PRINT Last name: KEY First name:

Signature: Student ID:

Math 430 Exam 1, Fall 2006

I. Examples, Counterexamples and short answer. (7 pts ea.) Do not give proofs, but clearly indicate your proposed example or counterexample, or short answer where appropriate.

1. The United Parcel Service pickup record number consists of 9 digits $a_1 \cdots a_9$ plus 1 check digit a_{10}, defined by $a_{10} = a_1 \cdots a_9 \mod 7$. Give an example of an identification number and an error in that number which the check digit does not catch.

   a_1 ... a_9 a_{10}

   000000000 0

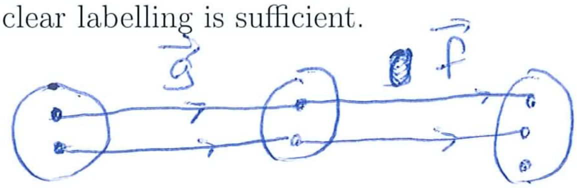
   000000007 0

   ~~a_1 1 2 3 4 5 6 7~~

2. Give an example of functions f and g such that

   (i) f is 1-1
   (ii) g is onto, and
   (iii) the composition fg is not onto.

   Be sure to specify the domain, range, and rule for each function. Or, a pictorial representation with clear labelling is sufficient.



3. Choose a group G. **(i)** Give two elements of the G which commute (i.e., $ab = ba$). **(ii)** Give two elements of G which do not commute. Clearly describe G and the elements a and b.

   $G = D_4 = \{R_0, R_{90}, R_{180}, R_{270}, D, H, V, D'\}$

   (ii) $R_{90}D \neq DR_{90}$

   (i) $R_{90}R_{180} = R_{180}R_{90}$

4. Give an example of a group G and elements $a, b \in G$ such that $a^{-1} = a$ but $b^{-1} \neq b$.

$\mathbb{Z}_4$ under addition mod n. $a = 2$, $b = 1$

$2^{-1} = -2 = 2 \mod 4$

$1^{-1} = -1 = 3 \mod 4$

and $1 \neq 3$.

5. **(i)** Give an example of a group G and an element $a \in G$ such that $G = \langle a \rangle$. **(ii)** Give an example of a group G such that for all $a \in G$, $G \neq \langle a \rangle$ (no element generates the group).

(i) $\mathbb{Z}_4 = \langle 1 \rangle$

(ii) D_4 or $U(8)$ (for example)

or any non-abelian group.

6. Give an example of an infinite group G with a finite subgroup H, which contains more than just the identity e of G. (That is, $\{e\} \subsetneq H \subsetneq G$.)

Let $G = GL(\mathbb{R}, 2)$

$= 2\times 2$ matrices over reals with nonzero determinant.

Let $H = \left\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$.

or Let $G = \mathbb{C}^*$ under multiplication

and $H = \langle i \rangle$. etc.

III. Proofs. (15 pts ea.) Part of the score is determined by careful formatting of the proof (forward and reverse directions, assumptions, conclusions, stating whether you are proving by direct proof, contrapositive, contradiction, induction, etc.). Partial credit will be awarded for this as well.

Prove ***ONE*** of 9-10. Clearly indicate which proofs you want graded.

9. Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. You may assume that $(a^{-1})^{-1} = a$ for all $a \in G$.

10. Prove that for any elements a, b in a group G and any integer n, $(a^{-1}ba)^n = a^{-1}b^n a$. (Hint: induction is a possible proof strategy.)

9. ($\Rightarrow$) Assume G is abelian.
By definition, $ab = ba$ $\forall a, b \in G$.
Let $a, b \in G$.
$(ab)^{-1} = (ba)^{-1}$ since $ab = ba$.
$= a^{-1}b^{-1}$ since $baa^{-1}b^{-1} = a^{-1}b^{-1}ba = e$.

($\Leftarrow$) Assume $(ab)^{-1} = a^{-1}b^{-1}$ $\forall a, b \in G$.
Let $a, b \in G$.
$ab = ((ab)^{-1})^{-1}$ by assumption in problem
$= (a^{-1}b^{-1})^{-1}$ by hypothesis
$= ba$ since $a^{-1}b^{-1}ba = baa^{-1}b^{-1} = e$ $\square$

10. Proof is by induction on $|n|$.
Base case $n=0$. $(a^{-1}ba)^0 = e$ by definition,
and $a^{-1}b^0a = a^{-1}a = e$.

assume true for $|n| = k \in \mathbb{Z}^+$.

