ON THE POINT SPECTRUM OF SOME PERTURBED DIFFERENTIAL OPERATORS WITH PERIODIC COEFFICIENTS

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Abstract

Finiteness of the point spectrum of linear operators acting in a Banach space is investigated from point of view of perturbation theory. In the first part of the paper we present an abstract result based on analytical continuation of the resolvent function through continuous spectrum. In the second part, the abstract result is applied to differential operators which can be represented as a differential operator with periodic coefficients perturbed by an arbitrary subordinated differential operator.

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1 Introduction

One fundamental problem in spectral analysis of linear operators is finiteness of the point spectrum, which is of great interest for various problems from mathematical physics, quantum mechanics and related topics as well as for spectral theory itself. Unlike the case of selfadjoint operators, in which various methods of investigation were elaborated thanks to the fundamental spectral theorem, in the case of nonselfadjoint operators this problem is typically reduced to the uniqueness theorem of analytic functions. From this point of view, the problem of finiteness of the point spectrum (i.e. the set of all eigenvalues including those contained in the continuous spectrum) of nonselfadjoint operators has been studied in many papers, where concrete classes of operators have been considered: differential operators (of the second order) [4, 17, 23], the Friedrics model [13, 21], the finite-difference operators [12], perturbed Winer-Hopf integral operators [7]. Some general results have been presented in [14], where the nonselfadjoint operator is considered as a perturbation of a selfadjoint operator acting in some Hilbert space.

In the present paper, using direct methods of perturbation theory, we propose a generalization of the so-called method of analytic continuation of the resolvent of the unperturbed operator across its continuous spectrum for the case of Banach spaces and unbounded operators. It should be mentioned that the results obtained here are framed in the theory of large perturbations.

In Section 2 the problem of finiteness of the point spectrum is solved in abstract settings. Conditions on the linear operators H_0 and B are given such that the perturbed operator H_0+B has a finite set of eigenvalues. The obtained results actually guarantee the finiteness of the spectral singularities which were investigated for the first time by M. A. Naimark [16], and which are also very closely related to the problem of eigenfunction expansion and generalized Parseval identity. The abstract results proposed here can be applied to different classes of unbounded operators, among which we can mention ordinary differential operators, integro-differential operators, pseudo-differential operators, etc. We want to mention that similar problem for bounded (nonselfadjoint) operators has been studied by author in [1, 2, 3]. The abstract results, similar to those presented here, have been applied to finite-difference perturbed operator (of any order), Wiener-Hopf perturbed operators (abstract, discrete, and integral), operators generated by Jacobi matrices, etc.

In Section 3, we obtain some new results about finiteness of the point spectrum of the following class of differential operators: the unperturbed operator is a differential operator with periodic coefficients of any order and the perturbation is a subordinated differential operator. These operators act in one of the spaces $L_p(\mathbb{R})$ or $L_p(\mathbb{R}_+)$ $(1 \leq p < +\infty)$. We want to stress out that while these results are of independent interest, and could be obtained separately, we derived them from the abstract results presented in Section 2. Also, we note that the unperturbed operator can be of any order and can be nonselfadjoint itself. The results agree with known ones. For example, if the unperturbed operator is Hill operator, then the finiteness of the point spectrum is guaranteed if the potential in the perturbation part has exponential decay at infinity (see the concluding remarks of this paper).

2 Abstract Results

Let H_0 and B be linear operators acting in a Banach space $\mathcal{B}$ such that the following assumptions are fulfilled:

(i) The spectrum $\sigma(H_0)$ of the operator H_0 is a simple rectifiable curve and the point spectrum of H_0 is absent, i.e. $\sigma_p(H_0) = \emptyset$;

(ii) The operator B can be represented in the form $B = RTS$, where S is an operator acting from $\mathcal{B}$ into $\mathcal{B}_1$ with $\mathrm{Dom}(S) \supset \mathrm{Dom}(H_0)$, the operator R acts from $\mathcal{B}_1$ into $\mathcal{B}$, T acts in $\mathcal{B}_1$, and $\mathcal{B}_1$ is a Banach space (possibly different from $\mathcal{B}$).

