I. Short answer (2 pts each). No partial credit – only the response will be graded. Suggested time 55 minutes.

1. Compute the following: 23 mod 5 = 3　　　　-40 mod 7 = 2

$23 = 4 \cdot 5 + 3$　　　　$-40 = -6 \cdot 7 + 2$

2. Suppose that $P(n)$ is the statement "$n+1 = n+2$." What is wrong with the following "proof" that the statement $P(n)$ is true for all nonnegative integers n:

You assume that $P(k)$ is true for some positive integer k; that is, $k+1 = k+2$. Then you add 1 to both sides of this equation to obtain $k+2 = k+3$; therefore $P(k+1)$ is true. By the principle of mathematical induction $P(n)$ is true for all nonnegative integers n.

(a) There is nothing wrong with this proof.

(b) The proof is incorrect because the statement used in the inductive hypothesis is incorrect.

(c) The proof is incorrect because there is no basis step.

(d) The proof is incorrect because you cannot add 1 to both sides of the equation in the inductive step.

3. For which of the following is the set S equal to the set of positive integers not divisible by 3?

(a) $1 \in S$; $x \in S \to 2x + 2 \in S$　　**(b)** $1 \in S$; $x \in S \to x + 3 \in S$

(c) $1 \in S$; $2 \in S$; $x \in S \to x + 3 \in S$.　　**(d)** $1 \in S$; $x \in S \to 3x + 1 \in S$

4. Which of the following is a recursive definition for $a_n = 4n + 3$, for $n \in \mathbb{N}$?

(a) $a_n = 2a_{n-1} + 1$, $a_0 = 3$.　　**(b)** $a_n = a_{n-1} + 4n$, $a_0 = 3$.

(c) $a_n = a_{n-2} + 8$, $a_0 = 3$, $a_1 = 7$.　　**(d)** $a_n = a_{n-1} - a_{n-2}$, $a_0 = 3$, $a_1 = 7$.

5. How many permutations of the set $\{A, B, C, D, E, F, G\}$ either begin or end with a vowel (A or E)? (Note: a permutation of a set contains each letter exactly once.)

(a) $2 \cdot 6! + 2 \cdot 6!$.

(b) $2 \cdot 6! + 2 \cdot 6! - 2 \cdot 5!$.

(c) $6! + 6! - 5!$.

(d) $6! + 6!$.

(e) $7! - 5!$.

A _ _ _ _ _ _ : 6!
E _ _ _ _ _ _ : 6!
_ _ _ _ _ _ A : 6!
_ _ _ _ _ _ E : 6! +
A _ _ _ _ _ E : 5! −
E _ _ _ _ _ A : 5! −

6. Consider the statement "If the product of two integers is even, then their sum is also even." Circle the letter of the correct assertion.

   (a) The statement is false as can be shown by finding a counterexample.

   (b) The statement is true and can be proved most easily by using a direct proof.

   (c) The statement is true and can be proved most easily using a proof by contradiction.

   (d) The statement is true and can be proved most easily using a proof by contraposition.

7. Circle the letter of the statement that is true for all sets S and T.

   (a) $S \cap \overline{T} = \overline{S} \cap T$.

   (b) $\overline{S \cup T} = \overline{S} \cup \overline{T}$.

   (c) $\overline{\overline{S} \cap \overline{T}} = S \cup T$.

   (d) $(S - T) \cup (T - S) = \overline{S \cup T}$.

8. Suppose $f : A \to \mathbb{R}$ has the rule $f(x) = \dfrac{1}{x^3 - x}$. Circle the letter of the set that could be the domain of A.

   (a) $\mathbb{R}$.

   (b) $\mathbb{R} - \{0, 1\}$.

   (c) $\mathbb{R} - \{-1, 1\}$.

   (d) $\mathbb{R} - \{-1, 0, 1\}$.

9. Suppose you wanted to prove that the square of every even positive integer ends in 0, 4, or 6. Circle the letter of the proof type that would be easiest to use for this purpose.

   (a) Proof by contraposition.

   (b) Direct proof.

   (c) Proof by cases.

10. (Circle the letter of the correct response.) Suppose you are examining a conjecture of the form $\forall x(P(x) \to Q(x))$. If you are looking for a counterexample, you need to find a value of x such that:

    (a) $P(x)$ and $Q(x)$ are both true.

    (b) $P(x)$ and $Q(x)$ are both false.

