ILLINOIS INSTITUTE OF TECHNOLOGY

Department of Applied Mathematics and IIT SIAM Student Chapter

Math Weekly Problem Competition

Tuesday, September 21, 2010: Solution

Let $b > a > 0$. Evaluate the following integral

$$\int_0^1 \frac{x^b - x^a}{\ln x} dx\,.$$

Solution.

$$\begin{aligned}\int_0^1 \frac{x^b - x^a}{\ln x} dx &= \int_0^1 dx \int_a^b x^y dy \\ &= \int_a^b dy \int_0^1 x^y dx = \int_a^b \frac{x^{y+1}}{1+y}\Big|_0^1 dy \\ &= \int_a^b \frac{1}{1+y} dy = \ln(1+y)\big|_a^b \\ &= \ln(1+b) - \ln(1+a)\end{aligned}$$

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Good Luck! Have fun and enjoy Mathematics!

Tuesday, September 28, 2010: Solution

A student has 11 weeks before taking GRE test. To prepare for the exam, he decided to solve at least one test each day before the exam, but in order not to get 'tired' he agrees to solve no more than twelve tests during any given week. Prove that there exists a succession of days during which the student solved exactly 20 tests.
To avoid confusions, assume that he starts practicing and counting on Monday.

Solution. Assume that the student solves a_1 tests on Monday (first day), a_2 test during the first two days, and so on, he solves a_{77} tests during all 77-th days. Consider the following sequences of integers:

$$a_1 + a_2 + \ldots + a_{77}$$
$$a_1 + 20 + a_2 + 20 + \ldots + a_{77} + 20.$$

Total, we have $77 + 77 = 154$ numbers. Note that according to the assumption of the problem $a_{77} \leq 132 = 11 \cdot 12$ Thus, each integer from those 154 listed above does not exceed $132 + 20 = 152$, and hence at least two of them coincide. Also note, that $a_1, a_2, \ldots, a_{77}$ are distinct numbers, since every day the student solves at least one problem. This implies that none of the integers $a_1 + 20, a_2 + 20, \ldots, a_{77} + 20$, can be the same. Hence, there exists a_k and a_s such that

$$a_k = a_s + 20.$$

This yields, that $a_k - a_s = 20$, i.e. from day $s + 1$ till day k, the student solve exactly 20 tests.

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Tuesday, October 5, 2010

Assume that $a,\ b,\ c$ are reals that satisfy the following inequalities:

$$\begin{aligned} |a-b| &\geq |c| \\ |b-c| &\geq |a| \\ |c-a| &\geq |b|. \end{aligned}$$

Prove that one of these three numbers is equal to the sum of other two.

Solution. Square both sides of first inequality, and decompose it as follows:

$$(a-b-c)(a-b+c) \geq 0,$$

or equivalently

$$(a-b-c)(b-a-c) \leq 0.$$

Similarly, we deduce

$$\begin{aligned} (b-c-a)(c-a-b) &\leq 0 \\ (c-a-b)(a-b-c) &\leq 0. \end{aligned}$$

Multiplying the last three inequalities, we have

$$(a-b-c)^2(b-c-a)^2(c-a-b)^2 \leq 0.$$

Since each factor is nonnegative, and their product is non-positive, it follows that at least on of them is equal to zero, which consequently gives that one number is equal to sum of other two.

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Tuesday, October 12, 2010

The stock (closing) price of company XYZ is rasing or dropping each day by $n\%$, where n is a fixed positive integer smaller than 100. The stock price is not rounded, i.e. it is a real number. Does it exist an integer n, such that the stock price will take the same values in two different days?

Solution. Note that if stock price increases k times and decreases j times, and the price becomes equal to initial prices, then we have

$$S_0\left(1+\frac{n}{100}\right)^k\left(1-\frac{n}{100}\right)^j = S_0,$$

where S_0 denotes the initial stock price. From here we deduce

$$(100+n)^k(100-n)^j = 100^{k+j}.$$

Since the RHS is even, the LHS is even too, and thus n has to be even too. Similarly, one can show that since RHS is divisible by 5, the LHS is divisible by 5, and hence n is also divisible by 5. Thus, n is a multiple of 10, i.e. 10, 20, 30, 40, 50, 60, 70 80, and 90 are all possible values of n. It easy to show that n can not take any of these values. For example, if $n = 10$, then LHS is divisible by 11, but RHS is not divisible by 11, so n can not be equal to 10.

