Extremal Positive Semidefinite Matrices for graphs without K_5 minors

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Outline

1. Series-Parallel Graphs
2. Three Convex Bodies
3. Facet-Ray Identification Property
4. Open problems

Series-Parallel Graph

Definition

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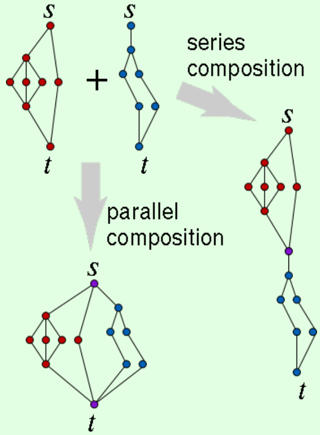

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Cut Polytopes

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A **cut** of the graph G is a bipartition of the vertices, (U, U^c), and its associated **cutset** is the collection of edges $\delta(U) \subset E$ with one endpoint in each block of the bipartition. To each cutset we assign a (± 1)-vector in $\mathbb{R}^E$ with a -1 in coordinate e if and only if $e \in \delta(U)$. The (± 1)-cut polytope of G is the convex hull in $\mathbb{R}^E$ of all such vectors.

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- Maximizing over the polytope $\text{cut}^{\pm 1}$ (G) is equivalent to solving the max-cut problem for G.
- The max-cut problem is known to be NP-hard.
- However, it is possible to optimize in polynomial time over a (often times non-polyhedral) positive semidefinite relaxation of $\text{cut}^{\pm 1}$ (G), known as an **elliptope**.

Elliptopes

Elliptopes

Let $\mathbb{S}^p$ denote the real vector space of all real $p \times p$ symmetric matrices, and let $\mathbb{S}^p_{\succeq 0}$ denote the cone of all positive semidefinite matrices in $\mathbb{S}^p$. The ***p*-elliptope** is the collection of all $p \times p$ correlation matrices, i.e.

$$\mathcal{E}_p = \{X \in \mathbb{S}^p_{\succeq 0} | X_{ii} = 1 \text{ for all } i \in [p]\}.$$

The elliptope $\mathcal{E}_G$ is defined as the projection of $\mathcal{E}_p$ onto the edge set of G. That is,

$$\mathcal{E}_G = \{\mathbf{y} \in \mathbb{R}^{\mathbf{E}} | \ \exists Y \in \mathcal{E}_p \text{ such that } Y_e = y_e \text{ for every } e \in E(G)\}.$$

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Notes

The elliptope $\mathcal{E}_G$ is a positive semidefinite relaxation of the cut polytope $\text{cut}^{\pm 1}$ (G), and thus maximizing over $\mathcal{E}_G$ can provide an approximate solution to the max-cut problem.

Cone of Concentration Matrices

Concentration Matrices

Consider the **Graphical Gaussian model** $N(\mu, \Sigma)$ where $\mu \in \mathbb{R}^p$ is the mean and $\Sigma \in \mathbb{R}^{p \times p}$ is the correlation matrix for the model. The concentration matrix of Σ is $K = \Sigma^{-1}$.

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Definition

The **sparsity order** of G is defined as the maximum rank of an extremal matrix in $\mathcal{K}_G$.

$\text{cut}^{\pm 1}$ (G) and $\mathcal{K}_G$

Goal

Want to show that the facets of $\text{cut}^{\pm 1}$ (G) identify extremal rays of $\mathcal{K}_G$ for any graph G without K_5 minor and to compute the sparsity order of any series-parallel graph G.

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Show these identifications arises via the geometric relationship that exists between the three convex bodies $\text{cut}^{\pm 1}$ (G), $\mathcal{E}_G$, and $\mathcal{K}_G$.

