STUDY OF DEPENDENCE FOR SOME STOCHASTIC PROCESSES: SYMBOLIC MARKOV COPULAE

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Abstract. We study dependence between components of multivariate (nice Feller) Markov processes: what conditions need to be satisfied by a multivariate Markov process so that its components are Markovian with respect to the filtration of the entire process and such that they follow prescribed laws? To answer these question we introduce a symbolic approach, which is rooted in the concept of pseudo-differential operator (PDO). We investigate connections between dependence, in the sense described above, and the PDOs. In particular we study the problem of constructing a multivariate nice Feller process with given marginal laws in terms of symbols of the related PDOs. This approach leads to relatively simple conditions, which provide solutions to this problem.

1. Introduction

This paper continues studies presented in [2] and [3], where the problem of constructing a multivariate stochastic process with components following prescribed laws was investigated. In a nutshell, we can say that those papers studied the problem of modeling dependence between random processes subject to prescribed marginal laws. Thus, in a sense, they extended the classical finite-dimensional copula theory encapsulated in the famous theorem due to Abe Sklar (cf. [17]) to the infinite-dimensional realm of stochastic processes. In this paper we focus on study of dependence between (nice Feller) Markov processes. It needs to be stressed that in the context of multivariate Markov processes two problems are actually studied: what conditions need to be satisfied by a multivariate Markov process so that its components are Markovian in the filtration of entire process, and thus in their own filtrations (this is the first problem), and such that they follow prescribed laws (this is the second problem).

The first approach to construct a copula between some Markov processes was given in Bielecki et al. in [2], and it was then extended in Bielecki et al. in [3] to the case of general, real-valued Feller processes. In [3] the copula between Markov processes was given in terms of the infinitesimal generators. The approach taken here is complementing the one taken in [3]. Our *symbolic approach* is rooted in the concept of a pseudo-differential operator (PDO). This approach appears to be more transparent and gives relatively simple conditions guaranteeing that a multivariate (nice Feller) Markov process has (nice Feller) Markovian components with respect to the filtration of entire process, and thus with respect to their own filtrations. Moreover we give examples of construction of a (nice Feller) Markov process with prescribed marginal laws.

In our approach to constructing the symbol corresponding to a Markov copula, one just has to construct nonnegative definite functions satisfying appropriate conditions, whereas in the approach of [3] one has to construct an operator acting on functions. In particular, in the symbolic approach one avoids using tensor products of infinitesimal generators.

To avoid confusion we stress that the problem we study is different from the one considered, for example, in Lagerås [11], where results of Darsow et al. [5] are extended. Those papers aim at relating the classical concept of copula and the concept of Markov property. In this context they

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investigate dependence along the time line in the case of a one-dimensional Markov process, and characterize the Markov property in terms of copulae.

The paper is organized as follows: In Section 2 we introduce so called strong Markovian consistency properties and we study their connection with the corresponding characteristic functions as well as with the corresponding PDOs.

In Section 3 we study the question of constructing a multivariate nice Feller process with given marginal laws in terms of symbols of the related PDOs. In other words, we show how to construct an n-dimensional nice Feller process such that its marginal laws, that is, the laws of its components, agree with the laws of a given collection of n one-dimensional nice Feller processes.

Without loss of generality we shall only consider time-homogeneous (nice Feller) Markov processes in this paper.

2. Dependence and Symbols

Consider $X = (X^j,\, j = 1, \dots, n)$, a time-homogenous Markov process, defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in $\mathbb{R}^n$. It is well known that, in general, the coordinates of X are not Markovian (neither with respect to filtration of entire process X nor in their own filtrations).

Remark 2.1. In order to simplify presentation we fix $j \in \{1, \dots, n\}$ for most of the discussion in the rest of this paper. Thus the discussion and the results apply to an arbitrary $j \in \{1, \dots, n\}$.

We are interested in the following problems P0 and P1:

P0 : Provide necessary and sufficient conditions so that for each X^j the following property holds: For every $B \in \mathcal{B}(\mathbb{R})$ and all $t, s \geq 0$ we have

$$\mathbb{P}\left(X^j_{t+s} \in B | \mathcal{F}^X_t\right) = \mathbb{P}\left(X^j_{t+s} \in B | X^j_t\right) \tag{2.1}$$

or equivalently

$$\mathbb{P}\left(X^j_{t+s} \in B | X_t\right) = \mathbb{P}\left(X^j_{t+s} \in B | X^j_t\right), \tag{2.2}$$

which means that X^j is a Markov process with respect to the filtration $\mathbb{F}^X$.

Note that if the above conditions hold then

$$\mathbb{P}\left(X^j_{t+s} \in B | \mathcal{F}^{X^j}_t\right) = \mathbb{P}\left(X^j_{t+s} \in B | X^j_t\right), \tag{2.3}$$

which means that X^j is a Markov process with respect to its own filtration.

Definition 2.2. i) We say that a Markov process X satisfies the *strong Markovian consistency condition* with respect to X^j if (2.1) (or equivalently (2.2)) holds. We say that a Markov process X satisfies the *weak Markovian consistency condition* with respect to X^j if (2.3) holds.
ii) If X satisfies the strong (weak) Markovian consistency condition with respect to X^j for each $j = 1, \dots, n$, then we say that X satisfies the strong (weak) Markovian consistency condition.

Remark 2.3. (i) It is rather clear that condition (2.3) needs not to imply condition (2.1). In this paper we carry out a study of the strong Markovian consistency condition (2.1) and the corresponding Markov copulae. In the follow up paper [1] we carry out a study of the weak Markovian consistency condition (2.3) and the corresponding Markov copulae. In particular, in [1] we give an example of a nice Feller process, which satisfies the weak Markovian consistency condition but it does not satisfy the strong Markovian consistency condition.
(ii) Sufficient conditions for strong Markov consistency in terms of infinitesimal generators can be deduced from Dynkin [6], Theorem 10.13, by taking transformation γ there to be a coordinate projection.

P1 : Provide necessary and sufficient conditions, which guarantee that

(2.4a) the strong Markovian consistency condition holds with respect to X^j,

(2.4b) $\mathcal{L}(X^j) = \mathcal{L}(Y^j)$ for a given one-dimensional Markov process Y^j defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$.

Definition 2.4. We say that a Markov process X satisfies the *strong Markovian consistency condition with respect to X^j relative to Y^j* if (2.4a) and (2.4b) hold.

Remark 2.5. Observe that if condition (2.4b) is satisfied then X^j is a Markov process with respect to its own filtration. However, the strong Markovian consistency condition with respect to X^j may not be satisfied.

Remark 2.6. Starting from Section 2.3 we shall study problems P0 and P1 with regard to a nice Feller Markov process X.

Remark 2.7. The above two problems can be extended to the case of an arbitrary subset of components of the process X.

2.1. **Preliminary Discussion of Problem P0.**

Here we shall provide sufficient and necessary conditions for the strong Markovian consistency condition to hold in terms of relevant characteristic functions.

Note that property (2.1) (or, equivalently (2.2)) is a property of conditional probabilities $\mu_{t-s}(x,\cdot)$ and $\mu^j_{s,t}(x_j,\cdot)$ defined as follows

$$\mu_{t-s}(x,B) := \mathbb{P}(X_t \in B | X_s = x), \quad \mu^j_{s,t}(x_j, B_j) := \mathbb{P}(X^j_s \in B_j | X^j_s = x_j),$$
$$t \geq s \geq 0,\ B_j \in \mathcal{B}(\mathbb{R}),\ B \in \mathcal{B}(\mathbb{R}^n),\ x_j \in \mathbb{R}.$$

Indeed, denoting

$$B^j = \mathbb{R} \times \ldots \times B_j \times \ldots \times \mathbb{R}.$$

we see that property (2.1) reads

$$\mu_{t-s}(X_s, B^j) = \mu^j_{s,t}(X^j_s, B_j).$$

Consequently, we can formulate property (2.1) in terms of conditional characteristic functions of X_t and X^j_t, which are defined by

$$\bar{\lambda}_{t-s}(x,\xi) := \mathbb{E}\left(e^{i(X_t,\xi)} | X_s = x\right), \quad \bar{\lambda}^j_{s,t}(x^j, \xi_j) := \mathbb{E}\left(e^{iX^j_t \xi_j} | X^j_s = x_j\right),$$

where $\xi \in \mathbb{R}^n$ and $\xi_j \in \mathbb{R}$. Indeed, denoting by e_j the standard unit vector in $\mathbb{R}^n$ with 1 in the j-th position and defining $\bar{\lambda}^j_t(x_j,\xi_j) := \bar{\lambda}^j_{0,t}(x_j,\xi_j)$, we have following result

Proposition 2.8. *(i) The strong Markovian consistency property for X^j implies that*

$$\bar{\lambda}^j_{s,t}(X^j_s,\xi_j) = \bar{\lambda}_{t-s}(X_s, e_j\xi_j), \qquad \forall\ \xi_j \in \mathbb{R},\ t \geq s \geq 0. \tag{2.5}$$

(ii) Assume that X^j is a time homogenous Markov process with respect to its own filtration and assume that

$$\bar{\lambda}^j_t(x_j,\xi_j) = \bar{\lambda}_t(x, e_j\xi_j), \qquad \forall\ \xi_j \in \mathbb{R},\ x \in \mathbb{R}^n,\ t \geq 0. \tag{2.6}$$

Then the strong Markovian consistency condition with respect to X^j is satisfied.

