ARBITRAGE PRICING OF DEFAULTABLE GAME OPTIONS AND APPLICATIONS

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1 Introduction

1.1 General Motivation

It is widely acknowledged (see, for instance, [13, 21, 24]) that a convertible bond has a natural interpretation as a defaultable bond supplemented with an option to exchange this defaultable bond for a given number κ of shares. Thus, convertible bonds are often advertised as products with upside potential and limited downside risk. However, after years of steady growth, the market of convertible bonds has suffered an unprecedented drawback in April–May 2005. Many hedge funds closed their convertible bond positions, while new convertible bond issues became more and more rare. This was largely due to persistently low credit default swap (CDS) spreads and low volatilities that limited the potential benefit of convertible bond arbitrage, and to regulatory changes that made financing by means of convertible bond a less attractive alternative to straight bond financing than before. In addition, some practitioners blamed this crisis on inadequate understanding of the product, that let people think for a while that convertible bonds were a win-win mixture to both issuers and holders, up to the point where disappointment changed their mind the other way around. So, many actors in the equity-to-credit universe closed their positions after the unexpected simultaneous rise in the General Motors CDS spreads and stock price in May 2005 (cf. [25]). Associated with this lack of understanding, deficiency of convertible bond software caused unexpected losses, which hastened sell-off of convertible bonds.

In this paper, we attempt to shed more light on the mathematical modeling of convertible bonds, thus continuing the previous research presented, for instance, in [1, 3, 11, 13, 15, 21, 22, 23, 24]. In particular, we consider the problem of the decomposition of a convertible bond into bond component and option component. This decomposition is indeed well established in the case of an 'exchange option', when the conversion can only occur at maturity (see [21]). However, it was not yet studied in the case of a real-life convertible bond.

More generally, we shall consider generic *defaultable game options* (GO), and *convertible securities* (CS), encompassing defaultable convertible bonds (and also more standard American or European options) as special cases. Moreover, we shall examine such contracts in a general framework of a fairly general market model in which prices of primary assets are assumed to follow semimartingales (see [12] or [15]) and the random moment of default is exogenously given.

1.2 Main Results

The main result of this work is Theorem 5.1, which furnishes a rigorous decomposition of the arbitrage price of a CS. In particular, it allows us to give a definite meaning to commonly used terms, such as: the *spread* and the *implied volatility* of a convertible bond (see Definition 5.1).

As a prerequisite, we provide in Theorem 3.1 a characterization of the set of ex-dividend arbitrage prices of a GO in terms of related Dynkin games, based on Kallsen and Kühn [15, Theorem 2.9].

This paper provides theoretical underpinning for a more extensive research continued in Bielecki et al. [5, 6], where more specific market models are introduced, and more explicit valuation and hedging results are established. In [5], we derive valuation results for a GO in the framework of a default risk model based on the hazard process, and we provide a characterization of minimal super-hedging strategies of a GO within the default intensity set-up through solutions of doubly reflected backward stochastic differential equations. In [6], we introduce a Markovian jump-diffusion model of credit risk, and we analyze the related variational inequalities.

1.3 Primary Market Model

We assume throughout that the evolution of the primary market can be modeled in terms of stochastic processes defined on a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$, where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ satisfies the usual conditions. Here, we denote by $\mathbb{P}$ the statistical (objective) probability measure. We assume that the primary market is composed of the savings account B and of a finite non-null number d of risky assets, such that, given a finite horizon date $T > 0$:

– the *discount factor* process β, that is, the inverse of the savings account B, is a $\mathbb{G}$-adapted, finite

variation, continuous, positive and bounded process;
– the risky assets are $\mathbb{G}$-semimartingales with càdlàg sample paths.

A primary risky asset with price process X^i may pay dividends whose cumulative value process, denoted by D^i, is assumed to be a $\mathbb{G}$–adapted, càdlàg and finite variation process. Given the price X^i, we define the *cumulative price* $\widehat{X}^i$ of the asset as

$$\widehat{X}^i_t := X^i_t + \widehat{D}^i_t, \tag{1}$$

where $\widehat{D}^i_t := B_t \int_{[0,t]} \beta_u dD^i_u$. In the financial interpretation, $\widehat{D}^i_t$ represents the current value at time t of all dividend payments of the asset over the period $[0,t]$, under the assumption that all dividends are immediately reinvested in the savings account.

Let $(X_t)_{t\in[0,T]}$, $(D^X_t)_{t\in[0,T]}$ and $(\widehat{X}_t)_{t\in[0,T]}$ denote the $\mathbb{R}^d$-valued price, cumulative dividend and cumulative price processes of the primary risky assets. A $\mathbb{G}$-predictable trading strategy (ζ^0, ζ) built on the primary market has the wealth process Y given as (denoting the transposition operator by $^\mathsf{T}$):

$$Y_t = \zeta^0_t B_t + \zeta^\mathsf{T}_t X_t, \quad t \in [0,T]. \tag{2}$$

Accounting for dividends, we say that a portfolio (ζ^0, ζ) is *self-financing* whenever it satisfies, for $t \in [0,T]$,

$$dY_t = \zeta^0_t \, dB_t + \zeta^\mathsf{T}_t (dX_t + dD^X_t),$$

or, equivalently,

$$d(\beta_t Y_t) = \zeta^\mathsf{T}_t d(\beta_t \widehat{X}_t). \tag{3}$$

In (3), we recognize the standard self-financing condition for a trading strategy (ζ^0, ζ) in non dividend paying primary risky assets (call them *the equivalent non-dividend-paying synthetic assets*) with price vector $\widehat{X}$. In view of this equivalence, the following definition is natural.

Definition 1.1 We say that $(X_t)_{t\in[0,T]}$ *is an arbitrage price for our primary market with dividend-paying assets*, if and only if $(\widehat{X}_t)_{t\in[0,T]}$ is an arbitrage price for the equivalent market with non-dividend-paying synthetic assets, in the sense that $(\widehat{X}_t)_{t\in[0,T]}$ satisfies the standard *No Free Lunch with Vanishing Risk* (NFVLR) condition of Delbaen and Schachermayer [12].

Then, by application of the main theorem in [12], we have that $(X_t)_{t\in[0,T]}$ is an arbitrage price for the primary market if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ for which $\beta\widehat{X}$ is a $\mathbb{G}$-martingale transform (also called σ-martingale, see [12]) under $\mathbb{Q}$. In the sequel, we assume that $(X_t)_{t\in[0,T]}$ is an arbitrage price for the primary market, and we denote by $\mathcal{M}$ the set of *risk-neutral measures* on the primary market, defined as the set of probability measures $\mathbb{Q} \sim \mathbb{P}$ for which $\beta\widehat{X}$ is a $\mathbb{G}$-martingale transform under $\mathbb{Q}$.

2 Game Options

2.1 Payoffs of a Game Option

As it is well known (see, for instance, Kifer [17]), a convertible bond with no call notice period can be formally seen as a special case of the so-called *game option*, which was introduced in Kifer [17] (see also Kallsen and Kühn [15]).

Let 0 (respectively T) stand for the *inception date* (respectively the *maturity date*) of a game option. For any $t \in [0,T]$, we write $\mathcal{G}^t_T$ to denote the set of all $\mathbb{G}$-stopping times with values in $[t,T]$.

Definition 2.1 A *game option* is a contract with the terminal payoff at time $\tau_p \wedge \tau_c$ given by, as seen from the perspective of the holder,

$$\mathbb{1}_{\{\tau_p \le \tau_c\}} \mathcal{L}_{\tau_p} + \mathbb{1}_{\{\tau_p > \tau_c\}} \mathcal{U}_{\tau_c}, \tag{4}$$

where $\tau_p, \tau_c \in \mathcal{G}^0_T$ are stopping times under the control of the holder and the issuer of a game option respectively. Additionally, a game option pays dividends with process D, a $\mathbb{G}$-adapted, càdlàg, real-valued process with finite variation. The *put payoff process* $\mathcal{L} = (\mathcal{L}_t)_{t\in[0,T]}$ and the *call payoff process* $\mathcal{U} = (\mathcal{U}_t)_{t\in[0,T]}$ are $\mathbb{G}$-adapted, càdlàg, $\mathbb{R} \cup \{+\infty\}$-valued processes, such that $\mathcal{L} \le \mathcal{U}$ and $\mathcal{L}_T = \mathcal{U}_T$. Moreover, defining the *cumulative payoffs* of a game option with dividends as the processes $\widehat{\mathcal{L}} := \mathcal{L} + \widehat{D}$ and $\widehat{\mathcal{U}} := \mathcal{U} + \widehat{D}$, where $\widehat{D}_t := B_t \int_{[0,t]} \beta_u \, dD_u$, we assume that there exists a constant c such that

$$-c \le \beta_t \widehat{\mathcal{L}}_t \quad \text{for} \quad t \in [0,T]. \tag{5}$$

We refer to τ_c (respectively τ_p) as the moment of *call* (respectively *put*) of a game option.

Remarks 2.1 (i) The case of dividend-paying game options is not explicitly treated in Kifer [17] or Kallsen and Kühn [15]. However, we shall see shortly that all the results in [15] can be immediately extended to this situation.
(ii) In [15], the payoff processes $\mathcal{L}$ and $\mathcal{U}$ are implicitly assumed to be specified in relative terms with respect to a certain numeraire. In the present work, we prefer to make explicit the presence of the discount factor β.
(iii) Kallsen and Kühn [15] postulate that the lower payoff process $\mathcal{L}$ is non-negative. However, as long as the process $\mathcal{L}$ is bounded from below, all their results are applicable, by a simple shift argument.
(iv) One can deduce from (4) that we impose the priority of τ_p over τ_c, in the sense, that the terminal payment equals $\mathcal{L}_{\tau_p}$ (rather than $\mathcal{U}_{\tau_p}$) on the event $\{\tau_p = \tau_c\}$. We thus follow here Kallsen and Kühn [15], from which we will deduce Proposition 2.1 below. Note, however, that in the general context of game options, this assumption is known to be essentially immaterial, in the sense, that is has no bearing neither on the price of a game option nor on the optimal stopping rules (cf. [17]).