~~case 1~~ → Now suppose $|n| = k+1$.

case 1: $n = k+1$.
$(a^{-1}ba)^{k+1} = (a^{-1}ba)^k a^{-1}ba = a^{-1}b^k a a^{-1}ba$ by inductive hypothesis
$= a^{-1}b^{k+1}a$.

case 2: $n = -k-1$. $(a^{-1}ba)^{-k-1} = ((a^{-1}ba)^{k+1})^{-1} = (a^{-1}b^{k+1}a)^{-1}$ by case 1
$= a^{-1}b^{-k-1}a$ $\square$

II. Constructions and Algorithms. (14 pts ea.) Do not write proofs, but do give clear, concise answers, including steps to algorithms where applicable.

7. Find $\gcd(354, 126)$ using the Euclidean algorithm. Find $s, t \in \mathbb{Z}$ such that $\gcd(354, 126) = 354 \cdot s + 126 \cdot t$.

$$\begin{array}{r} 354 \\ -252 \\ \hline 102 \end{array}$$

$354 = 126 \cdot 2 + 102$
$126 = 102 \cdot 1 + 24$
$102 = 24 \cdot 4 + 6$
$24 = 6 \cdot 4 + 0$

$\therefore \gcd(354, 102) = 6$

$6 = 102 - 24 \cdot 4$
$= 102 - (126 - 102 \cdot 1) \cdot 4$
$= 102 \cdot 5 - 126 \cdot 4$
$= (354 - 126 \cdot 2) \cdot 5 - 126 \cdot 4$
$= 354 \cdot 5 - 126 \cdot 14$
$(s = 5,\ t = -14)$

8. The group of Quaternions Q can be viewed as a subgroup of the 2×2 complex matrices with determinant 1 under matrix multiplication. The 8 elements of Q are $\pm U$, $\pm I$, $\pm J$ and $\pm K$, where

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

(i) Generate the Cayley table for Q (score = percentage of correct entries – you don't have to show work).
(ii) Identify the center $Z(G)$ of Q on the table.
(iii) Identify the centralizer $C(I)$ of I on the table.

$Z(G)$

	U	I	J	K	$-U$	$-I$	$-J$	$-K$
U	U	I	J	K	-U	-I	-J	-K
I	I	-U	K	-J	-I	+U	-K	J
J	J	-K	-U	I	-J	K	U	-I
K	K	J	-I	-U	-K	-J	I	U
$-U$	-U	-I	-J	-K	U	I	J	K
$-I$	-I	U	-K	J	I	-U	K	-J
$-J$	-J	K	U	-I	J	-K	-U	I
$-K$	-K	-J	I	U	K	J	-I	-U

$C(I)$

note $\epsilon_j A \cdot \epsilon_k B = \epsilon_j \epsilon_k AB$ where ϵ_j, ϵ_k are $+1, -$

Prove ***ONE*** out of 11-12. Clearly indicate which proof you want graded.

11. Let H be a subgroup of $\mathbb{R}$ under *addition*. Let $K = \{2^a : a \in H\}$. Prove that K is a subgroup of $\mathbb{R}^*$ under *multiplication*.

12. Let $H = \{a + bi : a, b \in \mathbb{R}, ab \geq 0\}$. Prove or disprove that H is a subgroup of $\mathbb{C}$ under addition.

11. By 1-step subgroup test.
K is nonempty: since $H \leq \mathbb{R}$, $0 \in H$.
Therefore $2^0 = 1 \in K$ and K is nonempty.
Let $x, y \in K$. Then $x = 2^a$ and $y = 2^b$ for some $a, b \in H$.
Since H is a subgroup, $a - b \in H$.
Therefore $2^{a-b} = 2^a 2^{-b} = 2^a (2^b)^{-1}$
$= xy^{-1} \in K$. □

12. H is not a subgroup of $\mathbb{C}$ under addition.
$x = 1 + 0i \in H$ since $1 \cdot 0 = 0 \geq 0$.
$y = 0 - 1i \in H$, since $0(-1) = 0 \geq 0$.
But $x + y = 1 - 1i \notin H$
since $1(-1) = -1 < 0$.
Therefore H fails to be closed with respect to the operation of $\mathbb{C}$ and is not a subgroup.

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