Friday, April 12, 2013

Solve the system

$$\begin{cases} \sin^5 x + \cos^5 x = 1 \\ 0 \le x \le \pi/2. \end{cases}$$

Solution. If $x \in [0, \pi/2]$, then $0 \le \sin^5 x \le \sin^2 x$ and $0 \le \cos^5 x \cos^2 x$. Thus, $\sin^5 x + \cos^5 x \le \sin^2 x + \cos^2 x = 1$, with equality if and only if $x = 0$ or $x = \pi/2$.

Good Luck! Have fun and enjoy Mathematics!

Friday, April 05, 2013

Assume that q is a positive rational number that approximates $\sqrt{2}$. Prove that $\frac{q+2}{q+1}$ is a better approximation of $\sqrt{2}$.

Solution.

$$\left|\frac{q+2}{q+1}-\sqrt{2}\right| = \left|\frac{q+2-\sqrt{2}(q+1)}{q+1}\right| = \frac{(\sqrt{2}-1)|q-\sqrt{2}|}{1+q} \leq |q-\sqrt{2}|.$$

Good Luck! Have fun and enjoy Mathematics!

Friday, March 29, 2013

Assume that n people have n distinct pieces of information, one piece of info per person. Every time a person, say A, calls person B, it tells B all the info he knows, but not vis versa (B does not tell A anything). What is the minimum number of calls needed such that everyone knows everything.

Solution.

Suppose that a sequence of calls makes everyone fully informed. Take the critical call, at which the first person, say p, is getting fully informed (everybody else is not fully informed). At least $n-1$ calls have been made, else p would not be fully informed. Also, each of these $n-1$ should receive at least one call each, after the critical one was med, to get fully informed. So, the number of calls made is at least $2(n-1)$.

Note that at least one configuration will lead to $2n-2$ calls. Indeed Denote by p_i the i-th person. Then the sequence

$$p_1 \to p_n, p_2 \to p_n \dots, p_{n-1} \to p_n, p_n \to p_{n-1}, \dots, p_n \to p_1,$$

leaves everyone knowing everything.

Good Luck! Have fun and enjoy Mathematics!

Friday, March 15, 2013

Consider n points in the plane, such that the distance between them is distinct. Joint each point by a straight segment to its nearest neighbor. Can the obtained configuration contain triangles?

Solution. Consider any three points in that set, call them A,B,C. Assume that AB is the longest distance between these three points. Then A can not be connected to B.

Good Luck! Have fun and enjoy Mathematics!

Friday, March 08, 2013

Given an even number of points in the plane, does there exist a straight line such that on each side of the line (half plane generate by that line) has half of the points?

Solution. Yes, it is possible. Indeed. Consider all possible lines that contain these given points. Note that there is a finite number of such lines. Hence, there exist a line that is not parallel to neither of these lines. Eventually, we can shift this line such that all points are on one side of that line. Translate this line perpendicular to the set of the points (or parallel shift towards the points). By the contraction, we will pass one point at a time. Stop when half of the points are passed.

Good Luck! Have fun and enjoy Mathematics!

Friday, March 01, 2013

A plane flies from point A to B and back with constant engine speed. Compare the travel time between the case of still air and of the case in which the wind flows with constant speed (say in the direction from A to B). You may neglect the turning time of the plane.

Solution. Let us use the following notations d - the distance between A and B, v - the speed of the engine and u - the speed of the wind. The time for the round trip in still air is

$$\frac{2d}{v}.$$

The time in the windy situation is equal to

$$\frac{d}{v+u} + \frac{d}{v-u} = \frac{2d}{v}\frac{v^2}{v^2-u^2} > \frac{2d}{v}.$$

Good Luck! Have fun and enjoy Mathematics!

Friday, February 22, 2013

Let $x_1, \ldots, x_n \in (0, \infty)$, and denote by $S = x_1 + x_2 + \ldots x_n$. Prove that

$$\frac{S}{S - x_1} + \frac{S}{S - x_2} + \cdots + \frac{S}{S - x_n} \geq \frac{n^2}{n - 1}.$$

Find necessary and sufficient conditions for x_i's such that the above inequality is an identity.

Solution. Recall the inequality for Harmonic-Arithmetic Means (for positive numbers)

$$\frac{n}{\frac{1}{a_1} + \cdots \frac{1}{a_n}} \leq \frac{a_1 + \ldots + a_n}{n}$$

with equality if and only if $a_1 = \ldots = a_n$. Take $a_i = \frac{S - x_i}{S}$ and the above inequality of interest follows immediately (we remark that $\sum_{i=1}^n a_i = n - 1$). Moreover, the equality will be achieved if and only if all x_i's are the same.