We denote by $\mathrm{Dom}(A)$ and $\mathrm{Ran}(A)$ the domain and the range of the operator A, by $\mathbb{B}(\mathcal{B})$ the set of all linear and bounded operators on $\mathcal{B}$, and by $\mathbb{B}_\infty(\mathcal{B})$ the class of all compact operators defined on $\mathcal{B}$. Also, we denote by $Q(\lambda)$, $\lambda \in \mathbb{C}$, the operator $S(H_0 - \lambda I)^{-1}RT$ defined on the set $\mathcal{L}_\lambda := \{u \in \mathcal{B}_1 \ : \ RTu \in \mathrm{Ran}(H_0 - \lambda I)\}$.

Under the above assumptions the following statement holds true.

Proposition 1. If $\lambda \in \sigma_p(H_0+B)$, then there exists $\varphi \in \mathcal{L}_\lambda,\ \varphi \neq 0$, such that

$$(I+Q(\lambda))\varphi = 0\,. \tag{1}$$

The proof is based on the following argument. Suppose that $\lambda \in \sigma_p(H_0+B)$. Then, there exists a vector $u \in \mathcal{B},\ u \neq 0$, such that $(H_0-\lambda)u+RTSu=0$. Note that $RTSu \in \mathrm{Ran}(H_0-\lambda I)$, hence $u+(H_0-\lambda I)^{-1}RTSu=0$. Consequently, $Su+S(H_0-\lambda I)^{-1}RTSu=0$. Put $\varphi = Su$ and equality (1) follows. Note that $\varphi \neq 0$, since otherwise $Su=0$, and then $(H_0-\lambda I)u=0$, which is a contradiction with initial assumption $\sigma_p(H_0)=\emptyset$.

It should be mentioned that the operator-valued function $Q(\lambda)$ plays a key role in perturbation theory and scattering theory. Proposition similar to Proposition 1 show up in many problems of spectral analysis (see for instance [10, 11, 20] where the selfadjoint case is considered). Note that (1) does not imply that corresponding λ belongs to the point spectrum of H_0. Actually, λ that satisfies (1) is called spectral singularity, and is related to eigenfunction expansion problem and generalized Parseval identity (see for instance [16]).

Due to Proposition 1, to establish that the point spectrum of the operator $H=H_0+B$ is a finite set, it suffices to show that the equality (1) holds for a finite set of numbers $\lambda \in \mathbb{C}$, and non-zero vectors $\varphi \in \mathcal{L}_\lambda$. Consequently, using the theorem of uniqueness of analytic operator-valued functions, it is sufficient to establish the analyticity of the function $Q(\lambda)$ with respect to λ. Generally speaking, $Q(\lambda)$ is analytic only on the resolvent set $\rho(H_0)$ (for instance if R, S, T are bounded operators), and the analyticity is lost in the neighborhood of $\sigma(H_0)$. In connection with this, we suppose that there exists an analytic continuation of the function $Q(\lambda)$ across $\sigma(H_0)$ (of course on Riemann surface). Namely, we suppose that the following assumption is satisfied.

Let $\lambda_0 \in \sigma(H_0)$, and let $U(\lambda_0)$ be a neighborhood of the point λ_0. We denote by $U_0(\lambda_0)$ one of the connected components of the neighborhood $U(\lambda_0)$ with respect to $\sigma(H_0)$. In other words, the curve $\sigma(H_0)$ divides the set $U(\lambda_0)$ in several parts, and by $U_0(\lambda_0)$ we denote the interior of one of these parts. For example if H_0 is a selfadjoint operator, and $\Pi_\pm$ denotes upper/lower half complex plane, then $U_0(\lambda_0) := U(\lambda_0)\cap\Pi_+$ or $U_0(\lambda_0) := U(\lambda_0)\cap\Pi_-$.

(iii) For every $\lambda_0 \in \mathbb{C}$ there exists a neighborhood $U(\lambda_0)$ such that for every $U_0(\lambda_0)$ there exists a neighborhood $\widehat{U}(\lambda_0)$ (maybe on a Riemann surface) of the point λ_0, and an operator-valued function $\widehat{Q}(\lambda) : \widehat{U}(\lambda_0) \to \mathbb{B}_\infty(\mathcal{B}_1)$ such that $U_0(\lambda_0) \subset \widehat{U}(\lambda_0)$, $\widehat{Q}(\lambda)$ is analytic on $\widehat{U}(\lambda_0)$ and $\widehat{Q}(\lambda) \supset Q(\lambda)$, $\lambda \in U_0(\lambda_0)$.

