## Motivation

Just as definite integrals can be solved using antiderivatives by the Fundamental Theorem of Calculus, we also want a method of solving finite summations of the form $\sum_{i=1}^{n} f(i)$. The summations appear frequently in mathematics: in finite difference methods used to solve differential equations numerically on a computer, in combinatorial identities that count discrete objects by grouping them according to type, and in countless theorems for which a closed form of a summation is useful.

## Finite Difference Formula

In calculus we have the familiar derivative formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For discrete applications, we may not be able to let $h$ go to 0. For example, $f$ might be defined only for integer $x$, or perhaps we are implementing a numerical procedure which samples $f$ only at values of $x$ with spacing like the integers. In this case we may alternatively set $h = 1$ and define the finite (forward) difference

$$\Delta f(x) = f(x+1) - f(x).$$

If we take, for integer $m$, the definition of the falling power of $x$ to be

$$x^{\underline{m}} = \begin{cases} x(x-1)\cdots(x-m+1) & \text{if } m > 0, \\ 1 & \text{if } m = 0, \\ \frac{1}{(x+1)\cdots(x+(-m))} & \text{if } m < 0; \end{cases}$$

then it is a straightforward exercise in algebra to derive the following finite difference analogy to the power rule $(d/dx)x^m = mx^{m-1}$:

$$\Delta x^{\underline{m}} = \begin{cases} m x^{\underline{m-1}} & \text{if } m \neq 0, \\ 0 & \text{if } m = 0. \end{cases} \tag{1}$$

## A Word on Summation Notation

We will be using two forms of summation notation in what follows. First, is the notation we are accustomed to, namely

$$\sum_{k=1}^{n} g(k) = g(1) + g(2) + \cdots + g(n-1) + g(n).$$

Here the index $k$ ranges from 1 up to $n$, hitting every integer value in between and both endpoints 1 and $n$.

The second notation is a formal notation for defining the analogy to a definite integral, namely the definite sum

$$\sum_{a}^{b} g(x)\delta x.$$

We don't know this now before reading the rest of these notes, but the relationship between these two sums is

$$\sum_{1}^{n} g(x)\delta x = \sum_{x=1}^{n-1} g(x) = g(1) + g(2) + \cdots + g(n-1).$$

In other words, the top index $n$ is missing when we transfer from the second formal notation to the usual summation notation we already know about.

## Antiderivatives and Antidifferences

In calculus we define antiderivatives by

$$g(x) = f'(x) \qquad \text{if and only if} \qquad \int g(x)\,dx = f(x) + C,$$

*Project Sponsor:* Robert Ellis, Asst. Prof. of Applied Mathematics, E1 105C, `rellis@math.iit.edu`

where $C$ is any constant. In other words, $f(x) + C$ is the class of functions whose derivatives are all $g(x)$. By analogy, we formally define the indefinite sum $\sum g(x)\delta x$ by

$$g(x) = \Delta f(x) \qquad \text{if and only if} \qquad \sum g(x)\delta x = f(x) + C;$$

where $C$ could be a constant, or since the finite difference is defined with $h = 1$ spacing, $C$ could be any periodic function with $C(x) = C(x+1)$ for all $x \in \mathbb{R}$, such as $\sin 2\pi x$. The formal sum $\sum g(x)\delta x$ is still vague, but we can make it more concrete by making an analogy with definite integrals.

## Definite Integrals and Definite Sums

In calculus we define definite integrals by

$$\int_a^b g(x)\,dx = f(x)|_a^b = f(b) - f(a),$$

whenever $f(x)$ is an antiderivative of $g(x)$. Now suppose $f(x)$ is an antidifference of $g(x)$, and let us consider what we might get for finite calculus. First we need notation. Let us decide by fiat that the notation we want is $\sum_a^b g(x)\delta x$, and we want to define this sum so that, analogous to definite integrals,

$$\sum_a^b g(x)\delta x = f(b) - f(a).$$

When $b = a$, we would like to get $\sum_a^a g(x)\delta x = f(a) - f(a) = 0$, analogously to $\int_a^a g(x)\,dx = 0$. When $b = a+1$, we would like to get $\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a)$, which by the definition of $g(x) = \Delta f(x)$, gives

$$\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a).$$

Now say that $b = a + k$, where $m$ is some positive integer. Then we have

$$\begin{aligned}
\sum_a^{a+m} g(x)\delta x &= f(a+m) - f(a) \\
&= [f(a+m) - f(a+m-1)] + [f(a+m-1) - f(a+m-2)] + \\
&\quad \cdots + [f(a+2) - f(a+1)] + [f(a+1) - f(a)] \\
&= g(a+m-1) + g(a+m-2) + \cdots + g(a+1) + g(a).
\end{aligned}$$