Since we wish also to deal with the practically important case of a convertible bond with a positive call notice period, we need also to introduce a more general concept of an *extended game option.*

Definition 2.2 An *extended game option* with time constraints (g, h) is a contract with the terminal payoff occurring either at time τ_1, if $\{\tau_p \le \tau_c\}$, or at time τ_2, if $\{\tau_p > \tau_c\}$, and given by, as seen from the perspective of the holder,

$$\mathbb{1}_{\{\tau_p \le \tau_c\}} \mathcal{L}_{\tau_1} + \mathbb{1}_{\{\tau_p > \tau_c\}} \mathcal{U}_{\tau_2}, \tag{6}$$

where $\tau_p, \tau_c \in \mathcal{G}_T^0$ are stopping times under the control of the holder and the issuer of an extended game option respectively, and $\tau_1, \tau_2 \in \{\tau_p, \tau_c\}$. Moreover, on the event $\{\tau_p \le \tau_c\}$ we have $\tau_c \le g(\tau_p)$ and on the event $\{\tau_c < \tau_p\}$ we have $\tau_p \le h(\tau_c)$, where the Borel functions $g, h : [0,T] \to [0,T]$ are such that $g(t) \ge t$, $h(t) \ge t$ for $t \in [0,T]$. Additionally, a game option pays dividends with cumulative process D, a real-valued, $\mathbb{G}$-adapted, càdlàg process with finite variation. Finally, the *terminal payoff processes* $\mathcal{L} = (\mathcal{L}_t)_{t\in[0,T]}$ and $\mathcal{U} = (\mathcal{U}_t)_{t\in[0,T]}$ are $\mathbb{G}$-adapted, càdlàg processes taking values in $\mathbb{R} \cup \{+\infty\}$.

In this rather abstract situation, we still find it convenient to refer to τ_c (respectively τ_p) as the moment of *call* (respectively *put*) of an extended game option. Thus dividends are paid while the option is alive, and if the contract is put by the holder prior to being called by the issuer (i.e., when $\tau_p \le \tau_c$) then the holder receives a terminal payment $\mathcal{L}_{\tau_1}$ at time τ_1; otherwise, a terminal payment $\mathcal{U}_{\tau_2}$ is received at time τ_2. We may formally distinguish four classes of extended game options, corresponding to $\tau_1 = \tau_2 = \tau_p$, $\tau_1 = \tau_2 = \tau_c$, $\tau_1 = \tau_p$ and $\tau_2 = \tau_c$, or $\tau_1 = \tau_c$ and $\tau_2 = \tau_p$, respectively. The third class — for the choice of admissible functions g and h given as $g(t) = h(t) = T$ for $t \in [0,T]$ — corresponds to game options as defined above. Therefore, a game option can be seen as a special case of an extended game option.

2.2 Valuation of a Game Option

The concept of an arbitrage price of a game option can be introduced in various ways. Kallsen and Kühn [15] make the distinction between a *static* and a *dynamic* approach. The former point of view corresponds to the assumption that only a buy-and-hold strategy in the derivative asset is allowed, whereas the primary assets can be traded dynamically. In the latter approach, it is assumed that a derivative asset becomes liquid and negotiable asset, so that it can be traded together with the primary assets during the whole period $[0,T]$. Consequently, in a dynamic approach, in order to determine a price process of a derivative asset, it is postulated that the extended market, including this derivative asset, remains arbitrage-free. In this work, we shall adopt the dynamic point of view.

For the formal definition of a (dynamic) arbitrage price process of a game option, we refer the reader to Kallsen and Kühn [15, Definition 2.6]. As explained in [15], this definition is based on an extension to markets containing game options of the *No Free Lunch with Vanishing Risk* condition, introduced by Delbaen and Schachermayer [12, Definition 2.8], using the notion of an admissible trading strategy involving primary assets and the game option. Without entering into details, let us note that admissible strategies in this sense include, in particular, trading strategies in the primary assets only, provided that the corresponding wealth process is bounded from below. The case of dividend-paying primary assets and/or game option is not explicitly treated in [15]. However, the results of [15] can be applied to the case of dividend-paying primary assets and/or game option by resorting to the transformation of prices into cumulative prices described in Section 1.3, and that we already used to characterize no-arbitrage prices in our primary risky market with dividends.

As a reality check of pertinency of Kallsen and Kühn's definition of an arbitrage price of a game option and of our extension to the case of dividend-paying assets, we show in forthcoming papers [5, 6] that in more specific models, in which we are able to identify well determined processes as arbitrage prices in the sense of this definition, these processes can alternatively be characterized as minimal super-hedging prices.

We decided not reproduce here the full statement of Definition 2.6 in [15], since it is rather technical and will not be explicitly used in the sequel. To proceed, it will be enough for us to make use of the following characterization of an arbitrage price.

We are interested in studying a problem of time evolution of an arbitrage price of a game option. Therefore, we shall formulate the problem in a dynamic way by allowing for any time $t \in [0,T]$ to serve as the contract's initiation date. Given $t \in [0,T]$ and stopping times $\tau_p, \tau_c \in \mathcal{G}_T^t$, let the *ex-dividend cumulative cash flow of the game option at time* t stand for the $\mathcal{G}_\tau$-measurable random variable θ_t such that

$$\beta_t \theta_t(\tau_p,\tau_c) := \beta_\tau \widehat{D}_\tau - \beta_t \widehat{D}_t + \beta_\tau \Big(\mathbb{1}_{\{\tau=\tau_p\}} \mathcal{L}_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}} \mathcal{U}_{\tau_c} \Big),$$

with $\tau = \tau_p \wedge \tau_c$. We shall see shortly that $\theta_t(\tau_p,\tau_c)$ represents the terminal cash flow paid at time τ of a non-dividend paying game option equivalent to the original game option with dividends. Note that the random variable $\theta_t(\tau_p,\tau_c)$ is not $\mathcal{G}_t$-measurable for $t<T$. It is only $\mathcal{G}_\tau$-measurable. This is, of course, expected, since it represents payments occurring between t and τ.

Proposition 2.1 (By application of Kallsen and Kühn [15, Theorem 2.9]) *If a process* $(X_t,\Theta_t)_{t\in[0,T]}$ *satisfies the following two conditions:*
(i) Θ *is a* $\mathbb{G}$*-semimartingale, and*
(ii) *there exists* $\mathbb{Q} \in \mathcal{M}$ *such that* Θ *is the* $\mathbb{Q}$*-value process of the Dynkin game related to the game option, in the sense that*

$$\begin{aligned} \operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t} \operatorname{essinf}_{\tau_c\in\mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\theta_t(\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) &= \Theta_t \\ = \operatorname{essinf}_{\tau_c\in\mathcal{G}_T^t} \operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}}\big(\theta_t(\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big),& \quad t\in[0,T]\,, \end{aligned} \tag{7}$$

then (X,Θ) *is an (ex-dividend) arbitrage price for the extended market composed of the primary market and the game option. Moreover, the converse holds true under the following integrability condition:*

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T]} \widehat{\mathcal{L}}_t \,\Big|\, \mathcal{G}_0 \Big) < \infty, \quad \text{a.s.} \tag{8}$$

Recall that the fact that the Dynkin game has a (conditional) *value* at time t means that we have equality between the lower value of the game, corresponding to the left-hand side of (7), and the upper value, as given by its right-hand side. It is well known that the lower value of a game is always less or equal to the upper value, but they do not need to coincide, in general. For general results on Dynkin games, see, for instance, Dynkin [14], Kifer [18], Lepeltier and Maingueneau [20].

Proof of Proposition 2.1. By definition of prices in markets with dividends, $(X_t, \Theta_t)_{t\in[0,T]}$ is an arbitrage price for the extended market with dividends, if and only if $(\widehat{X}_t, \widehat{\Theta}_t)_{t\in[0,T]}$ is an arbitrage price for the equivalent extended market without dividends, where $\widehat{\Theta}_t := \Theta_t + \widehat{D}_t$. Now, by an application of Kallsen and Kühn [15, Theorem 2.9], under condition (8) (which is actually only used for the converse part of the Theorem), this is equivalent to the fact that $\beta\widehat{X}$ is a $\mathbb{G}$-martingale transform under some $\mathbb{P}$-equivalent probability measure $\mathbb{Q}$, and that $\widehat{\Theta}$ is a $\mathbb{G}$-semimartingale equal to the $\mathbb{Q}$-value of the Dynkin game without dividends and with terminal payoffs $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{U}}$. Specifically, $\widehat{\Theta}$ satisfies, for $t \in [0, T]$,

$$\begin{aligned}\operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t}\operatorname{essinf}_{\tau_c\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\widehat{\theta}_t(\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) &= \widehat{\Theta}_t \\ &= \operatorname{essinf}_{\tau_c\in\mathcal{G}_T^t}\operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\widehat{\theta}_t(\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)\end{aligned} \tag{9}$$

with $\widehat{\theta}_t(\tau_p, \tau_c) = \theta_t(\tau_p, \tau_c) + \widehat{D}_t$, or equivalently,

$$\beta_t\widehat{\theta}_t(\tau_p,\tau_c) = \beta_\tau\Big(\mathbb{1}_{\{\tau=\tau_p\}}\widehat{\mathcal{L}}_{\tau_p} + \mathbb{1}_{\{\tau<\tau_p\}}\widehat{\mathcal{U}}_{\tau_c}\Big).$$

It now suffices to observe that (9) is equivalent to (7). □

This very general result essentially reduces the study of an arbitrage price of a game option to the study of the value, under a risk-neutral measure $\mathbb{Q}$, of the corresponding Dynkin game, with the issuer playing the role of the minimizer and the holder being the maximizer. Note that this result covers in particular the case of American options and European options, as we now see.

Definition 2.3 An *American option* is a game option with $\mathcal{U}_t = \infty$ for $t \in [0, T)$. A *European option* is an American option such that

$$\beta_t\widehat{\mathcal{L}}_t \le \beta_T\widehat{\mathcal{L}}_T, \quad t \in [0, T]. \tag{10}$$

By applying Proposition 2.1, we deduce that the $\mathbb{Q}$-value $\widehat{\Theta}_t$ of an American option becomes the essential supremum with respect to stopping times $\tau_p \in \mathcal{G}_T^t$, specifically,

$$\widehat{\Theta}_t = \operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\widehat{\theta}_t(\tau_p,T)\,\big|\,\mathcal{G}_t\big) = B_t\operatorname{esssup}_{\tau_p\in\mathcal{G}_T^t}\mathbb{E}_{\mathbb{Q}}\big(\beta_{\tau_p}\widehat{\mathcal{L}}_{\tau_p}\,\big|\,\mathcal{G}_t\big),$$

whereas for a European option it reduces to the following conditional expectation

$$\widehat{\Theta}_t = \mathbb{E}_{\mathbb{Q}}\big(\widehat{\theta}_t(T,T)\,\big|\,\mathcal{G}_t\big) = B_t\,\mathbb{E}_{\mathbb{Q}}\big(\beta_T\widehat{\mathcal{L}}_T\,\big|\,\mathcal{G}_t\big).$$

A natural question arises whether it is possible to generalize Proposition 2.1 to the case of an extended game option. We shall not address this pertinent issue at this stage of our research, but we shall rather focus on specific subclasses of game options (convertible securities and, as a special case, convertible bonds with no call notice period) and extended game options (convertible bonds with a positive call notice period).