Proof. (i) Equality of conditional distributions is equivalent to equality of conditional characteristic functions (see Loeve [12, p. 30] or Karatzas and Shreve [16, Lemma 6.13, p. 85]).
(ii) In view of (2.6) we have

$$\mu^j_{0,t}(x_j, B_j) = \mu_t(x, B^j) \qquad \forall B_j \in \mathcal{B}(\mathbb{R}),\ x \in \mathbb{R}^n,\ t \geq 0.$$

This and the assumed time homogeneity properties implies

$$\mathbb{P}\left(X^j_t \in B_j | X_s\right) = \mu_{t-s}(X_s, B^j) = \mu^j_{0,t-s}(X^j_s, B_j) = \mathbb{P}\left(X^j_t \in B_j | X^j_s\right)$$

for every $B \in \mathcal{B}(\mathbb{R})$. This completes the proof. ☐

Following Jacob [10] we define for $t \geq s \geq 0$

$$\lambda_{t-s}(x,\xi) := \mathbb{E}\left(e^{i(X_t - x,\xi)}\Big| X_s = x\right), \quad \lambda^j_{s,t}(x_j,\xi_j) := \mathbb{E}\left(e^{i(X^j_t - x_j)\xi_j}\Big| X^j_s = x_j\right) \tag{2.7}$$

and we denote $\lambda^j_t(x_j,\xi_j) := \lambda^j_{0,t}(x_j,\xi_j)$.

Remark 2.9. (i) For $x \in \mathbb{R}^n$, let $A^x = \{\omega \in \Omega : X_0(\omega) = x = (x_1, \dots, x_n)\}$. If condition (2.5) is satisfied then, for $\omega \in A^x$,

$$\lambda_{t-s}(x,\xi_j e_j) = \bar{\lambda}_{t-s}(X_0,\xi_j e_j)(\omega)e^{-i(x,\xi_j e_j)} = \bar{\lambda}^j_{s,t}(X^j_0,\xi_j)(\omega)e^{-ix_j\xi_j} = \lambda^j_{s,t}(x_j,\xi_j).$$

(ii) If

$$\lambda^j_{s,t}(x_j,\xi_j) = \lambda_{t-s}(x,\xi_j e_j), \quad \forall x \in \mathbb{R}^n,\ \xi_j \in \mathbb{R},\ t \geq s \geq 0, \tag{2.8}$$

then condition (2.5) is satisfied.

2.2. **Preliminary Discussion of Problem P1.**

We define

$$\psi^j_t(y_j,\xi_j) := \widetilde{\mathbb{E}}\left(e^{i\xi_j(Y^j_t - y_j)}\Big| Y^j_0 = y_j\right).$$

We have the following result regarding Problem P1:

Proposition 2.10. *(i) Suppose condition* (2.4b) *is satisfied. Then* (2.4a) *holds if for all $x \in \mathbb{R}^n$, $\xi_j \in \mathbb{R}$, and $t \geq 0$ the following equality holds:*

$$\lambda_t(x,e_j\xi_j) = \psi^j_t(x_j,\xi_j). \tag{2.9}$$

(ii) Suppose (2.4a) *holds. Also, suppose that $\mathcal{L}(X^j_0) = \mathcal{L}(Y^j_0)$. Then* (2.4b) *is satisfied if for every $\xi_j \in \mathbb{R}$ and $t \geq 0$ the following equality holds:*

$$\lambda^j_t(x_j,\xi_j) = \psi^j_t(x_j,\xi_j). \tag{2.10}$$

Proof. (i) Since Y^j is a Markov process with respect to its own filtration and since condition (2.4b) holds, it follows that, as already observed earlier, the process X^j is a time homogenous Markov with respect to its own filtration, and we have

$$\lambda^j_t(x_j,\xi_j) = \psi^j_t(x_j,\xi_j). \tag{2.11}$$

This together with (2.9) implies (2.8), which in turn implies (2.5) by Remark 2.9.ii. Thus, the result follows in view of Proposition 2.8(ii).
(ii) This follows from the fact that for a Markov processes the initial distribution and the transition laws determine the entire law of the process. □

2.3. **Connection with PDOs and their symbols.**

From now on we focus on nice Feller processes.[1] By a Feller process we mean a conservative Markov processes whose corresponding semigroup is a strongly continuous semigroup on $C_0(\mathbb{R}^n)$, the family of real valued continuous functions vanishing at infinity. A Feller process with strong generator A is called nice, if $C^\infty_c(\mathbb{R}^n) \subset D(A)$ (cf. e.g. [15]).

Study of problems P0 and P1 in terms of the families of functions λ_t, λ^j_t and ψ^j_t is equivalent to study of these problems in terms of the Markov semigroups corresponding to the processes X, X^j and Y^j. This follows from Jacob [9]: Specifically, it is shown there that for a family of functions λ_t given in (2.7) we can determine the semigroup $(T_t)_{t\geq 0}$, corresponding to X, in the following way:

$$T_t u(x) := \mathbb{E}^x u(X_t) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i(x,\xi)}\lambda_t(x,\xi)\hat{u}(\xi)d\xi, \tag{2.12}$$

[1]It is well known that a Feller process admits a càdlàg modification. Thus we implicity assume that processes considered from now on are càdlàg.

where $\hat{u}$ denotes the Fourier transform of function $u : \mathbb{R}^n \to \mathbb{R}$, that is,

$$\hat{u}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x,\xi)} u(x) dx.$$

Analogous properties hold for the families λ_t^j and ψ_t^j and the corresponding semigroups, say $(T_t^j)_{t\geq 0}$ and $(S_t^j)_{t\geq 0}$.

It is rather clear though that providing conditions, with regard to Problems P0 and P1, in terms of the entire families of operators $(T_t)_{t\geq 0}$, $(T_t^j)_{t\geq 0}$ and $(S_t^j)_{t\geq 0}$, or equivalently in terms of the entire families λ_t, λ_t^j and ψ_t^j, is quite inconvenient.[2]

That is why, in [2] and [3], Problems P0, P1 were studied in terms of the infinitesimal generators, say A, A^j and B^j, of the relevant Markov processes X, X^j and Y^j respectively.[3] Here we take an alternative approach, and pursue the study of these problems in terms of the derivatives of λ_t, λ_t^j and ψ_t^j at zero, i.e.

$$\begin{aligned} q(x,\xi) &= -\lim_{t\to 0} \frac{\lambda_t(x,\xi) - 1}{t}, \\ q_j(x_j,\xi_j) &= -\lim_{t\to 0} \frac{\lambda_t^j(x_j,\xi_j) - 1}{t}, \\ \varrho_j(x_j,\xi_j) &= -\lim_{t\to 0} \frac{\psi_t^j(x_j,\xi_j) - 1}{t}. \end{aligned} \tag{2.13}$$

For the in-depth discussion of these derivatives we refer to Jacob [9], Schilling [13] or Jacob [10]. These derivatives play a role similar to the role played by the generator of a Feller semigroup. In particular, they lead to the symbol of the corresponding semigroup and to the corresponding infinitesimal operator.