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Theorem [Solus, Uhler, Y. 2015]

The dual body of the elliptope $\mathcal{E}_G$ is

$$\mathcal{E}_G^{\vee} = \{x \in \mathbb{R}^E \mid \exists\, X \in \mathcal{K}_G \text{ such that } X_E = x \text{ and } \operatorname{tr}(X) = 2\}.$$

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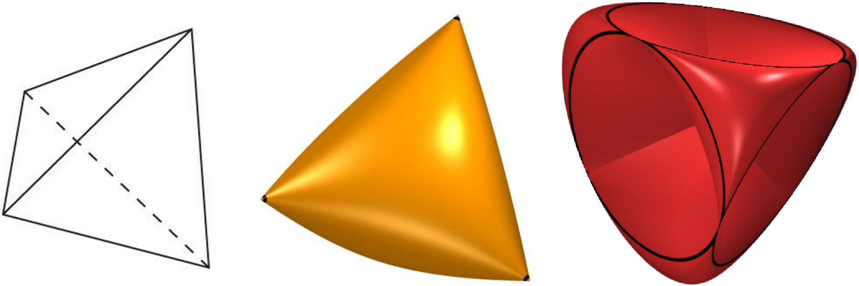
Example: 3-cycle



(a) $CUT^{\pm 1}(G)$ (b) $\mathcal{E}_G$ (c) $\mathcal{E}_G^V$

$$\mathrm{CUT}^{\pm 1}(G) = \mathrm{conv}((1,1,1),(-1,-1,1),(-1,1,-1),(1,-1,-1))$$

$$\mathcal{E}_G = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ x_1 & 1 & x_2 \\ x_3 & x_2 & 1 \end{pmatrix} \succeq 0 \right\} \qquad \mathcal{E}_G^V = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} : \begin{pmatrix} a & y_1 & y_3 \\ y_1 & b & y_2 \\ y_3 & y_2 & 2-a-b \end{pmatrix} \succeq 0 \right\}$$

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- The PD-completability problem would become easier for G with smaller sparsity order (i.e. where the max rank of an extremal ray is small).
- Our computations of the facets of $\text{cut}^{\pm 1}$ (G) for G series-parallel together with the proof of facet-ray identification tells us all these ranks are encoded nicely in the supporting hyperplanes of $\text{cut}^{\pm 1}$ (G).

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Facet-Ray Identification Property

Definition

Let G be a graph. For each facet F of $\text{cut}^{\pm 1}(G)$ let $\alpha^F \in \mathbb{R}^E$ denote the normal vector to the supporting hyperplane of F. We say that G has the **facet-ray identification property** (or FRIP) if for every facet F of $\text{cut}^{\pm 1}(G)$ there exists an extremal matrix $M = [m_{ij}]$ in $\mathcal{K}_G$ for which either $m_{ij} = \alpha^F_{ij}$ for every $\{i, j\} \in E(G)$ or $m_{ij} = -\alpha^F_{ij}$ for every $\{i, j\} \in E(G)$.

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Let G be a series-parallel graph. The constant terms of the facet-defining hyperplanes of $\text{cut}^{\pm 1}(G)$ characterize the ranks of extremal rays of $\mathcal{K}_G$. These ranks are 1 and $p - 2$ where C_p is any minimal cycle in G. Moreover, the sparsity order of G is $p^* - 2$ where p^* is the length of the largest minimal cycle in G.

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Theorem [Solus, Uhler, Y. 2015]

Graphs without K_5 minors have the facet-ray identification property.

Cut Polytope for the 4-cycle: an example

$G := C_4$, identify $\mathbb{R}^{E(G)} \simeq \mathbb{R}^4$ by identifying edge $\{i, i+1\}$ with coordinate i for $i = 1, 2, 3, 4$. The cut polytope of G is the convex hull of $(-1, 1)$-vectors in $\mathbb{R}^4$ containing precisely an even number of -1's.

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Facets

$\text{cut}^{\pm 1}$ (G) is the 4-cube $[-1, 1]^4$ with truncations at the eight vertices containing an odd number of -1's with sixteen facets supported by the hyperplanes

$$\pm x_i = 1, \qquad \text{and} \qquad \langle v_T, x \rangle = 2,$$

where T is an odd cardinality subset of [4], and v_T is the corresponding vertex of $[-1, 1]^4$ with an odd number of -1's.