    (c) $Q(x)$ is true and $P(x)$ is false.

    (d) $P(x)$ is true and $Q(x)$ is false.

11. Suppose $f : \mathbb{R} \to \mathbb{R}$ has the rule $f(x) = \left\lfloor \frac{x}{2} \right\rfloor$ and suppose $g : \mathbb{R} \to \mathbb{R}$ has the rule $g(x) = \left\lceil \frac{5-2x}{3} \right\rceil$. Find $(f \circ g)(5)$.

    (a) 1. **(b)** -1.

    (c) 0. **(d)** None of these.

$g(5) = \lceil -\frac{5}{3} \rceil = -1$

$f(-1) = \lfloor -\frac{1}{2} \rfloor = -1$

12. Let $f : A \to B$ where $A = \{-3, -2, -1, 0, 1, 2\}$ and f is defined by the rule $f(x) = x^2$. For which set B is the function onto B?

    (a) $\{-9, -4, -1, 0, 1, 4\}$. **(b)** $\{0, 1, 4, 9\}$.

    (c) $\{0, 1, 4\}$. **(d)** $\{-3, -2, -1, 0, 1, 2\}$.

13. Find the sum of the geometric series $\displaystyle\sum_{i=1}^{\infty} \frac{3 \cdot 4^i}{5^i}$.

    (a) 12. **(b)** 15.

    (c) 3. **(d)** ∞.

$$\frac{3 \cdot \frac{4^1}{5^1}}{1 - \frac{4}{5}} = \frac{\frac{12}{5}}{\frac{1}{5}} = 12$$

14. Which is the smallest integer n such that $f(x)$ is $O(x^n)$, where $f(x) = (x^3 + 1)(x + 2)(x + 2) + x^3(x + 7)$?

    (a) 4. **(b)** 5.

    (c) 6. **(d)** 7.

15. Compute $\text{lcm}(2^3 \cdot 3^2 \cdot 5^2, 2^2 \cdot 3^4 \cdot 7^4)$.

    Answer: $2^3\, 3^4\, 5^2\, 7^4$

16. Suppose $P(x, y)$ is a predicate where the universe for x and y is $\{1, 2, 3\}$. Also suppose that the predicate is true in the following cases – $P(1, 2)$, $P(2, 1)$, $P(2, 2)$, $P(2, 3)$, $P(3, 1)$, $P(3, 2)$ – and false otherwise. Circle the letter of the quantified statement that is FALSE.

    (a) $\forall y \exists x\ \neg P(x, y)$

    (b) $\exists y \forall x\ P(x, y)$

    (c) $\forall x \exists y\ (x \neq y \wedge P(x, y))$

    (d) $\exists x \forall y\ P(x, y)$

17. Circle the letter of the negation of $\exists x \forall y\ \neg P(x, y)$.

    (a) $\neg \forall x \exists y\ \neg P(x, y)$ **(b)** $\exists x \forall y\ P(x, y)$ **(c)** $\forall x \exists y\ P(x, y)$ **(d)** $\forall y \exists x\ P(x, y)$

18. Circle the letter of the statement that is true for all sets S and T.

    (a) $\emptyset \in S \cup T$. **(b)** $(S - T) \cap (T - S) = \emptyset$. **(c)** $\overline{S - T} = \emptyset$.

    (d) $\overline{S \cup T} = \emptyset$. **(e)** $S \cap T \neq \emptyset$.

19. Circle the letter of the contrapositive of the statement "If it is dark outside, then I stay at home."

    (a) If I stay at home, then it is dark outside.

    (b) It is dark outside and I do not stay at home.

    (c) If I do not stay at home, then it is not dark outside.

    (d) If it is not dark outside, then I do not stay at home.

20. Assume that x, y, and z are all real numbers. Circle the letter of the negation of the statement "x is positive, or y is negative, or z is positive."

    $\neg(x>0 \vee y<0 \vee z>0)$

    $\equiv x \le 0 \wedge y \ge 0 \wedge z \le 0$

    (a) If $x \leq 0$, then $y \geq 0$ or $z \geq 0$.

    (b) $x \leq 0$ and $y \geq 0$ and $z \leq 0$.

    (c) x is negative and y and z are positive.

    (d) $x \leq 0$, and either $y \geq 0$ or $z \geq 0$.

    (e) Either $x \leq 0$, or $y \geq 0$ and $z \geq 0$.

