PRINT Last name: KEY First name: ____________

Signature: ____________ Student ID: ____________

Math 152 Exam 1, Fall 2005

Instructions. *Part I* is multiple choice. There will be no partial credit. Clearly indicate your answer, especially if you change an answer. Problems with two indicated answers will receive no credit.
Part II is work-out problems. You must show work in order to receive full credit. Partial credit is possible for work which makes positive progress toward the solution.
Part III consists of the more conceptual problems; otherwise the instructions are the same as Part II.

Conditions. No calculators, notes, books, or scratch paper. By writing your name on the exam you certify that all work is your own, under penalty of all remedies outlined in the TAMU student rules. Please do not talk until after leaving the room.

Time limit: 2 hours (strict).

NOTE: The topics may not be in order either of increasing difficulty or of the order they were covered in the course.

POSSIBLY USEFUL FORMULAS

$\sec^2 x = \tan^2 x + 1$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

$PV = nRT$

$F = \rho g A d$

$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$

$M_n = \Delta x \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$

$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$

$\int \tan x\, dx = -\ln|\cos x| + C$

$|E_M| < \frac{K(b-a)^3}{24n^2} \quad (K \ge f''(x))$

$|E_T| < \frac{K(b-a)^3}{12n^2} \quad (K \ge f''(x))$

$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0) = 0$

$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

$|E_S| < \frac{K(b-a)^5}{180n^4} \quad (K \ge f^{(4)}(x))$

$\int_{n+1}^{\infty} f(x)\,dx \le s - s_n \le \int_n^{\infty} f(x)\,dx$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$\sin 2x = 2\sin x \cos x$

$\text{Vol} = \int_a^b 2\pi\left[(f(x))^2 - (g(x))^2\right]\,dx$

SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. Compute the derivative of $f(x) = \ln\left(\dfrac{x+3}{\sin(x^2)}\right)$

   Soln. Use the rule of logarithms to simplify: $\ln\frac{a}{b} = \ln a - \ln b$.

$$\frac{d}{dx}\left[\ln(x+3) - \ln(\sin(x^2))\right] = \boxed{\frac{1}{x+3} - \frac{2x\cos(x^2)}{\sin(x^2)}}$$

2. In a radioactive decay model, the amount of an isotope $A(t)$ in kilograms at time t years is given by

$$A(t) = A_0 e^{-(\ln 2)t/H},$$

   where A_0 is the initial amount at $t = 0$ and H is a positive constant called the "half-life."

   (a) Rearrange the right-hand side using properties of $y = \exp(x)$ and $y = \ln x$ so that the formula more clearly indicates why H is called the "half-life."

   (b) Find the time t at which 10% of the original amount remains.

   (a) $A_0 e^{-(\ln 2)t/H} = A_0(e^{(\ln 2^{-1})})^{t/H} = \boxed{A_0\left(\frac{1}{2}\right)^{t/H}}$. (When $t = H$ the quantity A_0 is multiplied by 1/2.)

   (b) Solve the equation for t:

$$\begin{aligned} A(t) = A_0 e^{-(\ln 2)t/H} &= .1A_0 \\ e^{-(\ln 2)t/H} &= .1 \\ -(\ln 2)t/H &= \ln .1 \\ t &= -H\frac{\ln .1}{\ln 2} = \boxed{H\frac{\ln .1}{\ln .5}}. \end{aligned}$$

3. Find an exponential function of the form $f(x) = C \cdot a^x$ which satisfies $f'(2) = 75\ln 5$ and $f(0) = 3$.

   First note that $f'(x) = C \cdot a^x \cdot \ln a$.

$$\begin{aligned} f(0) = 3 &= Ca^0, \text{ and so } \boxed{C = 3}. \\ f'(2) = 75\ln 5 &= C \cdot a^2 \ln a \\ 25\ln 5 &= a^2 \ln a, \text{ and so } \boxed{a = 5} \text{ is a solution.} \end{aligned}$$

   The question did not ask to find all possible solutions. In fact this is the only solution since in the region around $25\ln 5$, the function $g(x) = x^2 \ln x$ is strictly increasing.