Solution. The key identity to be used is

$$\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}.$$

Using this identity in chains, will lead to infinitely many representations

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1806} \\ &= \ldots \end{aligned}$$

Good Luck! Have fun and enjoy Mathematics!

Friday, February 15, 2013

Is it possible to represent the number 1 as a sum of reciprocal of distinct integers larger or equal than 2 in infinitely many ways?

Solution. The key identity to be used is

$$\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}.$$

Using this identity in chains, will lead to infinitely many representations

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1806} \\ &= \dots \end{aligned}$$

Good Luck! Have fun and enjoy Mathematics!

Friday, February 08, 2013

Let us consider the polynomial

$$x_n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n,$$

and assume that it has only real roots.

Prove that

$$(n-1)a_1^2 \geq 2na_2.$$

Find necessary and sufficient conditions such that the last inequality becomes identity.

Solution. Since all the roots are real, the polynomial can be represented as

$$(x+r_1)(x+r_2)\ldots(x+r_n)$$

for some real numbers $r_1, \ldots, r_n$. Then,

$$a_1 = \sum_{k=1}^{n} r_k, \quad a_2 = \sum_{1\leq i<j\leq n} r_i r_j.$$

This implies that

$$a_1^2 = \sum_{i=1}^{n} r_i^2 + 2a_2.$$

Hence the inequality of interest is equivalent to

$$(n-1)\left(\sum_{i=1}^{n} r_i^2 + 2a_2\right) \geq 2na_2,$$

or

$$(n-1)\sum_{i=1}^{n} r_i^2 \geq 2\sum_{1\leq i<j\leq n} r_i r_j,$$

which is equivalent to

$$\sum_{1\leq i<j\leq n} (r_i - r_j)^2 \geq 0,$$

which obviously holds true. Moreover, the equality is achieved if and only if all roots are equal.

Good Luck! Have fun and enjoy Mathematics!

Friday, February 01, 2013

Let

$$s_n := 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}, \quad n \in \mathbb{N}.$$

Prove that

$$2\sqrt{n+1} - 2 < s_n < 2\sqrt{n} - 1.$$

Solution. Observe that

$$\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1} + \sqrt{k}}.$$

Hence,

$$\frac{1}{2\sqrt{k+1}} < \sqrt{k+1} - \sqrt{k} < \frac{1}{2\sqrt{k}}.$$

Summing up the left inequalities for $k = 1, \ldots, n-1$, we get

$$\frac{1}{2}(s_n - 1) < \frac{1}{\sqrt{n}} - 1,$$

which implies the right side inequality from the problem ($s_n < 2\sqrt{n} - 1$). Similarly, summing up the right inequalities for $k = 1, \ldots, n$, we deduce the left inequality in the original statement.

Good Luck! Have fun and enjoy Mathematics!

Friday, November 16, 2012

Let $a_1, a_2, \ldots, a_{31}$ be some given real numbers. Show that

$$a_1 \cos(x) + a_2 \cos(2x) + \ldots + a_{31} \cos(31x) + \cos(32x)$$

as function of x takes positive as well as negative values.

Solution. Let us consider the function

$$f(x) = a_1 \cos(x) + a_2 \cos(2x) + \ldots + a_{31} \cos(31x) + \cos(32x), \quad x \in \mathbb{R}.$$

Since, $\int_0^{2\pi} f(x)dx = 0$, and since f is continuous, we conclude that f must take positive as well as negative values.

Good Luck! Have fun and enjoy Mathematics!

Friday, November 09, 2012

Each of the expression $(1+x^2-x^3)^{1000}$ and $(1-x^2+x^3)^{1000}$ is expanded and then reduced to the simplest form, hence we get two polynomials. Compare the coefficients next to x^{20} in these two polynomials.

Solution. Note that after substitution x by $(-x)$ the coefficients of even powers are not changing. Hence the coefficients next to x^{20} of new polynomials are the same as of original polynomials. However, clearly the polynomial $(1+x^2+x^3)^{1000}$ has a larger coefficient next to x^{20} than the polynomial $(1-x^2-x^3)^{1000}$.

Good Luck! Have fun and enjoy Mathematics!