21. How many permutations of the letters in the word ARKANSAS are there?

    (a) $P(8,5) \cdot P(3,3) \cdot P(2,2)$. (b) $C(8,5) \cdot C(5,3)$. (c) $8!/(3!+2!)$.

    (d) $P(8,3) \cdot P(8,2)$. (e) $8!/5!$. (f) $8!/(3! \cdot 2!)$.

    total: 8 A: 3 S: 2

22. How many distinct integers are necessary to guarantee that two of them have the same remainder when divided by 4?

    (a) 1 (b) 3

    (c) 5 (d) 7

    (e) 9 (f) 13

23. Using the Binomial Theorem, find the sum of the series $\sum_{j=0}^{n} \binom{n}{j} 2^{n-j}$.

    (a) 0 (b) 1

    (c) 2^n (d) $2^n/2$

    (e) 3^n (f) $3^n/2$

24. If two fair 6-sided dice are rolled, what is the probability of rolling a multiple of 4?

    (a) 1/4 (b) 1/3 (c) 1/2

    (d) 2/3 (e) 3/4 (f) 7/36

roll	2	3	$\boxed{4}$	5	6	7	$\boxed{8}$	9	10	11	$\boxed{12}$
36· prob	1	2	$\boxed{3}$	4	5	6	$\boxed{5}$	4	3	2	$\boxed{1}$

$\frac{3}{36} + \frac{5}{36} + \frac{1}{36} = \frac{1}{4}$

25. (2pts) Recall the following properties of a relation $R \subseteq A \times A$ (saying $a\,R\,b$ is the same as saying $(a,b) \in R$:

    Reflexive: For all $a \in A$, $a\,R\,a$.
    Symmetric: For all $a, b \in A$, if $a\,R\,b$, then $b\,R\,a$.
    Antisymmetric: For all $a, b \in A$, if $a\,R\,b$ and $b\,R\,a$, then $a = b$.
    Transitive: For all $a, b, c \in A$, if $a\,R\,b$ and $b\,R\,c$, then $a\,R\,c$.

    Check the box of each property satisfied by the relation $R = \{(x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid x \text{ divides } y\}$.

    - [x] reflexive
    - [ ] symmetric
    - [x] antisymmetric
    - [x] transitive

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Part II. Computation, Algorithms, and Examples (6 pts ea.). Show work for full credit. Suggested time 30 minutes.

26. Draw the "dots and arrows" representation of a relation R from the set $X = \{x, y, z\}$ to the set $A = \{a, b, c\}$ that is not a function from X to A. (An arrow from dot u to dot v means that u is related to v.)

27. Trace through the Bubble Sort algorithm on the list 2, 4, 3, 5, 1 by writing down the order of the list just before i is incremented. (Part of the question is to determine the last value of i, when the list is in sorted order.)

procedure bubblesort($a_1, \ldots, a_n$: reals with $n \geq 2$)
for $i := 1$ **to** $n - 1$
  for $j := 1$ **to** $n - i$
    if $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}

<u>Trace</u>

initial list:	2	4	3	5	1
$i = 1$:	2	3	4	1	5
$i = 2$:	2	3	1	4	5
⋮ $i = 3$:	2	1	3	4	5
$i = 4$:	1	2	3	4	5

28. An experiment consists of rolling two fair 6-sided dice and flipping one fair coin (a fair coin is equally likely to come up "heads" or "tails").
(a) Write down the sample space S for this experiment in Cartesian product notation.
(b) What is the cardinality of S?
(c) What is the probability of rolling a 12 and obtaining a "tails"?

(a) $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} \times \{H,T\} = S$

(b) $|S| = 6 \cdot 6 \cdot 2 = 72$

(c) $(6,6,T) \in S$, only. $\boxed{\dfrac{1}{72}}$

29. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is strictly increasing but not onto.

$f(x) = e^x$

30. Determine whether the following two propositions are logically equivalent:

$$r \to (\neg s \to t), \quad r \to (\neg s \land t)$$

r	s	t	$\neg s \to t$	$r \to (\neg s \to t)$	$\neg s \land t$	$r \to (\neg s \land t)$
T	T	T	T	T	F	F
T	T	F	T	T	F	F
T	F	T	T	T	T	T
T	F	F	F	F	F	F
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	T	T
F	F	F	F	T	F	T

not same

so not logically equiv.

or, set $s, t, r = $ True.
then $T \to (\neg T \to T)$ is T
but $T \to (\neg T \land T)$ is F.