Theorem 2. *If the operators H_0 and B satisfy conditions (i)-(iii), then the perturbed operator $H = H_0 + B$ has a finite set of eigenvalues. Moreover, the possible eigenvalues have finite multiplicity.*

Proof. By Assumption (iii), $\widehat{Q}$ is uniquely defined on entire Riemann surface. Moreover, one can formally write $\widehat{Q}(\lambda) \supset Q(\lambda)$, $\lambda \in \mathbb{C}$, meaning that for every $\lambda \in \mathbb{C}$, there exists μ on the Riemann surface, such that $Q(\lambda) = \widehat{Q}(\mu)$. According to the theorem about holomorphic operator-valued functions with values in $\mathbb{B}_\infty$ (see, for example, [8], Chapter VII.1.3 or [18], theorem XII.13), the function $\widehat{Q}(\cdot)$ has a finite number of zeros in every neighborhood on the Riemann surface. Consequently $Q(\cdot)$ has a finite number of zeros in $\mathbb{C}$. The possibility of existence of sequence $\lambda_n \in \mathbb{C}$, $\lambda_n \to \infty$, $Q(\lambda_n) = 0$, is ruled out by the analyticity of the resolvent function $(H_0 - \lambda I)^{-1}$. Thus, (1) is satisfied for a finite number of values λ. Moreover, for every λ, the subspace $\mathcal{B}_1(\lambda)$ generated by the corresponding vectors $\varphi \in \mathcal{L}_\lambda$ that satisfy (1) has a finite dimension. By Proposition 1, $\sigma_p(H_0 + B)$ is finite, and every eigenvalues has finite multiplicity. Theorem is proved. □

In many applications, usually H_0 is a selfadjoint operator, so $\sigma(H_0) \subset \mathbb{R}$. Hence, (i) is satisfied if H_0 has no eigenvalues. Condition (ii) is a technical condition, but strongly related to (iii). In particular, (ii) and (iii) holds true if one may find the operators R and S such that $R(H_0 - \lambda I)^{-1} S$ has analytic continuation and $T := S^{-1} B R^{-1}$ is a compact operator. The hardest to check is condition (iii), and verification depends on the class of operators to be considered, and the general rule is to have an explicit or manageable form of the resolvent function of the unperturbed operator.

3 Application to Differential Operators

In this section we will present one application of the general results from Section 2. We will consider some perturbations of differential operators with periodic coefficients of arbitrary order acting in $L_p(\mathbb{R})$.

Let H be the differential operator of the following form

$$H\varphi(t) = \sum_{k=0}^{n} h_k(t) \frac{d^k \varphi(t)}{dt^k}\,, \tag{2}$$

where $h_k(t) = a_k(t) + q_k(t)$ $(k = 0, 1, \dots n;\ t \in \mathbb{R}$ or $\mathbb{R}_+)$, $a_k(t)$ $(k = 0, 1, \dots, n;\ a_n \equiv 1)$ are periodic functions (with the same period T) and $q_k(t)$ $(k = 0, 1, \dots, n;\ q_n \equiv 0)$ are functions (generally speaking, complex-valued) vanishing for $t \to \infty$. Assume that $a_k(t)$ $(k = 0, 1, \dots, n)$ are as smooth as required. The operator H is supposed to act in the spaces $L_p(\mathbb{R})$ or $L_p(\mathbb{R}_+)$ $(1 \leq p < +\infty)$. The domain of the operator H consists of all functions $\varphi \in L_p(\mathbb{R})$ $(L_p(\mathbb{R}_+))$ having absolutely continuous derivatives of order $n-1$ on each bounded interval of the real axis (semiaxis) and derivative of the n-th order belonging to $L_p(\mathbb{R})$ $(L_p(\mathbb{R}_+))$.

