PRINT Last name:________________ First name:________________

Signature:________________ Student ID:________________

Math 430 Final, Fall 2006

I. Examples, Counterexamples and short answer. (4 pts ea.) Do not give proofs, but clearly indicate your proposed example or counterexample, or short answer where appropriate.

1. Define what makes a group cyclic. Give an example of a cyclic group with order > 1.

2. Define what makes a group Abelian. Give an example of an Abelian group which is not cyclic.

3. Define the center of a group. Give an example of a non-Abelian group G such that its center $Z(G)$ has more than one element.

4. At least two types of groups we have studied have order > 1 and trivial center, i.e., $|Z(G)| = 1$. Give an example of such a group. (We could say that examples 1-4 represent a decreasing hierarchy of "Abelian-ness." The next step might be *simple groups*, which have no normal subgroups other than $\{e\}$ and the group itself.)

5. What is the order of the element $6 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{48}/\langle 8 \rangle$?

6. Find all elements $g \in U(9)$ such that $\langle g \rangle = U(9)$.

7. Find a homomorphism ϕ from $\mathbb{Z}$ to $\mathbb{Z}$ such that $\mathbb{Z}/\mathrm{Ker}\phi \approx \mathbb{Z}_6$.

8. Find the number of order 2 elements in $D_6 \oplus D_4$. For partial credit, it is necessary to describe each element you find, or at least explain the combinatorial enumeration.

9. Write the permutation (12345) as a product of 2-cycles.

10. Write 1 as an integer combination of 11 and 19.

II. Constructions and Algorithms. (12 pts ea.) Do not write proofs, but do give clear, concise answers, including steps to algorithms where applicable.

11. **(a)** On the left side, list all Abelian groups of order 180 up to isomorphism, by expressing them as external direct products of cyclic groups of prime power order.
(b) On the right side, for each group in your list on the left, re-express it in the form $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ where $n_i | n_{i-1}$ for all $1 < i \leq k$.

12. A certain group of order 6 is generated by the elements a and b, where $|a| = 3$, $|b| = 2$ and $ab = ba^2$. Complete the Cayley table for the group, expressing each group element in the form $a^n b^m$ or $b^m a^n$ with $n, m \geq 0$.

	e	a	b			
e	e	a	b			
a	a					
b	b					

III. Proofs. (12 pts ea.) Part of the score is determined by careful formatting of the proof (forward and reverse directions, assumptions, conclusions, stating whether you are proving by direct proof, contrapositive, contradiction, induction, etc.). Partial credit will be awarded for this as well.

Prove ***ONE*** of 13-14. Clearly indicate which proof you want graded.

13. Assume that whenever a, b, c belong to the group G, if $ab = ca$, then $b = c$. Prove as a result that G is Abelian.

14. Prove that if $a_1, \ldots, a_n$ are distinct elements of the group G, and $b \in G$, then $ba_1, \ldots, ba_n$ are distinct elements of G.

Prove ***ONE*** of 15-17. Clearly indicate which proof you want graded.

15. Let $\mathbb{C}^*$ be the nonzero complex numbers under multiplication, and let $U = \{a + bi \in \mathbb{C} : \sqrt{a^2 + b^2} = 1\}$. Assuming that $\mathbb{C}^*$ is a group, prove that U is a subgroup of $\mathbb{C}^*$.

16. Prove that there is no largest finite subgroup of the nonzero complex numbers $\mathbb{C}^*$ under multiplication.

17. Let $G = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$. Prove that G under multiplication is isomorphic to $\mathbb{Z}$ under addition. (Don't skimp on the steps!)

Prove ***ONE*** of 18-19. Clearly indicate which proof you want graded.

18. The elements of the Quaternion Q group are $\{\pm U, \pm I, \pm J, \pm K\}$ with identity U, where $I^2 = J^2 = K^2 = -U$, $IJ = K$, $JK = I$, $KI = J$, $JI = -K$, $KJ = -I$, $IK = -J$, and $-$'s can be factored through; i.e., $(-G)(-H) = GH$ and $-(-G) = G$. Classify the possible homomorphic images of Q up to isomorphism.

19. Let $S = \left\{\dfrac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \text{ is odd}\right\}$. Prove or disprove that S is a subring of $\mathbb{Q}$, the rationals under addition and multiplication.