4. Differentiate the function $y = x^{\tan x}$.

   Soln. By logarithmic differentiation: $\ln y = \ln\left(x^{\tan x}\right) = \tan x \ln x$.

$$\begin{aligned}
\frac{d}{dx}\ln y &= \frac{d}{dx}\tan x \ln x \\
\frac{y'}{y} &= \sec^2 x \ln x + \frac{\tan x}{x} \\
y' &= y\left(\sec^2 x \ln x + \frac{\tan x}{x}\right) \\
y' &= \boxed{x^{\tan x}\left(\sec^2 x \ln x + \frac{\tan x}{x}\right)}.
\end{aligned}$$

5. Find the exact value of $\tan^{-1}(\tan(3\pi/4))$.

   Computing from the inside, $\tan(3\pi/4) = -1$. Then $\tan^{-1}(-1) = \boxed{-\pi/4}$, since the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$.

6. Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$.

   Solution 1. Let $u = x^2$. Then $du = 2x\,dx$ or $\frac{du}{2} = x\,dx$. The integral becomes

$$\begin{aligned}
\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} &= \frac{1}{2}\sin^{-1} u + C \qquad \text{by the formula on the cover.} \\
&= \boxed{\frac{1}{2}\sin^{-1}\left(x^2\right) + C}.
\end{aligned}$$

   Solution 2. After the u-substitution you can make the trig substitution $u = \sin\theta$, $du = \cos\theta\,d\theta$, $-\pi/2 < x < \pi/2$. Then

$$\begin{aligned}
\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} &= \frac{1}{2}\int \frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{2}\int \frac{\cos\theta\,d\theta}{\sqrt{\cos^2\theta}} \\
&= \frac{1}{2}\int \frac{\cos\theta\,d\theta}{|\cos\theta|} = \frac{1}{2}\int d\theta\,, \qquad \text{since } |\cos\theta| = \cos\theta \text{ on } -\pi/2 < x < \pi/2 \\
&= \frac{\theta}{2} + C = \frac{\sin^{-1} u}{2} + C = \boxed{\frac{1}{2}\sin^{-1}\left(x^2\right) + C}.
\end{aligned}$$

7. Evaluate the limit $\displaystyle\lim_{x\to 0} \frac{e^{4x} - 1 - 4x}{x^2}$.

Soln. Plugging in 0 to the expression gives the indeterminate form $\frac{0}{0}$, so we can use L'Hospital's Rule.

$$\lim_{x\to 0} \frac{e^{4x} - 1 - 4x}{x^2} = \lim_{x\to 0} \frac{4e^{4x} - 4}{2x}$$

Again, plugging in $x = 0$ gives the indeterminate form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{4e^{4x} - 4}{2x} = \lim_{x\to 0} \frac{16e^{4x}}{2} = \boxed{8}.$$

8. Evaluate the limit $\displaystyle\lim_{x\to 0^+} e^{-1/x}\ln x$.

$\lim_{x\to 0^+} e^{-1/x} = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$, giving the indeterminate form $0 \cdot (-\infty)$. Rewrite $e^{-1/x}\ln x = \frac{\ln x}{e^{1/x}}$. Using L'Hospital's Rule,

$$\lim_{x\to 0^+} \frac{\ln x}{e^{1/x}} = \lim_{x\to 0^+} \frac{1/x}{-e^{1/x}/x^2} = \lim_{x\to 0^+} -\frac{x}{e^{1/x}} = \boxed{0},$$

since $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} e^{1/x} = \infty$.