To apply the abstract scheme from Section 2, we consider the operator H as a perturbation of the operator

$$H_0 = \sum_{k=0}^{n} a_k(t) \left(\frac{d}{dt}\right)^k$$

by the differential operator

$$B = \sum_{k=0}^{n-1} q_k(t) \left(\frac{d}{dt}\right)^k.$$

The spectral properties of the unperturbed operator H_0 have been investigated by many authors (see for instance [15, 19] and the references therein). In [19] the operator H_0 is considered in the space $L_2(\mathbb{R})$, while in [15] in $L_p(\mathbb{R})$ $(1 \leq p \leq \infty)$. In these papers it is shown that the spectrum of the operator H_0 is continuous, coincides with the set of those $\lambda \in \mathbb{C}$ for which the equation $H_0\varphi = \lambda\varphi$ has a non-trivial solution (so-called zones of relative stability), and it is bounded in $\mathbb{C}$. Moreover, the unperturbed operator H_0 has no eigenvalues and satisfies condition (i) from Section 2. Actually this statement will also follow from our derivations related to properties (ii) and (iii). In what follows we suppose that the operator H is acting in the space $L_p(\mathbb{R})$, but all results (with obvious changes) hold true for $L_p(\mathbb{R}_+)$.

As we mentioned before, the key point in our abstract scheme is to find an analytical extensions of function $Q(\lambda)$, for which we need to have at hand a manageable representation of the resolvent function $(H_0 - \lambda I)^{-1}$. Although the spectrum of the operator H_0 is well-known (see for instance [15, 19]), we will present here a different method for describing explicitly the resolvent of H_0, suitable for our goal to verify the abstract conditions (ii) and (iii). The representation relies on Floquet-Liapunov theory about linear differential equations with periodic coefficients (see for instance [6, 22]).

Without loss of generality we can assume that $T = 1$.

Let us consider the equation

$$H_0 \varphi = \lambda \varphi \,, \tag{3}$$

where λ is a complex number, or in vector form

$$\frac{dx}{dt} = A(t, \lambda)\, x \,, \tag{4}$$

where

$$A(t,\lambda) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \lambda - a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix}, \quad x = \begin{pmatrix} \varphi \\ \varphi\prime \\ \vdots \\ \varphi^{(n-1)} \end{pmatrix}.$$

Denote by $U(t)$ $(= U(t, \lambda))$ the matriciant of the equation (4), i.e., the matrix which satisfies the following system of differential equations

$$\frac{dU(t)}{dt} = A(t, \lambda)\, U(t), \quad U(0) = E_n \,,$$

where E_n is $n \times n$ identity matrix. The matrix $U(1)$ is called the monodromy matrix of the equation (4) and the eigenvalues $\rho_1(\lambda), \dots, \rho_m(\lambda)$ of the matrix $U(1)$ are called the multiplicators. Also, we will say that $U(1)$ is the monodromy matrix and $\rho_1(\lambda), \dots, \rho_m(\lambda)$ are multiplicators of the operator $H_0 - \lambda I$.

Let $\Gamma = \ln U(1)$ be a solutions of the matrix equation $\exp(\Gamma) = U(1)$. Note that this equation has solutions since the monodromy matrix $U(1)$ is nonsingular. Due to Floquet theory (see for instance [6]), the matrix $U(t)$ has the following representation

$$U(t) = F(t)\exp(t\Gamma)\,, \tag{5}$$

where $F(t)$ is a nonsingular differentiable matrix of period $T = 1$.

Let us describe explicitly the structure of the matrix $\exp(t\Gamma)$. For this, we write Γ in its Jordan canonical form, $\Gamma = GJG^{-1}$, where $J = \mathrm{diag}[J_1, \ldots, J_m]$ and $J_k,\ k = 1, \ldots, m$, is the Jordan canonical block corresponding to the eigenvalue μ_k. Hence, $\exp(t\Gamma) = G\,\exp(tJ)\,G^{-1}$, $\exp(tJ) = \mathrm{diag}[\exp(J_1 t), \ldots, \exp(J_m t)]$, and

$$\exp(J_k t) = \exp(t\mu_k)\begin{pmatrix} 1 & t & \ldots & \frac{t^{p_k-1}}{(p_k-1)!} \\ 0 & 1 & \ldots & \frac{t^{p_k-2}}{(p_k-2)!} \\ . & . & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

where $p_k,\ k = 1, \ldots, m$, is the dimension of the Jordan block J_k.

Since $U(t)$ is the matriciant of the equation (4), it follows that every solution of this equation has the form

$$x(t) = U(t)\,x_0\,, \tag{6}$$

where x_0 is an arbitrary vector from $\mathbb{R}^n$.

Thus, from (4)-(6), we conclude that the components of the vector $x(t)$, and consequently the solution of the equation (3), are linear combinations of $\exp(\mu_k t)\ \ (k = 1, \ldots, m)$ with polynomial coefficients.

