Semimartingales and Shrinkage of Filtration

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Abstract

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is endowed with two filtrations, \mathbb{G} and \mathbb{F} , assumed to satisfy the usual conditions and such that $\mathbb{F} \subset \mathbb{G}$. On this probability space we consider a real valued \mathbb{G} -semimartingale X. The results can be generalized to the case of \mathbb{R}^n valued semimartingales, in a straightforward manner.

The purpose of this work is to study the following two problems:

- A. If X is \mathbb{F} -adapted, compute the \mathbb{F} -semimartingale characteristics of X in terms of the \mathbb{G} -semimartingale characteristics of X.
- B. If X is a special \mathbb{G} -semimartingale but not \mathbb{F} -adapted, compute the \mathbb{F} -semimartingale

characteristics of \mathbb{F} -optional projection of X in terms of the \mathbb{G} -canonical decomposition and \mathbb{G} -semimartingale characteristics of X.

In this paper problem B is solved under the assumption that the filtration \mathbb{F} is immersed in \mathbb{G} . Beyond the obvious mathematical interest, our study is motivated by important practical applications in areas such as finance and insurance (cf. [BJN19]).

Keywords: Semimartingale, special semimartingale, filtration shrinkage, semimartingale characteristics

Mathematics Subjects Classification (2010): 60G99, 60H99

1 Introduction

This paper is meant to initiate a systematic study of the change of properties of semimartingales under shrinkage of filtrations and, when appropriate, under respective projections. The paper does not aim at a complete and comprehensive study of the topic. Nevertheless, our study contributes, we believe, to understanding of these problems and to giving, in the some specific cases, explicit solutions.

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is endowed with two filtrations, \mathbb{G} and \mathbb{F} , assumed to satisfy the usual conditions and such that $\mathbb{F} \subset \mathbb{G}$. On this probability space we consider a real valued \mathbb{G} -semimartingale X. The results can be generalized to the case of \mathbb{R}^n valued semimartingales, in a straightforward manner. We fix a truncation function with respect to which the semimartingale characteristics are computed.

The purpose of this work is to study the following two problems:

- A. If X is \mathbb{F} -adapted, compute the \mathbb{F} -semimartingale characteristics of X in terms of the \mathbb{G} -semimartingale characteristics of X.
- B. If X is a special G-semimartingale but not \mathbb{F} -adapted, compute the \mathbb{F} -semimartingale characteristics of the \mathbb{F} -optional projection of X in terms of the G-canonical decomposition and G-semimartingale characteristics of X.

Note that \mathbb{G} -semimartingale characteristics of X are unique up to equivalence, even though they may not determine the law of X uniquely. Thus, the above two problems are well posed.

So, in a sense, we study problems, which are complementary to problems that arise when one studies what happens to a semimartingale under enlargement of filtration, where the main object of interest is study how martingales in a given filtration behave when they are considered in a larger filtration. The goal there is to give, under adequate conditions on the filtration enlargement, their semimartingale decomposition in this larger filtration. The literature regarding enlargement of filtrations is quite abundant (see, e.g., the recent monograph [AJ17] and the references therein). On the contrary, the literature regarding the shrinkage of filtration and its effect on the properties of a semimartingale is essentially non-existent. One can quote the seminal paper of Stricker [Str77] who establishes that a G-semimartingale which is \mathbb{F} -adapted (with $\mathbb{F} \subset \mathbb{G}$) is an \mathbb{F} -semimartingale, emphasizing that a G-local martingale which is \mathbb{F} -adapted may fail to be an \mathbb{F} -local martingale. But the problem of how the semimartingale characteristics change under shrinkage of the filtration is not addressed there. Two notable exceptions are Chapter 4. §6 in [LS89] and Section IX.2 in [Jac79], that feature partial versions of some of our results. Related study is also done in [BY78] where, however, a different, from our special semimartingales, class of processes was investigated (called semi-martingales there). Special cases of our Lemma 3.4 are present in the literature in the context of the filtering theory; see for example Lemma 8.4 in [LS01]. It needs to be stressed that, in general, the problems that we study with regard to shrinkage of filtration are, in general, different from problems studied by the theory of filtering, where, due to the noise in the observation, the observation filtration is not included in the signal filtration.