In the situation of Proposition 2.1, we shall briefly say in the sequel that $(\Theta_t)_{t\in[0,T]}$ is an *arbitrage price for the game option*, whenever $(X_t, \Theta_t)_{t\in[0,T]}$ is an arbitrage price for the extended market consisting of the primary market and the game option.

3 Defaultable Game Options and Convertible Securities

In this section, we introduce fairly general subclasses of game options, namely *defaultable game option* and *defaultable convertible securities* (GO and CS, for short), which encompass as special cases such financial instruments as convertible bonds, which will be discussed in some detail in Section 4.1 below, or convertible preferred stocks, as well as defaultable American or European options.

3.1 Defaultable Game Options

Let an $\mathbb{R}_+ \cup \{\infty\}$-valued $\mathbb{G}$-stopping time τ_d represent the *default time* of a reference entity. In broad terms, a defaultable game option (GO) is a game option with the following cash flows that are paid by the issuer of the contract and received by the holder of the contract:
– a *put payment* L_t made at the put time $t = \tau_p$ (chosen by the holder), given that $\tau_p \in [t, \tau_c \wedge \tau_d \wedge T]$; the rules governing the determination of the amount L_t are specified in the contract;
– a *call payment* U_t made at time $t = \tau_c$ (chosen by the issuer), given that $\tau_c \in [t, \tau_p \wedge \tau_d \wedge T)$; moreover, the call time may be subject to the constraint that $\tau_c \geq \bar{\tau}$, where $\bar{\tau}$ is the lifting time of the call protection; the rules governing the determination of the amount U_t are specified in the contract,
– a *dividend stream* D^{go}_t, subject to rules specified in the contract,
– a *payment at maturity* ξ made at time T provided that $T < \tau_d$ and $T \leq \tau_p \wedge \tau_c$.

The contract is terminated at default time, modelled as the $\mathbb{R}_+ \cup \{\infty\}$-valued $\mathbb{G}$-stopping time τ_d. In particular, there are no more cash flows related to this contract after the default time. In this setting the dividend stream D additionally includes a possible recovery payment made at the default time.

Of course, there is also a cash flow, namely the purchasing price of the contract, which is paid at the initiation time by the holder and received by the issuer.

Let H denote the *default indicator process* $H_t = \mathbb{1}_{\{\tau_d \leq t\}}$, where the $\mathbb{R}_+ \cup \{\infty\}$-valued $\mathbb{G}$-stopping time τ_d represents the default time of the reference entity.

Definition 3.1 A *defaultable game option* (GO) is a game option with cumulative put and call payoff processes $\widehat{\mathcal{L}}^{go} = (\widehat{\mathcal{L}}^{go}_t)_{t\in[0,T]}$ and $\widehat{\mathcal{U}}^{go} = (\widehat{\mathcal{U}}^{go}_t)_{t\in[0,T]}$ given by

$$\widehat{\mathcal{L}}^{go}_t = \widehat{D}^{go}_t + \mathbb{1}_{\{\tau_d > t\}}\big(\mathbb{1}_{\{t<T\}}L_t + \mathbb{1}_{\{t=T\}}\xi\big), \tag{11}$$

$$\widehat{\mathcal{U}}^{go}_t = \widehat{D}^{go}_t + \mathbb{1}_{\{\tau_d > t\}}\big(\mathbb{1}_{\{t<T\}}\bar{U}_t + \mathbb{1}_{\{t=T\}}\xi\big), \tag{12}$$

where:
– $\widehat{D}^{go}_t = B_t \int_{[0,t]} \beta_u \, dD^{go}_u$, where the *dividend process* $D^{go} = (D^{go}_t)_{t\in[0,T]}$ equals

$$D^{go}_t = \int_{[0,t]} (1 - H_u)\, dC_u + \int_{[0,t]} R_u \, dH_u \ ;$$

here, the *coupon process* $C = (C_t)_{t\in[0,T]}$ is a $\mathbb{G}$-adapted càdlàg process with bounded variation, and the *recovery process* $R = (R_t)_{t\in[0,T]}$ is a real-valued, $\mathbb{G}$-adapted process;
– the *put/conversion payment* process $L = (L_t)_{t\in[0,T]}$ is a $\mathbb{G}$-adapted, real-valued, càdlàg process;
– the process $\bar{U} = (\bar{U}_t)_{t\in[0,T]}$ equals

$$\bar{U}_t = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\geq\bar{\tau}\}}U_t,$$

where the lifting time of a *call protection* is modeled as a given stopping time $\bar{\tau} \in \mathcal{G}^0_T$, and where the *call payment* $U = (U_t)_{t\in[0,T]}$ is a $\mathbb{G}$-adapted, real-valued, càdlàg process, such that $L_t \leq \bar{U}_t$ for $t \in [0, \tau_d \wedge T)$, or equivalently,

$$L_t \leq U_t \quad \text{for} \quad t \in [\tau_d \wedge \bar{\tau}, \tau_d \wedge T) \ ; \tag{13}$$

– the *payment at maturity* ξ is a $\mathcal{G}_T$-measurable real random variable, such that $L_T \leq \xi \leq U_T$.

$\mathcal{G}^t_T$ denoting as above the set of all $\mathbb{G}$-stopping times with values in $[t, T]$, for any $t \in [0, T]$, let also $\bar{\mathcal{G}}^t_T$ stand for $\{\tau \in \mathcal{G}^t_T \, ; \, \tau \wedge \tau_d \geq \bar{\tau} \wedge \tau_d\}$, where the *lifting time of a call protection of a GO*, $\bar{\tau}$, is given in $\mathcal{G}^0_T$. Note that in the case of a GO, given the specification (11)-(12) of $\widehat{\mathcal{L}}^{go}$ and $\widehat{\mathcal{U}}^{go}$ with $\bar{U}_t = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\geq\bar{\tau}\}}U_t$, condition (9) can be rewritten as

$$\begin{aligned} &\operatorname{esssup}_{\tau_p\in\mathcal{G}^t_T} \operatorname{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big) = \Pi_t \\ &\quad = \operatorname{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T} \operatorname{esssup}_{\tau_p\in\mathcal{G}^t_T} \mathbb{E}_{\mathbb{Q}}\big(\pi(t;\tau_p,\tau_c) \,\big|\, \mathcal{G}_t\big), \quad t \in [0,T] \ , \end{aligned} \tag{14}$$

where for $t \in [0,T]$ and $(\tau_p, \tau_c) \in \mathcal{G}_T^t \times \bar{\mathcal{G}}_T^t$, the *ex-dividend cumulative cash flow of a GO* is given by

$$\beta_t \pi_t(\tau_p, \tau_c) := \beta_\tau \widehat{D}_\tau^{go} - \beta_t \widehat{D}_t^{go} + \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau < \tau_p\}} U_{\tau_c} + \mathbb{1}_{\{\tau = T\}} \xi \Big) ,$$

with $\tau = \tau_p \wedge \tau_c$. We thus have the following Theorem, as a consequence of Proposition 2.1.

Theorem 3.1 *If a process* Π *is a* $\mathbb{G}$*-semimartingale and if there exists* $\mathbb{Q} \in \mathcal{M}$ *such that (14) is verified, then* Π *is an arbitrage price for the GO with ex-dividend cumulative cash flow* π*. Moreover, the converse holds true provided*

$$\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \Big(\sup_{t \in [0,T]} \widehat{\mathcal{L}}_t^{go} \,\Big|\, \mathcal{G}_0 \Big) < \infty \ , \ \ a.s. \tag{15}$$

Remarks 3.1 (i) So the restriction that the issuer of GO is prevented from making a call on some random time interval $[0, \bar{\tau})$, where $\bar{\tau} \in \mathcal{G}_T^0$ (cf. the non-mathematical description of a GO above), is implicitly enforced in Definition 3.1 by putting $\bar{U}_t = \infty$ on the random interval $[0, \bar{\tau})$.
(ii) Note that $\pi_t(\tau_p, \tau_c) = 0$ for any $t \geq \tau_d$. Therefore an (ex-dividend) arbitrage price of a GO is necessarily equal to 0, for $t \geq \tau_d$. In what follows, an arbitrage price associated with a risk-neutral measure $\mathbb{Q}$ will be called the $\mathbb{Q}$*-price* of a GO.
(iii) In view of our formulation of the problem, the put or call decisions may take place after the default time τ_d. Nevertheless, the discounted cumulative payoff processes $\beta \widehat{\mathcal{L}}^{go}$ and $\beta \widehat{\mathcal{U}}^{go}$ are constant on the set $\{t \geq \tau_d\}$ (note that the processes D^{go} and $\beta \widehat{D}^{go}$ are stopped at τ_d). Thus, effectively, the GO game is stopped at the default time τ_d, unless the decision to stop it was already made prior to τ_d.

Definition 3.2 We also have the companion concepts of American GO (AO) and European GO (EO), namely, GO that are American or European options in the sense of Definition 2.3. An AO, namely a GO with $\bar{U} = \infty$, can equivalently be seen as a non-callable GO, namely a GO with $\bar{\tau} = T$. It is also worth noting that when D^{go} has bounded variation on $[0,T]$ and ξ is bounded from below, an AO becomes an EO provided that L is chosen to be a negatively large enough constant (depending on the other data of the AO). In the special case where ξ is bounded from below and from above, such an EO will be referred to as an *elementary security* (ES).

Remarks 3.2 Consider a defaultable coupon-paying bond with (positive or negative) bounded coupons, bounded recovery payoff, and a (finite or at least bounded from below and from above) face value. Such a bond can be formally treated as an ES, provided that we take $\bar{U} = \infty$ and we additionally introduce the constant process L which makes the inequality $\beta_t \widehat{\mathcal{L}}_t^{go} \leq \beta_T \widehat{\mathcal{L}}_T^{go}$ hold for every $t \in [0,T]$. Of course, the choice of L is somewhat arbitrary, in the sense that L will not appear explicitly in the valuation formula for the bond (see part (ii) in Theorem 3.2).

We shall now apply Theorem 3.1 in order to characterize arbitrage prices of an AO and a EO.