Remark that $|t^k \exp(t\lambda_j)| \to \infty,\ t \to \infty$, if $\mathrm{Re}\lambda_j \neq 0,\ k = 0, 1, \ldots,$ or $\mathrm{Re}\lambda_j = 0,\ k = 1, 2, \ldots$. Also note that $|t^k \exp(t\lambda_j)| = 1$ for $k = 0,\ \mathrm{Re}\lambda_j = 0$. From this we conclude that the only solution of equation (3) belonging to $L_p(\mathbb{R})$ $(1 \leq p < +\infty)$ is the function $\varphi \equiv 0$, which yields that $\sigma_p(H_0) = \emptyset$.

Since $\rho_j = \exp(\mu_j),\ j = 1, \ldots, m$, we conclude that $|\rho_j| < 1,\ |\rho_j| > 1$ or $|\rho_j| = 1$, if and only if $\mathrm{Re}\lambda_j < 0,\ \mathrm{Re}\lambda_j > 0$ or $\mathrm{Re}\lambda_j = 0$, respectively.

Now we are ready to solve explicitly equation $H_0 u - \lambda u = v$, where $v \in \mathrm{Ran}(H_0 - \lambda I)$. In matrix form this equation becomes

$$\frac{dx}{dt} = A(t, \lambda)\,x + f\,, \tag{7}$$

where $f = (0, 0, \ldots, v)^{\perp}$, $A(t,\lambda)$ and x are the same as in (4), and $\perp$ stands for the transposed vector. According to the Floquet representation of the matriciant (5), and making the substitution $x = F(t)\, y$ in (7), we get

$$\frac{dy}{dt} = \Gamma\, y + F^{-1}(t) f\,. \tag{8}$$

Assume that $|\rho_k| \neq 1,\ k = 1, \ldots, m$, and suppose that ρ_k are numbered such that $|\rho_k| > 1$ for $k = 1, \ldots, l$, and $|\rho_k| < 1$ for $k = l+1, \ldots, m$. Denote by P_1 the projection in $L_p^n(\mathbb{R})$ of the form $P_1\, y = (0, 0, \ldots, y_{j+1}, \ldots, y_n)$, where $y = (y_1, \ldots, y_n) \in L_p^n(\mathbb{R}),\ j = p_1 + \cdots + p_l$, and put $P_2 := I - P_1$.

An easy computation shows that the vector-valued function

$$y(t) = \int\limits_{-\infty}^{t} \exp(\Gamma(t-s))\, P_1\, F^{-1}(s)\, f(s) ds - \int\limits_{t}^{+\infty} \exp(\Gamma(t-s))\, P_2\, F^{-1}(s)\, f(s) ds \tag{9}$$

is a solution of the equation (8).

Since $x(t) = F(t)\, y(t)$, it follows that

$$\begin{aligned} x(t) &= F(t) \int\limits_{-\infty}^{t} \exp(\Gamma(t-s))\, P_1\, F^{-1}(s)\, f(s) ds - \\ &- F(t) \int\limits_{t}^{+\infty} \exp(\Gamma(t-s))\, P_2\, F^{-1}(s)\, f(s) ds\,, \end{aligned} \tag{10}$$

and taking into account (7) one obtains

$$\begin{aligned} (H_0 - \lambda I)^{-1} v(t) &= \sum_{r=l+1}^{m} \sum_{k=0}^{p_r} q_{rk}(t) \int\limits_{-\infty}^{t} \exp(\mu_r(t-s))(t-s)^k h_{rk}(s) v(s) ds + \\ &+ \sum_{r=1}^{l} \sum_{k=0}^{p_r} q_{rk}(t) \int\limits_{t}^{+\infty} \exp(\mu_r(t-s))(t-s)^k h_{rk}(s) v(s) ds, \end{aligned} \tag{11}$$

where $v \in \mathrm{Ran}(H_0 - \lambda I)$, q_{rk} and h_{rk} are some continuous periodic functions.

It is easy to show that under assumption $|\rho_k| \neq 1,\ k = 1, \ldots, m$, the operator defined in (11) is bounded, hence $\lambda \in \rho(H_0)$. Moreover, $\lambda \in \sigma(H_0)$ if there exists at least one multiplicator which lie on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. It should be mentioned, since $a_n(t) \equiv 1$, it is not possible to have a multiplicator ρ_k that belongs to $\mathbb{T}$ and is independent of λ (see for instance [19]).