Part III. Proofs (5 pts ea.). Write complete line-by-line proofs for full credit. Substantial partial credit for good proof structure. Suggested time 35 minutes.

31. Let U, V, and W be sets. Prove or disprove that $(U - V) \cup (U - W) = U - (V \cap W)$. A Venn diagram is helpful, but not a proof.

By set membership table

U	V	W	U-V	U-W	(U-V)∪(U-W)	(V∩W)	U-(V∩W)
1	1	1	0	0	0	1	0
1	1	0	0	1	1	0	1
1	0	1	1	0	1	0	1
1	0	0	1	1	1	0	1
0	1	1	0	0	0	1	0
0	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

Same, so sets are equal.

32. Prove that there are no positive integer solutions to the equation $4x^2 + 2y^2 = 28$.

WLOG, $4x^2 \le 28$

$x^2 \le 7$

$x \le \sqrt{7}$

$1 \le x \le \lfloor \sqrt{7} \rfloor$ since $x \in \mathbb{Z}^+$

$1 \le x \le 2$

WLOG $2y^2 \le 28$

$y^2 \le 14$

$y \le \sqrt{14}$

$1 \le y \le \lfloor \sqrt{14} \rfloor$ since $y \in \mathbb{Z}^+$

$1 \le y \le 3$

Check all cases.

x	y	$4x^2+2y^2$
1	1	6
1	2	12
1	3	22
2	1	18
2	2	24
2	3	34

28 never appears.

$\therefore$ no positive integer solution x, y. $\square$

33. Let m be a positive integer. Prove that for all integers a, b, and c, if $a \equiv b \pmod{m}$, then $a \cdot c \equiv b \cdot c \pmod{m}$.

(Direct) Let $a, b, c \in \mathbb{Z}$, and $m \in \mathbb{Z}^+$.
Assume $a \equiv b \pmod{m}$
By definition, $m \mid (a-b)$
By defn of divides, $\exists k \in \mathbb{Z}$ $m \cdot k = a-b$
add + subtract c from RHS:
$$m \cdot k = a-c-(b-c)$$
Thus $m \mid (a-c)-(b-c)$
and so $a-c \equiv b-c \pmod{m}$.

34. A sequence is defined by $a_1 = 1$, $a_2 = 16$, and $a_n = 3a_{n-1} + 4a_{n-2}$ for all $n \geq 3$. Prove that $a_n \leq 4^n$ for all positive integers n.

Base case $n=1$: $1 \leq 4^1 = 4$ true
$n=2$: $16 \leq 4^2 = 16$ true

Inductive Step
Now let $k \in \mathbb{Z}^+$ with $k \geq 2$.
Assume that the inequality is true for indices $1, \ldots, k$.
Now consider a_{k+1}. Since $k+1 \geq 3$,
$$a_{k+1} = 3a_k + 4a_{k-1}$$
$$\leq 3 \cdot 4^k + 4 \cdot 4^{k-1}$$ since $1 \leq k, k-1 \leq k$ + by inductive assumption
$$a_{k+1} \leq 3 \cdot 4^k + 4^k$$
$$a_{k+1} \leq (3+1)4^k = 4^{k+1}$$, as desired.
Therefore for all $n \in \mathbb{Z}^+$, $a_n \leq 4^n$. $\square$

I. Short answer (2 pts each). No partial credit – only the response will be graded. Suggested time 55 minutes.

1. Using the Binomial Theorem, find the sum of the series $\sum_{j=0}^{n} \binom{n}{j} 2^{n-j}$.

   (a) 0 (b) 1

   (c) 2^n (d) $2^n/2$

   (e) 3^n (f) $3^n/2$

2. If two fair 6-sided dice are rolled, what is the probability of rolling a multiple of 4?

   (a) 1/4 (b) 1/3 (c) 1/2

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3. (2pts) Recall the following properties of a relation $R \subseteq A \times A$ (saying $a\,R\,b$ is the same as saying $(a, b) \in R$:

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Transitive:	For all $a, b, c \in A$, if $a\,R\,b$ and $b\,R\,c$, then $a\,R\,c$.

   Check the box of each property satisfied by the relation $R = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid x \text{ divides } y\}$.