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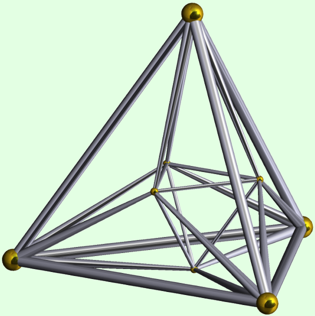

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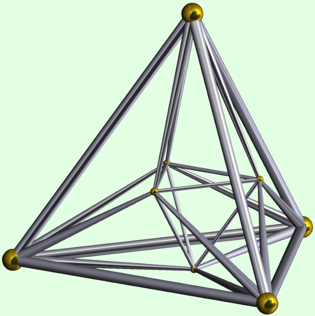

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Notes

It has 8 demicubes (tetrahedra) 8 tetrahedra as its facets.

4-cycle: an example

The facets supported by the hyperplanes $\pm x_1 = 1$ correspond to the rank 1 extremal matrices

$$Y = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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The facets $\langle v_T, x \rangle = 2$ for $v_T = (1, -1, 1, 1)$ and $v_T = (1, -1, -1, -1)$ respectively correspond to the rank 2 extremal matrices

$$Y = \frac{1}{3}\begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad Y = \frac{1}{3}\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

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These four matrices respectively project to the four extreme points in $\mathcal{E}_G^\vee$

$$(1, 0, 0, 0), \quad (-1, 0, 0, 0), \quad \frac{1}{3}(-1, 1, -1, -1), \quad \text{and} \quad \frac{1}{3}(-1, 1, 1, 1),$$

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Open problems

Problem

Determine all graphs G with the facet-ray identification property for which the facets of $\text{cut}^{\pm 1}(\text{G})$ characterize all extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$.

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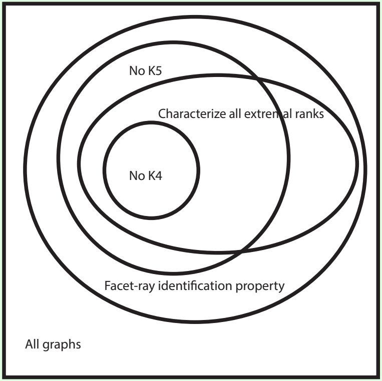

THANK YOU FOR YOUR ATTENTION!

Questions?

Reference: http://arxiv.org/abs/1506.06702

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Grone and Pierce (1990) characterized the extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$, and it is shown that G has extremal rays of ranks $1, 2$, and 3.

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Grone and Pierce (1990) characterized the extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$, and it is shown that G has extremal rays of ranks $1, 2$, and 3.

However, with the help of `Polymake` we see that the facet-supporting hyperplanes of $\text{cut}^{\pm 1}$ (G) are $x_e = \pm 1$ for each edge $e \in E(G)$ together with $\langle v^F, x \rangle = m - 2$ as C_m varies over the nine (chordless) 4-cycles within G. Thus, the constant terms of the facet-supporting hyperplanes only capture extreme ranks 1 and 2, but not 3.

Graph G with a K_4 minor but no K_5 minor

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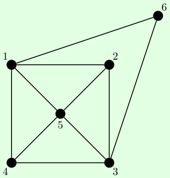
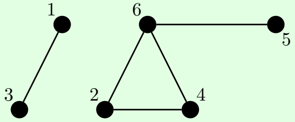


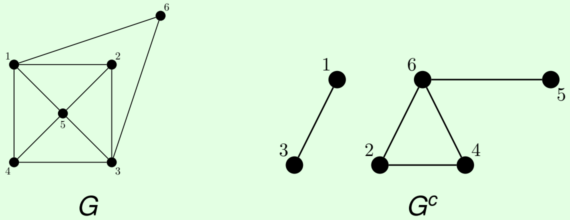
G



G^c

Graph G with a K_4 minor but no K_5 minor

This is an example of a graph G with a K_4 minor but no K_5 minor for which the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ are characterized by the facets of $\mathrm{cut}^{\pm 1}$ (G).



