Dynamic Hedging of Counterparty Exposure*

Tomasz R Bielecki† and Stéphane Crépey‡

Abstract We study mathematical aspects of dynamic hedging of Credit Valuation Adjustment (CVA) in a portfolio of OTC financial derivatives. Since the sub-prime crisis, counterparty risk and wrong way risk are a crucial issue in connection with valuation and risk management of credit derivatives. In this work we first derive a general, model free equation for the dynamics of the CVA of a portfolio of OTC derivatives. We then particularize these dynamics to the counterparty risk of a portfolio of credit derivatives including, for instance, CDSs and/or CDOs, possibly netted and collateralized, considered in the so called Markovian copula model. Wrong way risk is represented in the model by the possibility of simultaneous defaults. We establish a rigorous connection between the CVA, which represents the price of the counterparty risk, and a suitable notion of Expected Positive Exposure (EPE). Specifically, the EPE emerges as the key ingredient of the min-variance hedging ratio of the CVA by a CDS on the counterparty. Related notions of EPE have actually long been used in an ad-hoc way by practitioners for hedging their CVA. Our analysis thus justifies rigorously this market practice, making also precise the proper

Tomasz R. Bielecki
Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA, e-mail: bielecki@iit.edu

Stéphane Crépey
Equipe Analyse et Probabilité, Université d'Évry Val d'Essonne, 91025 Évry Cedex, France, and CRIS Consortium, e-mail: stephane.crepey@univ-evry.fr

* The authors warmly thank Giovanni Cesari for stimulating discussions throughout the preparation of this work. We also thank an anonymous referee for a very careful reading of the manuscript and for valuable comments and remarks.

† The research of T.R. Bielecki was supported by NSF Grant DMS–0603789 and NSF Grant DMS–0908099.

‡ The research of S. Crépey benefited from the support of the 'Chaire Risque de crédit', Fédération Bancaire Française, and of the DGE.

definition of the EPE which should be used in this regard, and the way in which the EPE should be used in the hedging strategy.

Key words: Counterparty Risk, Credit Risk, Credit Valuation Adjustment, Expected Positive Exposure, Collateralization, Markov Copula, Joint Defaults, Hedging.

1 Introduction

Counterparty risk is the most primitive risk in any financial contract involving cash-flows/liabilities distributed over time. This is the risk that the future contractual obligations will not be fulfilled by at least one of the two parties to such a financial contract.

There has been a lot of research activity in the recent years devoted to valuation of counter-party risk (we refer to [1] for a comprehensive survey of literature). In contrast, almost no attention has been devoted to quantitative studies of the problem of dynamic hedging of this form of risk. There is some discussion devoted to dynamic hedging of counterparty exposure in Cesari *et al.* [10] and in Gregory [16].

In this paper we build upon the model developed in [1] for the purpose of valuation of CVA, and we present formal mathematical results that provide analytical basis for the quantitative methodology of dynamic hedging of counterparty risk.

In Sect. 2 and 3 we recall and give new ramifications to the general CVA results of [1], integrating to the set-up important practical notions related to the modeling of the collateral. This is a key counterparty risk modeling issue since, for instance, AIG's bailout was largely triggered by its inability to face increasing margin calls on its sell-protection CDS positions (on the distressed Lehman in particular). In Sect. 4 we present a variant of the common shocks portfolio credit risk model of [2], more specifically tailored to the application of valuation and hedging of the counterparty risk on a portfolio of credit derivatives. We proceed, in Sect. 5, with a mathematical study of dynamic hedging of counterparty risk on a portfolio of credit derivatives, in the common shocks model of Sect. 4. In particular, we provide a formula for the risk-neutral min-variance delta of the portfolio CVA with respect to a counterparty clean CDS on the counterparty which is used to hedge the counterparty's jump-to-default exposure component of the CVA. Notably, we establish the connection between this delta, and a suitable notion of Expected Positive Exposure (EPE), providing ground to the market intuition of using EPE to hedge CVA. We make precise the proper definition of the EPE which should be used in this regard, and the way in which EPE should be used in the hedging strategy. Implementation issues and numerics will be considered in a follow-up paper.

1.1 General Set-Up

We consider two parties of a financial contract. We call them *the investor* and *the counterparty*. We denote by τ_{-1} and τ_0 the default times of the investor and of the counterparty, respectively. In [1] (see also [12]) we studied the problem of valuation of the unilateral counterparty risk (as seen from the perspective of the investor, i.e. $\tau_{-1} = \infty$ and $\tau_0 < \infty$), as well as valuation of the bilateral counterparty risk (i.e. $\tau_{-1} < \infty$ and $\tau_0 < \infty$). In particular, we formulated various ways to represent and to compute the counterparty value adjustment (CVA).

Here we focus on the problem of dynamic hedging of the counterparty risk. CVA can be thought of as the price of an exotic derivative, sometimes referred to as the contingent credit default swap (CCDS, see *e.g.* [10], [16]). In this paper, by hedging of the counterparty risk, we shall mean dynamic hedging of CVA (or, dynamic hedging of the corresponding CCDS).

We start by recalling from [1] a general representation formula for bilateral counterparty risk valuation adjustment, for a fully netted and collateralized portfolio of contracts between the investor and his/her counterparty. This result can be considered as general since, for any partition of a portfolio into netted sub-portfolios, the results of this section may be applied separately to every sub-portfolio. The exposure at the portfolio level is then simply derived as the sum of the exposures of the sub-portfolios. Moreover, this holds for a general portfolio, not necessarily made of credit derivatives.

It needs to be emphasized that we do not exclude simultaneous defaults of the investor and his/her counterparty, since in Sect. 4-5, we shall actually use simultaneous defaults, in the manner of [1], to implement defaults dependence and wrong way risk. We do assume however that the default times cannot occur at fixed times, which is for instance satisfied in all the intensity models of credit risk.

For $i = -1$ or 0, representing the two counterparties, let H^i stand for the default indicator processes of τ_i, so $H_t^i = \mathbb{1}_{\tau_i \le t}$. By default time, we mean the effective default time in the sense of the time at which promised dividends and margin calls, cease to be paid by the distressed party. We also denote $\tau = \tau_{-1} \wedge \tau_0$, with related default indicator process denoted by H. In the case where unilateral counterparty risk is considered, one simply sets $\tau_{-1} = +\infty$, so in this case $\tau = \tau_0$. We fix the portfolio time horizon $T \in \mathbb{R}_+$, and we fix an underlying risk-neutral pricing model $(\Omega, \mathbb{F}, \mathbb{P})$ such that τ_{-1} and τ_0 are $\mathbb{F}$-stopping times. All processes are $\mathbb{F}$-adapted.[4]

We assume that all the random times are $[0,T] \cup \{+\infty\}$-valued. We denote by $\mathbb{E}_\theta$ the conditional expectation under $\mathbb{P}$ given A_θ, for any $\mathbb{F}$ – stopping time θ. All the cash flows and prices (mark-to-market values of cash flows) are considered from the perspective of the investor. In accordance with the usual convention regarding *ex-dividend* valuation, $\int_a^b$ is to be understood as $\int_{(a,b]}$, so in particular $\int_a^b = 0$ whenever $a \ge b$.

In the rest of the paper, β will denote a finite variation and continuous *risk-free discount factor* process.

[4] See Remark 3.2 for filtration issues.

2 Cashflows

We let D and $\mathscr{D}$ represent, respectively, the counterparty clean and the counterparty risky cumulative dividend processes of the portfolio over the time horizon $[0,T]$, assumed to be of finite variation. For future convenience, we extend these processes to the interval $[0,\infty]$ by constancy, that is setting them equal to D_T and $\mathscr{D}_{T+\delta}$ on the intervals $(T,\infty]$ and $(T+\delta,\infty]$, respectively.

By *counterparty clean cumulative dividend process* we mean the cumulative dividend process that does not account for the counterparty risk, whereas by *counterparty risky cumulative dividend process* we mean the cumulative dividend process that does account for the counterparty risk.

We shall consider collateralized portfolios. In this regard we shall consider a *cumulative margin process* and we shall assume that no lump margin cash-flow can be asked for at time τ. Accordingly, given a finite variation cumulative margin process γ, we define the *cumulative discounted margin process* by

$$\beta\Gamma = \int_{[0,\cdot)} \beta_t(1-H_t)d\gamma_t \ . \tag{1}$$

So, in particular, $\Gamma_0 = \gamma_0 - \gamma_{0-}$, and one has for $\tau < \infty$,

$$\beta_\tau \Gamma_\tau = \int_{[0,\tau)} \beta_t d\gamma_t \ .$$

In our notation the collateral process Γ is the algebraic amount given to the investor -1 by the counterparty 0 at time τ. Thus, a positive Γ_t means cash and/or collateral assets already transferred to the account of the investor but still owned by the counterparty.[5] These funds will actually become property of the investor in case of default of the counterparty at time τ. It is worth stressing that, according to industry standards, in case of default of the investor at time τ, these funds will also become property of the investor, unless a special segregation procedure is in force (see Sect. 2.1). Symmetric remarks apply to negative Γ_t (swap the roles of the counterparty and investor in the above description).

Three reference collateralization schemes are the naked scheme $\Gamma = 0$, and the so-called perfect scheme and ISDA scheme to be defined in Sect. 3.2.

We assume for notational simplicity that γ and Γ are killed at T (so $\Gamma_t = \gamma_t = 0$ for $t \geq T$) and we define an $\mathscr{F}_\tau$-measurable random variable χ as

$$\chi = P_{(\tau)} + \Delta D_\tau - \Gamma_\tau \ , \tag{2}$$

in which, for $\tau < \infty$, $\Delta D_\tau = D_\tau - D_{\tau-}$ denotes the jump of D at τ, and where the so called *legal value* $P_{(\tau)}$ is a A_τ-measurable random variable representing the

[5] Consequently, any cash flows, such as dividends paid by the collateral assets, are thus channeled back to the counterparty.

