Friday, April 13, 2012

You are given the following 1000 weights: 1^2g, 2^2g,..., 1000^2g. Prove that you can split these weights into two groups such that there are 500 weights in each group, and the two groups have equal combined weight.

Solution. The key point is to note that squares of any eight consecutive numbers

$$x^2, (x+1)^2, \ldots, (x+7)^2$$

can be split into two groups with four elements in each group, and of equal combined weight. For example, this can be done as follows:
Group 1: $x^2, (x+3)^2, (x+5)^2, (x+6)^2$
Group 2: $(x+1)^2, (x+2)^2, (x+4)^2, (x+7)^2$.
Having this, the problem follows immediately, after splitting all 1000 numbers into groups of eight as above with $x = 8k+1,\ k = 0, \ldots, 124$.

Good Luck! Have fun and enjoy Mathematics!

Friday, April 6, 2012

A student during a year solves each day at least one problem. Each week he solves no more than 12 problems. Prove that there exist several consecutive days during which he solved exactly 20 problems.

Solution. Let a_j be the number of problems the student solved during first j days. Then, $a_{77} \leq 12 \cdot 11 = 132$. Consider the numbers

$$a_1, \ldots, a_{77}, a_1 + 20, \ldots a_{77} + 20.$$

These 154 numbers do not exceed 132+20=152, hence there exist two among them that are equal. Since $a_1, \ldots, a_{77}$ are pairwise distinct, then we can find two numbers l, k such that $a_k = a_l + 20$, and thus $a_k - a_l = 20$.

Good Luck! Have fun and enjoy Mathematics!

Friday, March 30, 2012

Two friends, John and Bob, have the following habits: John was lying on Mondays, Tuesdays and Wednesdays, while Bob was lying on Tuesdays, Thursdays and Saturdays. All other times they were telling the truth.

A man asked one of them "what is your name?"

- John, - replied one of the boys.

- What day is today?

- Yesterday was Sunday.

The second guy added: "But tomorrow is going to be Friday". Then, the man asked the second guy: "Are you telling the truth?"

- I always tell the truth on Wednesdays, - replied the guy.

Who is John, who is Bob and on what day of the week the discussion happened?

Solution. First guy is Bob, second is John, and the day of the week is Tuesday.

Good Luck! Have fun and enjoy Mathematics!

Friday, March 9, 2012

Two points are moving in the same direction on a circle of length 60 meters. One point makes a full rotation by 5 seconds faster than the other point. Moreover, the points are meeting each minutes only ones. Find the velocity of each point.

Solution. Let $v_1,\ v_2$ the velocities of the points, in m/s. Then $60/v_1,\ 60/v_2$ would be the times for making one full rotation for each point respectively. Hence

$$\frac{60}{v_1} - \frac{60}{v2} = 5.$$

Also, note that $60v_1,\ 60v_2$ is the distance each point travels during one minute. Hence,

$$60v_1 - 60_v 2 = 60.$$

Solving these two equations, we get $v_1 = 4$ m/s, and $v_2 = 3$m/s.

Good Luck! Have fun and enjoy Mathematics!

Friday, March 2, 2012

Find the remainder from dividing 67^{2012} by 16.

Solution. $67^{2012} = (67^{503})^4 = (2n+1)^4 = (4n(n+1)+1)^2 = 16n^2(n+1)^2 + 8n(n+1) + 1 = 16p + 1$. Thus, the remainder is 1.

Good Luck! Have fun and enjoy Mathematics!

Friday, February 24, 2012

A CS undergraduate wrote a piece of code that takes as its input a three digit number, and gives as an output the difference of this number and the sum of the cubes of the digits. For example, if the input is 123, then the output is $123-(1^3+2^3+3^3)=87$. Find the input that will give the maximum output. Of course, one can write another piece of code with one for-loop, that will check all possible inputs/outputs, and pick up the largest output. However, this time, we are looking for a mathematical solution, i.e. without using any computers.

Solution. Let $\overline{abc}$ be the input. Then, we want to maximize the following

$$m = 100a + 10b + c - (a^3 + b^3 + c^3).$$

Note that $m = (100a - a^3) + (10b - b^3) + (c - c^3)$ will be maximum if and only if each term will be maximum. This implies $a = 6,\ b = 2,\ c = 0$ or $c = 1$, and hence, the inputs are 620, and 621.

Good Luck! Have fun and enjoy Mathematics!

Friday, February 17, 2012

Assume that a, b, c are real numbers such that

$$ab + bc + ca > 0$$

and

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} > 0.$$

Show that a, b, c have the same sign (positive or negative).

