PRINT Last name: KEY First name:

Signature: Student ID:

Math 152 Exam 2, Fall 2005

Instructions. You must show the mathematical steps of the solution in order to receive full credit; writing only the answer will receive no credit. Written explanations for each step are not required (due to time constraint) but an incorrect solution with a correct written explanation might receive partial credit.

Conditions. No calculators, computers, notes, books, or scratch paper. By writing your name on the exam you certify that all work is your own, under penalty of all remedies outlined in the IIT student rules. **Please do not talk until you are away from the room.**

Time limit: 50 minutes (strict).

NOTE: The topics may not be in order either of increasing difficulty or of the order they were covered in the course.

POSSIBLY USEFUL FORMULAS

$\sec^2 x = \tan^2 x + 1$

$M_n = \Delta x \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$

$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

$\int \tan x \, dx = -\ln|\cos x| + C$

$PV = nRT$

$|E_M| < \frac{K(b-a)^3}{24n^2} \quad (K \geq f''(x))$

$F = \rho g A d$

$|E_T| < \frac{K(b-a)^3}{12n^2} \quad (K \geq f''(x))$

$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$

$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

$|E_S| < \frac{K(b-a)^5}{180n^4} \quad (K \geq f^{(4)}(x))$

$\int_{n+1}^{\infty} f(x)\,dx \leq s - s_n \leq \int_n^{\infty} f(x)\,dx$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$\sin 2x = 2 \sin x \cos x$

$\text{Vol} = \int_a^b 2\pi \left[(f(x))^2 - (g(x))^2\right] \, dx$

SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. Find the partial fraction decomposition of $\dfrac{x^3+x^2}{(x^2+1)^2}$.

The denominator is the repeated quadratic irreducible factor $(x^2+1)\cdot(x^2+1)$. Therefore the partial fraction decomposition form is

$$\frac{x^3+x^2}{(x^2+1)^2} = \boxed{\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}}.$$

Cross-multiply and solve for A, B, C, D.

$$\begin{aligned} x^3+x^2 &= (Ax+B)(x^2+1)+Cx+D \\ x^3+x^2 &= Ax^3+Bx^2+(A+C)x+(B+D). \end{aligned}$$

For polynomials to be equal, the coefficients must be equal. Therefore $\boxed{A=1}$ and $\boxed{B=1}$. Plugging these in, we have

$$\begin{aligned} x^3+x^2 &= x^3+x^2+(1+C)x+(1+D) \\ 0 &= (1+C)x+(1+D), \end{aligned}$$

from which $\boxed{C=-1}$ and $\boxed{D=-1}$.

2. Given the integral $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$,
(a) Find a value of p so that the integral diverges.
(b) Find a value of p so that the integral converges, and compute the integral in this case.
(Hint for both parts: Find the *indefinite* integral only once using the substitution $u = \ln x$.)

First the indefinite integral using $u = \ln x$ and $du = dx/x$.

$$\int \frac{dx}{x(\ln x)^p} = \int \frac{du}{u^p} = \begin{cases} \frac{u^{-p+1}}{-p+1} + C, & \text{if } p \neq 1, \\ \ln|u| + C, & \text{if } p = 1 \end{cases} = \begin{cases} \frac{(\ln x)^{-p+1}}{-p+1} + C, & \text{if } p \neq 1, \\ \ln|\ln x| + C, & \text{if } p = 1\,. \end{cases}$$

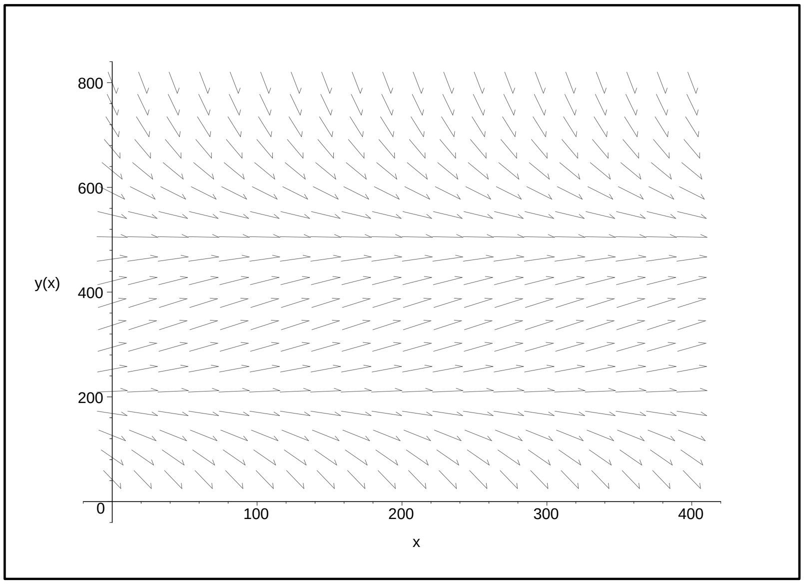
Soln. to (a). Any value $p \leq 1$ will work. Here is the justification for $p = 1$:

$$\lim_{t\to\infty} \int_e^t \frac{dx}{x\ln x} = \lim_{t\to\infty} \ln|\ln x|\Big|_e^t = \lim_{t\to\infty} (\ln\ln t - \ln\ln e) = \boxed{\infty}.$$

Soln. to (b). Any value $p > 1$ will work. Here is the justification for $p = 2$:

$$\lim_{t\to\infty} \int_e^t \frac{dx}{x\ln x} = \lim_{t\to\infty} \frac{(\ln x)^{-1}}{-1}\Bigg|_e^t = \lim_{t\to\infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln e}\right) = \boxed{1}.$$

3. Sketch four particular solutions to the differential equation whose slope field is below: **(a)** 2 equilibrium solutions, **(b)** 1 increasing solution, and **(c)** 1 decreasing solution.



(a) The two equilibrium solutions are the two horizontal lines at approximately $y = 200$ and $y = 500$, respectively.

(b) Any solution with initial condition $y(0)$ between 200 and 500 will yield an increasing solution.

(c) Any solution with initial condition $y(0)$ between 0 and 200 or between 500 and 800 will yield a decreasing solution.

4. Plutonium-238, a satellite power source, has a half-life of 88 years. Compute the time at which 10% of a sample of plutonium-238 remains. You may start from the general solution for exponential decay: $y(t) = A \cdot e^{kt}$.

Starting from the general solution, solve for k by plugging in $y(88) = A/2$.

$$\begin{aligned} y(88) &= \frac{A}{2} = Ae^{k\cdot 88}, \qquad \text{or } \frac{1}{2} = e^{k\cdot 88} \\ \ln\frac{1}{2} &= 88k, \qquad \text{or } \boxed{k = \frac{\ln\frac{1}{2}}{88}}. \end{aligned}$$

Now the model is $y(t) = A \cdot \exp(t\frac{\ln\frac{1}{2}}{88})$. We use this to solve $y(t) = .1A$ for t:

$$\begin{aligned} y(t) = .1A &= A \cdot \exp(t\frac{\ln\frac{1}{2}}{88}) \\ .1 &= \exp(t\frac{\ln\frac{1}{2}}{88}) \\ \ln .1 &= t\frac{\ln\frac{1}{2}}{88} \\ t &= \boxed{88\frac{\ln .1}{\ln .5}}. \end{aligned}$$

5. For this problem, use the differential equation for Newton's Law of Cooling:

$$\frac{dT}{dt} = k(T - T_s),$$

where T is the temperature, t is time, T_s is the ambient temperature, and k is a constant.

Question: Water at temperature 60°F is placed in a freezer which is at 30°F. After 1 hour, the water is at temperature 40°F. How long does it take the water to freeze (i.e., at 32°F)?

First solve the differential equation by separation of variables.

$$\begin{aligned} \frac{dT}{dt} &= k(T-30) \\ \frac{dT}{T-30} &= k\,dt \\ \ln|T-30| &= kt + C \\ |T-30| &= e^{kt+C} \\ T - 30 &= \pm e^C e^{kt} = Ae^{kt} \\ T(t) &= \boxed{30 + Ae^{kt}}. \end{aligned}$$

Now find k using the data $T(0) = 60$ and $T(1) = 40$:

$$\begin{aligned} T(0) = 60 &= 30 + Ae^{k\cdot 0} \\ 30 &= 30 + A, \quad \text{so } \boxed{A = 30}. \\ T(1) = 40 &= 30 + 30e^{k\cdot 1} \\ \frac{1}{3} &= e^k \\ k &= \boxed{\ln\frac{1}{3}}. \end{aligned}$$

Finally, using the above solve $T(t) = 32$ for t.

$$\begin{aligned} T(t) = 32 &= 30 + 30\exp(t\ln\frac{1}{3}) \\ \frac{2}{30} &= \exp(t\ln\frac{1}{3}) \\ \ln\frac{1}{15} &= t\ln\frac{1}{3}, \qquad \text{so } \boxed{t = \frac{\ln(1/15)}{\ln(1/3)}}. \end{aligned}$$

6. Find the general solution to the differential equation

$$x^2 y' + 3xy \;=\; e^x\,,$$

where $x > 0$ (this means the general solution is defined only for $x > 0$).