Also, contrary to the theory of the enlargement of the filtrations, where only initial and progressive enlargements are studied, here we do not make any specific restrictions regarding relation between the filtrations \mathbb{G} and \mathbb{F} , except for the inclusion condition $\mathbb{F} \subset \mathbb{G}$, and perhaps some additional conditions, such as the immersion condition in Section 4.

An important motivation behind the study originated in this paper is coming from the theory of stochastic structures that has been under works in recent years (cf. [BJN19]). One of the problems arising in this theory can be summarized as follows: Suppose that $S = (S^1, \ldots, S^n)$ is a multivariate semimartingale. Suppose that (B^i, C^i, ν^i) are the semimartingale characteristics of the semimartingale S^i in the natural filtration of S. The problem is to find the semimartingale characteristics of S^i in the filtration of a sub-group of coordinates $S^{i_1}, \ldots, S^{i_k}, i_1, \ldots, i_k \in \{1, \ldots, n\}$, of S, in terms of (B^i, C^i, ν^i) . Once it is understood how to do this, then one can proceed with construction of semimartingale structures, which, by definition, are multivariate semimartingales whose components are semimartingales with predetermined marginal characteristics in their own filtrations. In a sense, this corresponds to what is being done in the realm of finite dimensional probability distributions via the classical copula theory, where a multivariate distribution is constructed with given margins. This allows for modeling dependence between components of a multivariate random variable, with preservation of the predetermined marginal distributions. Semimartingale structures find applications in areas such as finance and insurance. For example, in insurance, when designing claim policies for a group of claimants it is important to model dependence between multiple claim processes, subject to idiosyncratic statistical properties of these claim processes. In finance, semimartingale structures come in handy for traders who trade basket derivatives, as well as the individual constituents of these derivatives, and need to make sure that respective models for evolution of the basket price process and the price processes of individual constituents are calibrated in a consistent way. In this regard, quite importantly, semimartingale structures allow for separation of estimating (calibrating) individual (idiosyncratic) characteristics of the components of the structure, from estimation (calibration) of the stochastic dependence between the components of the structure. We refer to [BJN19] for more applications of stochastic structures.

The paper is organized as follows. In Section 2 we formulate the mathematical set-up for our study and we recall some useful concepts and results. In Section 3 we study problem A. In Section 4 we study problem B. In Section 5 we provide several examples illustrating and complementing our theoretical developments. The complexity of the examples varies. But all of them are meant to illustrate our theoretical developments, even though results presented in some of the examples might possibly be obtained directly.

Finally, in Section 6 we formulate some non-trivial open problems, solution of which will require more in-depth understanding of subject matters discussed in this paper.

2 Preliminaries

We begin with recalling the concept of characteristics of a semimartingale. These characteristics depend on the choice of filtration and the choice of so called truncation function. In what follows, we will use the standard truncation function $\chi(x) = x \mathbb{1}_{|x| \leq 1}$. Given the truncation function χ , the \mathbb{G} -characteristic triple $(B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$ of a \mathbb{G} -semimartingale X is given in the following way. First we define the process $X(\chi)$ by

$$X_t(\chi) = X_t - X_0 - \sum_{0 < s \le t} (\Delta X_s - \chi(\Delta X_s)), \quad t \ge 0.$$
 (2.1)

Since $X(\chi)$ has bounded jumps it is a special G-semimartingale. Thus, it admits a unique canonical decomposition

$$X(\chi) = B^{\mathbb{G}} + M^{\mathbb{G}}, \qquad (2.2)$$

where $M^{\mathbb{G}}$ is a \mathbb{G} -local martingale such that $M_0^{\mathbb{G}} = 0$, and $B^{\mathbb{G}}$ is a \mathbb{G} -predictable process with finite variation and $B_0^{\mathbb{G}} = 0$. The process $B^{\mathbb{G}}$ is called the first characteristic of X, and this is the only characteristic that depends on the truncation function.