'fair value', in a sense to be agreed upon between the two parties at the contract's inception, of the portfolio at time τ.

From the point of view of financial interpretation, χ represents the *(algebraic) debt* of the counterparty to the investor at the first time of default τ of either party, accounting for the legal value of the portfolio at that time, plus any bullet dividend which should be paid at time τ by the counterparty to the investor, less the margin amount Γ_τ which is already in the hands of the counterparty (cf. term Γ_τ in the first line of equation e̊:divsup below).

Let D^* denote the dividend process corresponding to the cash flows of D 'stopped at $\tau-$', that is

$$D^* = (1-H)D + HD_{\tau-} \ .$$

We model the counterparty risky portfolio cumulative dividend process as

$$\begin{aligned}\mathscr{D} &= D^* + \\ &\mathbb{1}_{\tau<T}\big(\Gamma_\tau H + \big(R_0\chi^+ - \chi^-\big)[H,H^0] - \big(R_{-1}\chi^- - \chi^+\big)[H,H^{-1}] - \chi[[H,H^0],H^{-1}]\big),\end{aligned} \tag{3}$$

where in the *close out cash-flow* corresponding to the second line of (3), the $[0,1]$-valued A_{τ_0} – and $A_{\tau_{-1}}$ – measurable random variables R_0 and R_{-1}, respectively denote the recovery rates of the investor and of its counterparty upon default, and $[\cdot,\cdot]$ is the covariation process, which in the present case of default indicator processes, reduces to the indicator process of the simultaneous default.

So, if the investor defaults first at time $\tau_{-1} < \tau_0$, then, at time $\tau = \tau_{-1}$, the close out cash-flow takes place in the amount of $\Gamma_\tau - \big(R_{-1}\chi^- - \chi^+\big)$; if the investor's counterparty defaults first at time $\tau_0 < \tau_{-1}$, then, at time $\tau = \tau_0$, the close out cash-flow takes place in the amount of $\Gamma_\tau + R_0\chi^+ - \chi^-$; if the investor and the counterparty default simultaneously at time $\tau_0 = \tau_{-1}T$, then, at time $\tau = \tau_0 = \tau_{-1}$ the close out cash-flow takes place in the amount of $\Gamma_\tau + R_0\chi^+ - R_{-1}\chi^-$.

2.1 Re-hypothecation Risk and Segregation

Re-hypothecation refers to the possibility for the investor (the symmetric issue arises relatively to her counterparty) to use as collateral, assets that were already posted to her as collateral, in the context of another transaction with a third party. In this case, setting-up the collateral is at no cost for the investor, and can even be beneficial in certain cases. This explains the popularity of re-hypothecation among market participants. But, on the other hand, re-hypothecation raises a new counterparty risk, namely the risk of not getting back one's collateral at time where this should be the case (one's position having appreciated), because the collateral that one has posted to a counterparty is not by this counterparty anymore, but stuck to some third party to which it has been re-hypothecated. This practically means that the counterparty defaults at this time, and that the colletaral is then lost, up to a fractional recovery.

However, in practice, collateral is typically kept in a segregated, third-party account, and under certain collateral conventions, clauses are that, should the counterparty default first at time $\tau = \tau_0 < \tau_{-1}$ and the investor be in-the money at that time because of the collateralization scheme, so $\chi = P_{(\tau)} + \Delta D_\tau - \Gamma_\tau > 0$ but $\chi^0 := P_{(\tau)} + \Delta D_\tau < 0$, then the investor will be fully compensated on the segregated collateral and will incur no loss at default in this case (see Durand and and Rutkowski [15]).

This means in this case that the collateral posted in excess by the investor will be returned to her, and that the close out cashflow will be $P_{(\tau)} + \Delta D_\tau$, instead of $\Gamma_\tau H + \big(R_0\chi^+ - \chi^-\big) < P_{(\tau)} + \Delta D_\tau$ (assuming a 'nominal' recovery rate $R_0 < 1$).

Note that this can be accounted for in the above formalism, by working with an 'effective' (as opposed to nominal) recovery rate R_0 of the counterparty, equal to one on the event that $P_{(\tau)} + \Delta D_\tau$ is negative.

Segregation in this sense thus eliminates the investor's re-hypothecation risk.

Likewise, the symmetric case regarding the counterparty can be accounted for by letting an effective recovery rate R_{-1} be equal to one on the event that $P_{(\tau)} + \Delta D_\tau$ is positive, to the effect of eliminating the counterparty's re-hypothecation risk.

2.2 Cure Period

In practice there is a time lag $\delta > 0$, called the cure period, and typically taken to be $\delta =$ two weeks, between the default time τ and the close out cash flow, which thus occurs at time $t + \delta$. The exact interpretation of the cure period depends on the CSA (Credit Support Annex) which is in force regarding the particular portfolio at hand.

More generally, one calls *margin period of risk*, the time lag between the last margin call preceding τ, and the time $\tau + \delta$ of the close-out cash flow. The cure period thus constitutes the second part of the margin period of risk, the first part of the margin period of risk consisting of the time lag between the default time τ and the last margin call preceding it. These two components of the margin period of risk play rather distinct roles in the modeling. The role of the first component will be analyzed in Sect. 3.2.

Let $\hat{H}_t = \mathbb{1}_{t \geq \tau + \delta}$, and let similar notations $\hat{H}^0$ and $\hat{H}^{-1}$ hold for H^0 and H^{-1}.

In a first interpretation, the cure period accounts for the time that is needed to liquidate collateral assets in case of the default of one of the two parties, so

$$
\begin{aligned}
\mathscr{D} = D^* + &\\
&\mathbb{1}_{\tau<T}\big(\Gamma_\tau \hat{H} + \big(R_0\chi^+ - \chi^-\big)[\hat{H}, \hat{H}^0] \\
&- \big(R_{-1}\chi^- - \chi^+\big)[\hat{H}, \hat{H}^{-1}] - \chi[[\hat{H}, \hat{H}^0], \hat{H}^{-1}]\big) \\
&=: D^* + \Upsilon \; .
\end{aligned}
\tag{4}
$$

For example, if the investor defaults first a time $\tau = \tau_{-1} < \tau_0 \wedge T$, then, at time $\tau + \delta$ the close out cash-flow takes place in the amount of $\Gamma_\tau - \left(R_{-1}\chi^- - \chi^+\right)$.

In a second interpretation, the cure period represents a time period δ between the 'effective' default time τ in the sense of the time at which promised dividends and margin calls actually cease to be paid by the distressed party, and the 'legal' default time $\tau + \delta$ of the close-out cashflow (whereas the effective and the legal default time are both equal to τ in the first interpretation). The counterparty risky cash-flows are thus still given by (4), but for χ in (4) now given by, instead of (2),

$$\beta_{\tau+\delta}\chi = \beta_{\tau+\delta}P_{(\tau+\delta)} + \int_{[\tau,\tau+\delta]} \beta_t dD_t - \beta_{\tau+\delta}\Gamma_\tau , \tag{5}$$

for an $A_{\tau+\delta}$-measurable legal value $P_{(\tau+\delta)}$. Also, the recoveries R_0 and R_{-1} are now given as $A_{\tau_0+\delta}$ and $A_{\tau_{-1}+\delta}$ (instead of A_{τ_0} and $A_{\tau_{-1}}$ previously) – measurable random variables.

For example, if the investor stops payments a time $\tau = \tau_{-1} < \tau_0$, then, at time $\tau + \delta$ the close out cash-flow takes place in the amount of $\Gamma_\tau - \left(R_{-1}\chi^- - \chi^+\right)$, for χ therein given by (5).

Of course, in case $\delta = 0$, both interpretations reduce to the above no-cure-period case.

3 Pricing

The following definitions are consistent with the standard theory of arbitrage (cf. [14]).

Definition 3.1 **(i)** *The* counterparty clean price process, or counterparty clean mark-to-market process, *of the portfolio, is given by $P_t = \mathbb{E}_t[p^t]$, where the random variable $\beta_t p^t$ represents the* cumulative discounted cash flows of the portfolio *on the time interval $(t,T]$, not accounting for counterparty risk. So, for $t \in [0,T]$,*

$$\beta_t p^t = \int_t^T \beta_s dD_s \, . \tag{6}$$

The cumulative counterparty clean value process *of the portfolio is given by*

$$\widehat{P}_t = P_t + p_t , \tag{7}$$

where p_t represents the discounted cumulative dividend process up to time t, so

$$\beta_t p_t = \int_0^t \beta_s dD_s \, . \tag{8}$$

(ii) *The* counterparty risky mark-to-market *process of the portfolio is given by* $\Pi_t = \mathbb{E}_t\left[\pi^t\right]$, *where the random variable* π^t *represents the* cumulative discounted cash flows of the portfolio adjusted for the counterparty risk *on the time interval* $(t,T]$. *So, for* $t \in [0,T]$,

$$\beta_t \pi^t = \int_t^T \beta_s d\mathscr{D}_s \ . \tag{9}$$

The cumulative counterparty risky price process *of the portfolio is given by*

$$\widehat{\Pi}_t = \Pi_t + \pi_t, \tag{10}$$

where

$$\beta_t \pi_t = \int_0^t \beta_s d\mathscr{D}_s \ . \tag{11}$$

Recall $\tau = \tau_{-1} \wedge \tau_0, H_t = \mathbb{1}_{\tau \le t}$. In the counterparty risky case there are no cash flows after $\tau \wedge T$, so the (A_T-measurable) random variable π^t is in fact $A_{\tau\wedge T}$-measurable, and one has that $\pi^t = \Pi_t = 0$ for $t \ge \tau \wedge T$.