Theorem 3.2 (i) *If a process* $\bar{\Pi} = (\bar{\Pi}_t)_{t \in [0,T]}$ *is a* $\mathbb{G}$*-semimartingale and if there exists* $\mathbb{Q} \in \mathcal{M}$ *such that*

$$\bar{\Pi}_t = \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \mathbb{E}_{\mathbb{Q}} \big(\bar{\pi}_t(\tau_p) \,\big|\, \mathcal{G}_t \big), \quad t \in [0,T],$$

where the ex-dividend cumulative cash flow $\bar{\pi}_t(\tau_p)$ *of an AO can be represented as follows, for* $t \in [0,T]$:

$$\beta_t \bar{\pi}_t(\tau_p) = \beta_{\tau_p} \widehat{D}_{\tau_p}^{go} - \beta_t \widehat{D}_t^{go} + \mathbb{1}_{\{\tau_d > \tau_p\}} \beta_{\tau_p} \big(\mathbb{1}_{\{\tau_p < T\}} L_{\tau_p} + \mathbb{1}_{\{\tau_p = T\}} \xi \big) ,$$

then $\bar{\Pi}$ *is an arbitrage price of the related AO. Moreover, the converse holds true provided (15) is satisfied.*
(ii) *If there exists* $\mathbb{Q} \in \mathcal{M}$ *such that*

$$\Phi_t = \mathbb{E}_{\mathbb{Q}}(\phi_t \,|\, \mathcal{G}_t), \quad t \in [0,T],$$

where the ex-dividend cumulative cash flow ϕ_t of an EO can be represented as follows, for $t \in [0,T]$:

$$\beta_t \phi_t = \beta_T \widehat{D}^{go}_T - \beta_t \widehat{D}^{go}_t + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi \ , \tag{16}$$

then the process $\Phi = (\Phi_t)_{t\in[0,T]}$ is an arbitrage price of the related EO. Moreover, the converse holds true provided

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\Big(\widehat{\mathcal{L}}^{go}_T \,\Big|\, \mathcal{G}_0\Big) < \infty \ a.s. \tag{17}$$

where we recall that $\widehat{\mathcal{L}}^{go}_T = \widehat{D}^{go}_T + \mathbb{1}_{\{\tau_d > T\}}\xi$.

Proof. Since an AO and an EO are special cases of a GO, their ex-dividend cumulative cash flows are given by the general formula of Theorem 3.1. By saying that they can be represented as $\bar{\pi}_t(\tau_p)$ and ϕ_t, respectively, we mean that for the valuation purposes the general payoff $\pi_t(\tau_p,\tau_c)$ can be reduced to either $\bar{\pi}_t(\tau_p)$ or ϕ_t. Note that consistently with the notation, $\bar{\pi}_t(\tau_p)$ does not depend on τ_c and ϕ_t is independent of τ_p and τ_c.

Part (i) of the theorem follows by a straightforward application of Theorem 3.1. To prove part (ii) we observe that

$$\beta_t \Phi_t = \mathbb{E}_{\mathbb{Q}}(\beta_t \phi_t \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(\beta_T \widehat{D}^{go}_T + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi \,|\, \mathcal{G}_t) - \beta_t \widehat{D}^{go}_t,$$

and thus Φ given by (16)is a $\mathbb{G}$-semimartingale. □

3.2 Convertible Securities

Let us now introduce the concept of a *convertible security* (CS) with underlying S, a financial contract that can be situated somewhere between a GO and a much more specific *convertible bond*, which will be discussed in some detail in Section 4 below. Let S denote one of the primary risky assets, called the *underlying* to a CS. In broad terms, a *convertible security* (CS) with underlying S, is a GO such that:

– the put payment L_t represents in fact a *put/conversion payment* L_t made at the put/conversion time $t = \tau_p$; usually, payment L_t depends on the value S_t of the underlying asset, and corresponds to the right of the holder of the CS to convert it to a fixed number of units of this asset – hence the name of convertible – or to receive a predetermined cash flow;

– conversion is typically still possible at default time τ_d or at maturity time T, if the CS is still alive at these times.

The specific nature of CS payements motivates the following definition.

Definition 3.3 A *defaultable convertible security* (CS) with underlying S, is a GO such that the processes R, L and the random variable ξ satisfy the following inequalities, for some positive reals a, b, c:

$$\begin{aligned} -c &\le R_t \le a \vee bS_t, \quad t \in [0,T], \\ -c &\le L_t \le a \vee bS_t, \quad t \in [0,T], \\ -c &\le \xi \le a \vee bS_T \ . \end{aligned} \tag{18}$$

In the case of a CS, $\widehat{D}^{go}$ will be renotd as $\widehat{D}^{cs}$. Let $\widehat{\mathcal{L}}^{cs}$ denote the cumulative put payoff of a CS. Given our assumptions, we have for (modified) positive reals a, b :

$$\widehat{\mathcal{L}}^{cs}_t \le a \vee bS_{t\wedge\tau_d} \ , \ t \in [0,T] \tag{19}$$

so that in the case of a CS, the following condition enforces (15):

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\Big(\sup_{t\in[0,T\wedge\tau_d]} S_t \,\Big|\, \mathcal{G}_0\Big) < \infty \ , \quad \text{a.s.} \tag{20}$$

Also note that an ES (cf. Definition (3.2)) is a special case of a CS.

4 Convertible Bonds

We shall now address the issue of arbitrage valuation of a convertible bond with real-life features, including the call notice period. As will be explained in Section 4.4, it is rather difficult to directly value a convertible bond with a positive call notice period, since it does not fulfill conditions of the definition of a game option, in general. To circumvent this difficulty, we shall introduce the concept of a *reduced convertible bond* (RB), that is, a convertible bond whose value upon call is exogenously given as a certain stochastic process, so that we can assume, without loss of generality, that the bond has no call notice period. Since an RB is a special case of a CS, hence a GO, the valuation results in the previous sections are directly applicable to a reduced convertible bond.

4.1 Covenants of a Convertible Bond

To describe the covenants of a typical *convertible bond* (CB), we introduce the following additional notation:

$\bar{N}$: the par (nominal) value,

c_t^{cb}: the continuous coupon rate, a $\mathbb{G}$-adapted bounded process,

T_i, c_i, $i = 0, 1, \ldots, K$: the coupon dates $T_0, \ldots, T_K$ are deterministic fixed times with $T_0 = 0$ (for notational convenience) and $T_{K-1} < T \leq T_K$; the coupon amounts c_i are $\mathcal{G}_{T_{i-1}}$-measurable and bounded, for $i = 1, 2, \ldots, K$, and $c_0 = 0$,

A_t: the accrued interest at time t, specifically,

$$A_t = \frac{t - T_{i_t - 1}}{T_{i_t} - T_{i_t - 1}} \, c_{i_t},$$

where i_t is the integer satisfying $T_{i_t - 1} \leq t < T_{i_t}$; in view of our assumptions on the coupons, the process $(A_t)_{t \in [0,T]}$ is càdlàg and $\mathbb{G}$-adapted,

$\bar{R}_t$: the recovery process on the CB upon default of the issuer at time t, a $\mathbb{F}$-predictable bounded process,

κ : the conversion factor,

$R_t^{cb} = \bar{R}_t \vee \kappa S_t$: the effective recovery process,

D_t^{cb} : the cumulative dividend process (to be specified below),

$\xi^{cb} = \bar{N} \vee \kappa S_T + A_T$: the payoff at maturity,

$\bar{P} \leq \bar{C}$: the put and call nominal payments, respectively,

$\delta \geq 0$: the length of the call notice period (see the detailed description below),

$t^{\delta} = (t + \delta) \wedge T$: the end date of the call notice period started at t.

We shall now present a detailed description of specific CB covenants. Let us consider a CB at any date $t \in [0, T]$ at which it is still alive. Then we have the following provisions:

put/conversion provision – at any time $\tau_p \in [t, \tau_c \wedge \tau_d \wedge T]$, where τ_c is a stopping time under the discretion of the issuer, the bond holder may convert a CB to κ shares of equity. In addition, at any time $\tau_p \in [t, \tau_c \wedge \tau_d \wedge T)$, and possibly also at τ_c if $\tau_c < \tau_d \wedge T$, the holder may put (return) the bond to the issuer for a nominal put payment $\bar{P}$ pre-agreed at time of issuance. Only one of the two above decisions may be executed. Since the bond holder is also entitled to receive a relevant accrued interest payment, the *effective put/conversion payment* collected in case of put or conversion (depending on which one is more favorable to the holder) at time τ_p (if $\tau_p < T$) equals $L_{\tau_p}^{cb} = \bar{P} \vee \kappa S_{\tau_p} + A_{\tau_p}$, where κ denotes the conversion ratio. The effective put payment in case $\tau_p = T$ is considered separately (see the *promised payment* below).

call provision – the issuer has the right to call the bond at any time $\tau_c \in [t, \tau_p \wedge \tau_d \wedge T)$, where τ_p is a random time under the discretion of the holder, for a nominal call payment $\bar{C}$ pre-agreed at time of issuance. More precisely, there is a fixed call notice period $\delta \geq 0$ (typically, one month) such that if the issuer calls the bond at time τ_c, then the bond holder has either to redeem the bond for $\bar{C}$ or convert the bond into κ shares of stock, at any time u at its convenience in $[\tau_c, \tau_c^\delta]$, where $\tau_c^\delta = (\tau_c + \delta) \wedge T$. Accounting for accrued interest, the *effective call/conversion payment* to the holder at time u is $\bar{C} \vee \kappa S_u + A_u$.

call protection – typically, a CB also includes *call protections*, either *hard* or *soft*. For instance, the issuer's right to call a CB early becomes active only after a certain period of time has lapsed since the original issue date. A CB, which can't be called under any circumstances during the initial time period $[0, \bar{T})$, is subject to *hard* call protection. Alternatively, a CB that is non-callable unless the stock price reaches a certain predetermined level, say $\bar{S}$, is subject to *soft* call protection. The introduction of the stopping times $\bar{\tau}$ in $\mathcal{G}_T^0$, and of the associated class $\bar{\mathcal{G}}_T^t \subseteq \mathcal{G}_T^t$, allows one to model quite general kinds of call protections. So hard call protections correspond to $\tau_c \in \bar{\mathcal{G}}_T^t$ with $\bar{\tau} = \bar{T}$, and standard soft call protections to $\tau_c \in \bar{\mathcal{G}}_T^t$ with $\bar{\tau} = \inf\{t \in \mathbb{R}_+ \,;\, S_t \geq \bar{S}\} \wedge T$.

promised payment – the issuer agrees to pay to the bond holder, at any coupon date T_i prior to default time τ_d and to $\tau_p \wedge \tau_c \wedge T$, a bounded coupon amount c_i. He also agrees to pay the par value $\bar{N}$ at the maturity date T, provided that $T < \tau_d$ and $T \leq \tau_p \wedge \tau_c$. Since the bond holder may still convert at time T, we define the *effective payment at maturity* as $\xi^{cb} = \bar{N} \vee \kappa S_T + A_T$; it is collected at time T if the CB is still alive at T.

recovery structure at default – it is assumed throughout that in the case of default at time $\tau_d \leq \tau_p \wedge \tau_c \wedge T$, the *effective recovery* $R_{\tau_d}^{cb} = \bar{R}_{\tau_d} \vee \kappa S_{\tau_d}$ is recovered. Indeed, we assume that the CB can still be converted at default time τ_d.