The G-local martingale $M^{\mathbb{G}}$ can be decomposed uniquely into the sum of two orthogonal martingales $M^{\mathbb{G}} = M^{c,\mathbb{G}} + M^{d,\mathbb{G}}$, where $M^{c,\mathbb{G}}$ is a continuous G-local martingale and $M^{d,\mathbb{G}}$ is a purely discontinuous G-local martingale. It can be shown that $M^{c,\mathbb{G}}$ does not depend on the choice of truncation function and it is called the continuous G-martingale part of X and is denoted by $X^{c,\mathbb{G}}$. Then, the second characteristic of X is defined as $C^{\mathbb{G}} = \langle X^{c,\mathbb{G}} \rangle$, where $\langle X^{c,\mathbb{G}} \rangle$ is the predictable quadratic variation process of $X^{c,\mathbb{G}}$. Finally, the third characteristic of X is denoted as $\nu^{\mathbb{G}}$ and is defined as the G-predictable measure which is the G-compensator of μ - the jump measure of X as defined in Proposition II.1.16 in [JS03]. Clearly, $\nu^{\mathbb{G}}$ does not depend on a choice of truncation function. It is clear that, given a truncation function, the G-characteristic triple $(B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$ is unique (up to equivalence).

In view of Proposition II.2.9 in [JS03], there exists a \mathbb{G} -predictable, locally integrable increasing process, say $A^{\mathbb{G}}$, such that

$$B^{\mathbb{G}} = b^{\mathbb{G}} \cdot A^{\mathbb{G}}, \qquad C^{\mathbb{G}} = c^{\mathbb{G}} \cdot A^{\mathbb{G}}, \qquad \nu^{\mathbb{G}}(dt, dx) = K_t^{\mathbb{G}}(dx) dA_t^{\mathbb{G}}, \tag{2.3}$$

where

- i. $b^{\mathbb{G}}$ is an \mathbb{R} -valued and \mathbb{G} -predictable process,
- ii. $c^{\mathbb{G}}$ is an \mathbb{R}_+ -valued and \mathbb{G} -predictable process,
- iii. $K_t^{\mathbb{G}}(\omega, dx)$ is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P}_{\mathbb{G}})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, satisfying condition analogous to condition II.2.11 in [JS03], and where $\mathcal{P}_{\mathbb{G}}$ is the \mathbb{G} -predictable σ -field on $\Omega \times \mathbb{R}_+$,

and where \cdot denotes the stochastic or Stieltjes integral, wherever appropriate.

We will assume that

$$A_t^{\mathbb{G}} = \int_0^t a_u^{\mathbb{G}} du, \qquad (2.4)$$

where $a^{\mathbb{G}}$ is a \mathbb{G} -progressively measurable process. This assumption will be satisfied in examples studied in Section 5.

In what follows we use the following notions and notation:

1. For a given process Z, we denote by ${}^{o,\mathbb{F}}Z$ the optional projection of Z on \mathbb{F} defined in the sense of He et al. [HWY92], i.e., the unique \mathbb{F} -optional, finite valued process such that for every \mathbb{F} -stopping time τ we have

$$\mathbb{E}(Z_{\tau}\mathbb{1}_{\tau<\infty}|\mathcal{F}_{\tau}) = {}^{o,\mathbb{F}}Z_{\tau}\mathbb{1}_{\tau<\infty}.$$

Note that by Theorem 5.1 in [HWY92] this optional projection exists if Z is a measurable process such that $Z_{\tau} \mathbb{1}_{\tau < \infty}$ is σ -integrable with respect to \mathcal{F}_{τ} for every \mathbb{F} -stopping time τ . That is, there exists a sequence of sets $(A_n)_{n=1}^{\infty}$ such that $A_n \in \mathcal{F}_{\tau}$, $A_n \uparrow \Omega$ and $\mathbb{E}(Z_{\tau} \mathbb{1}_{\tau < \infty} \mathbb{1}_{A_n}) < \infty$ for $n = 1, 2, \ldots$

2. For a given process Z, we denote by ${}^{p,\mathbb{F}}Z$ the predictable projection of Z on \mathbb{F} defined in the sense of He et al. [HWY92], i.e., the unique \mathbb{F} -predictable, finite valued process such that for every \mathbb{F} -predictable stopping time τ we have

$$\mathbb{E}(Z_{\tau}\mathbb{1}_{\tau<\infty}|\mathcal{F}_{\tau-}) = {}^{p,\mathbb{F}}Z_{\tau}\mathbb{1}_{\tau<\infty}.$$