   ☒ reflexive

   ☐ symmetric

   ☒ antisymmetric

   ☒ transitive

4. How many permutations of the letters in the word ARKANSAS are there?

   (a) $P(8, 5) \cdot P(3, 3) \cdot P(2, 2)$. (b) $C(8, 5) \cdot C(5, 3)$. (c) $8!/(3! + 2!)$.

   (d) $P(8, 3) \cdot P(8, 2)$. (e) $8!/5!$. (f) $8!/(3! \cdot 2!)$.

5. How many distinct integers are necessary to guarantee that two of them have the same remainder when divided by 4?

   (a) 1 (b) 3

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6. Circle the letter of the contrapositive of the statement "If it is dark outside, then I stay at home."

   (a) If I stay at home, then it is dark outside.

   (b) It is dark outside and I do not stay at home.

   (c) If I do not stay at home, then it is not dark outside.

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7. Assume that x, y, and z are all real numbers. Circle the letter of the negation of the statement "x is positive, or y is negative, or z is positive."

   (a) If $x \leq 0$, then $y \geq 0$ or $z \geq 0$.

   (b) $x \leq 0$ and $y \geq 0$ and $z \leq 0$.

   (c) x is negative and y and z are positive.

   (d) $x \leq 0$, and either $y \geq 0$ or $z \geq 0$.

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8. Suppose $P(x, y)$ is a predicate where the universe for x and y is $\{1, 2, 3\}$. Also suppose that the predicate is true in the following cases – $P(1, 2)$, $P(2, 1)$, $P(2, 2)$, $P(2, 3)$, $P(3, 1)$, $P(3, 2)$ – and false otherwise. Circle the letter of the quantified statement that is FALSE.

   (a) $\forall y \exists x \; \neg P(x, y)$

   (b) $\exists y \forall x \; P(x, y)$

   (c) $\forall x \exists y \; (x \neq y \wedge P(x, y))$

   (d) $\exists x \forall y \; P(x, y)$

9. Circle the letter of the negation of $\exists x \forall y \; \neg P(x, y)$.

   (a) $\neg \forall x \exists y \; \neg P(x, y)$ (b) $\exists x \forall y \; P(x, y)$ (c) $\forall x \exists y \; P(x, y)$ (d) $\forall y \exists x \; P(x, y)$

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    (a) $\emptyset \in S \cup T$. (b) $(S - T) \cap (T - S) = \emptyset$. (c) $\overline{S - T} = \emptyset$.

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11. Find the sum of the geometric series $\sum_{i=1}^{\infty} \frac{3 \cdot 4^i}{5^i}$.

(a) 12.
(b) 15.
(c) 3.
(d) ∞.

12. Which is the smallest integer n such that $f(x)$ is $O(x^n)$, where $f(x) = (x^3+1)(x+2)(x+2) + x^3(x+7)$?

(a) 4.
(b) 5.
(c) 6.
(d) 7.

13. Compute $\text{lcm}(2^3 \cdot 3^2 \cdot 5^2, 2^2 \cdot 3^4 \cdot 7^4)$.

Answer: $2^3 \cdot 3^4 \cdot 5^2 \cdot 7^4$

14. Suppose $f : \mathbb{R} \to \mathbb{R}$ has the rule $f(x) = \left\lfloor \frac{x}{2} \right\rfloor$ and suppose $g : \mathbb{R} \to \mathbb{R}$ has the rule $g(x) = \left\lceil \frac{5-2x}{3} \right\rceil$.
Find $(f \circ g)(5)$.

(a) 1.
(b) -1.
(c) 0.
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15. Let $f : A \to B$ where $A = \{-3, -2, -1, 0, 1, 2\}$ and f is defined by the rule $f(x) = x^2$. For which set B is the function onto B?

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    (a) Proof by contraposition.

    (b) Direct proof.

    (c) Proof by cases.

17. (Circle the letter of the correct response.) Suppose you are examining a conjecture of the form $\forall x(P(x) \to Q(x))$. If you are looking for a counterexample, you need to find a value of x such that:

    (a) $P(x)$ and $Q(x)$ are both true.

    (b) $P(x)$ and $Q(x)$ are both false.

    (c) $Q(x)$ is true and $P(x)$ is false.

    (d) $P(x)$ is true and $Q(x)$ is false.

18. Consider the statement "If the product of two integers is even, then their sum is also even." Circle the letter of the correct assertion.

