Random Geometric Graph Diameter in the Unit Disk with ℓ_p Metric (Extended Abstract)

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Abstract. Let n be a positive integer, and $\lambda > 0$ a real number. Let V_n be a set of n points randomly located within the unit disk, which are mutually independent. For $1 \leq p \leq \infty$, define $G_p(\lambda, n)$ to be the graph with the vertex set V_n, in which two vertices are adjacent if and only if their ℓ_p-distance is at most λ. We call this graph a *unit disk random graph*. Let $\lambda = c\sqrt{\ln n/n}$ and let X be the number of isolated points in $G_p(\lambda, n)$. Let a_p be the (constant) ratio of the area of the ℓ_p-ball to the ℓ_2-ball of the same radius. Then, almost always, $X = 0$ when $c > a_p^{-1/2}$, and $X \sim n^{1-a_p c^2}$ when $c < a_p^{-1/2}$. Penrose proved that with probability approaching 1, the graph $G_p(\lambda, n)$ is connected when it has minimum degree 1. We extend Penrose's method to prove that if $G_p(\lambda, n)$ is connected, then there exists a constant K, independent of p, such that the diameter of $G_p(\lambda, n)$ is bounded above by K/λ. We show in addition that when c exceeds a certain constant depending on p, the diameter of $G_p(\lambda, n)$ is bounded above by $(2 \cdot 2^{1/2+1/p} + o(1))/\lambda$. More generally, there is a function $c_p(\delta)$ such that the diameter is at most $2^{1/2+1/p}(1 + \delta + o(1))/\lambda$ when $c > c_p(\delta)$.

1 Introduction

Let D be the unit disk in $\mathbb{R}^2$ and n a positive integer. Let V_n be a set of n points in D, distributed independently and uniformly with respect to the usual Lebesgue measure on $\mathbb{R}^2$. For $p \in [1, \infty]$, the ℓ_p *metric* on $\mathbb{R}^2$ is defined by

$$d_p((x_1, y_1), (x_2, y_2)) = \begin{cases} (|x_2 - x_1|^p + |y_2 - y_1|^p)^{1/p} & \text{for } p \in [1, \infty), \\ \max\{|x_2 - x_1|, |y_2 - y_1|\} & \text{for } p = \infty \ . \end{cases}$$

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For $\lambda \in (0, \infty)$, the *unit disk random graph* $G_p(\lambda, n)$ on the vertex set V_n is defined by declaring two vertices $u, v \in V_n$ to be adjacent if and only if $d_p(u, v) \leq \lambda$. Beside their purely graph-theoretical interest, unit disk random graphs have important applications to wireless communication networks; see, e.g., [1, 2, ?].

The first and third authors along with with X. Jia studied the case $p = 2$ in [3]. In this extended abstract, we generalize some of the results of that article to arbitrary p: namely, those concerning the threshold for connectedness and bounds on the graph diameter. Complete results with proofs will be included in [4].

We will say that $G_p(\lambda, n)$ has a property P *almost always* if

$$\lim_{n\to\infty} \Pr\left[G_p(\lambda, n) \text{ has the property } P\right] = 1 \ .$$

For the purpose of stating the results of the paper, we begin with some geometry. Denote by $B_p(u, r)$ the ℓ_p-ball of radius r about $u \in \mathbb{R}^2$. It is not hard to show that the area of $B_p(u, r)$ is $4r^2\Gamma((p+1)/p)^2/\Gamma((p+2)/p)$. We omit the calculation, which uses the beta function; see [5, §12.4]. An important quantity in our work will be the ratio

$$a_p := \frac{\text{Area}(B_p(u, r))}{\text{Area}(B_2(u, r))} = \frac{4\Gamma\left(\frac{p+1}{p}\right)^2}{\pi\Gamma\left(\frac{p+2}{p}\right)}$$

Another elementary calculation shows that the ℓ_p-diameter of the unit disk D is

$$\text{diam}_p(D) := \max_{u,v\in D} \{d_p(u, v)\} = \begin{cases} 2^{1/2+1/p}, \ 1 \leq p < 2, \\ 2, \ p \geq 2 \ . \end{cases}$$

The diameter is achieved by the points $(\sqrt{2}/2, \sqrt{2}/2)$ and $(-\sqrt{2}/2, -\sqrt{2}/2)$ when $1 \leq p \leq 2$, and by $(0, 1)$ and $(0, -1)$ when $p \geq 2$.