Remark 3. Summing up, we conclude: the point spectrum of the unperturbed operator H_0 is absent; $\sigma(H_0)$ consists from the set of all curves determined by the equation $\det(U(1,\lambda)-\rho I)=0,\ |\rho|=1$; for every regular point $\lambda \in \rho(H_0)$ the resolvent function $(H_0-\lambda I)^{-1}$ has the form (11).

Now we are ready to prove the main result of this section. In what follows we will preserve the same notations as we defined above.

Theorem 4. *If the functions $q_k(t),\ k=0,1,\dots,n-1$, are such that*

$$q_k(t)\,\exp(\tau|t|) \in L_\infty(\mathbb{R})\,, \tag{12}$$

for some $\tau>0$, then the point spectrum of the perturbed operator H is at most a finite set. Furthermore, the possible eigenvalues have finite multiplicity.

Proof. For an arbitrary $\lambda \in \mathbb{C}$, in the space $L_p^n(\mathbb{R})$ we consider the operator H_1 of the following form

$$H_1 x(t) = \left(\frac{d}{dt} - A(t,\lambda)\right) x(t) + B_1 x(t)\,, \tag{13}$$

where $A(t,\lambda)$ and x are as in (4), and

$$B_1 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \\ .. & .. & \dots & .. \\ \\ q_0(t) & q_1(t) & \dots & q_{n-1}(t) \end{pmatrix}.$$

Denote by P_1 the projection in $L_p^n(\mathbb{R})$ of the form $P_1\,y=(0,0,\dots,y_{l+1},\dots,y_n)$, and put $P_2:=I-P_1$. The index l will be specified latter on. To satisfy conditions (ii) and (iii) from the the abstract result, we factorize the operator B_1 as follows: $B_1=RTS$, where $R=\exp(-\delta|t|)\cdot P_2$, $S=\exp(-\delta|t|)\cdot P_1$, $T=\exp(\delta|t|)\cdot B_1\cdot\exp(\delta|t|)$, and $\delta>0$. Note that $T\in\mathbb{B}(L_p^n(\mathbb{R}))$, and condition (ii) is fulfilled.

By Remark 3, it is sufficient to show that the operator-valued function

$$Q(\lambda) = S\left(\frac{d}{dt} - A(t,\lambda)\right)^{-1} R\,T\,, \qquad \lambda \in \sigma(H_0)\,,$$

satisfies condition *(iii)*.

Let $\lambda_0 \in \sigma(H_0)$. Since $U(t)$ is the matriciant, the matrix $U(t) = U(t,\lambda)$ is analytic in λ (see for instance [22], p.71, Th.1.3). Therefore, the function $\det(U(1,\lambda) - \rho) = 0$ is also analytic in λ. By implicit function theorem, there exists a neighborhood $U(\lambda_0)$ of the point λ_0, such that the multiplicators $\rho_1(\lambda), \ldots, \rho_m(\lambda)$ are holomorphic on $U(\lambda_0)$. Let $U_0(\lambda_0)$ be one of the connected components of the neighborhood $U(\lambda_0)$, and assume that the multiplicators (including their multiplicity) are enumerated such that

$$\begin{aligned} &\mathrm{Re}(\mu_k) > 0, \qquad && \text{for all } \lambda \in U_0(\lambda_0);\ k = 1, \ldots, l, \\ &\mathrm{Re}(\mu_k) < 0, \qquad && \text{for all } \lambda \in U_0(\lambda_0);\ k = l+1, \ldots, m, \end{aligned}$$

where $\exp(\mu_k) = \rho_k \quad (k = 1, \ldots, n)$. Using (10), (11) and (13), we conclude that the operator $Q(\lambda),\ \lambda \in U_0(\lambda_0)$, is a linear combination of the following operators

$$\begin{aligned} \left(Q_1(\lambda)\varphi\right)(t) &= \int\limits_{-\infty}^{t} \exp(\mu_k(\lambda)(t-s))(t-s)^r \exp(-\tau|t|)\, T\varphi(s)\, ds, \\ &\qquad\qquad (k = l+1, \ldots, m;\ r \in \mathbb{N}) \qquad\qquad (14) \\ \left(Q_2(\lambda)\varphi\right)(t) &= \int\limits_{t}^{+\infty} \exp(\mu_k(\lambda)(t-s))(t-s)^r \exp(-\tau|s|)\, T\varphi(s)\, ds, \\ &\qquad\qquad (k = 1, \ldots, l;\ r \in \mathbb{N}). \end{aligned}$$

