Second Integral Estimation of Series Method

Here is a summary of the formula we derived for estimating a series derived from a continuous, non-negative, decreasing function. The corresponding section in the book is in the Series chapter, Integral Test section.

Suppose the following:

- $f(x)$ is a continuous, positive, decreasing function,
- the series $\sum_{i=1}^{\infty} a_i$ is defined by $a_i = f(i)$ for all i,
- we define $s_n = \sum_{i=1}^{n} a_i = a_1 + \cdots + a_n$ to be the nth partial sum, and
- both $\sum_{i=1}^{\infty} a_i$ and $\int_0^{\infty} f(x)\,dx$ converge, so that $\sum_{i=1}^{\infty} a_i = s$.

Then the quantity

$$s_n + \frac{\int_n^{\infty} f(x)\,dx + \int_{n+1}^{\infty} f(x)\,dx}{2}$$

is guaranteed to be within $\frac{a_n}{2}$ of s.[1]

How to use this method.

- Given a tolerance t (for example, $t = .0005$), solve the inequality $a_n/2 \leq t$ for n.
- Remember to round up the answer to the nearest integer. For example, $n \geq 4.781\ldots$ means $n \geq 5$.
- Plug in the value of n into the estimate. For example, if $n \geq 5$ was found in the previous step, the following estimate is within .0005 of the actual series sum s:

$$\begin{aligned} & s_5 + \frac{\int_n^{\infty} f(x)\,dx + \int_{n+1}^{\infty} f(x)\,dx}{2} \\ = {} & a_1 + a_2 + a_3 + a_4 + a_5 + \frac{\int_n^{\infty} f(x)\,dx + \int_{n+1}^{\infty} f(x)\,dx}{2} \end{aligned}$$

- Work out the integrals. Typically on an exam, you don't have to actually add up the first n terms, but unless otherwise instructed you should work out the integrals. Note that you only have to compute the antiderivative once.

[1] In our derivation, we actually guaranteed that it would be within

$$\frac{\int_n^{\infty} f(x)\,dx - \int_{n+1}^{\infty} f(x)\,dx}{2} = \frac{\int_n^{n+1} f(x)\,dx}{2}$$

of s, but because $f(x)$ is decreasing, we know that $\int_n^{n+1} f(x)\,dx \leq a_n$.