Recall that a k-block is a graph P of order k that has no proper induced subgraph of order k. Agler et al. characterized all 3-blocks in terms of their complements. It follows immediately from this theorem that G contains no induced 3-block. Thus, $\mathrm{ord}(G) \leq 2$, and since G is not a chordal graph we see that $\mathrm{ord}(G) = 2$. By Theorem the facets of $\mathrm{cut}^{\pm 1}$ (G) identify extremal rays of rank 1 and 2. Thus, all possible extremal ranks of G are characterized by the facets of $\mathrm{cut}^{\pm 1}$ (G).

Graph G with no $K_{3,3}$ minor

Are graphs with no $K_{3,3}$ minor the collection of graphs for which the facets characterize the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$? Answer is no.

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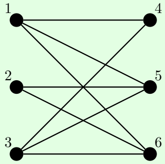

G

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Graph G with no $K_{3,3}$ minor

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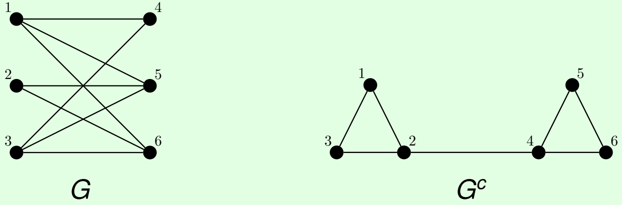


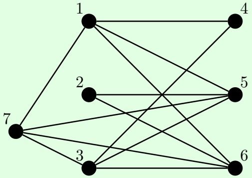
Notice that G contains no $K_{3,3}$ minor, but it does contain a K_4 minor.
By Agler et al, G is a 3-block since its complement graph is two triangles connected by an edge. Thus, G has an extremal ray of rank 3, but by Theorem, the facets of $\mathrm{cut}^{\pm 1}$ (G) only detect extremal rays of ranks 1 and 2.

Facet-ray identification property with K_5 minor

A graph with a K_5 minor whose facets characterize all extremal rays.

Facet-ray identification property with K_5 minor

A graph with a K_5 minor whose facets characterize all extremal rays.



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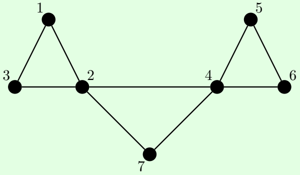
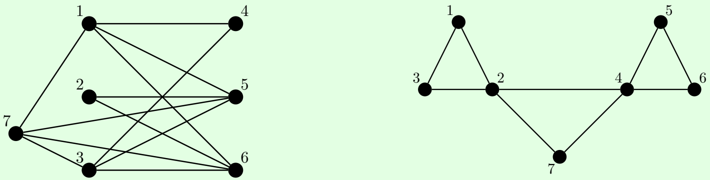


G^c

Facet-ray identification property with K_5 minor

A graph with a K_5 minor whose facets characterize all extremal rays.



G G^c

G has the facet-ray identification property, and the facets identify extreme rays of rank $1, 2$, and 3.

Graph without facet-ray identification property

Not all graphs have facet-ray identification propoerty.

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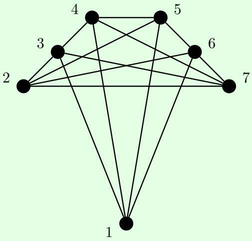
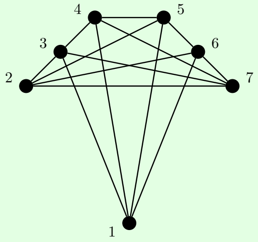


Graph without facet-ray identification property

Not all graphs have facet-ray identification propoerty.



$$x_{13} + x_{14} + x_{15} + x_{16} + x_{25} + x_{26} + x_{27} + x_{37} + x_{47} - x_{23} - x_{34} - x_{45} - x_{56} - x_{67} \leq 4$$

is not a facet-defining inequality.