Remark 3.2 *In principle, when dealing with CVA, one should consider not one but two filtered pricing models relatively to a given risk-neutral measure* $\mathbb{P}$*:* $(\Omega, \mathbb{F}, \mathbb{P})$ *and* $(\Omega, \widetilde{\mathbb{F}}, \mathbb{P})$. *Here the filtration* $\widetilde{\mathbb{F}} = (\widetilde{A}_t)_{t\in[0,T]}$ *would represent the counterparty risk free filtration, not carrying any direct information about the default times* τ_{-1} *and* τ_0, *nor about any factors that might be specific to evolution of credit standards ('ratings') of the counterparties. This is the proper filtration that would normally be used for pricing the counterparty risk free contracts,*[6] *which serve as a reference so to assess the counterparty riskiness of actual contracts being priced and hedged. Mathematically speaking, we have that* $\sigma(\tau_{-1} \wedge t) \nsubseteq \widetilde{A}_t$ *and* $\sigma(\tau_0 \wedge t) \nsubseteq \widetilde{\mathscr{F}_t}$. *The filtration* $\mathbb{F} = (A_t)_{t\in[0,T]}$ *would represent the counterparty risky filtration, and it is a filtration such that* $\widetilde{A}_t \vee \sigma(\tau_{-1} \wedge t) \vee \sigma(\tau_0 \wedge t) \subseteq A_t$.

The discount factor β*, the counterparty clean cumulative dividend process D and the cumulative margin process* γ*, would thus be assumed to be* $\widetilde{\mathbb{F}}$ *adapted, and the* counterparty clean price process (or counterparty clean mark-to-market process) *of the portfolio would be given by* $P_t = \widetilde{\mathbb{E}}_t[p^t]$, *where we denote by* $\widetilde{\mathbb{E}}_t$ *the conditional expectation under* $\mathbb{P}$ *given* $\widetilde{A}_t$.

The fact that we only work with one, counterparty risky, filtration $\mathbb{F}$ *in this paper, as, incidentally, is the case with all the counterparty risk literature that we know of, simply means that we work under the implicit assumption that, for any* $t \le T$, *the time-t price of any* $\widetilde{A}_T$ *measurable, integrable cumulative cash flow, can be computed by evaluating appropriate conditional expectation (under* $\mathbb{P}$*) either conditioned on* $\widetilde{A}_t$ *or conditioned on* A_t. *See discussion in [9].*

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[6] We stress again that the clean price process P above is the process of the 'clean contract,' that is the contract in which any counterparty risk is disregarded.

It thus holds for instance that $P_t = \widetilde{\mathbb{E}}_t p^t = \mathbb{E}_t p^t$ *. This property implies, in particular, that process* $\beta\widehat{P}$ *is an* $\widetilde{\mathbb{F}}$*- as well as an* $\mathbb{F}$*- martingale, whereas the process* $\beta\widehat{\Pi}$ *is only an* $\mathbb{F}$*-martingale.*

Models of this type are for example models where so called immersion property is satisfied between filtrations $\widetilde{\mathbb{F}}$ *and* $\mathbb{F}$ *(see [7] for a general reference). Another example is provided by* Markov copula models *[5, 6] such as the one to be considered in Sect. 4.*

3.1 CVA

We introduce now the (cumulative) CVA process on the time interval $[0, \tau \wedge T]$ (we do not define the CVA beyond $\tau \wedge T$ since it is not needed there).

Definition 3.3 *The CVA process* Θ *is given as, for* $t \in [0, \tau \wedge T]$,

$$\Theta_t = \hat{P}_t - \hat{\Pi}_t \ .$$

Lemma 1 *The martingale* $\beta\Theta$ *can be represented as, for every* $t \in [0, \tau \wedge T]$*:*

$$\beta_t \Theta_t = \mathbb{E}_t \left[\beta_{\tau+\delta} \mathbb{1}_{\tau<T} \xi \right] \ , \tag{12}$$

where:
(i) *In case* $\delta = 0$ *(no cure period),*

$$\xi = P_\tau - P_{(\tau)} + (1 - R_0)\mathbb{1}_{\tau=\tau_0}\chi^+ - (1 - R_{-1})\mathbb{1}_{\tau=\tau_{-1}}\chi^- \ ;$$

(ii) *In the first interpretation of a cure period* δ,

$$\xi = \beta^{-1}_{\tau+\delta}\beta_\tau \big(P_\tau + \Delta D_\tau \\ -B(\tau, \tau+\delta)\left(\Gamma_\tau + \mathbb{1}_{\tau=\tau_0}\left(R_0\chi^+ - \chi^-\right) - \mathbb{1}_{\tau=\tau_{-1}}\left(R_{-1}\chi^- - \chi^+\right) - \mathbb{1}_{\tau_0=\tau_{-1}}\chi\right)\big) \ ,$$

where $B(s,t)$ *is the time-s price of zero coupon bond expiring at time* t;
(iii) *In the second interpretation of a cure period* δ,

$$\xi = P_{\tau+\delta} - P_{(\tau+\delta)} + (1 - R_0)\mathbb{1}_{\tau=\tau_0}\chi^+ - (1 - R_{-1})\mathbb{1}_{\tau=\tau_{-1}}\chi^- \ ,$$

with χ *as of* (5) *therein.*

Proof. **(i)** See [1].
(ii) First observe that for $t \in [0, \tau \wedge T]$ we have (recalling that $dD_t = 0$ for $t > T$, so that $P_t = 0$ and $\Delta D_t = 0$ for $t > T$)

$$\mathbb{E}_\tau \int_t^\infty \beta_s \big(dD_s - dD^*_s\big) = \mathbb{E}_\tau \int_t^T \beta_s \big(dD_s - dD^*_s\big) = \mathbb{E}_\tau \int_{[\tau,T]} \beta_s dD_s = \beta_\tau \left(P_\tau + \Delta D_\tau\right) \ .$$

Consequently, in the first interpretation of a cure period δ, one has by Definition 3.1 and in view of (4), for $t \in [0, \tau \wedge T]$,

$$
\begin{aligned}
&\beta_t \Theta_t = \beta_t(\widehat{P}_t - \widehat{\Pi}_t) = \\
&= \mathbb{E}_t \left(\mathbb{E}_\tau \int_t^\infty \beta_s \big(dD_s - dD^*_s\big) - \int_t^\infty \beta_s d\Upsilon_s \right) = \mathbb{E}_t \left(\mathbb{E}_\tau \int_t^T \beta_s \big(dD_s - dD^*_s\big) - \int_t^\infty \beta_s d\Upsilon_s \right) \\
&= \mathbb{E}_t \left\{ \mathbb{1}_{\tau<T} \beta_\tau \big(P_\tau + \Delta D_\tau\big) - \mathbb{1}_{\tau<T} \beta_{\tau+\delta} \left(\Gamma_\tau + \mathbb{1}_{\tau=\tau_0} \big(R_0 \chi^+ - \chi^-\big) \right.\right. \\
&\qquad \left.\left. - \mathbb{1}_{\tau=\tau_{-1}} \big(R_{-1}\chi^- - \chi^+\big) - \mathbb{1}_{\tau_0=\tau_{-1}} \chi \right) \right\} \\
&= \mathbb{E}_t \left\{ \mathbb{1}_{\tau<T} \beta_\tau \big(P_\tau + \Delta D_\tau\big) \right\} - \mathbb{E}_t \left\{ \mathbb{1}_{\tau<T} \beta_\tau \left(\Gamma_\tau + \mathbb{1}_{\tau=\tau_0} \big(R_0 \chi^+ - \chi^-\big) \right.\right. \\
&\qquad \left.\left. - \mathbb{1}_{\tau=\tau_{-1}} \big(R_{-1}\chi^- - \chi^+\big) - \mathbb{1}_{\tau_0=\tau_{-1}} \chi \right) \mathbb{E}_\tau \beta_\tau^{-1} \beta_{\tau+\delta} \right\} \\
&= \mathbb{E}_t \left\{ \mathbb{1}_{\tau<T} \beta_\tau \big(P_\tau + \Delta D_\tau\big) \right\} - \mathbb{E}_t \left\{ \mathbb{1}_{\tau<T} \beta_\tau B(\tau, \tau+\delta) \left(\Gamma_\tau + \mathbb{1}_{\tau=\tau_0} \big(R_0 \chi^+ - \chi^-\big) \right.\right. \\
&\quad \left.\left. - \mathbb{1}_{\tau=\tau_{-1}} \big(R_{-1}\chi^- - \chi^+\big) - \mathbb{1}_{\tau_0=\tau_{-1}} \chi \right) \right\} .
\end{aligned}
$$

(iii) In the second interpretation of a cure period δ, the result follows by a straightforward adaptation of the no-cure-period computations of [1]. □

For simplicity we assume henceforth that $\delta = 0$. We also assume that the legal value of the portfolio is given by its counterparty clean value, so $P_{(\tau)} = P_\tau$. This simplifying assumption is common in the counterparty risk literature. Note however that in practice, the quantity $P_{(\tau)}$ should account not only for the clean mark-to-market value of the contract, but also for replacement costs as well as for the systemic risk (via modified funding rates).