Our approach in this paper is motivated by the fact that in view of the results of Courrège [4], the strong generator A of X, acting on $u \in C_c^\infty(\mathbb{R}^n)$, the space of infinitely differentiable functions with compact support, has a representation

$$Au(x) = -q(x,D)u(x) := -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x,\xi)} q(x,\xi) \widehat{u}(\xi) d\xi, \tag{2.14}$$

where $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is an analytic symbol, i.e. it is a measurable, locally bounded, continuous function in ξ, and for every x the function $q(x,\cdot)$ is negative definite[4]. In this context, the function $q(x,\xi)$ is called the symbol of the pseudo-differential operator $q(x,D)$ (cf. Jacob [10]), and it has the following form:

$$q(x,\xi) = -i(b(x),\xi) + (\xi, a(x)\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{i(y,\xi)} + \frac{i(y,\xi)}{1+|y|^2}\right) \mu(x,dy) \tag{2.15}$$

where a, b are Borel measurable functions, $b(x) \in \mathbb{R}^n$, $a(x)$ is a symmetric non-negative definite matrix, and $\mu(x,dy)$ is a Lévy kernel. Moreover, if q is continuous (in all variables) then q maps $C_c^\infty(\mathbb{R}^n)$ into $C(\mathbb{R}^n)$ (Jacob [10, Vol. 1, Theorem 4.5.7, page 337]). Schnurr [15, Theorem 3.10] in his PHD thesis gives a probabilistic interpretation of (b,a,μ) , namely (B,C,ν) is semimartingale characteristic of X, where

$$B_t := \int_0^t b(X_u)du, \quad C_t := 2\int_0^t a(X_u)du, \quad \nu(du,dy) := \mu(X_u,dy)du.$$

Remark 2.11. We shall distinguish the concept of analytic symbol from the concept of a probabilistic symbol i.e. the symbol of a Feller process. This is to stress that not every analytic symbol generates a Feller process.

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[2]Note that in the case of the Lévy processes these conditions can be significantly simplified in the sense that they can be reduced to considering $t = 1$ only.

[3]This also applies to Problem P2 discussed below.

[4]For definition of a negative definite function see for example [10, Vol I, Definition 3.6.5.]

Analogous results hold for q_j and ϱ_j. In particular, in the case of Y^j, the infinitesimal generator B^j, acting on $w \in C_c^\infty(\mathbb{R})$, satisfies

$$B^j w(x_j) = -(2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix_j\xi_j} \widehat{w}(\xi_j) \varrho_j(x_j, \xi_j) d\xi_j,$$

where

$$\varrho_j(x_j, \xi_j) = -i\, b_j(x_j)\xi_j + c_j(x_j)\xi_j^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{iz_j\xi_j} + \frac{iz_j\xi_j}{1+|z_j|^2}\right) \mu_j(x_j, dz_j).$$

We shall adopt the following convention:

Suppose that $f : \mathbb{R} \to \mathbb{R}$. Then we define $f_j : \mathbb{R}^n \to \mathbb{R}$ by $f_j(x) = f(x_j)$.

Note, however, that even though f may be of compact support, f_j is not a function of compact support.

In the remainder of the paper we shall need the following conditions[5]:

C1-a:

$$w_j \in D(\widetilde{A}), \text{ where } \widetilde{A} \text{ is the weak generator of } X, \text{ for all } w \in C_c^\infty(\mathbb{R}),$$

C1-b:

$$\widetilde{A} w_j(x) = -(2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix_j\xi_j} \widehat{w}(\xi_j) q(x, e_j\xi_j) d\xi_j, \text{ for all } w \in C_c^\infty(\mathbb{R}).$$

C2: The function $q(x, e_j\xi_j)$ as a function of x depends only on x_j.

Under **C2** we define

$$\tilde{q}_j(x_j, \xi_j) := q(x, e_j\xi_j), \tag{2.16}$$

and we postulate

C3: $\tilde{q}_j$ is a symbol of a nice Feller process.

Remark 2.12. i) We stress that condition **C2** is the central condition underlying in the property of strong Markovian consistency discussed in this paper.
ii) If condition **C2** is satisfied, and if q is an analytic symbol, then $\tilde{q}_j$ given by (2.16) is also an analytic symbol.

The following theorem is the first main result in this paper and it provides a solution to Problem P0.

Theorem 2.13. *Let X be a nice Feller process with symbol q satisfying **C1** and **C2**. In addition, assume that **C3** is satisfied. Then component X^j of X is a nice $\mathbb{F}^X$-Feller process with generator given by the symbol $q_j = \tilde{q}_j$.*

Proof. First, we observe that since X is a Feller process,

$$f(X_t) - \int_0^t \widetilde{A} f(X_u) du$$

is an $\mathbb{F}^X$-martingale for any function $f \in D(\widetilde{A})$. Consequently, for any $w \in C_c^\infty(\mathbb{R})$, we have by **C1-a** that the process

$$w_j(X_t) - \int_0^t \widetilde{A} w_j(X_u) du$$

is an $\mathbb{F}^X$-martingale.

Let us denote the strong generator corresponding to $\tilde{q}_j$ by A_j. We shall now verify that

$$\widetilde{A} w_j(x) = A_j w(x_j), \ \forall x \in \mathbb{R}^n. \tag{2.17}$$

[5] For definition and properties of a weak generator see Dynkin [6, Chapter I§6].

Indeed,

$$A_j w(x_j) = -(2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix_j\xi_j} \widehat{w}(\xi_j) \widetilde{q}_j(x_j,\xi_j) d\xi_j.$$

and by **C1-a**, **C1-b** and **C2**

$$\begin{aligned} \widetilde{A} w_j(x) &= -(2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix_j\xi_j} \widehat{w}(\xi_j) q(x, \mathbf{e_j}\xi_j) d\xi_j \\ &= -(2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix_j\xi_j} \widehat{w}(\xi_j) \widetilde{q}_j(x_j,\xi_j) d\xi_j \\ &= A_j w(x_j). \end{aligned}$$

which implies (2.17). Hence

$$w(X_t^j) - \int_0^t A_j w(X_u^j) du$$

is an $\mathbb{F}^X$-martingale, for any $w \in C_c^\infty(\mathbb{R})$.

Consequently, X^j is a solution to the martingale problem for A_j relative to the full filtration of process X, that is with respect to $\mathbb{F}^X$. Since $C_c^\infty(\mathbb{R}^n)$ is separating class then, in view of [7, Proposition 4.1.6], X^j is the unique solution to the martingale problem for A_j relative to the full filtration of process X. Thus X^j is a $\mathbb{F}^X$-Feller process with symbol $\tilde{q}_j$. □

Remark 2.14. The above theorem proves the strong Markovian consistency property with respect to the component X^j. As we already noticed before strong Markovian consistency implies weak Markovian consistency. Thus X^j is a $\mathbb{F}^{X^j}$-Feller process with symbol $\tilde{q}_j$. This can also be concluded by observing that

$$w(X_t^j) - \int_0^t A_j w(X_u^j) du$$

is also an $\mathbb{F}^{X^j}$-martingale, for any $w \in C_c^\infty(\mathbb{R})$.

Before we state the second main result of the paper (Theorem 2.18) we first prove the following two auxiliary results.

Proposition 2.15. *Let X be a nice Feller process with symbol q satisfying **C1-a** and **C1-b**. Assume that the component X^j of X is a nice $\mathbb{F}^X$-Feller processes with symbol q_j. Then*

$$q(x, e_j\xi_j) = q_j(x_j,\xi_j) \quad \textit{for all } x \in \mathbb{R}^n \textit{ and } \xi_j \in \mathbb{R}. \tag{2.18}$$

holds.

Proof. It is sufficient to consider the case when $j = 1$. By **C1-a** and **C1-b** and by the strong Markov consistency of X^1, for any $w \in C_c^\infty(\mathbb{R})$ we have

$$\begin{aligned} -(2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix_1\xi_1} \widehat{w}(\xi_1) q(x, e_1\xi_1) d\xi_1 &= \widetilde{A} w_1(x) = \lim_{t\to 0^+} \frac{\mathbb{E}^x\left(w_1(X_t) - w_1(x)\right)}{t} \\ = \lim_{t\to 0^+} \frac{\mathbb{E}^{x_1}\left(w(X_t^1) - w(x_1)\right)}{t} &= \widetilde{A}_1 w(x_1) = -(2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix_1\xi_1} \widehat{w}(\xi_1) q_1(x_1,\xi_1) d\xi_1. \end{aligned}$$

Therefore

$$q(x, e_1\xi_1) = q_1(x_1,\xi_1)$$

for all $x \in \mathbb{R}^n$ and $\xi_1 \in \mathbb{R}$. □

Remark 2.16. Note that in view of Theorem 2.13 and Proposition 2.15 conditions **C1**–**C3** are sufficient for (2.18) to hold.

