Math 577 — Homework Assignment 2, due Sept.28, 2006

1. Determine SVDs of the following matrices. Do not use a computer, and do not use the method for hand calculations discussed in class. Use only basic properties of the SVD and note that the matrices are either diagonal matrices or rank-1 matrices:

$$\text{(a)} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{(d)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{(e)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

2. In the discussion of matrix norms we claimed that the 2-norm of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is approximately 1.6180. Using the SVD, work out (the "by-hand" method is from now on allowed) the exact values of $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ for this matrix.

3. Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

   (a) Determine, on paper, a real SVD of A in the form $A = U\Sigma V^T$. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V.

   (b) List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices labeled.

   (c) What are the 1-, 2-, ∞-, and Frobenius norms of A?

   (d) Find A^{-1} not directly, but via the SVD.

   (e) Find the eigenvalues λ_1, λ_2 of A.

   (f) Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$.

   (g) What is the area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$?

4. Assume A is Hermitian and positive definite, i.e., A can be uniquely factored into $A = LL^*$ with L a lower triangular matrix with positive diagonal entries (Cholesky factorization). What is the SVD of A?

5. If P is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

6. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   Answer the following questions by hand calculation.

   (a) What us the orthogonal projector P onto range(A), and what is the image under P of the vector $[1, 2, 3]^*$?

   (b) Same questions for B.