Friday, March 20, 2015

If a, b, c are positive numbers such that $(1+a)(1+b)(1+c) = 8$, prove $abc \leq 1$.

Solution.

Assume that $(1+a)(1+b)(1+c) = 8$, and hence that

$$1 + a + b + c + ab + ac + bc + abc = 8\,.$$

By the arithmetic-mean-geometric-mean inequality

$$\begin{aligned} 1 = \frac{1 + a + b + c + ab + ac + bc + abc}{8} &\geq \sqrt[8]{1 \cdot a \cdot b \cdot c \cdot ab \cdot ac \cdot bc \cdot abc} \\ &= \sqrt[8]{a^4 b^4 c^4} = \sqrt{abc} \end{aligned}$$

from which $abc \leq 1$ follows easily.

Good Luck! Have fun and enjoy Mathematics!