PRINT Last name: Key First name:

Signature: Student ID:

Math 230 Exam 1, Spring 2008

1. (8pts) Determine whether or not $p \to (q \to r)$ is logically equivalent to $(p \to q) \to r$. Show your work or carefully describe your argument.

p	q	r	$q \to r$	$p \to (q \to r)$	$p \to q$	$(p \to q) \to r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

not same colums, so not logically equivalent.

2. (8pts) For which rows of the truth table is the compound proposition $(p \oplus q) \to (q \leftrightarrow r)$ false?

p	q	r	$p \oplus q$	$q \leftrightarrow r$	$(p \oplus q) \to (q \leftrightarrow r)$
T	T	T	F	T	T
T	T	F	F	F	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	F	T	T

False rows

3. (6pts) The original statement is "If $\underbrace{1 + 1 = 3}_{F}$ then $\underbrace{2 + 2 = 4}_{T}$." Circle the correct truth value of each of the following statements:

Contrapositive of the original statement (True / False) $F \to T$

Converse of the original statement (True / False) $T \to F$

Inverse of the original statement (True / False) $T \to F$

4. (4pts) Write the negation of the following statement (Do not write "It is not the case that ...").
"I will go to the movies or read a book but not both."
I will go to the movies if and only if I read a book.
– or –
Either I will go to the movies and read a book, or I will do neither.

5. (4pts) Is the following argument valid? (Circle Yes / No)

$$\begin{array}{l} p \to q \\ \underline{\quad\ \neg p} \\ \therefore\ \neg q \end{array}$$

6. (4pts) Is the following argument valid? (Circle Yes / No)

$$\begin{array}{l} p \to \neg q \\ \underline{\qquad\ q} \\ \therefore\ \neg p \end{array}$$

7. (8pts) Among these 5 propositions are exactly 1 tautology and exactly 1 contradiction. Write $\mathbb{T}$ next to the tautology. Write $\mathbb{F}$ next to the contradiction. Do nothing for the rest of the propositions.

_____(a) $p \vee q \vee r$

$\mathbb{T}$ (b) $p \vee (p \wedge \neg q) \vee \neg p$

_____(c) $p \vee (q \wedge r)$

_____(d) $q \to \neg q$

$\mathbb{F}$ (e) $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$

8. (8pts) Define $P(x, y)$ to be the predicate "$x + 2y = xy$". Circle the truth value of the following statements. (Recall that $\mathbb{Z}$ is the set of integers.)

(True / False) (a) $P(0, 2)$ $\quad 0 + 4 = 0$

(True / False) (b) $P(1, -1)$ $\quad 1 - 2 = -1$

(True / False) (c) $\exists y \in \mathbb{Z}\ P(3, y)$ $\quad 3 + 2y = 3y \Leftrightarrow 3 = y$

(True / False) (d) $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}\ P(x, y)$ $\quad$ set $y = 1$: $x + 2 = x \Leftrightarrow 2 = 0$ no integer solution

9. (5pts) Write the negation of the following proposition so that (i) All quantifiers are to the left of negations (this means no $\neg\forall$ or $\neg\exists$), and (ii) No negations appear outside of a set of parentheses (this means no $\neg(\cdots)$):

$$\exists x\ (P(x) \to (Q(x) \wedge \neg R(x)))$$

$\forall x\ \neg(P(x) \to (Q(x) \wedge \neg R(x)))$

$\equiv \forall x (P(x) \wedge \neg(Q(x) \wedge \neg R(x))) \equiv \boxed{\forall x (P(x) \wedge (\neg Q(x) \vee R(x)))}$

10. (4pts) The associative property of multiplication of the set of real numbers $\mathbb{R}$ says that you can multiply three real numbers in arbitrary order and get the same result. Express this property as a quantified statement.