Proposition 2.17. *Let X, Y be two nice Feller processes with symbols q^X and q^Y, respectively. Then*

$$q^X = q^Y \quad \textit{and} \quad \mathcal{L}(X_0) = \mathcal{L}(Y_0) \Longleftrightarrow \mathcal{L}(X) = \mathcal{L}(Y).$$

Proof. The result follows from [15, Corollary 1.21], [7, Proposition 4.1.6] and from the fact that $C_c^\infty(\mathbb{R}^n)$ is a separating class. □

Finally, we provide a solution to Problem P1. This is done in the following theorem, which combines the results of Theorem 2.13 and Propositions 2.15 and 2.17.

Theorem 2.18. *Let Y^j be a nice real valued Feller process with symbol ϱ_j, and let $X = (X_1, \ldots, X_n)$ be a nice Feller process with symbol q satisfying **C1**–**C3**. Then*

$$\mathcal{L}(X^j) = \mathcal{L}(Y^j) \tag{2.19}$$

if and only if

$$q(x, e_j\xi_j) = \varrho_j(x_j, \xi_j) \quad \text{for all } x \in \mathbb{R}^n \text{ and } \xi_j \in \mathbb{R}, \tag{2.20}$$

and

$$\mathcal{L}(X_0^j) = \mathcal{L}(Y_0^j) \tag{2.21}$$

hold. In particular the symbol of X^j is equal to ϱ_j .

Proof. First observe that **C1–C3** imply

$$q(x, e_j\xi_j) = q_j(x_j, \xi_j) \quad \text{for all } x \in \mathbb{R}^n \text{ and } \xi_j \in \mathbb{R}. \tag{2.22}$$

Thus if (2.20) holds and (2.21) then in view of Proposition 2.17 we have that (2.19) holds. Conversely if **C1–C3** and (2.19) hold then (2.21) and $q_j = \rho_j$, which in view of (2.22) implies that (2.20). ☐

Remark 2.19. It is interesting to verify whether conditions **C1–C3** in the formulation of Theorem 2.18 can be replaced with the assumption that process X is strongly Markovian consistent with respect to X^j. This is left for future work.

2.4. **Discussion of conditions C1 and C3.**

We start with following technical result.

Lemma 2.20. *Let X be a nice Feller process with symbol q and the corresponding generator A, and let $w \in C_c^\infty(\mathbb{R})$. Next, let us fix $j \in \{1, \ldots, n\}$, and $v \in C_c^\infty(\mathbb{R}^{n-1})$ such that $v(0) = 1$, $||v||_\infty \le 1$. Finally, define a function u^k as follows*

$$u^k(x) = w(x_j)v\left(\frac{1}{k}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\right), \quad \forall x \in \mathbb{R}^n. \tag{2.23}$$

Then

$$\lim_{k\to\infty} Au^k(x) = -(2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix_j\xi_j} \widehat{w}(\xi_j) q(x, e_j\xi_j) d\xi_j, \quad \forall x \in \mathbb{R}^n. \tag{2.24}$$

Proof. We shall consider the case of $j = 1$; the proof for arbitrary j is analogous. Note that

$$u^k(x) = w(x_1)v^k(\bar{x})$$

where

$$v^k(\bar{x}) = v\left(\frac{1}{k}\bar{x}\right)$$

and $\bar{x} = (x_2, \ldots, x_n)$. Clearly $(v^k)_{k\ge1}$ is a sequence of uniformly bounded (by 1) functions of class $C_c^\infty(\mathbb{R}^{n-1})$, that converges pointwise in $\mathbb{R}^{n-1}$ to 1. Using (2.14) and the fact that $\widehat{v^k}(\bar{\xi}) = k^{n-1}\widehat{v}(k\bar{\xi})$ we see that

$$\begin{aligned}
Au^k(x) &= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x,\xi)} q(x, \xi) \widehat{u^k}(\xi) d\xi \\
&= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x,\xi)} q(x, \xi) \widehat{w}(\xi_1) k^{n-1} \widehat{v}(k\bar{\xi}) d\bar{\xi} d\xi_1 \\
&= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix_1\xi_1 + } e^{i\frac{1}{k}(\bar{x},\bar{\xi})} q\left((x_1, \bar{x}), \left(\xi_1, \frac{\bar{\xi}}{k}\right)\right) \widehat{w}(\xi_1) \widehat{v}(\bar{\xi}) d\bar{\xi} d\xi_1.
\end{aligned}$$

We now observe that

$$\begin{aligned}\left|q\left((x_1,\bar{x}),\left(\xi_1,\frac{\bar{\xi}}{k}\right)\right)\widehat{w}(\xi_1)\widehat{v}(\bar{\xi})\right| &\le c(x)(1+|\xi_1|^2+\frac{1}{k^2}|\bar{\xi}|^2)|\widehat{wv}(\xi_1,\bar{\xi})| \\ &\le c(x)(1+|\xi_1|^2+|\bar{\xi}|^2)|\widehat{wv}(\xi_1,\bar{\xi})|,\end{aligned}$$

where the first inequality is implied by

$$|q(x,\xi)|\le c(x)(1+\|\xi\|^2),$$

which follows from the fact that q is a symbol of a Feller semigroup (see Jacob [10, Vol.1, Lemma 3.6.22 and Theorem 4.5.6]). In addition we note that since $wv\in C_c^\infty(\mathbb{R}^n)$ it follows that $\widehat{wv}\in\mathcal{S}(\mathbb{R}^n)$, which in turn implies that $\xi\mapsto(1+\|\xi\|^2)\widehat{u^k}(\xi)$ is in L^1. Thus invoking dominated convergence theorem we see that

$$\begin{aligned}\lim_{k\to\infty}Au^k(x) &= -(2\pi)^{-n/2}\int_{\mathbb{R}^n}\lim_{k\to\infty}e^{ix_1\xi_1+}e^{i\frac{1}{k}(\bar{x},\bar{\xi})}q\left((x_1,\bar{x}),\left(\xi_1,\frac{\bar{\xi}}{k}\right)\right)\widehat{w}(\xi_1)\widehat{v}(\bar{\xi})d\bar{\xi}d\xi_1 \\ &= -(2\pi)^{-1/2}\int_{\mathbb{R}}e^{ix_1\xi_1}\widehat{w}(\xi_1)q(x,\mathbf{e}_1\xi_1)d\xi_1,\end{aligned}$$

which demonstrates (2.24). □

Now we give sufficient conditions for **C1-a** and **C1-b** to hold.

Proposition 2.21. *Let X be a nice Feller process with symbol q. Assume that q is continuous and that*

$$|q(x,\xi)|\le c(1+|\xi|^2),\quad \forall x\in\mathbb{R}^n,\ \forall\xi\in\mathbb{R}^n. \tag{2.25}$$

*Then **C1-a** and **C1-b** holds.*

Proof. Without loss of generality we take $j=1$. We need to show that the following limit

$$\operatorname*{bp-lim}_{t\to0^+}\frac{T_tw_1-w_1}{t}$$

exists for each $w\in C_c^\infty(\mathbb{R})$. Towards this end we first note that the sequence $(u^k)_{k\ge1}$, defined in (2.23), is uniformly bounded and converges pointwise to $w_1(x)$. Therefore by the dominated convergence theorem we have

$$\lim_{k\to\infty}\mathbb{E}^xu^k(X_t)=\mathbb{E}^xw_1(X_t).$$

Using this we obtain

$$\frac{T_tw_1(x)-w_1(x)}{t}=\frac{\mathbb{E}^xw_1(X_t)-w_1(x)}{t}=\lim_{k\to\infty}\frac{\mathbb{E}^xu^k(X_t)-u^k(x)}{t}=\lim_{k\to\infty}\frac{1}{t}\mathbb{E}^x\left(\int_0^tAu^k(X_u)du\right),$$

where the second equality follows from Dynkin formula, since $u^k\in D(A)$. From (2.25) we deduce the following uniform boundedness

$$|Au^k(x)|\le K, \tag{2.26}$$

for a finite constant K. Applying again the dominated convergence theorem we obtain

$$\frac{\mathbb{E}^xw_1(X_t)-w_1(x)}{t}=\frac{1}{t}\mathbb{E}^x\left(\int_0^tBw(X_u)du\right),$$

where we denoted

$$Bw(x)=\lim_{k\to\infty}(Au^k)(x).$$

Assumption of continuity of q with respect to x, together with (2.24) and (2.26), implies that Bw is a bounded continuous function. Therefore, $Bw(X_u)$ is a right continuous function and

$$\lim_{t\to0^+}\frac{1}{t}\int_0^tBw(X_u)du=Bw(X_0).$$

Invoking the dominated convergence theorem gives

$$\lim_{t\to 0^+}\frac{\mathbb{E}^x w_1(X_t)-w_1(x)}{t}=\mathbb{E}^x\left(\lim_{t\to 0^+}\frac{1}{t}\int_0^t Bw(X_u)du\right)=\mathbb{E}^x Bw(X_0)=Bw(x).$$

