Critical exponents of graphs

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Working example

Question. Suppose

$$A = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}.$$

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Raise each entry to the αth power for some $\alpha > 0$.

When is the resulting matrix positive semidefinite?

Graphical models: Connections between statistics and combinatorics.

Let $X_1, \ldots, X_p$ be a collection of random variables.

- In a very large vector, it is rare that all the variables depend strongly on each other.

Motivation from high-dimensional statistics

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Important problem: Estimate Σ given data $x_1, \dots, x_n \in \mathbb{R}^p$ of $(X_1, \dots, X_p)$.

Covariance estimation

Classical estimator (sample covariance matrix):

$$S := \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \overline{x})(x_j - \overline{x})^T.$$

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Modern approach: Convex optimization: obtain *sparse* estimate of Σ (e.g., penalized likelihood methods)

- Works well for dimensions up to a few thousands.
- Does not scale to modern problems with $100,000+$ variables.

Covariance estimation (cont.)

Alternate approach: *Thresholding covariance matrices*

$$\text{True } \Sigma = \begin{pmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0.95 & 0.18 & 0.02 \\ 0.18 & 0.96 & 0.47 \\ 0.02 & 0.47 & 0.98 \end{pmatrix}$$

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- **Question:** When does this procedure preserve positive (semi)definiteness?

Critical for applications, since covariance matrices are positive semidefinite.

Entrywise functions preserving positivity

More generally: Given a function $f : I \to \mathbb{R}$, when is it true that

$$f[A] := (f(a_{jk})) \in \mathbb{P}_n \text{ for all } A \in \mathbb{P}_n(I)?$$

Problem: What kind of functions have this property?

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  (Applications use functions such as hard- and soft-thresholding, and powers, to regularize covariance matrices.)

Question: Which power functions applied entrywise preserve positivity on $\mathbb{P}_n$ (for fixed n)?

Powers preserving positivity

Theorem (FitzGerald and Horn, J. Math. Anal. Appl. 1977)

Let $n \geq 2$. *Then:*

1. $f(x) = x^\alpha$ *preserves positivity on* $\mathbb{P}_n((0,\infty))$ *if* $\alpha \geq n-2$.

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In other words, $f(x) = x^\alpha$ *preserves positivity on* $\mathbb{P}_n((0,\infty))$ *if and only if* $\alpha \in \mathbb{N} \cup [n-2, \infty)$.

Critical exponent: $n-2 =$ smallest α_0 such that $\alpha \geq \alpha_0$ preserves positivity.

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So for $A = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$, all powers $\alpha \in \mathbb{N} \cup [3, \infty)$ work.

Can we do better?

Matrices with structures of zeros

Refine the FitzGerald–Horn problem for matrices with zeros.

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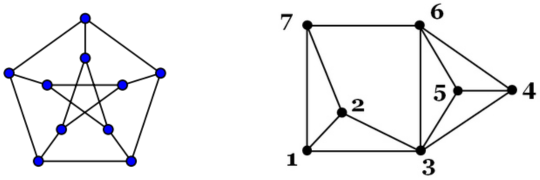
A *graph* $G = (V, E)$ is a set of *vertices* V and *edges* $E \subset V \times V$:



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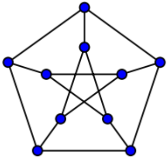
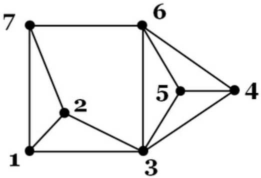
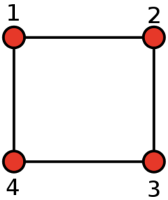




The *pattern of zeros* of a symmetric $n \times n$ matrix is naturally encoded by a graph on $V = \{1, 2, \ldots, n\}$:

$$\text{Edge between } j \text{ and } k \quad \Longleftrightarrow \quad a_{jk} \neq 0.$$

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 3 & 1 & 5 & 0 \\ 0 & 5 & 1 & 4 \\ 2 & 0 & 4 & 1 \end{pmatrix} \quad \longleftrightarrow$$



The cone $\mathbb{P}_G$

Given a graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ we define a subset of $\mathbb{P}_n$ by

$$\mathbb{P}_G := \{A \in \mathbb{P}_n : a_{jk} = 0 \text{ if } (j,k) \notin E \text{ and } j \neq k\}.$$

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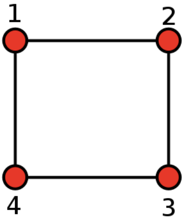

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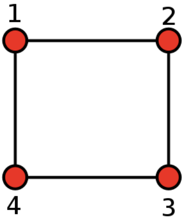

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Define the set of powers preserving positivity for G:

$$\mathcal{H}_G := \{\alpha \geq 0 : A^{\circ\alpha} \in \mathbb{P}_G \text{ for all } A \in \mathbb{P}_G([0,\infty))\}$$

$CE(G) :=$ smallest α_0 s.t. x^α preserves positivity on $\mathbb{P}_G, \forall \alpha \geq \alpha_0$.

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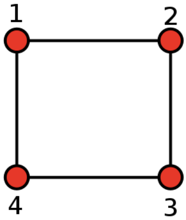

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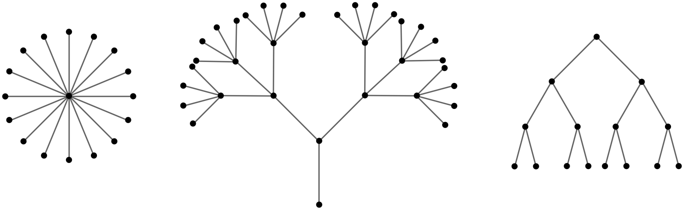
Problem: How does the structure of G relate to the set of powers preserving positivity? (FitzGerald-Horn studied the case $G = K_n$.)