The middle line is just a telescoping sum that we insert to make things work out. We see that this is exactly what we need to define the definite sum as a concrete object.

$$\sum_a^b g(x)\delta x := \sum_{x=a}^{b-1} g(x) = g(a) + g(a+1) + \cdots + g(b-1). \tag{2}$$

Notice the sum goes to $b-1$ instead of to $b$; this is required to make the analogy with definite integrals work.

## Table of Finite Differences and Indefinite Sums

Define the harmonic number function

$$H(x) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x},$$

when $x$ is an integer. Furthermore, define the "right-shift" operator

$$Ef(x) = f(x+1).$$

We have the following table of differences and sums, like an integral table.

Table of finite differences and indefinite sums

| $f$ | $\Delta f = g$ | $f$ | $\Delta f = g$ |
|---|---|---|---|
| $x^{\underline{0}} = 1$ | $0$ | $2^x$ | $2^x$ |
| $x^{\underline{1}} = x$ | $1$ | $c^x$ | $(c-1)c^x$ |
| $x^{\underline{m}}$ | $mx^{\underline{m-1}}$ | $c \cdot f$ | $c\Delta f$ |
| $x^{\underline{m+1}}/(m+1)$ | $x^{\underline{m}}$ | $f+g$ | $\Delta f + \Delta g$ |
| $H(x)$ | $x^{\underline{-1}}$ | $fg$ | $f\Delta g + Eg\Delta f$ |

Recognizable in the right-hand column are the constant, sum, and product rules of finite difference, analogous to differentiation. Also, recall that reversing the product rule gives us a sum by parts (cf. integration by parts) formula:

$$\sum u\Delta v = uv - \sum Ev\Delta u \tag{3}$$

which can be made into a definite sum formula:

$$\sum_a^b u\Delta v = u(b-1)v(b-1) - u(a)v(a) - \sum_a^b Ev\Delta u \tag{4}$$

## Binomial Coefficients

Finally, recall that the binomial coefficients are defined for integer $n$ and $k$ as

$$\binom{n}{k} = \begin{cases} \frac{n!}{(n-k)!}k! & \text{if } 0 \le k \le n, \\ 0 & \text{if } n < k. \end{cases}$$

We have the following simple binomial identity:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{5}$$

## Project Exercises

1. Prove the finite difference power rule in (1).
2. Prove the sum rule $\Delta(f+g) = \Delta f + \Delta g$ of the table.
3. Verify the identity $x^2 = x^{\underline{2}} + x^{\underline{1}}$, and use it to compute the sum $\sum_{k=0}^{n-1} k$ using finite calculus facts.
4. Prove $\Delta H(x) = x^{\underline{-1}}$ (you may assume $x$ is a positive integer).
5. Prove the two exponential rules in the table, i.e., $\Delta 2^x = 2^x$ and $\Delta c^x = (c-1)c^x$. Explain two connections (analogies or differences) with the usual calculus or formulas you are familiar with.
6. Use the definite sum by parts formula in (4) to evaluate $\sum_{k=1}^{n-1} k2^k$.
7. Compute $\Delta\binom{n}{k}$, and use (5) to compute $\sum_{x=0}^{n-1}\binom{x}{k}$.
8. Suppose that instead of defining the finite difference as $\Delta f(x) = f(x+1) - f(x)$, that we define $\nabla f(x) = f(x) - f(x-1)$. Can we define finite difference and indefinite sum formulas analogously to the table?
9. Suppose that instead of defining the finite difference as $\Delta f(x) = f(x+1) - f(x)$, we define it as the "middle difference" $\Delta f(x) = f(x+1/2) - f(x-1/2)$. What is the second finite difference $\Delta^2 f(x)$ of $f(x)$ (analogous to the second derivative)? Can we find any nice formulas for second derivatives under this assumption?

# References

[1] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete mathematics*. Addison-Wesley, Reading, MA, 1989.