Since

$$\sup_{t>0}\sup_{x\in\mathbb{R}}\left|\frac{\mathbb{E}^x w_1(X_t)-w_1(x)}{t}\right|\le K,$$

we conclude that $w_1\in D(\widetilde{A})$, and, in view of Dynkin [6, I.§6.1.15.D], we see that **C1-b** holds. □

Remark 2.22. By Schilling [14, Lemma 2.1] condition (2.25) is equivalent to

$$\|b\|_\infty+\|a\|_\infty+\left\|\int_{\mathbb{R}^n\setminus 0}\frac{|y|^2}{1+|y|^2}\mu(\cdot,dy)\right\|_\infty<\infty\ .$$

In order to give sufficient conditions for **C3** to hold, we first introduce some additional definitions and postulates (cf. Hoh [8, page 82]):

Let $\psi:\mathbb{R}^n\to\mathbb{R}$ be a continuous negative definite function such that for some positive constants r and c we have

$$\psi(\xi)\ge c\,\|\xi\|^r,\qquad \text{for}\quad \|\xi\|\ge 1.$$

Next, define

$$\lambda(\xi):=(1+\psi(\xi))^{1/2}.$$

Finally, let M be the smallest integer such that $M>(\frac{n}{r}\vee 2)+n$ and set $k=2M+1-n$.

Now, following Hoh [8], we introduce the following conditions for an analytic symbol $q:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$:

H0(n): The function q is continuous in both variables.

H1(n): The map $x\mapsto q(x,\xi)$ is k times continuously differentiable and

$$\left\|\partial_x^\beta q(x,\xi)\right\|\le c\lambda^2(\xi),\qquad \beta\in\mathbb{N}_0^n,\|\beta\|\le k.$$

H2(n): For some strictly positive function $\gamma:\mathbb{R}^n\to\mathbb{R}$

$$q(x,\xi)\ge\gamma(x)\lambda^2(\xi),\qquad \text{for}\quad \|\xi\|\ge 1, x\in\mathbb{R}^n.$$

H3(n):

$$\sup_{x\in\mathbb{R}^n}|q(x,\xi)|\underset{\xi\to 0}{\longrightarrow}0.$$

The following proposition is proved in Hoh [8, Theorem 5.24, page 82],

Proposition 2.23. *Under **H0**(n)–**H3**(n) the pseudo-differential operator $-q(x,D):C_c^\infty(\mathbb{R}^n)\to C_0(\mathbb{R}^n)$ has an extension, which generates a Feller semigroup given by*

$$P_tf(x)=\mathbb{E}^x f(X_t),$$

where $\mathbb{E}^x$ is expectation with respect to the solution of the associated well-posed martingale problem starting at x.

Given the above proposition we can now prove the following important result.

Proposition 2.24. *Assume that symbol q satisfies conditions **H0**(n)–**H3**(n) and **C2**. Then the pseudo-differential operator $-\tilde{q}_j(x,D):C_c^\infty(\mathbb{R})\to C_0(\mathbb{R})$, where $\tilde{q}_j$ is defined by* (2.16), *has an extension that generates a nice Feller process; in other words $\tilde{q}_j$ satisfies condition **C3**.*

Proof. By Remark (2.12) $\tilde{q}_j$ given by (2.16) is an analytic symbol. It is easy to verify that since q satisfies conditions **H0**(n)–**H3**(n), then $\tilde{q}^j$ satisfies conditions **H0**(1)–**H3**(1). Therefore the result follows from Proposition 2.23. □

Corollary 2.25. *Let X be a nice Feller process with symbol q satisfying conditions **H0**(n)–**H3**(n) and **C1**, **C2**. Then the component X^j of X is a nice Feller process with generator given by the symbol $\tilde{q}_j$.*

Proof. The result follows from Proposition 2.24 and Theorem 2.13. □

Proposition 2.26. *Let q be a symbol given by* (2.14). *Assume that the following conditions hold (cf. Stroock* [18]*)*

S1*(n): a is bounded, continuous and positive definite,*

S2*(n): b is bounded and continuous,*

S3*(n):*

$$\int_A \frac{y}{1+|y|^2}\ \mu(x,dy) \text{ is bounded and continuous,} \quad \forall A \in B(\mathbb{R}^n \setminus \{0\}), \tag{2.27}$$

S4*(n):*

$$\int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{i(y,\xi)} + \frac{i(y,\xi)}{1+|y|^2}\right) \mu(x,dy) \text{ is bounded and continuous in } x, \tag{2.28}$$

and that ***H3****(n) holds. Then the pseudo-differential operator $-q(x,D) : C_c^\infty(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ has an extension that generates a nice Feller process with symbol q satisfying* (2.25).

Proof. By results of Stroock [18], **S1**(n)–**S3**(n) imply that the martingale problem for q is well-posed. By **S1**(n)–**S4**(n) we have that q is continuous and that (2.25) is satisfied. Thus the result follows from [8, Theorem 5.23]. □

Remark 2.27. In case of nice Feller processes with continuous trajectories the result of Proposition 2.26 holds without assuming Condition **H3**(n). See [8, Proposition 5.18].

Proposition 2.28. *Assume that symbol q satisfies conditions* ***S1****(n)–****S4****(n),* ***H3****(n) and* ***C2****. Then the pseudo-differential operator $-\tilde{q}_j(x,D) : C_c^\infty(\mathbb{R}) \to C_0(\mathbb{R})$, where $\tilde{q}_j$ is defined by* (2.16)*, has an extension that generates a nice Feller process; in other words $\tilde{q}_j$ satisfies condition* ***C3****.*

3. Symbolic Markov Copulae

In this section we study the following problem:

P2 : Provide an algorithm for construction of an n-dimensional nice Feller process with given marginal distributions, and such that its components are also nice Feller processes.

In other words, we ask how to construct an n-dimensional nice Feller process such that its marginal laws, agree with the laws of a given collection of n one-dimensional nice Feller process. This leads to the following definition.

Definition 3.1 (Symbolic Markov copula for nice Feller processes)**.** A nice n-dimensional Feller process X with symbol q is a symbolic Markov copula for given nice Feller processes $Y^1,\ldots,Y^n$ with symbols $q_1,\ldots,q_n$, if X is strongly Markovian consistent and for all $j = 1,\ldots,n$,

$$q(x, e_j\xi_j) = q_j(x_j,\xi_j). \tag{3.1}$$

and

$$\mathcal{L}(X_0^j) = \mathcal{L}(Y_0^j) \tag{3.2}$$

hold.

Remark 3.2. i) For further reference we note that condition (3.1) implies condition **C2**.
ii) In order to simplify presentation we shall use the term copula with regard to symbols rather than with regard to processes. So, for example, we shall not say that *a nice n-dimensional Feller process X with symbol q is a symbolic Markov copula* but we shall simply say that *symbol q is a symbolic Markov copula.*

In view of Theorem 2.13 conditions **C1**–**C3** are sufficient for a nice Feller processes X to be strongly Markovian consistent. In addition, in view of Theorem 2.18, under conditions **C1**–**C3**, the sufficient and necessary conditions for

$$\mathcal{L}(X^j) = \mathcal{L}(Y^j) \tag{3.3}$$

are conditions (3.1) and (3.2).