It is typically assumed that $\bar{P} \leq \bar{N} \leq \bar{C}$, which we also suppose in the following.

Remarks 4.1 (i) As specified above, at maturity the bond holder is allowed to convert, but not to put, the bond. Some authors allow for a put decision at maturity date as well. In fact, allowing put decisions at maturity would not change anything, as long as one supposes (as we do) that $\bar{P} \leq \bar{N}$. Indeed, if $\bar{P} \leq \bar{N}$, we have $N_T = (\bar{N} \vee \kappa S_T) + A_T = (\bar{P} \vee \bar{N} \vee \kappa S_T) + A_T$.
(ii) It should be stressed that we do not consider the default decision to be a decision variable in the sense of "optimal default" studied in corporate finance. In other words, the default time is exogenously given random time, as opposed to call and put/conversion times. It would be possible to extend our study by allowing for two possible times of default: the exogenous time τ_d^{ex} chosen by the nature, and the endogenous default time τ_d^{en} which is optimally chosen by the bond issuer. Note that τ_d^{en} must not be identified with τ_c since call provisions are parts of the contract, whereas the bankruptcy provisions are not.
(iii) An important issue in the valuation of a CB is the so-called *dilution effect*. Dilution is the fact that the equity price may drop upon conversion, due to the sudden increase of the number of shares in circulation [10]. In practice, the importance of this effect depends on the number of bond holders, who decide to convert simultaneously. In our framework, we deal with a representative holder, who is supposed to make optimal decisions. Therefore, the whole issue of the convertible bond will be converted at the same time, so that a jump in the stock price upon conversion is expected. To account for dilution, one could introduce a fractional loss $0 \leq \nu \leq 1$ of the stock price at put, so that $\bar{P} \vee \kappa S_{\tau_p} = \bar{P} \vee \kappa(1-\nu) S_{\tau_p -}$. However, in the abstract framework considered in this paper, this would be immaterial.
(iv) A further possible covenant of a CB is *resettability*. *Resettability* means that to compensate for fluctuations in S, the conversion ratio κ may depend on S_t in a particular way specified in the bond indenture. It is straightforward to check that all the results in this paper remain valid, if one assumes that $\kappa_t = \kappa(S_t)$ for some bounded Borel function κ.
(v) There exist soft call protection clauses more sophisticated than the one mentioned above, such as clauses preventing the issuer to call a CB unless the stock has been above a certain level for a given amount of time. A soft call protection always introduces a certain path-dependency to the valuation problem (cf. [2, 19]). However, we shall see that it does not complicate much the analysis from a general point of view. Naturally, it makes computationally heavier the numerical resolution

of the pricing variational inequalities in a Markovian model (see [6]).
(vi) In practice, coupons of a CB are purely discrete. However, frequently, in the literature on CBs or in CB software pricing models, a simplifying assumption is made that coupons are paid continuously. Here both forms of coupons are represented.
(vii) In practice, $\bar{R}$ is generally specified as $\bar{X}\bar{Y}$, where:
– the *default claim* process $\bar{X}$ is specified in the indenture of a CB. Typically, $\bar{X}$ is simply equal to the bond par value, or the bond par value plus the accrued interest;
– the *recovery rate* process $\bar{Y}$ depends on legal specifications, such as the seniority of the related debt, etc. In practice, $\bar{Y}$ tends to be lower in periods with more defaults. However, this statistical observation holds under the real-world probability, with no obvious consequences under the market pricing measure [4]. A common recovery assumption is the so-called *face recovery* assumption, which means that $\bar{X}$ is equal to $\bar{N}$ and that $\bar{Y}$ is a given constant (typically, $\bar{Y} = 40\%$ for investment grade issues).
(viii) Upon default, the stock price process typically falls sharply. To account for this effect, one should introduce, in a model for the stock process S, a fractional loss upon default $0 \le \eta \le 1$, such that $S_{\tau_d} = (1-\eta)S_{\tau_d-}$ (see, for instance, [5]). However, in the abstract framework considered in this paper, this particular feature of the stock price is irrelevant.

Definition 4.1 In accordance with the CB covenants, the dividend process D^{cb} of a CB is given by the expression

$$D_t^{cb} = \int_0^{t\wedge\tau_d} c_u^{cb}\,du + \sum_{0\le T_i\le t,\,T_i<\tau_d} c_i + \mathbb{1}_{\{0\le\tau_d\le t\}} R_{\tau_d}^{cb}, \quad t\in[0,T]. \tag{21}$$

As in Section 3.1, we define the auxiliary process $\widehat{D}^{cb}$ representing the current value of past dividends of a CB by setting

$$\widehat{D}_t^{cb} = B_t \int_{[0,t]} \beta_u\, dD_u^{cb}, \quad t\in[0,T].$$

Recall also that we write

$$R_{\tau_d}^{cb} = \bar{R}_{\tau_d} \vee \kappa S_{\tau_d}, \quad L_t^{cb} = \bar{P} \vee \kappa S_t + A_t, \quad \xi^{cb} = \bar{N} \vee \kappa S_T + A_T.$$

4.2 Convertible Bonds without Call Notice Period

Assume first that a CB has no call notice period, so that $\delta = 0$.

Definition 4.2 A *convertible bond with no call notice period* is a convertible security with the cumulative put and call payoff processes $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{U}}$ given by the expressions

$$\widehat{\mathcal{L}}_t^{cb} = \widehat{D}_t^{cb} + \mathbb{1}_{\{\tau_d>t\}}\left(\mathbb{1}_{\{t<T\}} L_t^{cb} + \mathbb{1}_{\{t=T\}}\xi^{cb}\right), \tag{22}$$

$$\widehat{\mathcal{U}}_t^{cb} = \widehat{D}_t^{cb} + \mathbb{1}_{\{\tau_d>t\}}\left(\mathbb{1}_{\{t<T\}} \bar{U}_t^{cb} + \mathbb{1}_{\{t=T\}}\xi^{cb}\right), \tag{23}$$

where we set

$$\bar{U}_t^{cb} = \mathbb{1}_{\{t<\bar{\tau}\}}\infty + \mathbb{1}_{\{t\ge\bar{\tau}\}}(\bar{C}\vee\kappa S_t + A_t), \quad t\in[0,T]. \tag{24}$$

It is a routine task to check that the processes $\widehat{\mathcal{L}}^{cb}$ and $\widehat{\mathcal{U}}^{cb}$ satisfy all technical assumptions stated in Section 3.1.

The arbitrage valuation of a CB with no call notice period is covered by Theorem 3.1. Therefore, it remains to address the issue of valuation of convertible bonds with a positive call notice period.

4.3 Convertible Bonds with a Positive Call Notice Period

At the intuitive level, a convertible bond with a positive call notice period $\delta > 0$ can be seen as a contract involving the following decisions: the decision to call the bond by its issuer, the decision to put/convert a bond by its holder, provided that the bond was not yet called, and the decision to put/convert the bond by its holder during the call notice period $[t, t+\delta]$, assuming that the bond has been called at some date t. This rather loose description is made rigorous in the following definition.

Definition 4.3 A *convertible bond with a positive call notice period* is an extended game option with cumulative payoff paid at τ, as seen from the perspective of the holder, given by

$$\mathbb{1}_{\{\tau_p \le \tau_c\}} \widehat{\mathcal{L}}^{cb}_{\tau_p} + \mathbb{1}_{\{\tau_p > \tau_c\}} \widehat{\mathcal{L}}^{c}_{\tau_p}, \tag{25}$$

where τ_c, τ_p are $\mathbb{G}$-stopping time under the control of the issuer and the holder respectively, and $\tau_p \le (\tau_c + \delta) \wedge T$ on the event $\{\tau_c < \tau_p\}$. Moreover, $\widehat{\mathcal{L}}^{cb}$ is given by (22) and $\widehat{\mathcal{L}}^{c}$ is given by the formula

$$\widehat{\mathcal{L}}^{c}_t = \widehat{D}^{cb}_t + \mathbb{1}_{\{\tau_d > t\}} (\bar{C} \vee \kappa S_t + A_t). \tag{26}$$

Let us make few comments regarding the definition above.

On the one hand, is is worth noting that for $\delta = 0$ the definition above formally excludes $\tau_p > \tau_c$, so that formula (25) reduces to $\widehat{\mathcal{L}}^{cb}_{\tau_p}$. Consequently, for $\delta = 0$, Definition 4.3 (which would be inappropriate in this case) does not reduce to Definition 4.2.

On the other hand, in order to obtain the definition above, it suffices to set $\tau_1 = \tau_2 = \tau_p$, $g(t) = T$, $h(t) = (t+\delta) \wedge T$, $\widehat{\mathcal{L}} = \widehat{\mathcal{L}}^{cb}$ and $\widehat{\mathcal{U}} = \widehat{\mathcal{L}}^{c}$ in Definition 2.2 of an extended game option.

As already mentioned, we do not attempt here to directly value a convertible bond with a positive call notice, in the sense that we do not adapt Proposition 2.1 to the case of extended game options.

4.4 Reduced Convertible Bonds

An alternative approach proposed in this work is based on a conjecture that the valuation of a convertible bond with positive call notice period can be done recursively. In the first step, we shall value this bond upon call. In the second step, we use this price as the payoff at call time of a CB with no call notice period. This idea motivates the introduction of the following auxiliary concept.

Definition 4.4 A *reduced convertible bond* (RB) is a convertible security with the cumulative put payoff process $\widehat{\mathcal{L}}^{cb}$ given by (22) and the cumulative call payoff process $\widehat{\mathcal{U}}^{cb}$ given by (23) with

$$\bar{U}^{cb}_t = \mathbb{1}_{\{t < \bar{\tau}\}} \infty + \mathbb{1}_{\{t \ge \bar{\tau}\}} U^{cb}_t, \quad t \in [0, T],$$

where $(U^{cb}_t)_{t \in [0,T]}$ is a càdlàg process that is required to satisfy the following inequality

$$U^{cb}_t \ge \bar{C} \vee \kappa S_t + A_t, \quad t \in [0, T]. \tag{27}$$

The financial interpretation of the process U^{cb} is that U^{cb}_t represents the value of our reduced convertible bond upon a call at time t. As we shall see in what follows, one of the major problems arising here is that an arbitrage price of a CB upon call is not uniquely defined, unless we place ourselves in the framework of a complete market model.

Note that the only difference between Definitions 4.2 and 4.4 is that the latter definition postulates that the process U^{cb} is exogenous and satisfies inequality (27), whereas in the former this process is given by equality (24). It is thus trivial to observe that a CB with no call notice period is an RB. The same remark applies to a *puttable bond* (PB), that is, a CB with no call clause (formally, we set $\bar{\tau} = T$; so a puttable bond is a special case of an AO).