Friday, November 03, 2012

A factory can produce 100 items of type A or 300 items of type B during one day. The QA (quality assurance) department can check nor more than 150 items. A type A item cost twice as much as a type B item. How many items of each type should be produced per day such that the cost (combined value of all produced items) is maximal.

Solution. The item A is produced during 1/100 of the day, and the item B is produced during 1/300 of the day. Let x be the number of items A produced during that day. Hence, during the remain time of that day, the factory can produce

$$\frac{1 - x/100}{1/300} = 3(100 - x)$$

items of type B. Assume that the cost of one item of type A is 1, and thus one item of type B is equal B. Also, assume that the total number of produced items is not greater than 150, which implies that

$$x + 3(100 - x) = 300 - 2x \leq 150 \quad \Leftrightarrow x \geq 75.$$

In this case, the total cost of the produced items is equal to $2x + 3(100 - x) = 300 - x$. The maximal cost of 225 will be archived if $x = 75$. Hence, the factory has to produce 75 items of each type.

Good Luck! Have fun and enjoy Mathematics!

Friday, October 26, 2012

Prove that

$$\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\cdots\left(1+\frac{1}{2^n}\right) < 2,$$

for any $n \geq 2$.

Solution. Note that

$$\begin{aligned}&\ln\left[\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\cdots\left(1+\frac{1}{2^n}\right)\right]\\ &= \ln\left(1+\frac{1}{4}\right)+\ldots\ln\left(1+\frac{1}{2^n}\right)\\ &< \frac{1}{4}+\frac{1}{8}+\cdots\frac{1}{2^n}+\ldots\\ &= \frac{1/4}{1-1/2}=\frac{1}{2},\end{aligned}$$

where we used the fact that $\ln(1+x) < x,\ x > 0$. The former follows directly from the fact that $x - \ln(1+x)$ is an increasing function (just take the derivative). Hence,

$$\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\cdots\left(1+\frac{1}{2^n}\right) < \sqrt{e} < 2.$$

Good Luck! Have fun and enjoy Mathematics!

Friday, October 19, 2012

The decimals of a given natural number n contain the digits 1,3,7 and 9. Prove that it is possible to rearrange the digits of n such that the new number is divisible by 7.

Solution. Without loss of generality we can consider that n has the last digits equal to $1, 3, 7, 9$, else rearrange first. Hence, n can be written as a sum of 1379 and a where a has last four digits equal to zero. Note that, the following numbers

$$1379,\ 1793,\ 3719,\ 1739,\ 1397,\ 1937,\ 1973,$$

have the reminders of division by 7 equal to $0, 1, 2, 3, 4, 5, 6$. Thus, if a has a reminder after division by 7 equal to r, then take that four digit number from above, than gives the reminder $7 - r$. Now, a plus the chosen four digit number will be divisible by 7.

Good Luck! Have fun and enjoy Mathematics!

Friday, October 12, 2012

Prove that for any natural number $n > 2$, the following inequality holds true

$$\sqrt[n+1]{n+1} < \sqrt[n]{n}.$$

Solution. Rasing to the power $n(n+1)$ both sides of the inequality we get the following equivalent inequality

$$(n+1)^n < n^{n+1}.$$

Consequently, the last inequality is equivalent to

$$\left(1+\frac{1}{n}\right)^n < n.$$

The last one is easy to prove by induction.

For $n = 3$, we have $(1 + 1/3)^3 = 64/27 < 3$. Assume it is true for n. Then,

$$\begin{aligned}\left(1+\frac{1}{n+1}\right)^{n+1} &= \left(1+\frac{1}{n+1}\right)^n \left(1+\frac{1}{n+1}\right)\\ &< \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n+1}\right)\\ &< n\left(1+\frac{1}{n+1}\right) = n + \frac{n}{n+1}\\ &< n+1.\end{aligned}$$

Good Luck! Have fun and enjoy Mathematics!

Friday, October 05, 2012

Assume that all edges of a triangle are smaller than one. Prove that the area of the triangle is smaller than $\sqrt{3}/4$.

Solution. Let ABC be the triangle with all edges smaller than one, and assume that AB is the longest edge. Consider the triangle ABC_1, that is equilateral and has the common edge AB with ABC. Note that in this case, the altitude of ABC with base AB is smaller than the altitude of ABC_1, hence, the area of ABC is smaller than the area of ABC_1. But,

$$S_{ABC_1} = \frac{AB^2\sqrt{3}}{4} \leq \frac{\sqrt{3}}{4}.$$

Thus, $S_{ABC} \leq S_{ABC_1}$.