To ensure (3.2), having $Y_0^1, \ldots, Y_0^n$, we take X_0 as a random variable with cumulative distribution function

$$F : (x_1, \ldots, x_n) \to C(F_1(x_1), \ldots, F_n(x_n)),$$

where C is a standard copula function (cf. Sklar [17]), and F_j is a cumulative distribution function of Y_0^j. So, our aim is to give a recipe for constructing a nice Feller process with symbol q, starting from given nice Feller one-dimensional processes with symbols $q_1, \ldots, q_n$, such that q satisfies conditions **C1**, **C3** and (3.1). Examples of relevant constructions are given in Section 3.1.

Working with conditions **C1**, **C3** directly is rather difficult. However Propositions 2.24 or 2.28 provide conditions that are sufficient for **C1** and **C3** to hold, and that are much easier to work with. Consequently in the examples given below we will use Propositions 2.24 or 2.28 as modus operandi.

3.1. **Examples.** Recall that a generic analytic symbol q is expressed through a triple of coefficients: an $\mathbb{R}^n$-valued function b, a function a with values in the set of symmetric non-negative definite matrices, and a function $\mu(\cdot, dy)$ taking values in the class of Lévy measures on $\mathbb{R}^n$. We call (b, a, μ) the characteristic triple of q.

Example 1. [Product copula] Let $q_1, \ldots, q_n$ be analytic symbols with characteristic triples $(d_j, c_j, \mu_j)_{j=1}^n$ satisfying **S1**(1)–**S4**(1) and **H3**(1). Thus, in view of Proposition 2.26, the related PDOs have extensions that generate nice Feller processes, say $Y^1, \ldots, Y^n$, with symbols $q_1, \ldots, q_n$ satisfying (2.25).

A symbol q is the *product copula* for symbols $q_1, \ldots, q_n$ if its characteristic triple (b, a, μ) is constructed as

$$b_j(x) := d_j(x_j), \qquad a_{i,j}(x) := c_j(x_j)\mathbf{1}_{\{i=j\}},$$

$$\mu(x, dy) := \sum_{l=1}^{n} \left(\bigotimes_{k=1, k\neq l}^{n} \delta_{\{0\}}(dy_k) \right) \otimes \mu_l(x_l, dy_l).$$

In order to verify that q constructed above is indeed a Markov copula we will verify that (3.1) is satisfied, and that conditions **S1**(n)–**S4**(n), **H3**(n) hold. By Proposition 2.26 this will imply that q is a symbol of a nice Feller process satisfying (2.25) which by Proposition 2.21 will also imply **C1**. By Proposition 2.28 this will imply that **C3** holds.

First we note that by construction condition (3.1) is satisfied. For this we only need to observe that

$$\int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{iy_j\xi_j} + \frac{iy_j\xi_j}{1+|y|^2} \right) \mu(x, dy) = \int_{\mathbb{R} \setminus \{0\}} \left(1 - e^{iy_j\xi_j} + \frac{iy_j\xi_j}{1+|y_j|^2} \right) \mu_j(x_j, dy_j).$$

Next, we note that here q satisfies the following property

$$q(x, \xi) = \sum_{l=1}^{n} q_l(x_l, \xi_l). \tag{3.4}$$

Consequently, properties **S1**(1)–**S4**(1), **H3**(1) assumed for $q_1, \ldots, q_n$ imply that properties **S1**(n)–**S4**(n), **H3**(n) are satisfied for q. It is intuitively clear from (3.4) that the product copula constructed here represents a family of one-dimensional independent nice Feller processes. In this regard also see Schnurr [15, Lemma 4.7]. □

Example 2. [Diffusion copula] Consider analytic symbols $q_1, \ldots, q_n$ given by

$$q_j(x_j, \xi_j) = -i\xi_j d_j(x_j) + c_j(x_j)\xi_j^2,$$

where d_i, c_i are functions satisfying **S1**(1)–**S2**(1). In view of Remark 2.27 this is sufficient for the related PDOs to have unique extensions that generate nice Feller (diffusion) processes, say $Y^1, \dots, Y^n$, with symbols $q_1, \dots, q_n$ satisfying (2.25).

In this example we shall construct a symbolic Markov copula q for $q_1, \dots, q_n$, that will correspond to multivariate diffusion process. Towards this end we observe that it is natural to construct q as

$$q(x,\xi) := -i(b(x),\xi) + (\xi, a(x)\xi),$$

where the functions $b : \mathbb{R}^n \to \mathbb{R}^n$, $a : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n)$ satisfy

$$b_j(x) = d_j(x_j), \qquad a_{j,j}(x) = c_j(x_j), \qquad \forall j = 1, \dots, n, \tag{3.5}$$

and moreover a is symmetric and chosen so that **S1**(n) holds. It is clear that **S2**(n) holds in this construction. Consequently, in view of Proposition 2.26 and Remark 2.27, q is a symbol of a nice Feller process.

We shall now verify that (3.1) is also satisfied here. Indeed, one easily checks that conditions (3.5) imply

$$q(x, e_j\xi_j) = i(b(x), e_j\xi_j) + (e_j\xi_j, a(x)e_j\xi_j) = ib_j(x)\xi_j + a_{j,j}(x)\xi_j^2 = q_j(x_j, \xi_j),$$

which means that (3.1) is satisfied for q.

We conclude that q is a symbolic copula for $q_1, \dots, q_n$. □

Remark 3.3. i) For construction of a diffusion copula under weaker assumptions on marginal processes (indeed marginal symbols) we refer to [2].
ii) If a is chosen to be a diagonal matrix then this example is a special case of Example 1.

Example 3. [Poisson copula, cf. Section 3 in [2]] In this example verification of all technical conditions (**H** and **S**) is straightforward and therefore will be omitted.

Consider two one-dimensional Poisson processes Y^1, Y^2 with constant intensities η_1, η_2. In particular Y^1, Y^2 are nice Feller processes. From (2.13) it follows that their symbols are given by

$$q_i(x_i, \xi_i) = (1 - e^{i\xi_i})\eta_i, \quad i = 1, 2.$$

A natural candidate for a symbolic copula is

$$q(x,\xi) = (1 - e^{i\xi_2})\lambda_{(0,1)} + (1 - e^{i\xi_1})\lambda_{(1,0)} + (1 - e^{i(\xi_1+\xi_2)})\lambda_{(1,1)},$$

where $\lambda_{(0,1)}, \lambda_{(1,0)}, \lambda_{(1,1)}$ are nonnegative constants. However, observe that q is a symbolic copula for q_1, q_2 if and only if $\lambda_{(0,1)}, \lambda_{(1,0)}, \lambda_{(1,1)}$ satisfy the following system of linear equations:

$$\lambda_{(0,1)} + \lambda_{(1,1)} = \eta_2,$$
$$\lambda_{(1,0)} + \lambda_{(1,1)} = \eta_1.$$

The above system has infinitely many solutions, which can be parameterized by $\lambda_{(1,1)}$. In this case $\lambda_{(0,1)}$, $\lambda_{(1,0)}$ are given by

$$\lambda_{(0,1)} = \eta_2 - \lambda_{(1,1)},$$
$$\lambda_{(1,0)} = \eta_1 - \lambda_{(1,1)}.$$

Since we are interested in nonnegative solutions, we restrict $\lambda_{(1,1)}$ to the interval $[0, \eta_1 \wedge \eta_2]$. Generalization to the n-dimensional case is immediate. □

In Example 4 below we provide a generalization of Example 3 by allowing λ-s to depend on x. We need to precede the example with some important comments.

Suppose that we are given a family, parameterized by x, of finite positive measures $\mu(x, dy)$, that are locally bounded with respect to x. Then the function $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ defined by

$$q(x,\xi) := \int_{\mathbb{R}^n} \left(1 - e^{i(y,\xi)}\right) \mu(x, dy) \tag{3.6}$$

is a continuous and negative definite function in ξ, and therefore is a good candidate for a symbol of a nice Feller process. It is clear that properties of q are fully determined by the properties of

$\mu(\cdot, dy)$, and therefore the properties of $\mu(\cdot, dy)$ are decisive with regard to whether q is the symbol of a nice Feller process. Thus natural questions that one needs to consider are:

(1) For what families of measures $\mu(\cdot, dy)$ does there exist a nice Feller process corresponding to q?
(2) Is it unique in the sense of finite-dimensional distributions?
(3) What are the properties of processes corresponding to symbol q defined in such a way?