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Tuesday, October 19, 2010

Two hundred students are arranged in ten rows (hence 20 in each row or 20 columns). From each of the 20 columns the shortest student is selected, and the tallest of these 20 students is tagged A. Similarly, from each 10 rows, the tallest is selected, and from these 10 students the shortest is selected and tagged B. Who is taller A or B?

Solution. If A and B are from the same row, then B is taller than A, since B is the tallest in that row. If A and B are from the same column, then B is taller than A, since A is the shortest in that column. If A and B are from different columns and rows, then take C being the student that is in the same column with A and same row with B. Then B is taller than C and C is taller than A, hence B is taller than A. Overall, we conclude that B is taller than A.

Tuesday, October 26, 2010

Two brothers sold a herd of sheep they owned. For each sheep they have been paid as many \$'s as sheep originally in the herd. The brothers divided the revenue as follows: in turns, each of them would take \$10 with older brother being the first to start. At the very end of this procedure the younger brother found that fewer than \$10 are left, and took them. To make the division fair, the older brother gave to the younger one his swiss-army-knife. How much the knife was worth?

Solution. Denote by n the total number of sheep in the original herd. Then the brothers received n^2 dollars. Denote by r the reminder from division of n by 10. Then $n = 10d + e$ for some positive integer d. Hence,

$$N := n^2 = (10d + r)^2 = 100d^2 + 20dr + r^2.$$

By the conditions of the problem, the number N (total revenue) is an odd number of tens plus a reminder less than ten. On the other hand, $100d^2 + 20dr$ is divisible by 20, hence contains an even multiple of ten, and thus r^2 has to contain an odd number of multiples of ten. Since $r < 10$, the only possible values of r^2 are: 1, 4, 9, 16, 25, 36, 49, 64, 81. Among these numbers, only 16 and 36 contains an odd number of tens, both of them having the reminder from the division by 10 equal to 6. Thus, at the last step of dividing the revenue, the young brother took only \$6, and thus the older brother got \$4 more. To make the division fair, the older brother had to give back to younger one \$2, which means that the knife costs \$2.

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Tuesday, November 2, 2010

Does it exist a convex solid in $\mathbb{R}^3$ different from a sphere, such that its canonical projections (projections on XOY, YOZ, XOZ) are disks.

Solution. Yes, there exists such solid. Consider a sphere and take its canonical projections (which are disks). Consider three cylinders with bases being equal to these disks and perpendicular to the corresponding plane. The solid obtained from the intersection of these cylinders is the required solid.

Alternatively, consider a sphere, and a point on this sphere that can not be seen from the projections. There exists such point! Then a neighborhood of this point is also not seen from the projections. Cut out of the sphere that neighborhood.

Tuesday, November 9, 2010

A traveler has no money, but owns a seven link golden chain. He is staying at a hotel where he has to pay daily one golden link (from the chain.) He stays seven days, and he accepts change only in the form of links. What is the minimal number of cuts to be made to the original chain?

Solution. If we cut the 3rd link in the chain, we get 3 pieces of length 1, 2, 4.
Day one: pay with 1
Day two: give 2 get back 1
Day three: give 1
Day four: give 4 get back 1 and 2
Day five: five 1
Day six: give 2 get back 1
Day seven: give back 1

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Tuesday, November 16, 2010

In a country, the combined salary of the top 10% most paid people is equal to the combined salary of the rest 90% of population. Is it possible that in each state of this country the combined salary of any 10% people from this state does not exceed 11% of the entire state's salary.

Solution. Yes, it is possible. For example, consider a country where in all states people are paid equal salary, but one state, where 10% of people leave and are paid 90% of total country's salary.