Consequently the random variables χ and ξ are the values at time τ of the progressively measurable processes (χ_t) and (ξ_t) defined by, for $t \in [0, T]$,

$$
\begin{aligned}
&\chi_t = P_t + \Delta D_t - \Gamma_t \\
&\xi_t = (1 - R_0) \mathbb{1}_{t \geq \tau_0} \chi_t^+ - (1 - R_{-1}) \mathbb{1}_{t \geq \tau_{-1}} \chi_t^- \ .
\end{aligned}
\tag{13}
$$

In the theoretical part of the paper we assume henceforth nil interest rates so that the discount factor β is one. Time-deterministic interest-rates will be used in the numerical part, the extension of all results to constant or time-deterministic interest rates being straightforward (but more cumbersome notationally, especially regarding hedging).

3.1.1 CVA Dynamics

The next step consists in deriving dynamics of the CVA Θ, which, under the current zero interest rates environment, is a martingale over $[0, \tau \wedge T]$.

Lemma 2 *For any $t \in [0, \tau \wedge T]$, we have*

$$\begin{aligned} d\Theta_t &= (1-H_{t-})(d\hat{P}_t - d\hat{\Pi}_t) \\ &= (1-H_t)(dP_t - d\Pi_t) + (\Delta\hat{P}_\tau - \Delta\hat{\Pi}_\tau)dH_t \\ &= (1-H_t)(dP_t - d\Pi_t) + (\xi_\tau - \Theta_{\tau-})dH_t \\ &= (1-H_t)(dP_t - d\Pi_t) + (\xi_t - \Theta_{t-})dH_t. \end{aligned} \tag{14}$$

Proof. The first line holds by definition of Θ and by application of Itô's formula. The second one follows from the fact that $p^0 - p^t = \pi^0 - \pi^t$ for any $t < \tau$. The remaining three equalities follow easily. □

Equation (14) is the key to hedging of counterparty risk. The dynamics of Θ splits into the 'pre-counter-party-default' part $(1-H_t^0)(dP_t - d\Pi_t)$, and the 'at-counter-party-default' part $(\xi_t - \Theta_{t-})dH_t^0$.

3.2 Collateral Modeling

Three reference collateralization schemes are the *naked scheme* $\Gamma = 0$, the *perfect scheme* $\Gamma = P_{\cdot-}$, and the *ISDA scheme* to be studied now. According to ISDA document [17], page 57, the paradigm for the level of collateral amount is the following:

> "Collateral value = (i) the [Collateral Taker]'s Exposure plus (ii) the aggregate of all Independent Amounts applicable to the [Collateral Provider], if any, minus (iii) the aggregate of all Independent Amounts applicable to the Collateral Taker, if any, minus (iv) the [Collateral Provider]'s Threshold"

The 'exposure' in the above terminology refers to the counterparty risk free mark-to-market value of the reference portfolio. Here, we propose an algorithm that is meant to generate the collateral process, which, right after every margin call time, conforms to the above paradigm. That is to say, since there are no 'Independent Amounts' as of items (ii) and (iii) in our set-up, 'Collateral value = Mark-to-Market minus Threshold', where Threshold refers to bounds which are set on the admissible values of χ (so the parties need to adjust the collateral Γ in case χ leaves these bounds).

Towards this end, we denote by $t_0 = 0 < t_1 < \cdots < t_n < T$ the margin call dates. Thus, we assume, as it is done in practice that margin calls are executed according to a discrete tenor of dates. Note that the time interval $\tau - t_\iota$ between the effective default time τ and the last margin call date t_ι preceding it, constitutes the first part of the margin period of risk, the second part consisting of the cure period δ already dealt with in Sect. 2.2.

In order to construct the collateral process we need to introduce the following quantities,

- the nominal threshold for the counterparty: $\bar{\chi}^0 \geq 0$,
- the nominal threshold for the investor: $\bar{\chi}^{-1} \leq 0$,
- the minimum transfer amount for the counterparty: $\varepsilon^0 \geq 0$,
- the minimum transfer amount for the investor: $\varepsilon^{-1} \leq 0$

- the effective threshold for the counterparty: $\hat{\chi}^0 = \bar{\chi}^0 + \varepsilon^0$,
- the effective threshold for the investor: $\hat{\chi}^{-1} = \bar{\chi}^{-1} + \varepsilon^{-1}$.

In the ISDA collateralization scheme we construct the left continuous, piecewise-constant collateral process Γ by setting $\Gamma_0 = 0$ and by postulating that at every $t_i < \tau$,

$$\begin{aligned}\Delta\Gamma_{t_i} &:= \Gamma_{t_i+} - \Gamma_{t_i} = \mathbb{1}_{\chi_{t_i} > \hat{\chi}^0}(\chi_{t_i} - \bar{\chi}^0)^+ - \mathbb{1}_{\chi_{t_i} < \hat{\chi}^{-1}}(\chi_{t_i} - \bar{\chi}^{-1})^- - \Delta D_{t_i} \\ &= \mathbb{1}_{\chi_{t_i} > \hat{\chi}^0}(\chi_{t_i} - \bar{\chi}^0) + \mathbb{1}_{\chi_{t_i} < \hat{\chi}^{-1}}(\chi_{t_i} - \bar{\chi}^{-1}) - \Delta D_{t_i} \ .\end{aligned} \tag{15}$$

Then we let Γ be constant on every interval (t_i, t_{i+1}).

Note that the amount of collateral transferred at the call times according to the above collateralization scheme, that is the quantity $\Delta\Gamma_{t_i}$, satisfies natural properties. For instance, assuming no bullet dividend paid at t_i (so $\Delta D_{t_i} = 0$):

- $\Delta\Gamma_{t_i} > 0$ if the collateralized exposure χ_{t_i} exceeds the counterparty's threshold $\hat{\chi}^0$; this means that at time t_i the investor makes a margin call, and the counterparty delivers $\Delta\Gamma_{t_i}$ worth of (cash) collateral; intuitively, the counterparty thus brings χ_t down to $\bar{\chi}^0$ at t_i+ if it exceeded $\hat{\chi}^0$ at t_i,
- $\Delta\Gamma_{t_i} < 0$ if the collateralized exposure χ_{t_i} is less than the investor's threshold $\hat{\chi}^1$; this means that at time t_i the counterparty makes a margin call, and the investor delivers $-\Delta\Gamma_{t_i}$ worth of (cash) collateral; intuitively, the investor brings it up to $\bar{\chi}^{-1}$ at t_i+ if it was lower than $\hat{\chi}^{-1}$ at t_i,
- $\Delta\Gamma_{t_i} = 0$ if the collateralized exposure χ_{t_i} is within the bounds $[\hat{\chi}^{-1}, \hat{\chi}^0]$; this means that at time t_i no margin call is made and no collateral is transferred by any of the two parties; collateralized exposure remains unadjusted.

More generally, identity (16) in the following result indeed shows that, right after the margin call times, the ISDA collateralization scheme conforms to the requirements of ISDA.

Proposition 3 *One has, at every t_i,*

$$\Gamma_{t_i+} = P_{t_i} - \left(\mathbb{1}_{\chi_{t_i} > \hat{\chi}^0}\bar{\chi}^0 + \mathbb{1}_{\chi_{t_i} < \hat{\chi}^{-1}}\bar{\chi}^{-1} + \mathbb{1}_{\hat{\chi}^{-1} \le \chi_{t_i} \le \hat{\chi}^0}\chi_{t_i}\right) \ . \tag{16}$$

or, equivalently,

$$\chi_{t_i+} = \left(\mathbb{1}_{\chi_{t_i} > \hat{\chi}^0}\bar{\chi}^0 + \mathbb{1}_{\chi_{t_i} < \hat{\chi}^{-1}}\bar{\chi}^{-1} + \mathbb{1}_{\hat{\chi}^{-1} \le \chi_{t_i} \le \hat{\chi}^0}\chi_{t_i}\right) \ . \tag{17}$$

In particular,

$$\Gamma_\tau \in [P_{t_\iota} - \hat{\chi}^0, P_{t_\iota} - \hat{\chi}^{-1}] \ , \tag{18}$$

where t_ι denotes the greatest t_i less or equal to τ.

Proof. Recall (13): $\chi_t = P_t + \Delta D_t - \Gamma_t$. From (15), one thus has at every t_i,

$$\Gamma_{t_i+} = \Gamma_{t_i} - \left(\mathbb{1}_{\chi_{t_i} > \hat{\chi}^0} \bar{\chi}^0 + \mathbb{1}_{\chi_{t_i} < \hat{\chi}^{-1}} \bar{\chi}^{-1} + \mathbb{1}_{\hat{\chi}^{-1} \le \chi_{t_i} \le \hat{\chi}^0} \chi_{t_i} \right) + \chi_{t_i} - \Delta D_{t_i} \ ,$$

which is (16). Now, P does not jump at fixed times, and one has by càdlàg regularity of D that $D = D_{\cdot+} = (D_{\cdot-})_{\cdot+}$, so $(\Delta D)_{\cdot+} = D_{\cdot+} - (D_{\cdot-})_{\cdot+} = 0$. Thus

$$\chi_{t_i+} = \chi_{t_i} - \Delta D_{t_i} - (\Gamma_{t_i+} - \Gamma_{t_i}) = P_{t_i} - \Gamma_{t_i+} \ ,$$

hence (16) is equivalent to (17). Finally (18) is an immediate consequence of (16), which implies in particular

$$\Gamma_\tau = P_{t_\iota} - \left(\mathbb{1}_{\chi_{t_\iota} > \hat{\chi}^0} \bar{\chi}^0 + \mathbb{1}_{\chi_{t_\iota} < \hat{\chi}^{-1}} \bar{\chi}^{-1} + \mathbb{1}_{\hat{\chi}^{-1} \le \chi_{t_\iota} \le \hat{\chi}^0} \chi_{t_\iota} \right) \in [P_{t_\iota} - \hat{\chi}^0, P_{t_\iota} - \hat{\chi}^{-1}] \ .$$

□

Note that our construction above is implicitly cash based. Translations to cash from a portfolio of assets needs to be done via haircuts. That is, if the collateral transferred at time t_i is posted in some asset different form cash, then the total value of that asset that needs to be posted is $(1 + h_{t_i})\Delta\Gamma_{t_i}$, where h_{t_i} is the appropriate haircut to be applied at time t_i. In case of a portfolio of assets, one distributes $\Delta\Gamma_{t_i}$ among the assets and applies appropriate haircut to each portion.