Note that by Theorem 5.2 in [HWY92] this predictable projection exists if Z is a measurable process such that $Z_{\tau} \mathbb{1}_{\tau < \infty}$ is σ -integrable with respect to $\mathcal{F}_{\tau-}$ for every predictable \mathbb{F} -stopping time τ . That is, there exists a sequence of sets $(A_n)_{n=1}^{\infty}$ such that $A_n \in \mathcal{F}_{\tau-}$, $A_n \uparrow \Omega$ and $\mathbb{E}(Z_{\tau} \mathbb{1}_{\tau < \infty} \mathbb{1}_{A_n}) < \infty$ for $n = 1, 2, \ldots$

3. We will also need a notion of \mathbb{F} -optional and \mathbb{F} -predictable projections for any function $W: \widetilde{\Omega} \to \mathbb{R}$, which is measurable with respect to $\widetilde{\mathcal{F}}$, where

$$\widetilde{\Omega} := \Omega imes \mathbb{R}_+ imes \mathbb{R}, \quad \widetilde{\mathcal{F}} := \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}).$$

The \mathbb{F} -optional projection of such a function W is defined as the jointly measurable function ${}^{o,\mathbb{F}}W$ on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$, which is such that for all $x \in \mathbb{R}$ the process ${}^{o,\mathbb{F}}W(\cdot, x)$ is the optional projection on \mathbb{F} of the process $W(\cdot, x)$. Similarly, the \mathbb{F} -predictable projection of such a function W is defined as the function ${}^{p,\mathbb{F}}W$ on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$, which is such that for all $x \in \mathbb{R}$ the process ${}^{p,\mathbb{F}}W(\cdot, x)$ is the predictable projection on \mathbb{F} of the process $W(\cdot, x)$.

4. We denote by $\mathcal{O}_{\mathbb{F}}$ (resp. $\mathcal{P}_{\mathbb{F}}$), the F-optional (resp. the F-predictable) sigma-field on $\Omega \times \mathbb{R}_+$ generated by F-adapted càdlàg (resp. continuous) processes. Analogously we introduce the sigma fields $\widetilde{\mathcal{O}}_{\mathbb{F}}$ and $\widetilde{\mathcal{P}}_{\mathbb{F}}$ on $\widetilde{\Omega}$ defined by

$$\widetilde{\mathcal{O}}_{\mathbb{F}}:=\mathcal{O}_{\mathbb{F}}\otimes\mathcal{B}(\mathbb{R}),\qquad \widetilde{\mathcal{P}}_{\mathbb{F}}:=\mathcal{P}_{\mathbb{F}}\otimes\mathcal{B}(\mathbb{R}).$$

5. A random measure π on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ is \mathbb{F} -optional (resp. \mathbb{F} -predictable) if for any $\widetilde{\mathcal{O}}_{\mathbb{F}}$ -measurable (resp. $\widetilde{\mathcal{P}}_{\mathbb{F}}$ -measurable) positive real function W, the real valued process

$$V(\omega,t):=\int_{[0,t]\times\mathbb{R}}W(\omega,s,x)\pi(\omega;ds,dx)$$

is F-optional (resp. F-predictable); equivalently if for any positive real, measurable function W on $\widetilde{\Omega}$

$$\mathbb{E}\bigg(\int_{\mathbb{R}_+\times\mathbb{R}} W(s,x)\pi(ds,dx)\bigg) = \mathbb{E}\bigg(\int_{\mathbb{R}_+\times\mathbb{R}} q,\mathbb{F}W(s,x)\pi(ds,dx)\bigg),$$

where q = o (resp. q = p).

6. We say that a random measure π on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ is \mathbb{F} -optionally (resp. \mathbb{F} -predictably), σ -integrable if the measure M_{π} on $\widetilde{\mathcal{F}}$ defined by

$$M_{\pi}(\widetilde{B}) := \mathbb{E}\bigg(\int_{\mathbb{R}_{+} \times \mathbb{R}} \mathbb{1}_{\widetilde{B}}(\omega, t, x) \pi(\omega, dt, dx)\bigg), \qquad \widetilde{B} \in \widetilde{\mathcal{F}},$$
(2.5)

restricted to $\widetilde{\mathcal{O}}_{\mathbb{F}}$ (resp. to $\widetilde{\mathcal{P}}_{\mathbb{F}}$), is a σ -finite measure. In other words π is \mathbb{F} -optionally, resp. \mathbb{F} -predictable, σ -integrable if there exist a sequence of sets $(\widetilde{A}_k)_{k=1}^{\infty}$ such that $\widetilde{A}_k \in \widetilde{\mathcal{O}}_{\mathbb{F}}$ (resp. $\widetilde{A}_k \in \widetilde{\mathcal{P}}_{\mathbb{F}}$), with $M_{\pi}(\widetilde{A}_k) < \infty$ for each k and $\widetilde{A}_k \uparrow \widetilde{\Omega}$.