In Sect. 2, we show that almost always, $G_p(\lambda, n)$ has $n^{1-a_pc^2}(1+o(1))$ isolated vertices when $c < a_p^{-1/2}$ and no isolated vertices when $c > a_p^{-1/2}$. Penrose [6] has shown that almost always, $G_p(\lambda, n)$ is connected when it has no isolated points; combining this with our result, it follows that when $\lambda = c\sqrt{\ln n/n}$ and $c > a_p^{-1/2}$, the graph $G_p(\lambda, n)$ is almost always connected. In Sect.3, we show that when $G_p(\lambda, n)$ is connected, its diameter[3] is almost always $\leq K/\lambda$, where $K \approx 387.17\cdots$ is an absolute constant independent of p. In Sect. 4, we show that for c larger than a fixed constant, the graph diameter of $G_p(\lambda, n)$ is almost always $\leq 2 \cdot \text{diam}_p(D)(1 + o(1))/\lambda$. In fact, there is a function $c_p(\delta) > 0$ with the following property: almost always, if $c > c_p(\delta)$, then the diameter of $G_p(\lambda, n)$ is at most $\text{diam}_p(D)(1 + \delta + o(1))/\lambda$.

[3] The diameter of a graph—not to be confused with the diameter of a unit disk!—is defined as the maximum distance between any two of its vertices.

2 Isolated Vertices

Theorem 1. *Let $1 \le p \le \infty$, let $\lambda = c\sqrt{\ln n/n}$, and let X be the number of isolated vertices in the unit disk random graph $G_p(\lambda, n)$. Then, almost always,*

$$X = \begin{cases} 0 & \text{when } c > a_p^{-1/2}, \\ n^{1-a_p c^2}(1+o(1)) & \text{when } 0 < c < a_p^{-1/2} \end{cases} .$$

We sketch the proof, which uses the so-called second moment method [7] to show that the expected number of isolated vertices is $\mathbb{E}[X] = n^{1-a_p c^2}$, and that its variance is $var\mathrm{Var}[X] = o(\mathbb{E}[X]^2)$. When $c < a_p^{-1/2}$, an application of Chebyshev's inequality yields $X = n^{1-a_p c^2}(1+o(1))$. Let A_i be the event that vertex v_i has degree 0. Then

$$\frac{a_p}{2}\pi\lambda^2(1+O(\lambda)) \le \text{Area}\left(B_p(v_i,\lambda) \cap D\right) \le a_p\pi\lambda^2 \ ,$$

where the lower (resp. upper) bound is achieved when $B_p(v_i,\lambda) \subseteq D$ (resp. $B_p(v_i,\lambda) \not\subseteq D$). Conditioning on the event that $B_p(v_i,\lambda) \subseteq D$, we have

$$(1-a_p\lambda^2)^{n-1} \le \Pr[A_i] \le \Pr[B_p(v_i,\lambda) \subseteq D](1-a_p\lambda^2)^{n-1} + \Pr[B_p(v_i,\lambda) \not\subseteq D]\left(1 - \frac{a_p}{2}\lambda^2(1+O(\lambda))\right)^{n-1} .$$

By linearity of expectation, $\mathbb{E}[X] = n \cdot \Pr[A_i] = n^{1-a_p c^2}(1+o(1))$. The variance $\mathrm{Var}[X]$ is computed via $\Pr[A_i \wedge A_j]$, conditioned on $d_p(v_i, v_j)$. The rest of the proof is a straightforward computation.

Penrose [6, Theorem 1.1] proved that for every $t \ge 0$, the unit-*cube* random graph simultaneously becomes $(t+1)$-connected and achieves minimum degree $t+1$. Penrose's proof remains valid for the unit disk. The precise statement is as follows: for $t \ge 0$, almost always,

$$\min\{\lambda \mid G_p(\lambda,n) \text{ is } (t{+}1)\text{-connected}\} = \min\{\lambda \mid G_p(\lambda,n) \text{ has minimum degree } t{+}1\} \ .$$

In the case $t = 0$, combining Penrose's theorem with Theorem 1 yields the following.

Theorem 2. *Let $1 \le p \le \infty$ and $\lambda = c\sqrt{\ln n/n}$. If $c > a_p^{-1/2}$, then, almost always, the unit disk random graph $G_p(\lambda, n)$ is connected.*

3 Diameter of $G_p(\lambda, n)$ Near the Connectivity Threshold

Suppose that $G_p(\lambda, n)$ is connected by virtue of Theorem 2. In general, it will contain two vertices whose Euclidean distance is close to $\mathrm{diam}_p(D)$, so we conclude that the graph diameter of $G_p(\lambda, n)$ is at least $\mathrm{diam}_p(D)/\lambda$. It appears to be much more difficult to obtain an *upper* bound on diameter; however, the upper bound is a constant multiple of the lower bound, as we now explain.