    (a) The statement is false as can be shown by finding a counterexample.

    (b) The statement is true and can be proved most easily by using a direct proof.

    (c) The statement is true and can be proved most easily using a proof by contradiction.

    (d) The statement is true and can be proved most easily using a proof by contraposition.

19. Circle the letter of the statement that is true for all sets S and T.

    (a) $S \cap \overline{T} = \overline{S} \cap T$.

    (b) $\overline{S \cup T} = \overline{S} \cup \overline{T}$.

    (c) $\overline{\overline{S} \cap \overline{T}} = S \cup T$.

    (d) $(S - T) \cup (T - S) = \overline{S \cup T}$.

20. Suppose $f : A \to \mathbb{R}$ has the rule $f(x) = \dfrac{1}{x^3 - x}$. Circle the letter of the set that could be the domain of A.

    (a) $\mathbb{R}$.

    (b) $\mathbb{R} - \{0, 1\}$.

    (c) $\mathbb{R} - \{-1, 1\}$.

    (d) $\mathbb{R} - \{-1, 0, 1\}$.

21. For which of the following is the set S equal to the set of positive integers not divisible by 3?

(a) $1 \in S$; $x \in S \to 2x + 2 \in S$ (b) $1 \in S$; $x \in S \to x + 3 \in S$

(c) $1 \in S$; $2 \in S$; $x \in S \to x + 3 \in S$. (d) $1 \in S$; $x \in S \to 3x + 1 \in S$

22. Which of the following is a recursive definition for $a_n = 4n + 3$, for $n \in \mathbb{N}$?

(a) $a_n = 2a_{n-1} + 1$, $a_0 = 3$. (b) $a_n = a_{n-1} + 4n$, $a_0 = 3$.

(c) $a_n = a_{n-2} + 8$, $a_0 = 3$, $a_1 = 7$. (d) $a_n = a_{n-1} - a_{n-2}$, $a_0 = 3$, $a_1 = 7$.

23. How many permutations of the set $\{A, B, C, D, E, F, G\}$ either begin or end with a vowel (A or E)? (Note: a permutation of a set contains each letter exactly once.)

(a) $2 \cdot 6! + 2 \cdot 6!$.

(b) $2 \cdot 6! + 2 \cdot 6! - 2 \cdot 5!$.

(c) $6! + 6! - 5!$.

(d) $6! + 6!$.

(e) $7! - 5!$.

24. Compute the following: $23 \mod 5 =$ 3 $\qquad -40 \mod 7 =$ 2

$23 = 4 \cdot 5 + 3$ $\qquad -40 = -6 \cdot 7 + 2$

25. Suppose that $P(n)$ is the statement "$n + 1 = n + 2$." What is wrong with the following "proof" that the statement $P(n)$ is true for all nonnegative integers n:

You assume that $P(k)$ is true for some positive integer k; that is, $k + 1 = k + 2$. Then you add 1 to both sides of this equation to obtain $k + 2 = k + 3$; therefore $P(k + 1)$ is true. By the principle of mathematical induction $P(n)$ is true for all nonnegative integers n.

(a) There is nothing wrong with this proof.

(b) The proof is incorrect because the statement used in the inductive hypothesis is incorrect.

(c) The proof is incorrect because there is no basis step.

(d) The proof is incorrect because you cannot add 1 to both sides of the equation in the inductive step.

Part II. Computation, Algorithms, and Examples (6 pts ea.). Show work for full credit. Suggested time 30 minutes.

26. An experiment consists of rolling one fair 6-sided die and flipping two fair coins (a fair coin is equally likely to come up "heads" or "tails").
(a) Write down the sample space S for this experiment in Cartesian product notation.
(b) What is the cardinality of S?
(c) What is the probability of rolling a 1 and obtaining two "heads"?

(a) $S = \{1,2,3,4,5,6\} \times \{H,T\} \times \{H,T\}$

(b) $|S| = 6 \cdot 2 \cdot 2 = 24$

(c) $(1,H,H) \in S$ only. $\boxed{\frac{1}{24}}$

27. Draw the "dots and arrows" representation of a relation R from the set $A = \{a, b, c\}$ to the set $X = \{x, y, z\}$ that is not a function from A to X. (An arrow from dot u to dot v means that u is related to v.)