The answer to the first two questions is positive provided that the family of measures $\mu(\cdot, dy)$ is such that q given by (3.6) satisfies either conditions of Proposition 2.23 or Proposition 2.26. To answer the third question one should first plug the above formula for q into (2.14). Then, using the basic property of Fourier transforms,

$$e^{i(y,\xi)}\hat{u}(\xi) = \widehat{u(\cdot + y))}(\xi),$$

one concludes that

$$Au(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x))\, \mu(x, dy)$$

for every $u \in C_0^\infty(\mathbb{R}^n)$. This form of generator corresponds to a Markov jump process (see Ethier, Kurtz [7, Section 4.2, p.162]), and measures $\mu(\cdot, dy)$ are jump measures.

Example 4. [Generalized two-dimensional point Markov processes] Consider analytic symbols q_1 and q_2 given by

$$q_j(x_j, \xi_j) := \left(1 - e^{i\xi_j}\right) \eta_j(x_j), \quad j = 1, 2,$$

where η_1 and η_2 are assumed to be nonnegative continuous functions. This implies that assumptions **S3**(1), **S4**(1) and **H3**(1) are satisfied for q_1 and q_2. In view of Proposition 2.26 this is sufficient for the related PDOs to have unique extensions that generate nice Feller processes, say Y^1, Y^2, with symbols q_1, q_2 satisfying (2.25). Process Y^j is a counting process with $\mathbb{F}^{Y^j}$-intensity process $(\eta_j(Y^j_{t-}))_{t\geq 0}$, i.e.,

$$Y^j_t - \int_0^t \eta_j(Y^j_{s-})ds$$

is an $\mathbb{F}^{Y^j}$ local martingale. By analogy with Example 3, we define

$$\mu(x, dy_1, dy_2) = \delta_{(0,1)}(dy_1, dy_2)\lambda_{(0,1)}(x) + \delta_{(1,0)}(dy_1, dy_2)\lambda_{(1,0)}(x) + \delta_{(1,1)}(dy_1, dy_2)\lambda_{(1,1)}(x),$$

where $\lambda_{(0,1)}(x)$, $\lambda_{(1,0)}(x)$, $\lambda_{(1,1)}(x)$ are nonnegative continuous functions satisfying

$$\lambda_{(0,1)}(x) + \lambda_{(1,1)}(x) = \eta_2(x_2),$$
$$\lambda_{(1,0)}(x) + \lambda_{(1,1)}(x) = \eta_1(x_1).$$

Just as in Example 3, $\lambda_{(1,1)}(x)$ cannot be too large:

$$0 \leq \lambda_{(1,1)}(x) \leq \eta_i(x_i), \quad i = 1, 2.$$

So $\lambda_{(1,1)}$ is bounded, and also $\lambda_{(1,0)}$, $\lambda_{(1,0)}$ are bounded. Then the corresponding symbol given by

$$q(x, \xi) := \left(1 - e^{i\xi_2}\right) \lambda_{(0,1)}(x) + \left(1 - e^{i\xi_1}\right) \lambda_{(1,0)}(x) + \left(1 - e^{i(\xi_1+\xi_2)}\right) \lambda_{(1,1)}(x).$$

satisfies **S3**(n), **S4**(n) and **H3**(n). Therefore q is a symbol of a nice Feller process. One can easily check that this is indeed a symbol such that (3.1) holds, so it is a symbolic copula for q_1, q_2. □

Example 5. [Generalized n-dimensional Markov point processes] Now we generalize the above example to n dimensions. Thus, we consider a family of analytic symbols $q_1, \ldots, q_n$ given by

$$q_j(x_j, \xi_j) := \left(1 - e^{i\xi_j}\right) \eta_j(x_j), \quad j = 1, \ldots, n,$$

where η_j, $j = 1, \ldots, n$, are assumed to be bounded, nonnegative continuous functions. This implies that assumptions **S3**(1), **S4**(1) and **H3**(1) are satisfied for q_j,$j = 1, \ldots, n$. In view of Proposition 2.26 this is sufficient for the related PDOs to have unique extensions that generate nice Feller processes, say $Y^1, \ldots, Y^n$, with symbols $q_1, \ldots, q_n$ satisfying (2.25).

Now, in order to construct a relevant symbolic copula, we introduce some notation. We set $\mathcal{S} := 2^{\{1,\dots,n\}}$, $\mathcal{J}_k := \{S \in \mathcal{S} : \mathrm{card}(S) \geq k\}$. Given that, we construct a symbolic copula q using formula (3.6) with the family of jump measures $\mu(\cdot, dy)$ specified as

$$\mu(x, dy) := \sum_{S \in \mathcal{J}_1} \lambda^S(x) \bigotimes_{k \in S} \delta_{\{1\}}(dy_k) \otimes \bigotimes_{l \in S^c} \delta_{\{0\}}(dy_l),$$

where λ^S-s are nonnegative continuous functions indexed by $S \in \mathcal{S}$, chosen in such a way that for each $j = 1, \dots, n$ we have

$$\sum_{S \in \mathcal{J}_1 : j \in S} \lambda^S(x) = \eta_j(x_j), \qquad \forall x \in \mathbb{R}^n.$$

Hence,

$$\lambda^S(x) \leq \eta_j(x_j), \qquad \forall j \in S, \;\; \forall x \in \mathbb{R}^n,$$

which implies that λ^S is bounded. Thus q satisfies

$$q(x, \xi) = \sum_{S \in \mathcal{J}_1} \left(1 - e^{i \langle \mathbf{e}(S), \xi \rangle}\right) \lambda^S(x),$$

where $\mathbf{e}(S)$, for $S \in \mathcal{S}$, denotes the vector in $\mathbb{R}^n$ defined by

$$\mathbf{e}(S)_j := \begin{cases} 1 & \text{for } j \in S, \\ 0 & \text{for } j \notin S. \end{cases}$$

It is straightforward to verify that conditions **S3**(n), **S4**(n) and **H3**(n) are satisfied, thus q is a symbol of a nice Feller process. To check that this symbol gives a symbolic copula for $q_1, \dots, q_n$, we note that

$$\left(1 - e^{i \langle \mathbf{e}(S), e_k \xi_k \rangle}\right) \lambda^S(x) = \left(1 - e^{i \xi_k}\right) \lambda^S(x) 1_{\{k \in S\}},$$

hence

$$q(x, e_k \xi_k) = \sum_{S \in \mathcal{J}_1} \left(1 - e^{i \xi_k}\right) \lambda^S(x) 1_{\{k \in S\}} = \left(1 - e^{i \xi_k}\right) \sum_{S \in \mathcal{J}_1 : k \in S} \lambda^S(x) = \left(1 - e^{i \xi_k}\right) \eta_k(x_k) = q_k(x_k, \xi_k),$$

and thus (3.1) holds. □

Example 6. [Markov jump processes] We consider a family of analytic symbols $q_1, \dots, q_n$ of the form

$$q_j(x_j, \xi_j) := \eta^j(x_j) \int_{\mathbb{R}} \left(1 - e^{i \langle y_j, \xi_j \rangle}\right) r^j(x_j, dy_j), \tag{3.7}$$

where η_j, $j = 1, \dots, n$, are nonnegative, bounded continuous functions, and the r^j are probability measures. We assume that conditions **S3**(1), **S4**(1) and **H3**(1) hold for $q_1, \dots, q_n$.[6] In view of Proposition 2.26 this is sufficient for the related PDOs to have unique extensions that generate nice Feller processes, say $Y^1, \dots, Y^n$, with symbols $q_1, \dots, q_n$ satisfying (2.25). With the notation as in Example 5 we construct a symbolic copula q using formula (3.6) with the family of jump measures $\mu(\cdot, dy)$ specified as

$$\mu(x, dy) := \sum_{S \in \mathcal{J}_1} \lambda^S(x) \left(\bigotimes_{j \in S} r^j(x_j, dy_j) \otimes \bigotimes_{i \in S^c} \delta_{\{0\}}(dy_i) \right), \tag{3.8}$$

where the λ^S are nonnegative continuous functions indexed by $S \in \mathcal{S}$ such that for every $j = 1, \dots, n$ we have

$$\sum_{S \in \mathcal{J}_1 : j \in S} \lambda^S(x) = \eta_j(x_j), \qquad \forall x \in \mathbb{R}^n. \tag{3.9}$$

Again, for every $S \in \mathcal{S}$ we have

$$\lambda^S(x) \leq \eta_j(x_j), \qquad \forall j \in S, \forall x \in \mathbb{R}^n.$$

[6] Conditions for r^j-s, under which these assumption is satisfied will be investigated elsewhere.