Theorem 3. *Let $1 \le p \le \infty$ and $\lambda = c\sqrt{\ln n/n}$, where $c > a_p^{-1/2}$. Suppose that $K > 256\sqrt{2} + 8\pi \approx 387.17\ldots$. Then, almost always, the unit disk random graph $G_p(\lambda, n)$ has diameter $< K/\lambda$.*

The proof is based on the following fact.

Proposition 1. *Let $1 \le p \le \infty$, let $\lambda = c\sqrt{\ln n/n}$, and let $c > a_p^{-1/2}$. If $K_0 > 128/(\pi\sqrt{2}) \sim 28.180\cdots$, then $\lim_{n\to\infty} \Pr[A_n(K_0)] = 0$.*

For any two points $u, v \in D$, define

$$T_{u,v}(k) := \big(\text{convex closure of } B_2(u, k\lambda) \cup B_2(v, k\lambda)\big) \cap D$$

and impose upon this lozenge-shaped region a grid of squares of side length proportional to λ. Let $A_n(k)$ be the event that there exist two points $u, v \in V_n$ such that (i) at least one of u, v point lies in $B_2(O, 1-(k+\sqrt{2})\lambda)$, and (ii) there is no path of $G_p(\lambda, n)$ joining u to v that lies entirely inside $T_{u,v}(k)$. If the event $A_n(k)$ occurs, then by the argument of Penrose's theorem [6, p. 162], there exists a curve L separating v_i and v_j which intersects a large number of grid squares, none of which contains any vertex of V_n (see Fig. 1). Combining this fact with a Peierls argument as in [8, Lemma 3] leads to an upper bound on k:

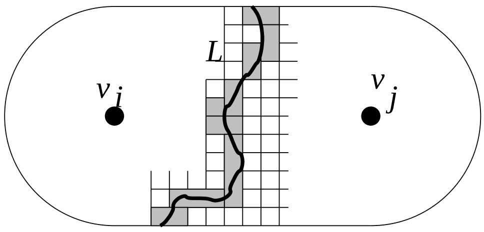

Fig. 1. Two vertices $u, v \in V_n$ which are not connected by any path in $T_{u,v}(k)$, and the "frontier" L separating them.

Consider any pair of vertices $u, v \in V_n$. If K_0 is large enough, then Proposition 1 guarantees the existence of a path from u to v inside $T_{u,v}(K_0)$. Comparing the total area of $T_{u,v}(K_0)$ to the area of the ℓ_p-balls around the vertices in a shortest path from u to v inside $T_{u,v}(K_0)$, one obtains the desired diameter bound on $G_p(\lambda, n)$. (Minor adjustments are needed if one or both of the vertices u, v is close to the boundary of D.) The following corollary will be used in the next section.

Corollary 1. *Let $1 \le p \le \infty$ and $\lambda = c\sqrt{\ln n/n}$, where $c > a_p^{-1/2}$. Suppose that $K > 256\sqrt{2} + 8\pi \approx 387.17\ldots$. Then, almost always, any two vertices u, v in the unit disk random graph $G_p(\lambda, n)$ are connected by a path of length at most $Kd_p(u, v)/\lambda$ in $G_p(\lambda, n)$.*

4 Diameter of $G_p(\lambda, n)$ for Larger c

In this section we obtain an upper bound for the diameter of the graph $G_p(\lambda, n)$ by means of a "spoke overlay construction." Roughly, a spoke consists of a number of evenly spaced, overlapping ℓ_p-balls whose centers lie on a diameter L of the unit disk D. We will superimpose several spokes on D so that the regions of intersection of the ℓ_p-balls are distributed fairly evenly around D. The idea is that if the constant c is large enough, then, almost always, every region of intersection contains at least one vertex of V_n, so that $G_p(\lambda, n)$ contains a path joining the antipodes of D on L. The length of this path, which may be calculated geometrically, will provide an upper bound for the diameter of $G_p(\lambda, n)$.

Definition 1 (Spoke construction). *Fix $1 \leq p \leq \infty$, $\theta \in (-\pi/2, \pi/2]$, and $r > 0$. Let D be the Euclidean unit disk. For $m \in \mathbb{Z}$, put*

$$u_m = u_m(r, \theta) = ((r/2 + rm)\cos\theta,\ (r/2 + rm)\sin\theta) \in \mathbb{R}^2 \ .$$

We now define the corresponding spoke *as the point set $U_{p,\theta}(r) = \{u_m\} \cap D$, together with an ℓ_p-ball of radius $\lambda/2$ centered at each $u_m \in U_{p,\theta}(r)$.*

The points u_m lie on the line L_θ through O at angle θ, and the Euclidean distance $d_2(u_m, u_{m'})$ equals $r|m - m'|$. By choosing r sufficiently small, we can ensure that each pair of adjacent ℓ_p-balls intersects in a set with positive area (the shaded rectangles in Fig. 2).