Solution. Because of the symmetry, without loss of generality we can assume that $a \le b \le c$. Assume that a, b, c are of different sign. Then, either $a < 0 < b \le c$ or $a \le b < 0 < c$. Let's us consider case first. Then

$$\begin{cases} a < 0 < b \le c \\ ab + bc + ca > 0 \\ \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} > 0 \end{cases}$$

equivalently

$$\begin{cases} a < 0 < b \le c \\ bc > |a|(b + c) \\ \frac{1}{bc} > \frac{b+c}{|a|bc} \end{cases}$$

and hence

$$\begin{cases} a < 0 < b \le c \\ bc > (b + c)^2 \\ |a| > (b + c). \end{cases}$$

From here, we find that the last system of inequalities does not have a solution. Contradiction. Second case is treated similarly.

Good Luck! Have fun and enjoy Mathematics!

Friday, February 10, 2012

The distance between two cities is equal to a. Two cars started to move from these cities would meet each other in the middle if the second car starts moving by t hours later than the first car. If the cars start moving at the same time, then they meet after $2t$ hours. Find the speed of each car.

Solution. Let u and v be the speeds of the cars. Then

$$\frac{a}{2} = ut + v\frac{a}{2v}$$
$$a = 2t(u + v).$$

Hence,

$$4t^2v^2 + 2atv - a^2 = 0.$$

This implies that $v = \frac{a}{4t}(\sqrt{5} - 1)$ and $u = \frac{a}{4t}(3 - \sqrt{5})$.

Good Luck! Have fun and enjoy Mathematics!

Friday, November 18, 2011

The boundaries of country SQRT form a square with sides equal to 1000 miles. There are 51 cities in this country. Is it possible to connect this cities by railroads such that the total length of the railroads is less or equal 11000 miles?

Solution. Let us divide the square in k equal bands, and in each of them construct a railroad of length 1000 miles. Also construct a railroad perpendicular to all bands (or to all already constructed rails) and that passes through one of the cities. Finally, construct a rail from each city to the closest existing rail. Overall, we will construct at most

$$1000 \cdot (k+1) + 50 \cdot 500/k$$

miles of roads. For $k = 5$ we get exactly 11000 miles.

Good Luck! Have fun and enjoy Mathematics!

Friday, November 11, 2011

A drill sergeant stays in front of a rank (aline, which is only one element in depth) of new recruits. The sergeant commands "Right, Face". [1] Some recruits turn 90° to the right, some of them 90° to the left and some of them turn by 180°. Can sergeant take a place in the rank such that the number of recruits facing him (the face is looking at the sergeant) from sergeant's left is equal to the number of recruits facing the sergeant from his right?

Solution. Denote by m the number of recruits from sergeant's left facing him, and by n the number of recruits from sergeant's right (also facing him). Assume that the sergeant chooses to stay at the very left side of the rank. Then $m = 0$. If none of the recruits is facing him, then $n = 0$, and the problem is solved. If not, then $n > 0$, and thus $m - n < 0$. Every time the sergeant is moving by one recruit to the right, then $m - n$ can change only by one (consider all three cases for the person passed by the sergeant). When the sergeant gets to the right end of the rank, then $n = 0$, and hence $m - n \geq 0$. Thus, we started with $m - n < 0$, it was increasing by one unit, and became nonnegative. This implies that at some moment $m - n = 0$, or $n = m$.

Good Luck! Have fun and enjoy Mathematics!

[1] On the command, "Right, face," the soldier
a. Slightly raises left heel and right toe and turns 90 degrees to the right on right heel, assisted by a slight pressure on the ball of the left foot.
b. Holds left leg straight.
c. Places left foot beside right foot, as in the position of attention.
d. Holds arms as at attention when performing this movement.

Friday, November 4, 2011

There are three floors in a department store. Customers can move between floors only by elevator. During one day, the following was recorded:

- half of the customers entering the elevator on the second floor went to the first floor, while the second half of the customers went to the third floor;
- among all the customers that exited an elevator, less that a third were exiting on third floor.

What number is larger: the number of customers going from the first to the third floor (entering on 1st exiting on 3rd floor), or from the first to the second floor?

Solution. Denote by n_{ij} the number of people moved from floor i to floor j. We will split all moves in three groups: $n_{13}+n_{23}$ is number of customers traveled to 3rd floor, $n_{31}+n_{n_{32}}$ - number of customers from 3rd floor, and $n_{12}+n_{21}$ - number of customers moved between 1st and 2nd floors. From the records, we have that $n_{31}+n_{n_{32}}$ is less than a third of all moves. On the other hand it is equal to $n_{31}+n_{n_{32}}$. Hence, the last group $n_{12}+n_{21}$ is the largest and hence

$$n_{31}+n_{32}=n_{13}+n_{23}<n_{12}+n_{21}.$$

From here, since $n_{21}=n_{23}$, we have $n_{13}<n_{12}$. Thus, more people moved from 1st to 2st floor than from 1st to 3rd.

Good Luck! Have fun and enjoy Mathematics!

Friday, October 21, 2011

Closing price of IIT-Plus-Forever stock is 17% larger or smaller than yesterday closing price. Can the closing stock price of this company be the same at two different days.