Good Luck! Have fun and enjoy Mathematics!

Friday, September 28, 2012

Find the smallest number A such that, for any quadratic polynomial $p(x)$ that satisfies the inequality

$$|p(x)| \leq 1, \quad 0 \leq x \leq 1,$$

we also have that $p'(0) \leq A$.

Solution. Assume that $p(x) = ax^2 + bx + c$, and it satisfies the inequality $|p(x)| \leq 1$ for all $x \in [0, 1]$. Then, in particular, we will have

$$|p(0)| \leq 1, \ |p(1/2)| \leq 1, \ |p(1)| \leq 1.$$

Since,

$$p(0) = c, \ p(1/2) = \frac{a}{4} + \frac{b}{2} + c, \ p(1) = a + b + c,$$

we have the following estimates

$$\begin{aligned} |p'(0)| = |b| &= \left| 4(\frac{a}{4} + \frac{b}{2} + c) - (a + b + c) - 2c \right| \\ &\leq 4\left| \frac{a}{4} + \frac{b}{2} + c \right| + |a + b + c| + 2|c| \\ &= 4|p(1/2)| + |p(1)| + 2|p(0)| \leq 4 + 1 + 3 = 8. \end{aligned}$$

Hence, we conclude that $A \leq 8$. On the other hand, it is easy to verify that the plynomial $p(x) = -8x^2 + 8x - 1$ satisfies all the conditions and $p'(0) = 8$. Thus, $A = 8$.

Good Luck! Have fun and enjoy Mathematics!

Friday, September 21, 2012

Find

$$x^{13} + \frac{1}{x^{13}}$$

in terms of $a := x + 1/x$.

Solution. Note that

$$\left(x^m + \frac{1}{x^m}\right)\left(x^n + \frac{1}{x^n}\right) = \left(x^{m+n} + \frac{1}{x^{m+n}}\right) + \left(x^{m-n} + \frac{1}{x^{m-n}}\right)$$

from where we have

$$x^{m+n} + \frac{1}{x^{m+n}} = \left(x^m + \frac{1}{x^m}\right)\left(x^n + \frac{1}{x^n}\right) - \left(x^{m-n} + \frac{1}{x^{m-n}}\right).$$

In particular, for $n = 1$ and $n = m$ we the following two identities

$$x^{m+1} + \frac{1}{x^{m+1}} = \left(x^m + \frac{1}{x^m}\right)\left(x + \frac{1}{x}\right) - \left(x^{m-1} + \frac{1}{x^{m-1}}\right) \quad (1)$$

$$x^{2m} + \frac{1}{x^{2m}} = \left(x^m + \frac{1}{x^m}\right)^2 - 2. \quad (2)$$

From (2), we get

$$x^2 + \frac{1}{x^2} = a^2 - 2$$

$$x^4 + \frac{1}{x^4} = \left(x^2 + \frac{1}{x^2}\right)^2 - 2 = a^4 - 4a^2 + 2.$$

From (1), using the above, we get

$$x^3 + \frac{1}{x^3} = a^3 - 3a.$$

Similarly, one can find $x^6 + 1/x^6$, and $x^7 + 1/x^7$, and finally, from here one finds

$$\begin{aligned} x^{13} + \frac{1}{x^{13}} &= \left(x^7 + \frac{1}{x^7}\right)\left(x^6 + \frac{1}{x^6}\right) - \left(x + \frac{1}{x}\right) \\ &= (a^7 - 7a^5 + 14a^3 - 7a)(a^6 - 6a^4 + 9a^2 - 2) - a \\ &= a^{13} - 13a^{11} + 65a^9 - 156a^7 + 182a^5 - 91a^3 + 13a. \end{aligned}$$

Good Luck! Have fun and enjoy Mathematics!

Friday, September 14, 2012

Prove that a square can be divided in any number of squares greater or equal to six.

Solution.

For any $n > 1$, divide each side of the square in n equal parts, and connect them so to get the original square divided in n^2 equal squares. Next, keep all the squares along two adjacent sides of the original square (that would be $2n - 1$ squares), and make one big square from the others (delete all remaining and keep on big square). Now, we have the original square divided in $2n$ squares, for any $n > 1$. Hence, we can divide in any even number of squares starting four. For any partition with even numbers, take any square and divide it in four equal squares. Then, the total number of squares increased by 3, hence we can get any odd number larger or equal 7.

Good Luck! Have fun and enjoy Mathematics!