The corresponding symbolic copula q is given by

$$q(x,\xi)=\sum_{S\in\mathcal{J}_1}\lambda^S(x)\int_{\mathbb{R}^n}\left(1-e^{i(y,\xi)}\right)\left(\bigotimes_{j\in S}r^j(x_j,dy_j)\otimes\bigotimes_{i\in S^c}\delta_{\{0\}}(dy_i)\right).$$

To see that this is indeed a symbolic copula for $q_1,\dots,q_n$ defined in (3.7), we first note that for each $S\in\mathcal{S}$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(1-e^{i(y,\mathbf{e}_k\xi_k)}\right)\left(\bigotimes_{j\in S}r^j(x_j,dy_j)\otimes\bigotimes_{i\in S^c}\delta_{\{0\}}(dy_i)\right)\\ &=\int_{\mathbb{R}^n}\left(1-e^{iy_k\xi_k}\right)\left(\bigotimes_{j\in S}r^j(x_j,dy_j)\otimes\bigotimes_{i\in S^c}\delta_{\{0\}}(dy_i)\right)=1_{\{k\in S\}}\int_{\mathbb{R}}\left(1-e^{iy_k\xi_k}\right)r^k(x_k,dy_k).\end{aligned}$$

The above calculation implies that for all $k\in\{1,\dots,n\}$ we have

$$\begin{aligned}q(x,\mathbf{e}_k\xi_k)&=\sum_{S\in\mathcal{J}_1}\lambda^S(x)1_{\{k\in S\}}\int_{\mathbb{R}}\left(1-e^{iy_k\xi_k}\right)r^k(x_k,dy_k)\\ &=\left(\sum_{S\in\mathcal{J}_1:k\in S}\lambda^S(x)\right)\int_{\mathbb{R}}\left(1-e^{iy_k\xi_k}\right)r^k(x_k,dy_k)\\ &=\eta_k(x_k)\int_{\mathbb{R}}\left(1-e^{iy_k\xi_k}\right)r^k(x_k,dy_k)=q_k(x_k,\xi_k),\end{aligned}$$

where the third equality follows from (3.9), and thus (3.1) holds. □

In the previous example we have constructed a copula for given Markov jump processes by adding the possibility of common jumps. Note that the distribution of these common jump sizes was taken to be the product of marginal distributions. In the next example we will show that it is also possible to introduce dependence between common jumps by using ordinary copulae between finite-dimensional random variables; however, we will have to sacrifice some generality of processes under consideration.

Example 7. [Markov jump processes with space homogeneous jump size distribution] We consider a family of analytic symbols $q_1,\dots,q_n$ of the form

$$q_j(x_j,\xi_j):=\eta^j(x_j)\int_{\mathbb{R}}\left(1-e^{i(y_j,\xi_j)}\right)r^j(dy_j),\tag{3.10}$$

where $\eta_j,\ j=1,\dots,n$, are nonnegative, bounded continuous functions, and the r^j are probability measures. In view of Proposition 2.26 this is sufficient for the related PDOs to have unique extensions that generate nice Feller processes, say $Y^1,\dots,Y^n$, with symbols $q_1,\dots,q_n$ satisfying (2.25).

Process Y^j is a Markov jump processes with jumps size distribution that is independent of x; we will call such processes *space homogeneous Markov jump processes*. Similarly as before, we construct a symbolic copula q for $q_1,\dots,q_n$ by exploiting formula (3.6). Here we specify the family of jump measures $\mu(\cdot,dy)$ as

$$\mu(x,dy):=\sum_{S\in\mathcal{J}_1}\lambda^S(x)\left(C^S(r^j(dy_j),j\in S)\otimes\bigotimes_{i\in S^c}\delta_{\{0\}}(dy_i)\right),\tag{3.11}$$

where for $S\in\mathcal{J}_1$, C^S is an ordinary copula on $[0,1]^S$, and where λ^S-s are nonnegative continuous functions indexed by $S\in\mathcal{S}$, and such that for every $j=1,\dots,n$ we have

$$\sum_{S\in\mathcal{J}_1:j\in S}\lambda^S(x)=\eta_j(x_j),\qquad\forall x\in\mathbb{R}^n.\tag{3.12}$$

The symbolic copula q that corresponds to μ via (3.6) is given by

$$q(x,\xi) := \sum_{S\in\mathcal{J}_1} \lambda^S(x) \int_{\mathbb{R}^n} \left(1-e^{i(y,\xi)}\right) \left(C^S(r^j(dy_j), j\in S) \otimes \bigotimes_{i\in S^c} \delta_{\{0\}}(dy_i)\right).$$

Again, to see that this is indeed a symbolic copula for $q_1,\ldots,q_n$ defined in (3.10), we first note that for each $S\in\mathcal{S}$,

$$\begin{aligned}&\int_{\mathbb{R}^n} \left(1-e^{i(y,\mathbf{e}_k\xi_k)}\right) \left(C^S(r^j(dy_j), j\in S) \otimes \bigotimes_{i\in S^c} \delta_{\{0\}}(dy_i)\right)\\ &= \int_{\mathbb{R}^n} \left(1-e^{iy_k\xi_k}\right) \left(C^S(r^j(dy_j), j\in S) \otimes \bigotimes_{i\in S^c} \delta_{\{0\}}(dy_i)\right) = 1_{\{k\in S\}} \int_{\mathbb{R}} \left(1-e^{iy_k\xi_k}\right) r^k(x_k, dy_k),\end{aligned}$$

where the last equality follows from the fact that $C^S(r^j(dy_j), j\in S)$ is a probability measure with given marginals $r^j(dy_j)$ for $j\in S$. Now by analogous arguments to the ones used in the previous example we find that $q(x,\mathbf{e}_k\xi_k) = q_k(x_k,\xi_k)$ for all $k\in\{1,\ldots,n\}$. Consequently (3.1) holds. □

Example 8. [Markov jump-diffusion processes with space homogeneous jump size distribution] We consider a family of analytic symbols $q_1,\ldots,q_n$ of the form

$$q_j(x_j,\xi_j) := -id_j(x_j)\xi_j + c_j(x)\xi_j^2 + \eta^j(x_j) \int_{\mathbb{R}} \left(1-e^{i(y_j,\xi_j)}\right) r^j(dy_j), \tag{3.13}$$

where d_i, c_i are functions satisfying $\mathbf{S1}(1)$–$\mathbf{S2}(1)$, $\eta_j,\ j=1,\ldots,n$, are nonnegative, bounded continuous functions, and the r^j are probability measures. In view of Proposition 2.26 this is sufficient for the related PDOs to have unique extensions that generate nice Feller processes, say $Y^1,\ldots,Y^n$, with symbols $q_1,\ldots,q_n$ satisfying (2.25).

Process Y^j is a Markov jump-diffusion processes with jumps size distribution that is independent of x. Similarly as before, we construct a symbolic copula q for $q_1,\ldots,q_n$ by exploiting formula (2.15), so the symbolic copula q is given by

$$q(x,\xi) := -i(b(x),\xi) + (\xi,a(x)\xi) + \sum_{S\in\mathcal{J}_1} \lambda^S(x) \int_{\mathbb{R}^n} \left(1-e^{i(y,\xi)}\right) \left(C^S(r^j(dy_j), j\in S) \otimes \bigotimes_{i\in S^c} \delta_{\{0\}}(dy_i)\right).$$

where the functions $b:\mathbb{R}^n\to\mathbb{R}^n$, $a:\mathbb{R}^n\to L(\mathbb{R}^n,\mathbb{R}^n)$ satisfy (3.5), and moreover a is symmetric and chosen so that $\mathbf{S1}(\mathrm{n})$ holds, and for each $S\in\mathcal{J}_1$, C^S is an ordinary copula on $[0,1]^S$, and λ^S satisfies (3.12). By combining calculations from Examples 2 and 7 we immediately obtain that q is a symbolic copula for $q_1,\ldots,q_n$ defined in (3.13). □

Remark 3.4. We stress that the above examples are motivated by those presented by Bielecki, Vidozzzi and Vidozzi [3] by using infinitesimal generators of Markov processes. It appears however that the approach based on symbols is more transparent and gives a relatively simple condition to verify.

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