4 Common Shock Model of Counterparty Credit Risk

4.1 Unilateral Counterparty Credit Risk

We shall more specifically focus henceforth on the issue of counterparty credit risk. We consider a bank that holds a portfolio of credit contracts referencing various credit names. This portfolio is subject to a counterparty credit risk with regard to a single counterparty. A bank typically disregards its own counterparty risk when assessing the counterparty risk of a portfolio with another party. Thus, we are led to considering unilateral counterparty risk from the perspective of the bank.

Towards this end, we postulate that the contracts comprising the portfolio between the investor (the bank) and its counterparty, reference defaultable credit names. We denote by τ_i, for $i \in \mathbb{N}_n^* = \{1, \ldots, n\}$, the default times of n credit names underlying the portfolio's contracts. For $i \in \mathbb{N}_n = \{0, 1, \ldots, n\}$, we let H^i stand for the default indicator process of τ_i, so $H_t^i = \mathbb{1}_{\tau_i \le t}$, and we denote $\mathbf{H} = (H^i)_{i \in \mathbb{N}_n}$.

More precisely, our final aim is to study the hedge of the unilateral counterparty risk exposure of a portfolio credit derivative by means of a counterparty clean CDS contract referencing the counterparty. We assume that the CDS contracts which are used therein for hedging are entered into with counterparties that are remote from default. So, there is not counterparty risk associated with the hedging instruments.

Let, for $t \in [0, T]$,

$$D_t = \int_{[0,t]} \varphi(\mathbf{H}_s)ds + \phi(\mathbf{H}_t),\ D_t^i = \int_{[0,t]} \varphi_i(H_s^i)ds + \phi_i(H_t^i)\ , \tag{19}$$

represent the cumulative cash flow processes of a portfolio credit derivative on all names, and of a single-name credit derivative on name $i \in \mathbb{N}_n$. Here the idea is that φ and φ_i correspond to the fees (also called premium) leg of a swapped credit derivative, with continuous-time premium payments for notational simplicity, whereas ϕ and ϕ_i correspond to the default leg.

A practically important category of portfolio credit derivatives consists of the *portfolio loss derivatives*, for which the cash flows in (19) only depend on $\mathbf{H}_t = (H_t^i)_{i\in\mathbb{N}_n}$ through the number of defaults $N_t = \sum_{i\in\mathbb{N}_n} H_t^i$ in the portfolio, or $N_t^* = \sum_{i\in\mathbb{N}_n^*} H_t^i$ in the more standard situation of a counterparty not belonging to the pool of credit names underlying the contracted derivative, so

$$\varphi(\mathbf{k}) = \hat{\varphi}(|\mathbf{k}|),\ \phi(\mathbf{k}) = \hat{\phi}(|\mathbf{k}|) \text{ or } \varphi(\mathbf{k}) = \hat{\varphi}(|\mathbf{k}|^*),\ \phi(\mathbf{k}) = \hat{\phi}(|\mathbf{k}|^*)\ , \tag{20}$$

where we let $|\mathbf{k}| = \sum_{i\in\mathbb{N}_n} k_i$, $|\mathbf{k}|^* = \sum_{i\in\mathbb{N}_n^*} k_i$, for every $\mathbf{k} = (k_i)_{i\in\mathbb{N}_n} \in \{0,1\}^n$. For instance, one has in the case of a payer CDO tranche on names 1 to n, with contractual spread Σ and normalized attachment/detachment point L/U:

$$\varphi(\mathbf{k}) = -\Sigma\big(U - L - \phi(\mathbf{k})\big),\ \phi(\mathbf{k}) = \Big((1-\mathscr{R})\frac{|\mathbf{k}|^*}{n} - L\Big)^+ \wedge (U-L) \tag{21}$$

where a constant and homogenous recovery $\mathscr{R}$ on the underlying CDSs is assumed.

As for single-name credit derivatives, one has in the case of a payer CDS with contractual spread S on name $i \in \mathbb{N}_n$:

$$\varphi_i(k_i) = -S(1-k_i),\ \phi_i(k_i) = (1-R_i)k_i\ . \tag{22}$$

Note that in the unilateral counterparty risk case we have $\tau = \tau_0$ and thus $H = H^0$. For simplicity we assume a constant recovery rate in case counterparty defaults; specifically we set $R_0 = R$. By application of (13), one thus has,

$$\Theta_t = \mathbb{E}_t\left[\mathbb{1}_{\tau<T}\,\xi\right] \tag{23}$$

with

$$\xi = (1-R)\chi^+,\ \chi = P_\tau + \Delta\phi(\mathbf{H}_\tau) - \Gamma_\tau \tag{24}$$

in which $P_t = \mathbb{E}_t(D_T - D_t)$ is the counterparty clean price process of the portfolio credit derivative, and Γ represents the collateral process.

4.2 Model of Default Times

We now propose a Markovian model of counterparty credit risk, that will be able to put the above general results to work. This model is a variant of the common shocks portfolio credit risk model of [2], more specifically tailored to the application of valuation and hedging of the counterparty risk on a portfolio of credit derivatives.

In order to describe the defaults we define a certain number m (typically small: a few units) of groups $I_l \subseteq \mathbb{N}_n$, of obligors who are likely to default simultaneously, for $l \in \mathbb{N}_m$. More precisely, the idea is that at every time t, there will be a positive probability that the survivors of the group of obligors I_l (obligors of group I_l still alive at time t) default simultaneously. Let $\mathscr{I} = \{I_0, \dots, I_m\}$, $\mathscr{Y} = \{\{0\}, \dots, \{n\}, I_0, \dots, I_m\}$. Let *group intensity processes* X^Y be given in the form of extended CIR processes as, for every $Y \in \mathscr{Y}$,

$$dX_t^Y = a(b_Y(t) - X_t^Y)dt + c\sqrt{X_t^Y}\, dW_t^Y \ , \tag{25}$$

where the Brownian motions $W^{\{i\}}$ s for $0 \le i \le n$ are correlated at the level ρ, and the Brownian motions W^Is for $I \in \mathscr{I}$ are independent between themselves and from everything else. Given $\mathbf{X} = (\mathbf{X}^Y)_{Y \in \mathscr{Y}}$s, we would like a model in which the predictable intensity of a jump of $\mathbf{H} = (H^i)_{i \in \mathbb{N}_n}$ from $\mathbf{H}_{t-} = \mathbf{k}$ to $\mathbf{H}_t = \mathbf{l}$, with $\mathrm{supp}(\mathbf{k}) \subsetneq \mathrm{supp}(\mathbf{l})$ in $\{0,1\}^{n+1}$, is given by

$$\sum_{\{Y \in \mathscr{Y};\, \mathbf{k}^Y = \mathbf{l}\}} X_t^Y \ , \tag{26}$$

where $\mathbf{k}^Y$ denotes the vector obtained from $\mathbf{k} = (k_i)_{i \in \mathbb{N}_n}$ by replacing the components k_i, $i \in Y$, by numbers one. The intensity of a jump of $\mathbf{H}$ from $\mathbf{k}$ to $\mathbf{l}$ at time t is thus equal to the sum of the intensities of the groups $Y \in \mathscr{Y}$ such that, if the default of the survivors in group Y occured at time t, the state of $\mathbf{H}$ would move from $\mathbf{k}$ to $\mathbf{l}$.

To achieve this, we classically construct $\mathbf{H}$ by an $\mathbf{X}$-related change of probability measure, starting from a continuous-time Markov chain with intensity one (see [2, 11]). As a result (see [2, 11]), the pair-process $(\mathbf{X}, \mathbf{H})$ is a Markov process with respect to the filtration $\mathscr{F}$ generated by the Brownian Motion $\mathbf{W}$ and the random measure counting the jumps of $\mathbf{H}$, with infinitesimal generator $\mathscr{A}$ of $(\mathbf{X}, \mathbf{H})$ given as, for $u = u(t, \mathbf{x}, \mathbf{k})$ with $t \in \mathbb{R}_+, \mathbf{x} = (\lambda_Y)_{Y \in \mathscr{Y}}$ and $\mathbf{k} = (k_i)_{i \in \mathbb{N}_n}$:

$$\begin{aligned}\mathscr{A}_t u(t,\mathbf{x},\mathbf{k}) = &\sum_{Y \in \mathscr{Y}} \Big(a(b_Y(t) - \lambda_Y)\partial_{\lambda_Y} u(t,\mathbf{x},\mathbf{k}) + \frac{1}{2}c^2\, \lambda_Y \partial^2_{\lambda_Y^2} u(t,\mathbf{x},\mathbf{k}) \Big) \\ &+ \sum_{0 \le i < j \le n} \rho_{i,j}(t) c^2 \sqrt{\lambda_{\{i\}}\lambda_{\{j\}}}\, \partial^2_{\lambda_{\{i\}},\lambda_{\{j\}}} u(t,\mathbf{x},\mathbf{k}) + \sum_{Y \in \mathscr{Y}} \lambda_Y \,\delta u^Y(t,\mathbf{x},\mathbf{k}) \ ,\end{aligned} \tag{27}$$

for non-negative constants a, c and non-negative functions $b_Y(t)$s, $[-1,1]$-valued correlation functions $\rho_{i,j}(t)$, and where we denote, for $Y \in \mathscr{Y}$,

$$\delta u^Y(t,\mathbf{x},\mathbf{k}) = u(t,\mathbf{x},\mathbf{k}^Y) - u(t,\mathbf{x},\mathbf{k}) .$$

