation are the marginal statistical properties of individual corporate bonds, and they are identical for all bonds in a given class.

Let us consider two different defaultable bonds, and let us denote the associated migration processes as C and C'. The respective default events are:

$$A = \bigcup_{t \leq T^*} \{C_{t-} \neq K,\, C_t = K\}, \quad A' = \bigcup_{t \leq T^*} \{C'_{t-} \neq K,\, C'_t = K\},$$

and the respective default times are:

$$\tau = \inf\{t \in [0, T^*] : C_t = K\}, \quad \tau' = \inf\{t \in [0, T^*] : C'_t = K\}.$$

We may study two types of default correlations: the correlation between random variables $\mathbb{1}_A$ and $\mathbb{1}_{A'}$ and the correlation between random variables τ and τ'. Various correlation coefficients, such as Pearson's (or linear) correlation coefficient, may be used to measure the strength of these correlations. Likewise, we may analyze the correlations between the *survival events* of the form: $S(t) = \{\tau > t\}$ and $S'(t) = \{\tau' > t\}$. Of course, the correlation structure will typically vary depending on whether one uses the risk-neutral probability $\mathbb{Q}^*$ or the real-world probability $\mathbb{Q}$.

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