We take the neighborhood $\widehat{U}(\lambda_0) \subset U(\lambda_0)$ such that $\mathrm{Re}(\mu_k(\lambda)) - \delta < 0$, $k = 1, \ldots, l;\ \lambda \in \widehat{U}(\lambda_0)$ and $\mathrm{Re}(\mu_k(\lambda)) + \delta > 0$, $k = l+1, \ldots, m$; $\lambda \in \widehat{U}(\lambda_0)$. For every $\lambda \in \widehat{U}(\lambda_0)$, we define the operator $\widehat{Q}(\lambda)$ by the same formula by which the operator $Q(\lambda)$ is defined on $U_0(\lambda_0)$ (i.e. integral operators generated by (14)). Under these conditions, the operator-valued functions (14) are analytic on $\widehat{U}(\lambda_0)$ and take values in $\mathbb{B}_\infty(L_p(\mathbb{R}))$. Hence, the same property holds true for the operator-valued function $\widehat{Q}(\lambda)$, $\lambda \in \widehat{U}(\lambda_0)$. By the definition of $\widehat{Q}$ we have $\widehat{Q}(\lambda) \supset Q(\lambda),\ \lambda \in U_0(\lambda_0)$. Thus, the condition *(iii)* of the abstract scheme is verified, and Theorem 4 is proved. $\square$

Remark 5. The initial spectral problem has been reduced to the corresponding system of first order differential equations (4) and (13). Moreover, we did not use the particular form and dimension of the matrices $A(t,\lambda)$ and $B_1(t)$. Actually, the Theorem 4 holds true for any matrices $A(t,\lambda)$ and $B_1(t)$, under condition that $A(t,\lambda)$ is periodic in t and analytic in λ, and the elements of the matrix $B_1(t)$ are such that $\exp(\tau|t|)|b_{jk}(t)| \in L_\infty(\mathbb{R})$, $\tau > 0$; $j,k = 1,\ldots,n$. The obtained results are also true if the operator (2) is a differential operator with matrix coefficients, i.e. $a_k(t)$ $(k = 0,1,\ldots,n)$ are periodic matrix-valued functions of dimension $r \times r$, $\det a_n(t) \neq 0$, $b_k(t)$, $k = 0,1,\ldots,n-1$, are measurable matrix-valued functions of the same dimension $r \times r$, and the operator H acts in the space $L_p(\mathbb{R},\mathbb{C}^n)$ $(1 \leq p < +\infty)$. In addition, the condition (12) should be replaced by the following one: $\exp(\tau|t|)|b_k| \in L_\infty(\mathbb{R})$, for some $\tau > 0$, where $|\cdot|$ is the operator matrix norm in $\mathbb{C}^n$.

Remark 6. It should be mentioned that similar results can be formulated for more general classes of operators. Namely, instead of periodicity of the unperturbed operator it is sufficient to suppose that the system of differential equations generated by the unperturbed operator is a reducible one.

Also, we want to mentioned that while our results cover a large class of operators, the conditions obtained for the Schrodinger operator are more restrictive than those obtained in [17, 23], where specific form of the operators have been used in deriving the corresponding results.

Example 7. As a concrete application of the previous results, consider the differential operator of the form

$$H = -\frac{d^2}{dt^2} + q_1(t)\frac{d}{dt} + p(t) + q_2(t)\,, \tag{15}$$

where $p(t+1) = p(t)$, q_k, $k = 1,2$, are measurable, complex-valued functions, and the operator H is acting in $L_2(\mathbb{R})$. The unperturbed operator is well-studied Hill operator (see for instance [5], p.281)

$$H_0 u = -\frac{d^2 u}{dt^2} + p(t)u\,.$$

Hence, for the perturbed Hill operator (15) we can formulate the following result.

Theorem 8. *If* $\exp(\tau|t|)q_k(t) \in L_\infty(\mathbb{R})$ $(k = 1,2;\ \tau > 0)$, *then the perturbed Hill operator (15) has a finite set of eigenvalues, each of them of finite* *multiplicity.*

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