A first example: trees

Definition: A tree is a connected graph containing no cycles.



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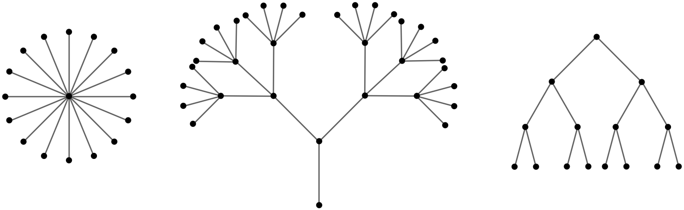

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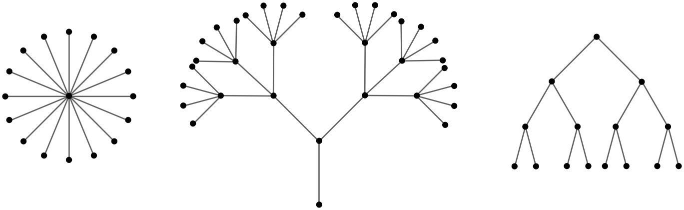

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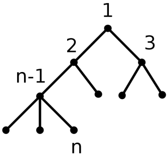
The proof uses induction on n, and Schur complements:

$$S_{M^{\circ\alpha}} - (S_M)^{\circ\alpha} \in \mathbb{P}_{n-1}.$$



$$M := \left(\begin{array}{ccc|c} & \mathbf{A}_{(n-1)\times(n-1)} & & \begin{matrix} 0 \\ \vdots \\ 0 \\ a_{n-1,n} \end{matrix} \\ \hline 0 & \cdots & 0 \;\; a_{n-1,n} & a_{nn} \end{array} \right)$$

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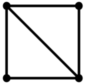
Answer: No. Counterexample: $G = K_4^{(1)}$ (K_4 minus a chord).



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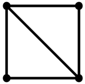
Answer: No. Counterexample: $G = K_4^{(1)}$ (K_4 minus a chord).



$K_4^{(1)}$

Clearly, the maximal clique is K_3. However, we can show that $\mathcal{H}_{K_4^{(1)}} = \{1\} \cup [2, \infty)$.

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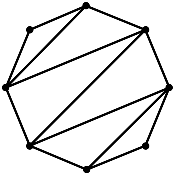

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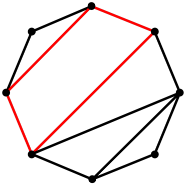

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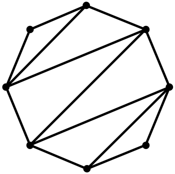

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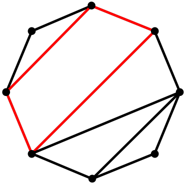

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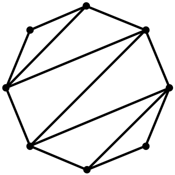

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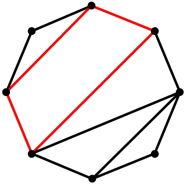

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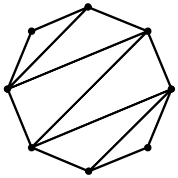

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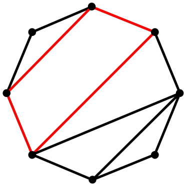

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- *Names:* Triangulated, decomposable, rigid circuit graphs...
- *Examples:* Trees, Complete graphs, Triangulation of any graph, Apollonian graphs, Band graphs, Split graphs, etc.
- Occur in many *applications*: positive definite completion problems, maximum likelihood estimation in graphical models, Gaussian elimination, etc.

Preserving positivity for chordal graphs

Theorem (Guillot, Khare, Rajaratnam, JCT-A 2015)

Let G be any chordal graph with at least 2 vertices and let r be the largest integer such that either K_r or $K_r^{(1)}$ is an induced subgraph of G. Then

$$\mathcal{H}_G = \mathbb{N} \cup [r-2, \infty).$$

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E.g., for band graphs with bandwidth d, $CE(G) = \min(d, n-2)$.

So for $A = \begin{pmatrix} 1 & 0.6 & 0.5 & 0 & 0 \\ 0.6 & 1 & 0.6 & 0.5 & 0 \\ 0.5 & 0.6 & 1 & 0.6 & 0.5 \\ 0 & 0.5 & 0.6 & 1 & 0.6 \\ 0 & 0 & 0.5 & 0.6 & 1 \end{pmatrix}$, all powers $\geq 2 = d$ work.

Preserving positivity for chordal graphs (cont.)

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$$f[A+B] - (f[A] + f[B]) \in \mathbb{P}_n \qquad \forall A, B \in \mathbb{P}_n.$$

Loewner super-additive powers classified in [Guillot, Khare, Rajaratnam], J. Math. Anal. Appl., 2015.

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3. Induction and properties of chordal graphs (decomposition, ordering of cliques, etc.).

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Theorem (Guillot, Khare, Rajaratnam, JCT-A 2015)

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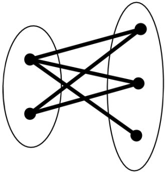
Suppose G *is a connected bipartite graph with at least* 3 *vertices. Then* $\mathcal{H}_G = [1, \infty)$.



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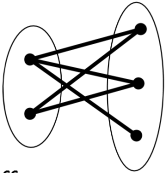

Proof uses a completely different approach based on the fact that,

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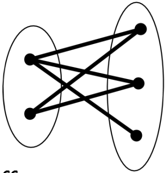

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5. Connections to other (purely combinatorial) graph invariants?

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Papers available at:
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