Example: two states. In 1st has population 1000, and in the second state population is 9000. Assume that each employee in first state is paid $\$x$ and each one in the second state is paid $\$y$, with $x > y$. Clearly, in each state, the combined salary of any 10% people of this sate is exactly 10% of the entire state salary, hence less than 11%. Clearly, the top 10% paid employee are those 1000 from the first state, with combined salary of $\$1000x$. The combined county wise salary is equal to $1000x + 9000y$, and thus 90% of it will be equal to $900x + 8100y$. Solving the equation

$$1000x = 900x + 8100y,$$

we get $x = 81y$. Hence, we can take $y = \$50,000/\text{year}$ and then $x = \$4,050,000/\text{year}$.

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Tuesday, November 23, 2010

Suppose that there are given thirteen gears, each of them weighting a integer number of grams. Assume that every twelve of them can be split in two equally weighted groups of six gears each. Prove that all gears have the same weight.

Solution. First, we claim that either all gears weight an even number of grams or all of them are of odd weight. Indeed, since any set of 12 gears can be divided in two groups of equal weight, then the 12 gears will weight an even number of grams. The total weight of 12 gears remain even if we exchange one gear with the thirteen one. This is possible if and only if the thirteen gear and the one that was removed from the twelve gears have both either even weight or odd weight. Since the removed gear was arbitrarily chosen, it follows that all twelve gears have the weight either even or odd depending on the weight of the thirteen gear (even or odd).

Subtract from each weight the weight of the lightest gear. Note that the new set satisfies the conditions of the problem too (considering the gear(s) of weight zero being part of the set of gears and having even weight). The gears in the new set have all even weight (zero is among the weights, and hence all have the same parity as zero). Form a new set of gears, by dividing the weights by 2 (can do that, since all weights are even).

Assume that the gears in the last obtained set are not of equal weight. Repeat the procedure above (subtract the lightest divide by two). This new set also satisfies the conditions of the problem. Eventually, some of the gears will have even weight (at least one weights zero grams), and some of them weight odd number of grams (dividing consequently by 2 one gets an odd number), that contradicts the fact that all must weight the same parity of grams.

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Friday, February 11, 2011

Is it enough to have 960 square inches of material in order to build a parallelepiped (box, closed from all sides) with integer edges of volume larger than 2000 cubic inches.

Solution. Consider the box of dimensions $11 \times 13 \times 14$. Its volume is equal to 2002. The total surface area is equal to $2(11 \cdot 13 + 11 \cdot 14 + 13 \cdot 14) = 958$, hence 960 square units will be enough.

To get to this numbers, one has to recall that the minimal surface area among all parallelepiped of fixed volume has cube. The cube with volume 2000 has edge length about 12.599. Thus, one has to look at the parallelepiped with integer edge's length 12, 13, 14.

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Friday, February 17, 2011

Consider a unit square and suppose that one of its diagonals is marked. The square is divided in a finite number of smaller squares (could be of different sizes). Is it possible that the sum of the perimeters of the smaller squares that contain part of the marked diagonal is larger than 2010?

Solution.

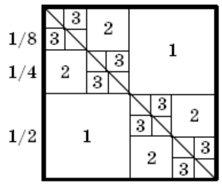

Figure 1: Division of unit square

Fix one diagonal of the unit square. Divide the unit square into four equal squares. The two squares that have only one common point with the diagonal are marked and called 'level one squares'. The other two are divided in four equal squares each, and out of these 8, four of which are having with the diagonal only one common point. Call them 'level two squares'. Continue the procedure 505 times (see the picture Fig. 1). Every time we get 2^k 'squares of level k', each of them of length $1/2^k$. Hence, the total perimeter of all 'squares of level k' is equal to $2^k \cdot 4 \cdot 1/2^k = 4$. Hence, the sum of perimeters of all obtained squares is equal to $4 \cdot 505 = 2020$ which is larger than 2010.