9. Compute the definite integral $\displaystyle\int_{-1}^{1} x\sin(\pi x)\,dx$.

Soln. Integration by parts.
$u = x \qquad dv = \sin(\pi x)\,dx$
$du = dx \quad v = -\frac{\cos(\pi x)}{\pi}$
Using the formula $\int u\,dv = uv - \int v\,du$,

$$\begin{aligned}\int_{-1}^{1} x\sin(\pi x)\,dx &= -\frac{x}{\pi}\cos(\pi x)\Big|_{-1}^{1} + \int_{-1}^{1} \frac{1}{\pi}\cos(\pi x)\,dx \\ &= \left(\frac{1}{\pi} + \frac{1}{\pi}\right) + \frac{1}{\pi^2}\sin(\pi x)\Big|_{-1}^{1} = \boxed{\frac{2}{\pi}}.\end{aligned}$$

10. Evaluate the indefinite integral $\int \tan^3(2x)\sec^5(2x)\,dx$.

Convert $\tan^2(2x)$ to $\sec^2(2x)-1$ (see cover).

$$\begin{aligned}\int \tan^3(2x)\sec^5(2x)\,dx &= \int \tan(2x)(\sec^2(2x)-1)\sec^5(2x)\,dx\\ &= \int \tan(2x)(\sec^7(2x)-\sec^5(2x))\,dx\,.\end{aligned}$$

Now let $u=\sec(2x)$, $du = 2\sec(2x)\tan(2x)\,dx$ or $\frac{du}{2} = \sec(2x)\tan(2x)\,dx$. Reserve a factor of $\sec(2x)\tan(2x)\,dx$ from the integrand.

$$\begin{aligned}\int \tan(2x)(\sec^7(2x)-\sec^5(2x)) &= \int (\sec^6(2x)-\sec^4(2x))\sec(2x)\tan(2x)\,dx\\ &= \frac{1}{2}\int (u^6-u^4)\,du \;=\; \frac{1}{2}\left(\frac{u^7}{7}-\frac{u^5}{5}\right)+C\\ &= \boxed{\frac{1}{2}\left(\frac{\sec^7(2x)}{7}-\frac{\sec^5(2x)}{5}\right)+C}\,.\end{aligned}$$

11. Evaluate the indefinite integral $\int \frac{dx}{\sqrt{x^2+4}}$.

Use trigonometric substitution $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$, where $-\pi/2 < x < \pi/2$.

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta\,d\theta}{\sqrt{4\tan^2\theta+4}} = \int \frac{2\sec^2\theta\,d\theta}{\sqrt{4(\tan^2\theta+1)}} = \int \frac{2\sec^2\theta\,d\theta}{\sqrt{4(\sec^2\theta)}}\\ &= \int \frac{2\sec^2\theta\,d\theta}{2|\sec\theta|} = \int \frac{\sec^2\theta\,d\theta}{\sec\theta}, \qquad \text{since } |\sec x| = \sec x \text{ on } -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}\\ &= \int \sec\theta\,d\theta \;=\; \ln|\sec\theta+\tan\theta|+C \;=\; \boxed{\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|+C}\,.\end{aligned}$$

12. Compute the partial fraction decomposition of $\dfrac{10}{(x^2-1)(x^2+9)}$, being sure to leave the final answer in terms of x.

Factor the denominator completely: $(x^2-1)(x^2+9) = (x-1)(x+1)(x^2+9)$.

$$
\begin{aligned}
\frac{10}{(x-1)(x+1)(x^2+9)} &= \boxed{\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+9}} \\
10 &= A(x+1)(x^2+9) + B(x-1)(x^2+9) + (Cx+D)(x-1)(x+1).
\end{aligned}
$$

Plugging in $x=-1$:

$$10 = B(-20) \qquad \text{and so } \boxed{B=-1/2}.$$

Plugging in $x=1$:

$$10 = A(20) \qquad \text{and so } \boxed{A=1/2}.$$

Plugging in $x = 3i = 3\sqrt{-1}$:

$$
\begin{aligned}
10 &= (3Ci+D)(3i-1)(3i+1) = (3Ci+D)(-9-1) \\
10 &= (3Ci+D)(-10),
\end{aligned}
$$

and so $\boxed{C=0}$ and $\boxed{D=-1}$.