Since an RB is a CS, hence a GO, in order to obtain a characterization of an arbitrage price of an RB, it suffices to make use of Theorem 3.1, with the properly modified notation.

In the next section, we shall examine a method of interpreting and valuing a CB with positive call notice period as an RB, based on an endogenous specification of the random variable U^{cb}_t as arbitrage price of a certain PB starting at time t.

4.5 Valuation of a Convertible Bond upon Call

It follows from (25) and (26) (see also Kwok and Lau [19]) that when a convertible bond with a positive call notice period is called at some date t, it is actually replaced by a puttable bond. This particular puttable bond, which is referred to as the t-PB in what follows, is endowed with the same characteristics as a considered CB, except that:
(i) the inception date of the t-PB is t, its maturity is $t^\delta = (t+\delta) \wedge T$, and its nominal is equal to the call payment $\bar{C}$,
(ii) the coupon schedule of the t-PB is the trace on $(t, t^\delta]$ of the coupon schedule of a CB,
(iii) the effective put/conversion payment of the t-PB is equal to the effective call/conversion payment $\bar{C} \vee \kappa S_u + A_u$ of a CB, at any date $u \in [t, t^\delta)$.

In (ii), we excluded t from the coupon schedule of the t-PB, because any coupon falling at call time is already paid to the bond holder via the CB.

Definition 4.5 A *t-PB* is a puttable bond with the inception date t, the maturity date t^δ, and the ex-dividend cumulative cash flow $\bar{\pi}^t_u(\tau_p)$, $u \in [t, t^\delta]$, given by

$$\beta_u \bar{\pi}^t_u(\tau_p) = \beta_{\tau_p} \widehat{D}^{cb}_{\tau_p} - \beta_u \widehat{D}^{cb}_u + \mathbb{1}_{\{\tau_d > \tau_p\}} \beta_{\tau_p} \big(\bar{C} \vee \kappa S_{\tau_p} + A_{\tau_p} \big),$$

where τ_p belongs to $\mathcal{G}^u_{t^\delta}$, that is, τ_p is a $\mathbb{G}$-stopping time taking values in $[u, t^\delta]$.

As we already noted, any PB can be seen as an example of an AO and a CS. Hence we may apply Theorem 3.2 and (19) in order to establish the following auxiliary result.

Proposition 4.1 *Assuming (20), let us fix $t \in [0,T]$. If $(\bar{\Pi}^t_u)_{u \in [t,t^\delta]}$ is an arbitrage-free price of the t-PB, then $\bar{\Pi}^t_t \geq \bar{C} \vee \kappa S_t + A_t$, on the event $\{\tau_d > t\}$.*

Proof. By part (i) in Theorem 3.2, there exists $\mathbb{Q} \in \mathcal{M}$ such that

$$\bar{\Pi}^t_t = \operatorname{esssup}_{\tau_p \in \mathcal{G}^t_{t^\delta}} \mathbb{E}_{\mathbb{Q}} \big(\bar{\pi}^t_t(\tau_p) \,\big|\, \mathcal{G}_t \big). \tag{28}$$

By considering the specific stopping time $\tau_p = t$ in the left-hand side of (28), we obtain the inequality $\bar{\Pi}^t_t \geq \bar{C} \vee \kappa S_t + A_t$ on the event $\{\tau_d > t\}$. □

Assume that for any $t \in [0,T]$, the price process $\bar{\Pi}^t$ is arbitrage-free. Since the price processes $(\bar{\Pi}^t_u)_{u \in [t,t^\delta]}$ of t-PBs constitute a whole family of processes indexed by $t \in [0,T]$, this assumption means that each of these price processes is arbitrage-free, in the sense of Definition 2.6 in [15].

It is not clear, however, whether the family of random variables $\bar{\Pi}^t_t$, $t \in [0,T]$, can be considered as a well-defined càdlàg process. Let us make a bold assumption that this is indeed the case. Then inequality (27) is manifestly satisfied by the process

$$U^{cb}_t = \mathbb{1}_{\{\tau_d > t\}} \bar{\Pi}^t_t + \mathbb{1}_{\{\tau_d \leq t\}} (\bar{C} \vee \kappa S_t + A_t).$$

In this way, we implement our conjecture of reducing the valuation problem for a CB with a positive call notice period to the already solved case of valuation of an RB.

5 Decompositions of Convertible Securities

We are ready to examine in more detail the optional features of convertible securities. To this end, we first introduce in Section 5.2 the pertinent decompositions of cash flows and prices of convertible securities with respect to some reference elementary security. In Section 5.4, we provide a (non-unique) decomposition of a reduced convertible bond into a *bond component* and a *game option component*. This representation allows us to give a definite meaning to commonly used terms of the 'CB spread' and the 'CB implied volatility' (see, for instance, Connolly [10]). To further motivate this point, let us consider some relevant market data (data provided by courtesy of Credit Agricole, Paris).

5.1 Spread and Implied Volatility of a Convertible Bond

Table 1 provides market quotes on CBs issued by the three companies of the CAC40 (French stock index) on May 10, 2005. The CB prices are Mid-Market Trading Euro Prices and *CB implied volatilities* (CB IV) are Offer-Side Implied Volatilities. In accordance with the French convention for quoting CBs, the bonds' nominals in Table 1 have been scaled by a factor κ^{-1}, so that the data in Table 1 correspond to a conversion ratio κ equal to 1. For instance, the price of the scaled Alcatel CB is equal to 17.42 euros. Immediate conversion would be for one share of stock priced at 8.39 euros, and the scaled nominal of the CB is equal to 16.18 euros.

CB	**Stock Price**	**Nominal**	**CB Price**	**Credit Spread**	**CB IV**
Alcatel 4.75% Jan-11	8.39	16.18	17.42	135 bp	30.2%
Pinault 2.50% Jan-08	77.80	90.97	93.98	65 bp	21.5%
Cap Gemini 2.00 % Jun-09	25.25	39.86	41.80	65 bp	33.9%

Table 1: CB data on names of the CAC40 on May 10, 2005

For comparison, Table 2 shows market quotes on the closest listed option for each case considered in Table 1. The 'closest listed option' means the listed vanilla option with strike and maturity as close as possible to the scaled nominal and to the 'CB expected life', i.e. the most likely time of call, put, conversion or default, as forecasted by financial analysts.

CB	**CB Expected Life**	**Option Strike and Expiry**	**Option IV**
Alcatel 4.75% Jan-11	Oct-10	13.0 Dec-09	30.7%
Pinault 2.50% Jan-08	Nov-07	90.0 Dec-07	20.5%
Cap Gemini 2.00 % Jun-09	May-09	40.0 Dec-08	35.6%

Table 2: CBs and the closest listed options

Investors are expected to use the information in Tables 1 and 2 to assess relative value of CBs and options, and to take positions as a consequence. For instance, in some circumstances traders used to say that buying a CB is a 'cheap way to buy volatility'. This means that in their view, the option component of a CB is 'cheaper' (has a lower Black-Scholes implied volatility) than the corresponding listed vanilla option. It is thus a bit surprising that, to the best of our knowledge, the exact meaning of a 'CB spread' and a 'CB implied volatility' (CB IV in Table 1) has not been discussed in the literature.

The decomposition of a CB into a bond and option components is well known in the 'exchange option' case when the conversion can only occur at maturity and there are no put or call clauses [21], but not in the general case of a defaultable CB with call and/or put covenants. In particular, at the intuitive level, it seems plausible that the strike of the option embedded into a general convertible bond is a floating strike equal to the current price of a defaultable bond. So, we conjecture that the implied volatilities for convertible bonds, as given in Table 1, are not directly comparable with the corresponding implied volatilities for the closest listed options, as given in Table 2. Of course, to examine this conjecture, we need to formally define the implied volatility of a convertible bond.

5.2 Decompositions of Cash Flows of a Convertible Security

Let us consider a convertible security corresponding to the data set $(D^{cs}, L, U, \bar{\tau}, \xi)$, as specified by Definition 3.3. Assume that we are given some *reference elementary security*, specifically, an ES in the sense of Definition 3.2, with the ex-dividend cumulative cash flow given by the expression

$$\beta_t \phi_t = \beta_T \widehat{D}^b_T - \beta_t \widehat{D}^b_t + \mathbb{1}_{\{\tau_d > T\}} \beta_T \xi^b, \tag{29}$$

where the dividend process D^b is supposed to have bounded variation on $[0, T]$ and the payment at maturity ξ^b is assumed to be bounded. The first goal is to describe the cash flow of the portfolio obtained by combining the long position in the convertible security with the short position in the reference ES (see formula (33) below).

Remarks 5.1 Assuming that D^{cs} is of bounded variation on $[0,T]$ and ξ is bounded, as the reference elementary security for the CS we may take the CS contract stripped of its game features, that is, the otherwise equivalent CS in which the only admissible decision times τ_p and τ_c are $\tau_p = \tau_c = T$. This is not, of course, the only possible choice for the reference ES, but in many instances this will be the most natural choice, provided that this reference security is indeed traded.

For any probability measure $\mathbb{Q} \in \mathcal{M}$, we define the process $\Phi_t = \mathbb{E}_{\mathbb{Q}}(\phi_t \,|\, \mathcal{G}_t)$ for $t \in [0,T]$. Note that by part (ii) in Theorem 3.2, the process Φ is actually an arbitrage price for the ES, associated with the probability measure $\mathbb{Q}$, that is, the $\mathbb{Q}$-price of the ES.