Solution.

After k increases and j decreases the price will be multiplied by the factor

$$\left(\frac{117}{100}\right)^k \left(\frac{83}{100}\right)^j.$$

If this number is equal to 1, then $117^k \cdot 83^j = 100^{k+j}$. However, the RHS is an even number, while the LHS is an odd number. Contradiction.

Good Luck! Have fun and enjoy Mathematics!

Friday, October 21, 2011

A student did not notice the multiplication sign between two three-digit numbers, and wrote a six-digit number. The six-digit number turned out to be seven times bigger than the product of original numbers. Find the initial three-digit numbers.

Solution. We have to solve the equation

$$7x \cdot y = 1000x + y, \quad x, y \in \mathbb{N}.$$

Divide both parts by x and get

$$7y = 1000 + \frac{y}{x}.$$

Since $y \leq 999$ and $x \geq 100$, we conclude $0 < y/x < 10$. Thus

$$1000 < 7y < 1010.$$

The last set of inequalities have only two solutions: $y = 143$, $y = 144$. If $y = 144$, then $x = 18$, and this solution does not satisfy the condition that x has three digits. For $y = 143$ we find $x = 143$, and this is the only solution to the problem.

Good Luck! Have fun and enjoy Mathematics!

Friday, October 14, 2011

Andrew leaves at noon and drives at constant speed from town A to B and back, and so on. Barbara also leaves at noon, driving at 40 miles/hour back and forth from town B to town A using the same highway as Andrew. Andrew arrives at B 20 minutes after passing Barbara, whereas Barbara arrives at town A 45 minutes after passing Andrew. At what time do Andrew and Barbara pass each other for the forth time.

Solution. The conditions of the problem give

$$(t+20)v_a = (t+45)40 = (v_a+40)t = 60d$$

where d is the distance between A and B, v_a is Andrew's speed, t is the time, in minutes, for first passing. This gives $d = 50$ miles, $t = 30$ minutes, and $v_a = 60$miles/hour.

After the first meeting the cars are separating at a relative velocity of 100 miles/hour for 20 minutes, and then, after Andrew reaches B then the cars are approaching at relative velocity 20 mph for 25 minutes until Barbara arrives at A. During this time they have been separated to a distance of 100/3 miles, then approaching to a distance of 25 miles. When Barbara leaves A she is approaching Andrew traveling from B with a relative velocity 100 mph. Hence, they will meet 15 minutes later, and the total time between 1st and 2nd meeting is 60 minutes. Similar argument establishes that the third meeting takes place at B 60 minutes later, and then by symmetry, subsequent meetings occur at 60 minutes intervals. Thus the two pass for the n-th time at $30 + 60(n-1)$ minutes past noon.

Good Luck! Have fun and enjoy Mathematics!

Friday, October 7, 2011

You have a calculator that does not work properly - it can not perform multiplication. However, it does correctly addition, substraction, and it can compute the reciprocal. Show that you can still perform multiplication of any numbers.

Solution. Assume that $u \neq 1$. Then one can compute u^2 by using the representation

$$u^2 = u - \left(\frac{1}{u} - \frac{1}{1-u}\right)^{-1}$$

and similarly

$$\frac{u}{2} = \left(\frac{1}{u} + \frac{1}{u}\right)^{-1}.$$

Having this, one can find the product of any two numbers

$$uw = \left(\frac{u+w}{2}\right)^2 - \left(\frac{u-w}{2}\right)^2.$$

Friday, September 30, 2011

A clock had stopped running and the owner did not have access to the correct time to set the clock. The clock was big so the man could not carry it anywhere to adjust it. The man walked to the home of a friend whose clock was correct, he stayed at the friends house for some time, and then walked back home (walking t he same amount of time as he did to his way to the friends house). Arriving at home, the man set up the correct time, even though he did not know how long he had walked. How did the man set up the correct time?

Solution. The trick is that the man could start running his clock before departure from his home. Upon arriving back home he could find how long he was away, say X minutes. Also, he knew how long he stayed at friends house, say Y minutes. Thus he spent walking one way $(X-Y)/2$ minutes. Adding this time to the time the man read when he departures from his friend gives the correct current time.

Good Luck! Have fun and enjoy Mathematics!

Friday, September 23, 2011

Prove that from any given 52 integers you can choose two numbers such that their difference or their sum is divisible by 100.

Solution. Let us divide the reminders of division by 100 into following groups

$$\{0\},\ \{1,99\},\ \{2,98\},\ldots,\{49,51\},\ \{50\}.$$

Since we have 51 groups and 52 integers, there exists at least two numbers among those 52 that have reminders from division by 100 in one of these groups. These are the numbers we are seeking. If their reminders are the same then the sum will be divisible by 100, if they have different reminders then the difference will be divisible by 100.

Good Luck! Have fun and enjoy Mathematics!