One also has the following expression for the predictable intensity ℓ_t^Z of the indicator process H_t^Z of the event of a joint default of names in set Z and only in Z, for every subset Z of $\mathbb{N}_n$ (see [2]):

$$\ell_t^Z = \ell_Z(t,\mathbf{X}_t,\mathbf{H}_{t-}) = \sum_{Y\in\mathscr{Y};\,Y_t=Z} X_t^Y , \tag{28}$$

where Y_t stands for the set of survivors of set Y right before time t, for every $Y\in\mathscr{Y}$. So $Y_t = Y\cap\mathrm{supp}^c(\mathbf{H}_{t-})$, where $\mathrm{supp}^c(\mathbf{k}) = \{i\in\mathbb{N}_n;\,k_i=0\}$, for $\mathbf{k}=(k_i)\in\{0,1\}^{n+1}$. One denotes by M^Z the corresponding *compensated set-event martingale*, so for $t\in[0,T]$,

$$dM_t^Z = dH_t^Z - \ell_Z(t,\mathbf{X}_t,\mathbf{H}_t)dt . \tag{29}$$

We refer the reader to [4] for a two-obligors preliminary version of this model dedicated to valuation and hedging of counterparty risk on a CDS. The numerical results of [4] illustrate that using such ‘fully stochastic’ specifications of the intensities potentially leads to a better behaved CVA than the intensities specification of [2], in which the X^Is are deterministic functions of time.

4.2.1 Markov Copula Properties

Note that the SDEs for factors X^Y have the same coefficients except for the $b_Y(t)$, to the effect that

$$X^i := \sum_{\mathscr{Y}\ni Y\ni i} X^Y = X^{\{i\}} + \sum_{\mathscr{I}\ni I\ni i} X^I ,$$

for $i\in\mathbb{N}_n$, is again an extended CIR process, with parameters a, c and

$$b_i(t) := \sum_{\mathscr{Y}\ni Y\ni i} b_Y(t) = b_{\{i\}}(t) + \sum_{\mathscr{I}\ni I\ni i} b_I(t) ,$$

driven by the Brownian motion W^i such that

$$\sqrt{X_t^i}dW_t^i = \sum_{Y\ni i}\sqrt{X_t^Y}dW_t^Y,\ dW_t^i = \sum_{Y\ni i}\frac{\sqrt{X_t^Y}}{\sqrt{\sum_{Y\ni i}X_t^Y}}dW_t^Y . \tag{30}$$

One can then check, as is done in [2], that the so-called *Markov copula property* holds (see [6]), in the sense that for every $i\in\mathbb{N}_n$, (X^i,H^i) is an $\mathscr{F}$ – Markov process admitting the following generator, for $u_i = u_i(t,\lambda_i,k_i)$ with $(\lambda_i,k_i)\in\mathbb{R}\times\{0,1\}$:

$$\mathscr{A}_t^i u_i(t,\lambda_i,k_i) = \tag{31}$$
$$a(b_i(t)-\lambda_i)\partial_{\lambda_i}u_i(t,\lambda_i,k_i) + \frac{1}{2}c^2\ \lambda_i\partial^2_{\lambda_i^2}u_i(t,\lambda_i,k_i) + \lambda_i\big(u_i(t,\lambda_i,1)-u_i(t,\lambda_i,k_i)\big) .$$

Also, the $\mathscr{F}$ – intensity process of H^i is given by $(1-H^i_t)X^i_t$. In other words, the process M^i defined by,

$$M^i_t = H^i_t - \int_0^t (1-H^i_s)X^i_s ds\ , \tag{32}$$

is an $\mathscr{F}$-martingale. Finally, the conditional survival probability function of name $i \in \mathbb{N}_n$ is given by, for every $t_i > t$,

$$\mathbb{P}(\tau_i > t_i \,|\, \mathscr{F}_t) = \mathbb{E}\left\{\exp\big(-\int_t^{t_i} X^i_s ds\big)\,|\, X^i_t\right\}\ , \tag{33}$$

so that in particular

$$\mathbb{E}\exp\big(-\int_0^t X^{\{i\}}_s ds\big) = \exp\left\{-\left(\Gamma_i(t) - \sum_{i\in I}\int_0^t X^I_s ds\right)\right\}\ , \tag{34}$$

where $\Gamma_i = -\ln \mathbb{P}(\tau_i > t)$ is the *hazard function* of name i.

4.3 Credit Derivatives Prices and Price Dynamics in the Common Shocks Model

The following pricing results also follow by straightforward adaptation to the proof of the analogous results in [2].

Let $\mathscr{Z}_t$ stand for the set of all non-empty sets of survivors of sets Y in $\mathscr{Y}$ right before time t. We denote $\nabla u(t,\mathbf{x},\mathbf{k}) = (\partial_{\lambda_Y} u(t,\mathbf{x},\mathbf{k}))_{Y\in\mathscr{Y}}$, and by $\sigma(t,\mathbf{x})$, the diagonal matrix with diagonal $(c\sqrt{x_Y})_{Y\in\mathscr{Y}}$.

Proposition 4 **(i)** *The price process P and the cumulative value $\hat{P}$ of the portfolio credit derivative are such that, for $t \in [0,T]$,*

$$\begin{aligned} P_t &= u(t,\mathbf{X}_t,\mathbf{H}_t) \\ d\hat{P}_t &= \nabla u(t,\mathbf{X}_t,\mathbf{H}_t)\sigma(t,\mathbf{X}_t)d\mathbf{W}_t + \sum_{Z\in\mathscr{Z}_t} \delta u^Z(t,\mathbf{X}_t,\mathbf{H}_{t-})dM^Z_t\ , \end{aligned} \tag{35}$$

where the pricing function $u(t,\mathbf{x},\mathbf{k})$ is given by

$$u(t,\mathbf{X}_t,\mathbf{H}_t) = \mathbb{E}[\int_t^T \varphi(\mathbf{H}_s)ds + \phi(\mathbf{H}_T) - \phi(\mathbf{H}_t)|\mathscr{F}_t]\ ; \tag{36}$$

(ii) *The price Q^i and the cumulative value $\hat{Q}^i$ of the single-name credit derivative on name i are such that, for $t \in [0,T]$,*

$$
\begin{aligned}
Q^i_t &= (1-H^i_t)v_i(t,X^i_t)\\
d\hat{Q}^i_t &= (1-H^i_t)\partial_{\lambda_i} v_i(t,X^i_t)\sigma_i(t,X^i_t)dW^i_t + \sum_{Z\in\mathscr{Z}_t} \mathbf{1}_{i\in Z}\left(\phi_i - v_i(t,X^i_t)\right)dM^Z_t \ ,
\end{aligned}
\tag{37}
$$

for a pre-default pricing function $v_i(t,\lambda_i)$ such that

$$
v_i(t,x_i) = \mathbb{E}\Big(\int_t^T e^{-\int_t^s X^i_\zeta d\zeta}\left(\varphi_i + \phi_i X^i_s\right)\Big| X^i_t = x_i\Big)\ . \tag{38}
$$

Now, an important practical point is that in the affine factor specification of this paper, all the expectations and conditional expectations that arise in the single-name formulas (33), (34) and (38), can be computed explicitly (see [4] for details).

Moreover, a common shocks interpretation of the model analogous to the one developed in [2], also allows one to compute the conditional expectations in e̊:excumbis in a fast and exact way, for all the *portfolio loss derivatives* as of (20).

To get the formulas for the conditional expectations in (36), simply ‘add’ expectations in front of the terms $\mathbb{E}\exp\left(\pm\int_0^t X^I_s ds\right)$ in the corresponding formulas in [2] (in which the X^Is are deterministic).

The min-variance hedging formula of [2] also still hold true, using the pricing functions of Proposition 4 therein.

5 Hedging Counterparty Credit Risk in the Common Shocks Model

Our final aim is the study of the hedging problem of the unilateral CVA on a portfolio credit derivative, in the common shocks model of the previous section.

5.1 Min-Variance Hedging by a rolling CDS on the Counterparty

We first study the problem of hedging the CVA by a single counterparty clean CDS on the counterparty. Note however that a fixed CDS (of a given contractual spread in particular) cannot be traded dynamically in the market. Indeed only freshly emitted CDSs can be entered into, at no cost and at the related fair market spread, at any given time. To address this issue we shall thus actually use a *rolling CDS* as our hedging instrument. The practical concept of a rolling CDS, introduced in [8] and already used for hedging purposes in [3], is essentially a self-financing trading strategy in market CDSs. So, much like with futures contracts, the value of a rolling CDS is null at any point in time, yet due to the trading gains of the strategy the related cumulative value process is not zero.

We now derive the dynamics of the CVA and of the rolling CDS in the common shocks model.

Note that $\mathbf{H}^Z_{t-}$ below stands for the vector obtained from $\mathbf{H}_{t-}$ by replacing its components with indices in Z, by numbers one (cf. the generic notation $\mathbf{k}^Y$ introduced with equation (26)), whereas M^Z_t is the compensated set-event martingale of (29).