Lemma 5.1 (i) *The ex-dividend cumulative cash flow of the CS can be decomposed as follows:*

$$\pi_t(\tau_p,\tau_c) = \phi_t + \varphi_t(\tau_p,\tau_c), \quad t \in [0,T], \tag{30}$$

where ϕ_t is given by (29)*, and thus it represents the ex-dividend cumulative cash flow of the reference ES, and $\varphi_t(\tau_p,\tau_c)$ is given by the formula*

$$\begin{aligned} \beta_t\varphi_t(\tau_p,\tau_c) &= \beta_\tau(\widehat{D}^{cs}_\tau - \widehat{D}^b_\tau) - \beta_t(\widehat{D}^{cs}_t - \widehat{D}^b_t) \\ &\quad + \mathbb{1}_{\{\tau_d>\tau\}}\,\beta_\tau\Big(\mathbb{1}_{\{\tau=\tau_p<T\}}\,(L_{\tau_p} - \phi_{\tau_p}) + \mathbb{1}_{\{\tau<\tau_p\}}\,(U_{\tau_c} - \phi_{\tau_c}) + \mathbb{1}_{\{\tau=T\}}(\xi - \xi^b)\Big). \end{aligned} \tag{31}$$

(ii) *Let $\mathbb{Q}$ be any probability measure from $\mathcal{M}$. Then we have*

$$\mathbb{E}_{\mathbb{Q}}(\varphi_t(\tau_p,\tau_c) \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(\psi_t(\tau_p,\tau_c) \,|\, \mathcal{G}_t), \quad t \in [0,T], \tag{32}$$

where $\psi_t(\tau_p,\tau_c)$ is defined by

$$\begin{aligned} \beta_t\psi_t(\tau_p,\tau_c) &= \beta_\tau(\widehat{D}^{cs}_\tau - \widehat{D}^b_\tau) - \beta_t(\widehat{D}^{cs}_t - \widehat{D}^b_t) \\ &\quad + \mathbb{1}_{\{\tau_d>\tau\}}\,\beta_\tau\Big(\mathbb{1}_{\{\tau=\tau_p<T\}}\,(L_{\tau_p} - \Phi_{\tau_p}) + \mathbb{1}_{\{\tau<\tau_p\}}\,(U_{\tau_c} - \Phi_{\tau_c}) + \mathbb{1}_{\{\tau=T\}}(\xi - \xi^b)\Big). \end{aligned} \tag{33}$$

Proof. The decomposition of cash flows stated in part (i) is straightforward. For part (ii), we recall from the proof of part (ii) in Theorem 3.2 that

$$\beta_t\Phi_t = \mathbb{E}_{\mathbb{Q}}(\beta_T\widehat{D}^b_T + \mathbb{1}_{\{\tau_d>T\}}\beta_T\xi^b \,|\, \mathcal{G}_t) - \beta_t\widehat{D}^b_t,$$

and thus

$$\beta_{\tau_p}\Phi_{\tau_p} = \mathbb{E}_{\mathbb{Q}}(\beta_T\widehat{D}^b_T + \mathbb{1}_{\{\tau_d>T\}}\beta_T\xi^b \,|\, \mathcal{G}_{\tau_p}) - \beta_{\tau_p}\widehat{D}^b_{\tau_p} = \mathbb{E}_{\mathbb{Q}}(\beta_{\tau_p}\phi_{\tau_p} \,|\, \mathcal{G}_{\tau_p}),$$

where the first equality follows from Doob's optional sampling theorem, and the second follows from the definition of ϕ. Hence, by taking iterated conditional expectations, we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau<\tau_d\}}\mathbb{1}_{\{\tau=\tau_p<T\}}\beta_\tau\phi_{\tau_p} \,|\, \mathcal{G}_t) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_p<\tau_d\}}\mathbb{1}_{\{\tau=\tau_p<T\}}\beta_{\tau_p}\phi_{\tau_p} \,\big|\, \mathcal{G}_{\tau_p}\big) \,\Big|\, \mathcal{G}_t\Big) \\ &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau_p<\tau_d\}}\mathbb{1}_{\{\tau=\tau_p<T\}}\beta_{\tau_p}\,\mathbb{E}_{\mathbb{Q}}\big(\phi_{\tau_p} \,\big|\, \mathcal{G}_{\tau_p}\big) \,\Big|\, \mathcal{G}_t\Big) \\ &= \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau<\tau_d\}}\mathbb{1}_{\{\tau=\tau_p<T\}}\beta_\tau\Phi_{\tau_p} \,\big|\, \mathcal{G}_t\big) \end{aligned}$$

where we have used the fact that the random variable $\mathbb{1}_{\{\tau_p<\tau_d\}}\mathbb{1}_{\{\tau=\tau_p<T\}}\beta_{\tau_p}$ is $\mathcal{G}_{\tau_p}$-measurable. Using the same arguments, we also get $\beta_{\tau_c}\Phi_{\tau_c} = \mathbb{E}_{\mathbb{Q}}(\beta_{\tau_c}\phi_{\tau_c} \,|\, \mathcal{G}_{\tau_c})$ and thus

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{\tau<\tau_d\}}\mathbb{1}_{\{\tau<\tau_p\}}\beta_\tau\phi_{\tau_c} \,|\, \mathcal{G}_t) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tau_c<\tau_d\}}\mathbb{1}_{\{\tau<\tau_p\}}\beta_{\tau_c}\,\mathbb{E}_{\mathbb{Q}}\big(\phi_{\tau_c} \,\big|\, \mathcal{G}_{\tau_c}\big) \,\Big|\, \mathcal{G}_t\Big) \\ &= \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau<\tau_d\}}\mathbb{1}_{\{\tau<\tau_p\}}\beta_\tau\Phi_{\tau_c} \,\big|\, \mathcal{G}_t\big). \end{aligned}$$

It is now easily seen that equality (32) is valid. □

Assume that we are given a CS and we have already chosen some reference ES. Then we define the $\mathbb{Q}$-*exchange CS*, as the CS with dividend process $D^{cs} - D^b$, put payment $L - \Phi$, call payment $U - \Phi$, call protection lifting time $\bar{\tau}$, and payment at maturity $\xi - \xi^b$. In other words, the $\mathbb{Q}$-exchange CS is the CS with ex-dividend cumulative cash flow $\psi_t(\tau_p,\tau_c)$ given by (33).

Remarks 5.2 (i) Note that, for a given CS, the cash flow of the $\mathbb{Q}$-exchange CS depends not only on the choice of the reference ES, but also on the choice of a probability measure $\mathbb{Q} \in \mathcal{M}$, through the definition of the $\mathbb{Q}$-price process Φ of the ES.
(ii) Since the process Φ is an arbitrage price for the reference ES, the $\mathbb{Q}$-exchange CS has the financial interpretation as the game option to exchange the reference ES for either L or U (as seen from the perspective of the holder), according to which player decides first to stop this game. This interpretation is particularly transparent when the reference ES is specified as in Remarks 5.1, since in that case (33) reduces to

$$\beta_t \psi_t(\tau_p, \tau_c) = \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} \left(L_{\tau_p} - \Phi_{\tau_p} \right) + \mathbb{1}_{\{\tau < \tau_p\}} \left(U_{\tau_c} - \Phi_{\tau_c} \right) \Big). \tag{34}$$

The contract with the cash flow given by the last formula can be seen as the pure game option component of the CS.

5.3 Price Decompositions of a Convertible Security

We are now in the position to derive the price decomposition of a convertible security with respect to some reference security. We assume that we are given a CS and the reference ES, as described in the previous section. The following result follows easily from Theorem 3.1 and Lemma 5.1.

Theorem 5.1 *Assuming (20), let $\mathbb{Q} \in \mathcal{M}$ be given and let Φ be the arbitrage $\mathbb{Q}$-price of the reference ES.*
(i) *If Π is an arbitrage $\mathbb{Q}$-price for the CS then $\Psi = \Pi - \Phi$ is an arbitrage $\mathbb{Q}$-price for the $\mathbb{Q}$-exchange CS.*
(ii) *If Ψ is an arbitrage $\mathbb{Q}$-price for the $\mathbb{Q}$-exchange CS then $\Pi = \Phi + \Psi$ is an arbitrage $\mathbb{Q}$-price for the CS.*

Proof. Let us prove (i). Using (14) and (30), we obtain

$$\Psi_t = \Pi_t - \Phi_t = \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}} \big(\pi_t(\tau_p, \tau_c) \,\big|\, \mathcal{G}_t \big) - \mathbb{E}_{\mathbb{Q}} \big(\phi_t \,\big|\, \mathcal{G}_t \big) =$$
$$\operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}} \big(\varphi_t(\tau_p, \tau_c) \,\big|\, \mathcal{G}_t \big) = \operatorname{esssup}_{\tau_p \in \mathcal{G}_T^t} \operatorname{essinf}_{\tau_c \in \bar{\mathcal{G}}_T^t} \mathbb{E}_{\mathbb{Q}} \big(\psi_t(\tau_p, \tau_c) \,\big|\, \mathcal{G}_t \big),$$

where the last equality is a consequence of (32). Moreover, the difference $\Psi = \Pi - \Phi$ is obviously a $\mathbb{G}$-semimartingale. Thus Ψ is an arbitrage $\mathbb{Q}$-price for the $\mathbb{Q}$-exchange CS, by the 'if' part of Theorem 3.1. The proof of part (ii) is similar to that of part (i). □

Let us stress that Ψ need not be positive, and thus Π need not be greater than Φ, in general. We have, however, the following result.

Corollary 5.1 *Under the assumptions of Theorem 5.1(i) or 5.1(ii), if the process $D^{cs} - D^b$ is non-decreasing, $U \geq \Phi$ on $[\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$, and $\xi \geq \xi^b$ then $\Psi \geq 0$ and thus $\Pi \geq \Phi$.*

Proof. Let us show that at any time $t < \tau_d \wedge T$ such that $L_t < \Phi_t$, exchanging the ES for the payoff L_t is suboptimal for the holder of the $\mathbb{Q}$-exchange CS. Towards this end, we define $\breve{\psi}_t(\tau_p, \tau_c)$ by the formula

$$\beta_t \breve{\psi}_t(\tau_p, \tau_c) = \beta_\tau (\widehat{D}_\tau^{cs} - \widehat{D}_\tau^b) - \beta_t (\widehat{D}_t^{cs} - \widehat{D}_t^b)$$
$$+ \mathbb{1}_{\{\tau_d > \tau\}} \beta_\tau \Big(\mathbb{1}_{\{\tau = \tau_p < T\}} \big(L_{\tau_p} - \Phi_{\tau_p} \big)^+ + \mathbb{1}_{\{\tau < \tau_p\}} \big(U_{\tau_c} - \Phi_{\tau_c} \big) + \mathbb{1}_{\{\tau = T\}} (\xi - \xi^b) \Big)$$

and we denote $\breve{\mathcal{G}}_T^t = \{\tau \in \mathcal{G}_T^t \,;\, L_\tau \geq \Phi_\tau \text{ if } \tau < T\}$. For any $\tau_p \in \mathcal{G}_T^t$, the stopping time $\breve{\tau}_p$, given by the formula

$$\breve{\tau}_p = \mathbb{1}_{\{L_{\tau_p} \geq \Phi_{\tau_p}\}} \tau_p + \mathbb{1}_{\{L_{\tau_p} < \Phi_{\tau_p}\}} T,$$

belongs to $\breve{\mathcal{G}}_T^t$. Since the process $D^{cs} - D^b$ is non-decreasing we have that the process $\beta_t (\widehat{D}_t^{cs} - \widehat{D}_t^b))$ is non-decreasing as well. We assumed also that $U \geq \Phi$ on $[\tau_d \wedge \bar{\tau}, \tau_d \wedge T)$ and $\xi \geq \xi^b$, so that the following inequalities hold, for any $\tau_p \in \mathcal{G}_T^t$ and $\tau_c \in \bar{\mathcal{G}}_T^t$,

$$\psi_t(\breve{\tau}_p, \tau_c) \geq \psi_t(\tau_p, \tau_c), \quad \breve{\psi}_t(\breve{\tau}_p, \tau_c) \geq \breve{\psi}_t(\tau_p, \tau_c).$$