28. Trace through the Bubble Sort algorithm on the list 2, 5, 3, 4, 1 by writing down the order of the list just before i is incremented. (Part of the question is to determine the last value of i, when the list is in sorted order.)

procedure bubblesort($a_1, \ldots, a_n$: reals with $n \geq 2$)
for $i := 1$ **to** $n - 1$
  for $j := 1$ **to** $n - i$
    if $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}

<u>Trace</u>

initial list:	2	5	3	4	1
$i = 1$:	2	3	4	1	5
$i = 2$:	2	3	1	4	5
⋮ $i=3$	2	1	3	4	5
$i=4$	1	2	3	4	5

29. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is strictly increasing but not onto.

$$f(x) = e^x$$

30. Determine whether the following two propositions are logically equivalent:

$$p \to (q \to \neg r), \quad p \to (q \wedge \neg r)$$

p	q	r	$q \to \neg r$	$p \to (q \to \neg r)$	$q \wedge \neg r$	$p \to (q \wedge \neg r)$
T	T	T	T	T	F	F
T	T	F	T	T	T	T
T	F	T	T	T	F	F
T	F	F	F	F	F	F
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	T	T	F	T
F	F	F	F	T	F	T

not same

not ~~tha~~ logically equivalent

Part III. Proofs (5 pts ea.). Write complete line-by-line proofs for full credit. Substantial partial credit for good proof structure. Suggested time 35 minutes.

31. Prove that there are no positive integer solutions to the equation $x^4 + y^4 = 100$.

WLOG, $y^4 \le 100$

$y^2 \le 10$

$y \le \sqrt{10}$

$1 \le y \le \lfloor \sqrt{10} \rfloor$ since $y \in \mathbb{Z}^+$

$1 \le y \le 3$.

By symmetry, $1 \le x \le 3$.

Now consider all possible cases.

$1^4 = 1$
$2^4 = 16$
$3^4 = 81$

x	y	$x^4 + y^4$
1	1	2
1	2	17
1	3	82
2	1	17
2	2	32
2	3	97
3	1	82
3	2	97
3	3	162

never $= 100$.

Therefore no positive solution $(x,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. $\square$

32. Let m be a positive integer. Prove that for all integers a, b, and c, if $a \equiv b \pmod{m}$, then $a - c \equiv b - c \pmod{m}$.

Direct proof

Let $m \in \mathbb{Z}^+$ and let $a, b, c \in \mathbb{Z}$.

Assume $a \equiv b \pmod{m}$

By defn of $\equiv$, $m \mid (a-b)$

By defn of divides, $\exists k \in \mathbb{Z}$ $mk = a - b$

multiplying through by c,

$$mkc = ac - bc$$

since $kc \in \mathbb{Z}$, $m \mid ac - bc$

by defn. of $\equiv$, $ac \equiv bc \pmod{m}$. $\square$

33. A sequence is defined by $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for all $n \geq 3$. Prove that $a_n \leq 3^n$ for all positive integers n.

(Strong Induction) Let $P(k)$ be statement $a_k \leq 3^k$, where $k \in \mathbb{Z}^+$.

Base cases. $P(1)$. $2 \leq 3^1 = 3$ true.

$P(2)$. $9 \leq 3^2 = 9$ true.

Inductive step Let $k \in \mathbb{Z}^+$ and $k \geq 2$.

Assume $P(1), \ldots, P(k)$ all true.

$a_{k+1} = 2a_k + 3a_{k-1}$ since $k+1 \geq 3$.

$\leq 2 \cdot 3^k + 3 \cdot 3^{k-1}$ since $P(k)$, $P(k-1)$ true.

$a_{k+1} \leq 2 \cdot 3^k + 3^k = (2+1)3^k = 3^{k+1}$.

Therefore $P(k+1)$ is true.

By strong induction, $\forall n \in \mathbb{Z}^+$ $P(n)$ is true. $\square$

34. Let A, B, and C be sets. Prove or disprove that $A - (B \cap C) = (A - B) \cup (A - C)$. A Venn diagram is helpful, but not a proof.

By Set membership table

A	B	C	$B \cap C$	$A - B \cap C$	$A - B$	$A - C$	$(A-B) \cup (A-C)$
1	1	1	1	0	0	0	0
1	1	0	0	1	0	1	1
1	0	1	0	1	1	0	1
1	0	0	0	1	1	1	1
0	1	1	1	0	0	0	0
0	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

same, so sets are equal.