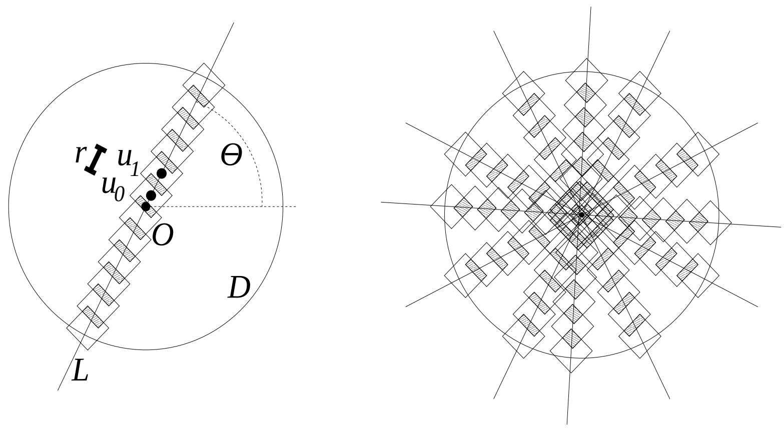

Fig. 2. The spoke overlay construction with $p = 1$, in the unit disk D. The left-hand figure shows a single spoke with parameters r, L, θ. The right-hand figure shows how spokes at different angles are superimposed on D.

Thus the two outermost points on each spoke are joined by a segmented path of Euclidean length approximately 2, which when $r = \min\{\lambda 2^{-1/2-1/p}, \lambda/2\}$

has approximately $2 \cdot \mathrm{diam}_p(D)/\lambda$ edges. Finally, we will need the quantity $A_p^*(r, \lambda/2)$, defined as the minimum area of intersection between two ℓ_p-balls in $\mathbb{R}^2$ of radius $\lambda/2$ whose centers are at Euclidean distance r. The general solution of this problem seems to involve integrals that cannot be evaluated exactly, except for very special cases such as $p = 1, p = 2, p = \infty$; however, for fixed r, $A_p^*(r, \lambda/2) = \Theta(\lambda^2)$.

Theorem 4. *Let $1 \le p \le \infty$ and let $\lambda = c\sqrt{\ln n/n}$. Choose $r = \min\{\lambda 2^{-1/2-1/p}, \lambda/2\}$, and let $A_p^*(r, \lambda)$ be the minimum area of intersection between two ℓ_p-balls in $\mathbb{R}^2$ of radius $\lambda/2$ whose centers are at Euclidean distance r. Suppose in addition that*

$$c > \sqrt{\pi\lambda^2/(2A_p^*(r, \lambda))} \ ,$$

a constant depending on p. Then, almost always, the unit disk random graph $G_p(\lambda, n)$ has diameter at most $(2 \cdot \mathit{diam}_p(D) + o(1))/\lambda$ as $n \to \infty$.

The spoke construction for Theorem 4 consists of $\sim \ln n$ spokes $U_{p,\theta}(r)$, at even angular spacings. Almost always, every region of intersection of two consecutive ℓ_p-balls of radius $\lambda/2$ in every spoke contains at least one vertex of V_n provided that $c > \sqrt{\pi\lambda^2/(2A_p^*(r, \lambda))}$. Given any two vertices $v_i, v_j \in V_n$, Corollary 1 provides paths of length $\Theta(1)$ to vertices v_i', v_j', respectively, which each lie inside a spoke. The path between v_i' and v_j' is obtained by travelling along those spokes to meet at the origin, where each edge covers an average Euclidean distance of $r = \min\{\lambda 2^{-1/2-1/p}, \lambda/2\}$.

We can make the average Euclidean distance covered in a path from v_i' to v_j' larger by increasing r. This causes the area of intersection of two consecutive ℓ_p-balls to decrease, which in turn requires an increase in c in order to guarantee a vertex of V_n in every region of intersection. This leads to the following corollary.

Corollary 2. *Let $1 \le p \le \infty$ and let $\lambda = c\sqrt{\ln n/n}$. For every $\delta \in (0, 1]$, there exists $c_p(\delta) > 0$ such that if $c > c_p(\delta)$, then the unit disk random graph $G_p(\lambda, n)$ is almost always connected with diameter $\le \mathit{diam}_p(D)(1+\delta+o(1))/\lambda$ as $n \to \infty$.*

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