Since obviously $\breve{\psi}_t(\tau_p,\tau_c) = \psi_t(\tau_p,\tau_c)$ for any $\tau_p \in \breve{\mathcal{G}}^t_T$, we obtain

$$\begin{aligned}
&\mathrm{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathrm{esssup}_{\tau_p\in\mathcal{G}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\breve{\psi}_t(\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) = \mathrm{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathrm{esssup}_{\tau_p\in\breve{\mathcal{G}}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\breve{\psi}_t(\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)\\
&\quad = \mathrm{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathrm{esssup}_{\tau_p\in\breve{\mathcal{G}}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\psi_t(\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big) = \mathrm{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathrm{esssup}_{\tau_p\in\mathcal{G}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\psi_t(\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)
\end{aligned}$$

and thus

$$\begin{aligned}
\Psi_t &= \mathrm{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathrm{esssup}_{\tau_p\in\mathcal{G}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\psi_t(\tau_p,\tau_c)\,\big|\,\mathcal{G}_t\big)\\
&= \mathrm{esssup}_{\tau_p\in\mathcal{G}^t_T}\mathrm{essinf}_{\tau_c\in\bar{\mathcal{G}}^t_T}\mathbb{E}_{\mathbb{Q}}\big(\breve{\psi}_t(\tau_p,\tau_c)\big|\,\mathcal{G}_t\big) \geq 0.
\end{aligned}$$

We conclude that $\Pi_t = \Phi_t + \Psi_t \geq \Phi_t$. □

5.4 Price Decompositions of a Reduced Convertible Bond

We shall now specialize our previous results to the case of a reduced convertible bond (hence, in particular, to the case of a convertible bond without call notice period). We thus postulate that the dividend process is of the form D^{cb} given by (21), that is,

$$D^{cb}_t = \int_0^{t\wedge\tau_d} c^{cb}_u\,du + \sum_{0\leq T_i\leq t,\,T_i<\tau_d} c_i + \mathbb{1}_{\{0\leq\tau_d\leq t\}}R^{cb}_{\tau_d}, \quad t\in[0,T].$$

In order to provide the most pertinent price decomposition of the RB, we choose as the reference instrument the ES with dividend process

$$D^{b}_t = \int_0^{t\wedge\tau_d} c^{cb}_u\,du + \sum_{0\leq T_i\leq t,\,T_i<\tau_d} c_i + \mathbb{1}_{\{0\leq\tau_d\leq t\}}R^{b}_{\tau_d}, \quad t\in[0,T],$$

that is, the ES with the same coupon process as the RB, and with R^b and ξ^b given as follows (see Section 4.1):

$$R^b_t = \bar{R}_t, \quad \xi^b = \bar{N} + A_T. \tag{35}$$

It is thus clear that

$$R^{cb}_t - R^b_t = (\kappa S_t - \bar{R}_t)^+ \geq 0, \quad \xi^{cb} - \xi^b = (\kappa S_T - \bar{N})^+ \geq 0.$$

So, in this case the reference security is the defaultable bond with ex-dividend cumulative cash flow ϕ_t given by the expression

$$\begin{aligned}
\beta_t\phi_t &= \beta_T\widehat{D}^b_T - \beta_t\widehat{D}^b_t + \mathbb{1}_{\{\tau_d>T\}}\,\beta_T\xi^b\\
&= \int_t^{T\wedge\tau_d}\beta_u c^{cb}_u\,du + \sum_{t<T_i\leq T,\,T_i<\tau_d}\beta_{T_i}c_i + \mathbb{1}_{\{t<\tau_d\leq T\}}\,\beta_{\tau_d}R^b_{\tau_d} + \mathbb{1}_{\{\tau_d>T\}}\,\beta_T\xi^b.
\end{aligned}$$

It is clear that this reference bond can be interpreted as the pure bond component of the RB, that is, the RB stripped of its optional clauses. Therefore, we shall call it the *bond embedded in the RB*, or simply the *embedded bond*. Given a probability measure $\mathbb{Q}\in\mathcal{M}$, the process $\Phi_t = \mathbb{E}_{\mathbb{Q}}(\phi_t\,|\,\mathcal{G}_t)$ is the $\mathbb{Q}$-price of the embedded bond.

Since the RB and the embedded bond have the same coupon schedule, the $\mathbb{Q}$-exchange CS is the zero-coupon CS, with the ex-dividend cumulative cash flow $\psi_t(\tau_p,\tau_c)$ given by the expression

$$\begin{aligned}
\beta_t\psi_t(\tau_p,\tau_c) &= \mathbb{1}_{\{t<\tau_d\leq\tau\}}\,\beta_{\tau_d}\big(R^{cb}_{\tau_d} - R^b_{\tau_d}\big)\\
&\quad + \mathbb{1}_{\{\tau_d>\tau\}}\,\beta_\tau\Big(\mathbb{1}_{\{\tau=\tau_p<T\}}\big(L^{cb}_{\tau_p} - \Phi_{\tau_p}\big) + \mathbb{1}_{\{\tau<\tau_p\}}\big(\bar{U}^{cb}_{\tau_c} - \Phi_{\tau_c}\big) + \mathbb{1}_{\{\tau=T\}}\big(\xi^{cb} - \xi^b\big)\Big).
\end{aligned}\tag{36}$$

This particular $\mathbb{Q}$-exchange CS will be referred to as the *embedded game $\mathbb{Q}$-exchange option*. As for any $\mathbb{Q}$-exchange CS (see Section 5.2), the cash flow of the embedded game $\mathbb{Q}$-exchange option depend on the choice of the probability measure $\mathbb{Q}$ through the definition of the price process Φ of the embedded bond.

The embedded game $\mathbb{Q}$-exchange option has the natural interpretation as a game option to exchange the embedded bond for either L^{cb} or U^{cb}, according to who decides first to stop this game. In the case of default, the contract stipulates that the recovery payoffs of the RB and the embedded bond are exchanged. Finally, if the embedded game contract is still alive at maturity date T, the terminal payoffs are exchanged.

Let Π and Ψ stand for the arbitrage $\mathbb{Q}$-price of the RB and the embedded game $\mathbb{Q}$-exchange option, respectively. Due to the game-theoretical features of the embedded option, Ψ need not be positive and thus Π need not be greater than Φ, in general.

Theorem 5.1(i) or 5.1(ii) can be directly applied to this specification of an RB. Since an explicit expression for the price decomposition of an RB with respect to the embedded bond is easy to obtain, it is not reported here. The following result is worth stating, however.

Corollary 5.2 *Consider an RB and the embedded bond with R^b and ξ^b given by (35). Then, under the assumptions of Theorem 5.1(i) or 5.1(ii), in the special case of a zero-coupon RB ($c^{cb} = 0$ and $c_i = 0$ for any i), assuming that the process β is non-increasing and that $\bar{R} \le \bar{N}$, we then have that $\Psi \ge 0$ and thus $\Pi \ge \Phi$.*

Proof. Under the present assumptions we have by (27)

$$\Phi_t \le \bar{N} \le \bar{C} \le U_t^{cb}, \quad \tau_d \wedge \bar{\tau} \le t < \tau_d \wedge T,$$

so that the inequality $\Pi \ge \Phi$ is an immediate consequence of Corollary 5.1. □

Remarks 5.3 (i) The possibility of the negative value of Ψ is related to the fact that we consider a **callable** RB, but as the reference security we choose the **non-callable** embedded bond. Hence, the value of the reference bond can be greater than the call price at the moment of call of the callable RB. In other words, the price of a callable and convertible bond can be either greater or less than the price of an equivalent non-callable and non-convertible bond.

It thus would be interesting to take as the reference security the callable version of the embedded bond. In that case, one would expect to have the positive value for the embedded game option, since this game option should reduce to a vulnerable American option with non-negative payoffs at default and at maturity.

(ii) Under the assumptions of Corollary 5.2, the reference zero-coupon bond is equivalent to a callable zero-coupon bond with the same nominal value $\bar{N}$ and call price $\bar{C} \ge \bar{N}$ (since in fact a callable zero-coupon bond with the call price $\bar{N} \ge \bar{C}$ will never be called if interest rates are non-negative). By contract, if we deal with a coupon-paying RB this argument breaks down, since now that assumption $\bar{N} \ge \bar{C}$ does not ensure that the callable version of the reference bond will never be called.

We can now formulate the definition of the spread and the implied volatility of an RB.

Definition 5.1 Let us consider an RB, under the assumptions of Theorem 5.1(i) or (ii). By the *RB spread* we mean the credit spread consistent with the price Φ for the embedded bond. By an *RB implied volatility* we mean any Black–Scholes volatility of the stock price process S, which is consistent with the price Ψ for the embedded game exchange option.

Remarks 5.4 (i) Note that it is not necessarily possible to map every possible arbitrage price process for the game exchange option to a well-defined and unique Black-Scholes implied volatility process.

(ii) The embedded game exchange option of an RB can be thought of as an equity option, but with a floating strike, equal at any date t to the current value Φ_t of the embedded bond. This clarifies the intuitive statements made at the beginning of this section, and confirms our conjecture that the implied volatility of a CB (when properly defined) in Table 1 and the implied volatility for the closest listed option in Table 2 are in fact of a quite different nature.

6 Conclusions

As announced in the introduction, this note provides a theoretical underpinning for a more extensive research continued in Bielecki et al. [5, 6], where more specific market models are introduced, and more explicit valuation and hedging results are established. Some open problems remain, however.

Most notably, it is not clear whether it is possible to generalize Proposition 2.1 to the case of extended game options, such as the ones that arise naturally in the study of real-life convertible bond with positive call notice period. In the present paper and the follow-up works, we develop an alternative approach based on a conjecture that the valuation of such a convertible bond can be done recursively. However, this conjecture remains to be justified, in general.

One of the major problems arising in this context is that an arbitrage price of such a convertible bond upon call is not uniquely defined, unless we work in the framework of a complete market model. Consequently, it is not clear whether the pricing measures associated to arbitrage prices of the convertible bond and of the embedded puttable bonds have to be the same, even if one assumes that all the instruments involved are liquidly traded. In the same line of thinking, the decomposition of a convertible bond as the portfolio comprising a bond and an option can only be used under an ad hoc assumption that the same pricing measure is used to value the convertible bond, the embedded bond and the embedded game exchange option. Since this is not a priori necessarily the case when one deals with an incomplete market model, a further study of this issue is needed.

In subsequent papers [5, 6], we shall see how these various questions can be solved, in the context of more specific market models.

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