First put the diff. eq. into linear form: $y' + \frac{3}{x}y = \frac{e^x}{x^2}$. Then the integrating factor is

$$I(x) \;=\; \exp\left(\int \frac{3}{x}\,dx\right) \;=\; \exp(3\ln|x|),$$

which since we are given $x > 0$ yields $\boxed{I(x) = \exp(3\ln x) = x^3}$. Multiplying through by the integrating factor, we have

$$\begin{aligned} x^3 y' + x^2 \frac{3}{y} &= xe^x \\ (yx^3)' &= xe^x \\ yx^3 &= \int xe^x\,dx \;=\; xe^x - e^x + C \\ y &= \boxed{\frac{xe^x - e^x + C}{x^3}}. \end{aligned}$$

The integral of xe^x is computed by parts.
$u = x \qquad dv = e^x\,dx$
$du = dx \quad v = e^x$
Using the formula $\int u\,dv = uv - \int v\,du$,

$$\int xe^x\,dx \;=\; xe^x - \int e^x\,dx \;=\; xe^x - e^x + C\,.$$

7. The plot below is of the parametric equations

$$\begin{aligned} x &= t \cdot \cos(3t) - \frac{1}{3}\sin(3t) \\ y &= \frac{t}{2} \cdot \sin(2t) + \frac{1}{4}\cos(2t) \\ & 0 < t < \pi \,. \end{aligned}$$

(a) Find all values of t in the interval $(0, \pi)$ which correspond to horizontal or vertical lines. (Do not check $t = 0$ or $t = \pi$.)
(b) Label the corresponding horizontal and vertical tangent lines on the plot above with the appropriate values of t.

Soln. to (a). We solve $x'(t) = 0$ and $y'(t) = 0$ and consider the results. Using the Chain Rule,

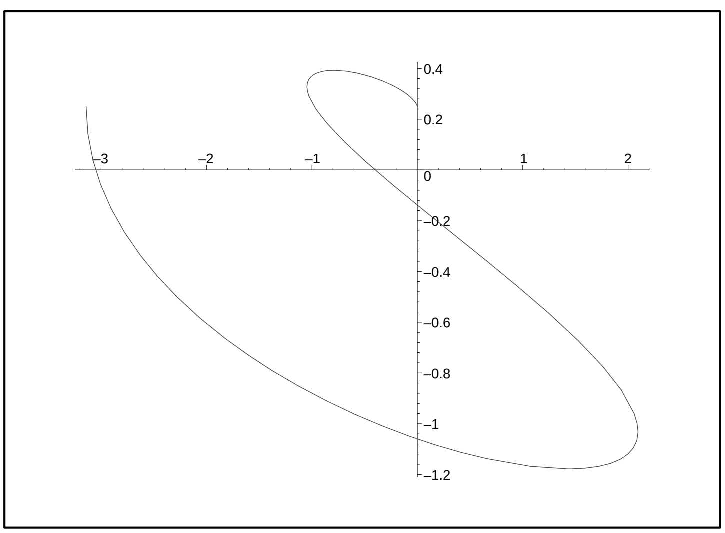
$$\frac{dx}{dt} = \cos(3t) - 3t\sin(3t) - \cos(3t) = -3t\sin(3t)\,.$$

$\frac{dx}{dt} = 0$ when either $t = 0$ or $\sin(3t) = 0$, which for $0 < t < \pi$ is when $\boxed{t = \pi/3, 2\pi/3}$.

$$\frac{dy}{dt} = \frac{t}{2}2\cos(2t) + \frac{1}{2}\sin(2t) - \frac{1}{4}2\sin(2t) = t\cos(2t)\,.$$

$\frac{dx}{dt} = 0$ when either $t = 0$ or $\cos(2t) = 0$, which for $0 < t < \pi$ is when $\boxed{t = \pi/4, 3\pi/4}$.

Because these zeros of dx/dt and dy/dt do not overlap, these zeros for dx/dt correspond to vertical tangent lines, and these zeros for dy/dt correspond to horizontal tangent lines. (Otherwise we would have to analyze further, maybe using L'Hospital's Rule.)



The sketch should have two horizontal and two vertical tangents. The horizontal tangents are at the points (approximated) $(-1.05, .328)$ and $(2.09, -1.03)$ and the vertical tangents are at the points (approximated) $(-.791, .393)$ and $(1.43, -1.18)$. These are the 4 readily visible horizontal and vertical tangencies on the plot of the parametric function.