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Friday, February 25, 2011

Prove that any function $f : \mathbb{R} \to \mathbb{R}$ that satisfies one of the following identities

$$f(x+y) = f(x) + f(y), \quad x, y \in \mathbb{R}$$
$$f(xy+x+y) = f(xy) + f(x) + f(y), \quad x, y \in \mathbb{R}$$

satisfies the other identity too.

Solution. If first identity is true then

$$f(xy+x+y) = f(xy) + f(x+y) = f(xy) + f(x) + f(y)$$

and hence the second identity is also satisfied.

Next assume that f satisfies second identity. Take $y = u + v + uv$, then

$$f(x+u+v+xu+xv+uv+xuv) = f(x) + f(u+v+uv) + f(xu+xv+xuv)$$

and hence

$$f(x+u+v+xu+xv+uv+xuv) = f(x) + f(u) + f(v) + f(uv) + f(xu+xv+xuv) \quad (1)$$

Taking $x := u$ and $u := x$ in the previous identity we get

$$f(x+u+v+xu+xv+uv+xuv) = f(x) + f(u) + f(v) + f(xv) + f(xu+uv+xuv) \quad (2)$$

. Combining (1) and (2) we get

$$f(uv) + f(xu+xv+xuv) = f(xv) + f(xu+xv+xuv) = f(xv) + f(xu+uv+xuv)$$

Take $x = 1$ in the last identity

$$f(uv) + f(u+v+uv) = f(v) + f(u+2uv)$$

and hence

$$f(uv) + f(u) + f(v) + f(uv) = f(v) + f(u+2uv)$$

or

$$f(u) + 2f(uv) = f(u+2uv). \quad (3)$$

Take $u = 1$ in (3) and deduce that $f(0) = 3f(0)$, and thus $f(0) = 0$. Put $v = -1$ in (3) and deduce that $f(-u) = f(u) + 2f(-u)$ or $f(-u) = -f(u)$. Put $v = -1/2$ in (3) and get

$$f(0) = f(u) + 2f(-u/2).$$

From above identities, we conclude

$$f(u) = 2f(u/2)$$

or

$$f(2u) = 2f(u).$$

From here and (3) we obtain

$$f(u + 2uv) = f(u) + f(2uv),$$

and finally substitute $2v = t$

$$f(u + ut) = f(u) + f(ut).$$

Note that for $x = 0$ the initial identity of interest is obvious, and for $x \neq 0$, using last identity,

$$f(x + y) = f(x + x \cdot \frac{y}{x}) = f(x) + f(x \cdot \frac{y}{x}) = f(x) + f(y).$$

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Friday, March 04, 2011

Assume that a flea is jumping on the set of positive integer numbers under the following assumptions: the flea can start (at time zero) from any positive integer; it jumps every second and it can jump only to next larger integer. For example it can start at 144 and in one second it will jump to 145 and then to 146 and so on. We don't know where it starts from, but after each jump we can try to hit it with a club. We can hit on any integer, and we will know whether we had hit the flea or not. However we do not see where the flea is sitting at a given moment nor from where it started. Provide a winning strategy to catch the flea.

Solution. We will solve the problem in a more general setup, assuming that it can start the jumping process at any integer, not necessarily positive ones. We will divide our hits into two sets.

1. After the first second we hit on 0 - if it started from -1 we're done. At our third second we hit on 1 if it started from -2 we've just caught him, and so on. After each of the k odd seconds we hit on the number $(k-1)/2$. If the flea started from a negative number t, it will slowly be catching up to our hits since we only cover the distance of one in two seconds. And we will catch it with our $(-2)t-1$th hit.

2. After the second second we hit on 2 - if it started from zero we hit it. After the fourth second we hit on 5, if it started from 1, we've got the little critter. After each of the k even seconds we hit on $3(k-2)-2$ and thus if it started from $k/2-1$ we'll catch it. If it really started from a positive integer we'll be catching up to it because we cover a distance of three during every two seconds while it will only cover two. Eventually if it started from $t \geq 0$ we'll catch it with our $2(t+1)$th hit.

Combining the two hit-sequences provided above, we will be catching the flea for sure. (More precisely, if it started from an integer with the absolute value z, we'll catch it within $2z+2$ seconds.

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