Proposition 5 **(i)** *One has, for $t \in [0, \tau \wedge T]$,*

$$\begin{aligned} \Theta_t &= \mathbb{E}_t\left[\mathbb{1}_{\tau<T}\xi\right] \text{ with } \xi = (1-R)\chi^+,\ \chi = u(\tau, \mathbf{X}_\tau, \mathbf{H}_\tau) + \Delta\phi(\mathbf{H}_\tau) - \Gamma_\tau \\ d\Theta_t &= (1-R)\left(\mu_t d\mathbf{W}_t + \sum_{Z \in \mathscr{Z}_t} \nu^Z_t dM^Z_t\right) \end{aligned} \tag{39}$$

for suitable integrands μ and ν^Ys with predictable ν^Ys. Moreover, for $t \in [0, \tau \wedge T]$,

$$\sum_{Z \in \mathscr{Z}_t; 0 \in Z} \nu^Z_t dM^Z_t = \sum_{Z \in \mathscr{Z}_t; 0 \in Z} \left(\chi^{Z,+}_t - (1-R)^{-1}\Theta_{t-}\right) dM^Z_t\ , \tag{40}$$

where, for every $Z \subseteq \mathbb{N}_n$, $\chi^{Z,+}_t$ stands for the positive part of χ^Z_t defined as, for $t \in [0, T]$,

$$\chi^Z_t = u(t, \mathbf{X}_t, \mathbf{H}^Z_{t-}) + \phi(t, \mathbf{H}^Z_{t-}) - \phi(t, \mathbf{H}_{t-}) - \Gamma_t\ . \tag{41}$$

(ii) *The value Q and the cumulative value $\hat{Q}$ of the rolling CDS on the counterparty are such that, for $t \in [0, \tau \wedge T]$,*

$$\begin{aligned} Q_t &= 0 \\ d\hat{Q}_t &= (1-R)\left(\partial_{x_0} v(t, X^0_t) c\sqrt{X^0_t} dW^0_t + \sum_{Z \in \mathscr{Z}_t; 0 \in Z} dM^Z_t\right)\ , \end{aligned} \tag{42}$$

where $\partial_{x_0} v(t, X^0_t)$ is a short-hand notation for $\partial_{x_0} p(t, X^0_t) - (1-R)^{-1} S(t, X^0_t)\partial_{x_0} f(t, X^0_t)$, and where p and f denote the pre-default pricing functions of the unit protection and fees legs of the CDS initiated at time t, so

$$f(t, X^0_t) = \mathbb{E}\left(\int_t^T e^{-\int_t^s X^0_\zeta d\zeta} ds \,|\, X^0_t\right),\ p(t, X^0_t) = \mathbb{E}\left(\int_t^T e^{-\int_t^s X^0_\zeta d\zeta} X^0_s ds \,|\, X^0_t\right)\ ,$$

and $S = p/f$ is the corresponding CDS fair spread function.

Proof. **(i)** Formula (39) is the predictable representation of the martingale Θ in our model. Note that this martingale representation indeed holds in virtue of our model construction by change of measure, starting from a measure under which $\mathbf{H}$ is a time-continuous Markov chain with intensity one: See for instance Proposition 24 in Crépey [11] or Proposition 7.6 in the online pre-print version, for analogous results with detailed proofs. Moreover, one has that $\Delta\Theta_\tau = \xi - \Theta_{\tau-}$ on $\{\tau \le T\}$. Recalling $P_t = u(t, \mathbf{X}_t, \mathbf{H}_t)$ and $\Delta\Gamma_\tau = 0$, also observe that $\chi = \chi^Z_\tau$ on the set $\{\Delta H^Z_\tau = 1\} \cap \{\tau \le T\}$, namely for $\tau \le T$ coinciding with the default time of the names in Z

(including the counterparty) and only them. The left-hand-side and the right-hand-side local martingales in (40) thus differ by an integral with respect to time, so that their difference is in fact constant.
(ii) In view of the Markov copula property of our model, this can be shown much as in Lemma 2.2 of [3] (see also Proposition 4(ii) for comparison with the case of a standard, non-rolling CDS on the counterparty). □

Now, let ζ be an $\mathbb{R}$-valued process, representing the number of units held in the rolling CDS which is used along with the constant asset in a self-financing hedging strategy for the counterparty risk of the portfolio credit derivative. Given (42) and (39), the tracking error (e_t) of the hedged portfolio satisfies $e_0 = 0$ and, for $t \in [0, \tau \wedge T]$,

$$\begin{aligned}(1-R)^{-1}de_t &= (1-R)^{-1}\left(d\Theta_t - \zeta_t d\hat{Q}_t\right)\\ &= \mu_t d\mathbf{W}_t - \zeta_t \partial_{x_0} v(t, X_t^0) c\sqrt{X_t^0} dW_t^0\\ &+ \sum_{Z\in\mathscr{Z}_t;0\in Z}\left(\nu_t^Z - \zeta_t\right)dM_t^Z + \sum_{Z\in\mathscr{Z}_t;0\notin Z}\nu_t^Z dM_t^Z \,, \end{aligned} \tag{43}$$

where the Brownian terms and the jump terms can be interpreted as the market and/or spread risk component and the jump-to-default risk component of the hedging error, the last sum representing the counterparty jump-to-default risk component of the hedging error.

Theorem 6. *The strategy which minimizes the risk-neutral variance of the jump-to-default risk component of the hedging error, or, equivalently, which minimizes the risk-neutral variance of the counterparty jump-to-default risk component of the hedging error, is given by, for $t \le \tau\wedge T$ (and $\zeta^{jd} = 0$ on $(\tau\wedge T, T]$)*

$$\zeta_t^{jd} = \sum_{Y\in\mathscr{Y};0\in Y_t} w_t^Y\left(\chi_t^{Y,+} - (1-R)^{-1}\Theta_{t-}\right) = \theta_t - (1-R)^{-1}\Theta_{t-} \,, \tag{44}$$

where $\theta_t = \sum_{Y\in\mathscr{Y};0\in Y_t} w_t^Y \chi_t^{Y,+}$, for weights w_t^Y defined as $\frac{X_t^Y}{\sum_{Z\in\mathscr{Y};0\in Z_t} X_t^Z}$, for every $Y\in\mathscr{Y}$ with $0\in Y_t$. In particular, on $\{\tau < T\}$,

$$\zeta_\tau^{jd} = \theta - (1-R)^{-1}\Theta_{\tau-} \,, \tag{45}$$

for the so called Expected Positive Exposure $\theta = \theta_\tau = \mathbb{E}(\chi^+ \,|\, \mathscr{F}_{\tau-})$.

Proof. The strategy minimizing the risk-neutral variance of the counterparty jump-to-default risk component of the hedging error is given by, for $t\le\tau$, $\zeta_t^{jd} = \frac{d\langle M,\hat{Q}\rangle_t}{d\langle\hat{Q}\rangle_t}$, with

$$M = \int_0^\cdot \sum_{Z\in\mathscr{Z}_t;0\in Z}\nu_t^Z dM_t^Z = \sum_{Z\in\mathscr{Z}_t;0\in Z}\int_0^\cdot\left(\chi_t^{Z,+} - (1-R)^{-1}\Theta_{t-}\right)dM_t^Z \,, \tag{46}$$

by (40). So, in view of the dynamics of $\hat{Q}$ in (42) (note that all the jump martingales integrands are predictable in e̊:dynPh and (46)) and of the expression (28) of the intensities ℓ^Zs of the M^Zs,

$$\zeta_t^{jd} = \frac{\sum_{Z\in\mathscr{Z}_t;0\in Z}\ell_t^Z(\chi_t^{Z,+}-(1-R)^{-1}\Theta_{t-})}{\sum_{Z\in\mathscr{Z}_t;0\in Z}\ell_t^Z} = \frac{\sum_{Y\in\mathscr{Y};0\in Y_t}X_t^Y(\chi_t^{Y_t,+}-(1-R)^{-1}\Theta_{t-})}{\sum_{Y\in\mathscr{Y};0\in Y_t}X_t^Y},$$

from which (44) follows by noting that one has $\chi^{Y_t}=\chi^Y$, for every $Y\in\mathscr{Y}$. Moreover, the M^Zs are pure jump processes which do not jump together, so $\hat{Q}$ is orthogonal to $\tilde{M}=\int_0^{\cdot}\sum_{Z\in\mathscr{Z}_t;0\in Z}\nu_t^Z dM_t^Z$. One thus also has that

$$\zeta_t^{jd} = \frac{d\langle M+\tilde{M},\hat{Q}\rangle_t}{d\langle\hat{Q}\rangle_t},$$

hence ζ^{jd} also minimizes the risk-neutral variance of the overall jump-to-default risk component of the hedging error.