$\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \forall z \in \mathbb{R}\ (x(yz) = (xy)z)$

11. (5pts) For this question, $F(A)$ is the predicate "A is a finite set," $S(A,B)$ is the predicate "A is a subset of B," and the domain of every quantifier is the universe of all sets. Translate the following statement into a concise, meaningful English sentence (Do not use "It is not the case that..."):

$\neg\exists A \exists B(\neg F(A) \wedge F(B) \wedge S(A,B))$

No finite set contains an infinite set.

12. (8pts) Among a certain group of 27 people, exactly 2 people were born on Sunday. Prove that at least 5 people were born on the same day of the week.

Assume 2 were born on Sunday.
Assume to the contrary that on no day were ≥ 5 people born.
Then the total number of people born on all days is

$$\leq \underbrace{2}_{\text{Sunday}} + \underbrace{6 \cdot 4}_{\text{Mon - Sat}} = 26.$$

But this contradicts there being 27 people #.
Therefore the original statement is true. □

13. (8pts) Prove the following statement. When n is an integer, the following are equivalent:
(1) n^2 is odd;
(2) $(n+1)^2$ is even;
(3) n is odd.

(1) → (2). Assume n^2 is odd.
Then $n^2 = 2k+1$ for some $k \in \mathbb{Z}$.
$(n+1)^2 = n^2 + 2n + 1 = 2k+1+2n+1$
$= 2(k+n+1)$
Therefore $(n+1)^2$ is even.

(2) → (3) By contrapositive.
Assume n is even.
Then $n = 2k$ for some $k \in \mathbb{Z}$.
$(n+1)^2 = n^2 + 2n + 1 = 2k + 2n + 1$
$= 2(k+n)+1$
Therefore $(n+1)^2$ is odd.

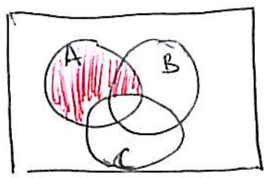
(3) → (1). Assume n is odd.
Then $n = 2k+1$ for some $k \in \mathbb{Z}$.
~~$(n+1)^2 =$~~ $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
Therefore n^2 is odd. □

14. (4pts) Write down the power set of $\{\emptyset, a\}$.

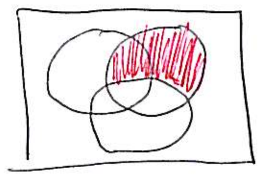
$\{\emptyset, \{\emptyset\}, \{a\}, \{a, \emptyset\}\}$

15. (8pts) Use Venn diagrams to justify which relationship ($\subseteq$, $=$, or $\supseteq$) is valid for the following pair of sets. Write the correct operator in the blank.

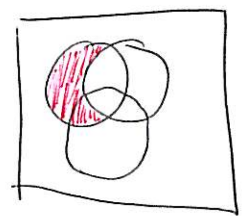
$(A - C) - (B - C)$ $\underline{\subseteq}$ $A - B$



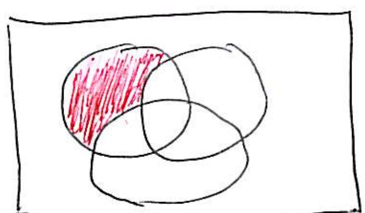
A - C



B - C



A - B



(A-C) - (B-C)

16. (8pts) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. There are several possible ways to do this. A Venn diagram can be helpful but is not a proof.

set membership table

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

same, so sets are equal.