7. For a random measure π on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ we denote by $\pi^{o,\mathbb{F}}$ the \mathbb{F} -dual optional projection of π on \mathbb{F} , i.e., the unique \mathbb{F} -optional measure on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ such that it is \mathbb{F} -optionally σ -integrable and for every positive $\widetilde{\mathcal{O}}_{\mathbb{F}}$ -measurable function W on $\widetilde{\Omega}$, we have

$$\mathbb{E}\Big(\int_{\mathbb{R}_+\times\mathbb{R}}W(t,x)\pi(dt,dx)\Big)=\mathbb{E}\Big(\int_{\mathbb{R}_+\times\mathbb{R}}W(t,x)\pi^{o,\mathbb{F}}(dt,dx)\Big).$$

The \mathbb{F} -dual predictable projection of π on \mathbb{F} , denoted by $\pi^{p,\mathbb{F}}$, is defined analogously, as the unique \mathbb{F} -predictable measure on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$ such that it is \mathbb{F} -predictably σ -integrable and for every positive $\widetilde{\mathcal{P}}_{\mathbb{F}}$ -measurable function W on $\widetilde{\Omega}$, we have

$$\mathbb{E}\Big(\int_{\mathbb{R}_+\times\mathbb{R}}W(t,x)\pi(dt,dx)\Big)=\mathbb{E}\Big(\int_{\mathbb{R}_+\times\mathbb{R}}W(t,x)\pi^{p,\mathbb{F}}(dt,dx)\Big).$$

We note that existence and uniqueness of $\pi^{o,\mathbb{F}}$ (resp. $\pi^{p,\mathbb{F}}(dt, dx)$) holds under assumption that π is \mathbb{F} -optionally (resp. \mathbb{F} -predictably), σ -integrable (see e.g. [HWY92, Theorem 11.8]).

- 8. For any process A and any (stopping) time ϑ , we denote by A^{ϑ} the process A stopped at ϑ .
- 9. We use the standard notation $[\cdot, \cdot]$ (resp. $[\cdot]$) for the quadratic co-variation (resp. variation) of real-valued semimartingales. We denote by $\langle \cdot, \cdot \rangle$ (resp. $\langle \cdot \rangle$) the F-predictable quadratic co-variation (resp. variation) of real-valued semimartingales. We denote by $\langle \cdot, \cdot \rangle^{\mathbb{G}}$ (resp. $\langle \cdot \rangle^{\mathbb{G}}$) the G-predictable quadratic co-variation (resp. variation) of real-valued semimartingales.

10. We use the usual convention that $\int_0^t = \int_{(0,t]}$, for any $t \ge 0$.

In the rest of the paper we shall study the \mathbb{F} -characteristics of X in the case when X is \mathbb{F} -adapted, and the \mathbb{F} -characteristics of the optional projection of X on \mathbb{F} in the case when X is not \mathbb{F} -adapted, providing conditions for such optional projection to exist.

3 Study of Problem A: The Case of X adapted to \mathbb{F}

In this section, we consider the case where X is a \mathbb{G} -semimartingale, which is \mathbb{F} -adapted. Thus, it is an \mathbb{F} -semimartingale (see [Str77, Theorem 3.1]).

We start with a result regarding semimartingales with deterministic G-characteristics, that is, semimartingales with independent increments (cf. [JS03, Theorem II.4.15]).

Proposition 3.1. If X has deterministic \mathbb{G} -characteristics, then they are also \mathbb{F} -characteristics of X.