Finally, to deduce (45) from (44), all one needs to show is that

$$\sum_{Y\in\mathscr{Y};0\in Y_\tau} w_\tau^Y\chi_\tau^{Y,+} = \theta = \mathbb{E}(\chi^+\,|\,\mathscr{F}_{\tau-}). \tag{47}$$

Now, by the classical expression for the conditional jump law of a finitely-valued pure jump process mitigated by a diffusion, the law of $\mathbf{H}_\tau$ conditional on $\mathscr{F}_{\tau-}$ is supported by $\{\mathbf{H}_{\tau-}^Z;Z\in\mathscr{Z}_\tau,0\in Z\}$, and it is given by, for every such Z (cf. (26))

$$\mathbb{P}\left(\mathbf{H}_\tau=\mathbf{H}_{\tau-}^Z\,|\,\mathscr{F}_{\tau-}\right) = \sum_{Y\in\mathscr{Y};Y_\tau=Z} w_\tau^Y.$$

So

$$\theta = \sum_{Z\in\mathscr{Z}_\tau;0\in Z}\left(\sum_{Y\in\mathscr{Y};Y_\tau=Z} w_\tau^Y\right)\left(u(\tau,\mathbf{X}_\tau,\mathbf{H}_{\tau-}^Z)+\phi(\tau,\mathbf{H}_{\tau-}^Z)-\phi(\tau,\mathbf{H}_{\tau-})-\Gamma_\tau\right)^+, \tag{48}$$

and (47) follows from the fact that $\mathbf{H}_{\tau-}^Z=\mathbf{H}_{\tau-}^Y$, for every $Y\in\mathscr{Y}$ with $Y_\tau=Z$. □

One thus retrieves in (45) the definition of the hedging ratio which is often advocated by CVA desks for hedging the counterparty jump-to-default component of the counterparty risk. In fact, this hedging ratio is commonly referred to as the Expected Positive Exposure, loosely defined as $\theta^\sharp=\mathbb{E}(\chi_t^+\,|\,\tau)$. But, the above min-variance hedging analysis reveals that from a dynamic hedging point of view, this hedging ratio should really be defined as $\theta-(1-R)^{-1}\Theta_{\tau-}$, where the second term accounts for the value of the portfolio CVA right before the default time τ of the counterparty, and where the Expected Positive Exposure corresponding to the first term should really defined as θ, rather than by its 'proxy' $\theta^\sharp$.

Note that in the course of the derivation of this result, Proposition 6 exploits two model-dependent features of our set-up:

• First, the fact that the cumulative value process of the rolling CDS only jumps at the default time τ of the counterparty, as opposed to jumps at other defaults too in a 'general' model of credit risk,
• Second, our assumption of a constant recovery of the counterparty (and of other obligors too, but this is irrelevant here).

Also observe that without hedging, so for $\zeta = 0$, one would have $\Delta e_\tau = \Delta e^0_\tau = \xi - \Theta_{\tau-}$ on $\{\tau < T\}$. With the ζ^{jd} strategy, one has $\Delta e_\tau = \Delta e^{jd}_\tau = (1-R)\theta - \Theta_{\tau-}$ on $\{\tau < T\}$. So, basically, the ζ^{jd} hedging strategy changes the counterparty jump-to-default exposure from ξ to $(1-R)\theta = \mathbb{E}(\xi_\tau \,|\, \mathscr{F}_{\tau-})$, the 'best guess' of ξ available right before τ. Note however that this strategy, which is 'optimal' as far as the counterparty jump-to-default component (or altogether jump-to-default component) of the counterparty risk is concerned, is disregarding the market risk component of the hedging error. In fact this strategy typically creates some additional market risk.

Given (30), one can also rewrite in (42), (43):

$$\partial_{x_0} v(t, X^0_t) c\sqrt{X^0_t} dW^0_t = \partial_{x_0} v(t, X^0_t) \sum_{Y \ni 0} c\sqrt{X^Y_t} dW^Y_t = \partial_{x_0} v(t, X^0_t) \Sigma_t d\mathbf{W}_t \ , \quad (49)$$

where Σ_t is the row-vector indexed by $\mathscr{Y}$ such that $\Sigma^Y_t = c\mathbb{1}_{Y \ni 0}\sqrt{X^Y_t}$, $Y \in \mathscr{Y}$. It is then rather straightforward to write the formula for the strategy which minimizes the risk-neutral variance of the hedging error altogether (see [2]). However in practice it will typically be difficult to compute all the terms that appear in this formula (unless we are in the pure jump case with no factors of a time-deterministic intensities model, for which $\zeta^{va} = \zeta^{jd}$).

Finally recall that Proposition 6 deals with the issue of hedging unilateral counterparty risk. The issue of hedging bilateral counterparty risk seems more involved, since in this case instruments sensitive to the default times of the investor and the counterparty should clearly be used (ideally, an instrument sensitive to their first default time, like a first-to-default swap on both names).

5.1.1 Case of one CDS

Let us consider the special case of a CDS on name one chosen as a special case of the above portfolio credit derivative, without collateralization. So $\Gamma = 0$, and

$$\varphi(\mathbf{k}) = f_1(k_1) = -S_1(1-k_1), \ \ \phi(\mathbf{k}) = g_1(k_1) = (1-R_1)k_1 \ . \quad (50)$$

In virtue of the Markov copula properties of the model one may and do forget about names 2 to n and take $\mathscr{Y} = \{\{0\},\{1\},\{0,1\}\}$, without loss of generality. Then (see Proposition 4(ii)),

$$u(t, \mathbf{X}_t, \mathbf{H}_t) = (1 - H_t^1) v_1(t, X_t^1)$$

with

$$v_1(t, X_t^1) = \mathbb{E}\left(\int_t^T e^{-\int_t^u X_v^1 dv} \left((1 - R_1) X_u^1 - S_1 \right) du \,|\, X_t^1 \right) . \quad (51)$$

Thus by (24), $\xi = (1-R)\chi^+$, in which χ here assumes the following form:

$$\chi = \mathbb{1}_{\tau_1 < \tau} v_1(\tau, X_\tau^1) + \mathbb{1}_{\tau_1 = \tau}(1 - R_1) . \quad (52)$$

Moreover, one has by application of formula (41):

$$\begin{aligned} \chi_t^{\{0\}} &= u(t, \mathbf{X}_t, \mathbf{H}_{t-}^{\{0\}}) + \phi(t, \mathbf{H}_{t-}^{\{0\}}) - \phi(t, \mathbf{H}_{t-}) = u(t, \mathbf{X}_t, \mathbf{H}_{t-}^{\{0\}}) = \mathbb{1}_{t \le \tau_1} v_1^+(t, X_t^1) \\ \chi_t^{\{0,1\}} &= u(t, \mathbf{X}_t, \mathbf{H}_{t-}^{\{0,1\}}) + \phi(t, \mathbf{H}_{t-}^{\{0,1\}}) - \phi(t, \mathbf{H}_{t-}) = 0 + (1 - R_1) - (1 - R_1) H_{t-}^1 \\ &= \mathbb{1}_{t \le \tau_1}(1 - R_1) . \end{aligned}$$

Therefore, (44) yields, for $t \le \tau \wedge \tau_1 \wedge T$ (and $\zeta^{jd} = 0$ on $(\tau \wedge \tau_1 \wedge T, \tau \wedge T]$),

$$\zeta_t^{jd} = \theta_t - (1 - R)^{-1} \Theta_{t-} ,$$

where

$$\theta_t = \sum_{Y \in \mathscr{Y}; 0 \in Y_t} w_t^Y \chi_t^{Y,+} = w_t^{\{0\}} v_1^+(t, X_t^1) + w_t^{\{0,1\}}(1 - R_1)$$

in which

$$w_t^{\{0\}} = \frac{X_t^{\{0\}}}{X_t^{\{0\}} + X_t^{\{0,1\}}}, \; w_t^{\{0,1\}} = \frac{X_t^{\{0,1\}}}{X_t^{\{0\}} + X_t^{\{0,1\}}} .$$

See also [4] for the entire specification of the dynamics of the CVA process on this example, in a related model with X^Is given as deterministic functions of time.

5.2 Multi-Instruments Hedge

We now consider the situation where additional instruments can be used for hedging the ‘market’ or ’spread risk’ component (diffusive part) of the counterparty risk exposure. More specifically, we suppose that there exists an $\mathbb{R}^m$-valued martingale price process $\mathbf{Q} = (Q^j)_{1 \le j \le m}$ of hedging instruments with $\mathbb{Q}$-dynamics

$$d\mathbf{Q}_t = \varsigma_t \, d\mathbf{W}_t , \quad (53)$$

for a left-invertible diffusion matrix-process ς_t, with left-inverse denoted by ς_t^{-1}. Let η be an $\mathbb{R}^{1 \otimes m}$-valued process, representing the number of units held in every of the Q^j’s, which are used along with the rolling CDS on the counterparty and the constant asset for hedging the counterparty risk exposure. The tracking error (e_t) of the hedged portfolio now satisfies $e_0 = 0$ and, for $t \in [0, \tau \wedge T]$,

$$de_t = d\Theta_t - \zeta_t d\hat{Q}_t - \eta_t d\mathbf{Q}_t \ . \tag{54}$$

Proposition 7 *The strategy which minimizes the risk-neutral variance of the hedging error is given by ζ^{jd} as of* (44)*, and, for $t \in [0, \tau \wedge T]$ (recall* (49)*),*

$$\eta_t^{va} = \left(\mu_t - \zeta_t^{jd} \partial_{x_0} v(t, X_t^0) \Sigma_t\right) \varsigma_t^{-1} \ . \tag{55}$$

The residual hedging error satisfies $e_0^{va} = 0$ and, for $t \in [0, T]$,

$$(1-R)^{-1} de_t^{va} = (\chi^+ - \theta) \sum_{Z \in \mathscr{Z}_t; 0 \in Z} v_t^Z dM_t^Z + \sum_{Z \in \mathscr{Z}_t; 0 \notin Z} v_t^Z dM_t^Z \ . \tag{56}$$

Proof. One has

$$\min_{\zeta, \eta} \mathbb{V}\mathrm{ar}\, e(\zeta, \eta) = \min_{\eta} \left(\min_{\zeta} \mathbb{V}\mathrm{ar}\, e(\zeta, \eta) \right) ,$$

where, given any η, the solution of the inner minimization problem is given by ζ^{jd}, independently of η. So

$$\min_{\zeta, \eta} \mathbb{V}\mathrm{ar}\, e(\zeta, \eta) = \min_{\eta} \mathbb{V}\mathrm{ar}\, e(\zeta^{jd}, \eta) ,$$

where the minimum in the right-hand-side is obviously achieved by η^{va}, the residual hedging error being then given by (56). □

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