PRINT Last name: Key First name: ____________

Signature: ____________ Student ID: ____________

Math 230 Exam 1, Spring 2008

1. (8pts) For which rows of the truth table is the compound proposition $(s \leftrightarrow t) \rightarrow (s \oplus r)$ false?

r	s	t	$s \leftrightarrow t$	$s \oplus r$	$(s \leftrightarrow t) \rightarrow (s \oplus r)$
T	T	T	T	F	F
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	F	T	T
F	F	T	F	F	T
F	F	F	T	F	F

False rows

2. (6pts) The original statement is "If $\overbrace{1+1=2}^{T}$ then $\overbrace{2+2=3}^{F}$." Circle the correct truth value of each of the following statements:

Contrapositive of the original statement (~~True~~ / False) $T \rightarrow F$

Converse of the original statement (True / False) $F \rightarrow T$

Inverse of the original statement (True / False) $F \rightarrow T$

3. (8pts) Determine whether or not $(r \rightarrow s) \rightarrow t$ is logically equivalent to $r \rightarrow (s \rightarrow t)$. Show your work or carefully describe your argument.

r	s	t	$r \rightarrow s$	$(r \rightarrow s) \rightarrow t$	$s \rightarrow t$	$r \rightarrow (s \rightarrow t)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

not same, so not logically equivalent

4. (8pts) Among these 5 propositions are exactly 1 tautology and exactly 1 contradiction. Write $\mathbb{T}$ next to the tautology. Write $\mathbb{F}$ next to the contradiction. Do nothing for the rest of the propositions.

_____ (a) $r \wedge s \wedge t$

_____ (b) $(s \wedge t) \vee (r)$

__$\mathbb{T}$__ (c) $(p \leftrightarrow q) \vee (p \leftrightarrow \neg q)$

__$\mathbb{F}$__ (d) $p \wedge (p \vee \neg q) \wedge \neg p$

_____ (e) $\neg r \rightarrow r$

5. (4pts) Write the negation of the following statement (Do not write "It is not the case that ...").
"I will ride my bike or drive my car but not both."

I will ride my bike if and only if I drive my car.

— or —

I will ride my bike and drive my car, or I will not ride my bike and not drive my car.

6. (4pts) Is the following argument valid? (Circle Yes / No)

$$\begin{array}{r} \neg r \rightarrow s \\ \neg s \\ \hline \therefore\ r \end{array}$$

7. (4pts) Is the following argument valid? (Circle Yes / No)

$$\begin{array}{r} \neg s \rightarrow \neg t \\ s \\ \hline \therefore\ t \end{array}$$

8. (5pts) Write the negation of the following proposition so that (i) All quantifiers are to the left of negations (this means no $\neg\forall$ or $\neg\exists$), and (ii) No negations appear outside of a set of parentheses (this means no $\neg(\cdots)$):

$\forall x\ (P(x) \rightarrow (\neg Q(x) \vee R(x)))$ negation: $\exists x\ \neg(P(x) \rightarrow (\neg Q(x) \vee R(x)))$

$\equiv \exists x\,(P(x) \wedge \neg(\neg Q(x) \vee R(x))) \equiv \boxed{\exists x\,(P(x) \wedge (Q(x) \wedge \neg R(x)))}$

9. (8pts) Define $Q(x, y)$ to be the predicate "$x + 2y = xy$". Circle the truth value of the following statements. (Recall that $\mathbb{Z}$ is the set of integers.)

(True / False) (a) $Q(1, -1)$ $1-2=-1$.

(True / False) (b) $Q(0, 2)$ $0+4=0$.

(True / False) (c) $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}\ Q(x, y)$ set $y=2$. $x+4=2x \Leftrightarrow 4=x$
set $y=1$. $x+2=x \Leftrightarrow 2=0$ no solution.

(True / False) (d) $\exists y \in \mathbb{Z}\ Q(3, y)$

$Q(3,y)$: $3+2y=3y$
$3=y$

10. (5pts) For this question, $F(A)$ is the predicate "A is a finite set." $S(A,B)$ is the predicate "A is a subset of B," and the domain of every quantifier is the universe of all sets. Translate the following statement into a concise, meaningful English sentence (Do not use "It is not the case that..."):

$$\neg\forall B\forall A(F(A)\wedge\neg F(B)\wedge S(A,B))$$

There is an infinite set that does not contain all finite sets.

11. (4pts) The associative property of multiplication of the set of integers $\mathbb{Z}$ says that you can multiply three integers in arbitrary order and get the same result. Express this property as a quantified statement. $\forall x\in\mathbb{Z}\,\forall y\in\mathbb{Z}\,\forall z\in\mathbb{Z}\,(x(yz)=(xy)z)$.

12. (8pts) Prove the following statement. When n is an integer, the following are equivalent:
(1) n^2 is even;
(2) $(n+1)^2$ is odd;
(3) n is even.

$(1)\to(2)$: Assume n^2 is even.
Then $n^2=2k$ for some $k\in\mathbb{Z}$.
$(n+1)^2=n^2+2n+1=2k+2n+1$
$=2(k+n)+1$.
Therefore $(n+1)^2$ is odd.

$(2)\to(3)$ by contrapositive.
Assume n odd.
Then $n=2k+1$ for some $k\in\mathbb{Z}$.
$(n+1)^2=n^2+2n+1=2k+1+2n+1$
$=2(k+n+1)$
Therefore $(n+1)^2$ is even.

$(3)\to(1)$. Assume n is even.
Then $n=2k$ for some $k\in\mathbb{Z}$.
$n^2=(2k)^2=4k^2=2(2k^2)$.
Therefore n^2 is even.

$\square$

13. (8pts) Among a certain group of 35 people, exactly 1 person was born in the month of January. Prove that there is a month in which at least 4 people were born.

Assume ~~to the contrary~~ that 1 person was born in January, and assume to the contrary that there is no month in which at least 4 people were born.
In Feb through Dec., at most 3 people we born each month.
In total $\leq \underbrace{1}_{\text{Jan}}+\underbrace{3\cdot 11}_{\text{Feb - Dec}}=34$ people were born.
This contradicts there being 35 people, # and so the original statement is true. $\square$

14. (8pts) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. There are several possible ways to do this. A Venn diagram can be helpful but is not a proof.

Set membership table

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$(A \cup B)$	$(A \cup C)$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

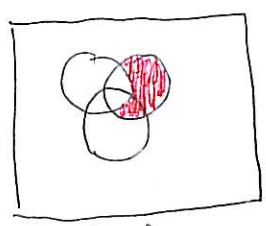
same, so sets equal.

15. (4pts) Write down the power set of $\{b, \emptyset\}$.

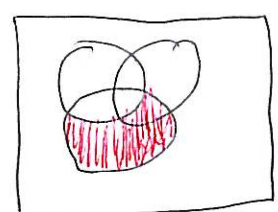
$\{\emptyset, \{b\}, \{\emptyset\}, \{b, \emptyset\}\}$

16. (8pts) Use Venn diagrams to justify which relationship ($\subseteq$, $=$, or $\supseteq$) is valid for the following pair of sets. Write the correct operator in the blank.

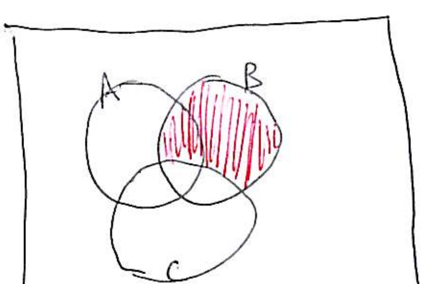
$B - C$ $\underline{\ \supseteq\ }$ $(B - A) - (C - A)$



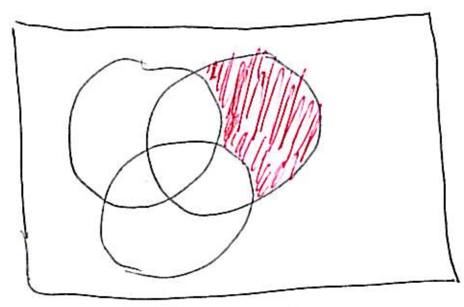
B-A



C-A



B-C



(B-A)-(C-A)