Proof. In the proof we will use the standard notation for integral of g with respect to a measure γ , that is

$$(g*\gamma)_t := \int_0^t \int_{\mathbb{R}} g(x)\gamma(ds, dx), \quad t \ge 0.$$

Note that $X(\chi)$ given in (2.1) is an \mathbb{F} -adapted process with jumps bounded by 1. From assumption (2.4) it follows that the process $B^{\mathbb{G}}$ does not have jumps. Given this, from [JS03, Theorem II.2.21] we know that the processes

$$M^{\mathbb{G}} = X(\chi) - B^{\mathbb{G}}; \quad Y = (M^{\mathbb{G}})^2 - C^{\mathbb{G}} - \chi^2(x) * \nu^{\mathbb{G}}; \quad g * \mu - g * \nu^{\mathbb{G}}, \quad g \in \mathcal{C}^+(\mathbb{R}), \quad (3.1)$$

are G-local martingales, where the class $\mathcal{C}^+(\mathbb{R})$ of functions is defined in [JS03, II.2.20]. These processes are F-adapted processes since the G characteristics of X are assumed to be deterministic. Thus, by [Str77, Theorem 3.1], $M^{\mathbb{G}}$ is an F-semimartingale. Moreover, since $M^{\mathbb{G}}$ has bounded jumps, it is a special F-semimartingale. So, by [Str77, Theorem 2.6] it is an F-local martingale. In order to analyze the G-local martingale Y in (3.1) let us take a localizing sequence of G-stopping times $(\tau_n)_{n=1}^{\infty}$. Then we have

$$\mathbb{E}(M^{\mathbb{G}})^{2}_{\tau_{n}\wedge\sigma} = \mathbb{E}(C^{\mathbb{G}} + \chi^{2} * \nu^{\mathbb{G}})_{\tau_{n}\wedge\sigma}, \quad n = 1, 2, \dots,$$

for every bounded \mathbb{G} -stopping time σ . This implies that

$$\mathbb{E}(C^{\mathbb{G}} + \chi^2 * \nu^{\mathbb{G}})_{\tau_n \wedge \sigma} \le \mathbb{E}(C^{\mathbb{G}} + \chi^2 * \nu^{\mathbb{G}})_T = C_T^{\mathbb{G}} + (\chi^2 * \nu^{\mathbb{G}})_T < \infty,$$

since the G-characteristics of X are deterministic. So, upon letting $n \to \infty$, we obtain

$$\mathbb{E}\Big((M^{\mathbb{G}})^2_{\sigma} - C^{\mathbb{G}}_{\sigma} - (\chi^2 * \nu^{\mathbb{G}})_{\sigma}\Big) = 0.$$

Since $\mathbb{F} \subset \mathbb{G}$ the above holds for all bounded \mathbb{F} -stopping times. Thus $(M^{\mathbb{G}})^2 - C^{\mathbb{G}} - \chi^2 * \nu^{\mathbb{G}}$ is an \mathbb{F} -martingale.

Similar reasoning can be used to show that the third \mathbb{G} -local martingale in 3.1 is an \mathbb{F} -martingale for every $g \in \mathcal{C}^+(\mathbb{R})$. Indeed, let is fix $g \in \mathcal{C}^+(\mathbb{R})$. Then there exist a localizing sequence of \mathbb{G} -stopping times $(\tau_n)_{n=1}^{\infty}$ such that

$$\mathbb{E}(g * \mu^X)_{\sigma \wedge \tau_n} = \mathbb{E}(g * \nu^{\mathbb{G}})_{\sigma \wedge \tau_n}, \quad n = 1, 2, \dots,$$

for an arbitrary bounded \mathbb{G} -stopping time σ . By letting $n \to \infty$ we obtain

$$\mathbb{E}(g*\mu)_{\sigma} = \mathbb{E}(g*\nu^{\mathbb{G}})_{\sigma} \le (g*\nu^{\mathbb{G}})_T < \infty.$$

This implies that $g * \mu^X - g * \nu^{\mathbb{G}}$ is an \mathbb{F} -martingale. Consequently all \mathbb{G} -local martingales defined in (3.1) are \mathbb{F} -local martingales. Using again [JS03, Theorem II.2.21] we finish the proof.

We now proceed to consider a general case of \mathbb{G} -characteristics of X. Towards this end, we make the following assumptions:

A1. For every $t \ge 0$ we have

$$\mathbb{E}\Big(\int_0^t |b_u^{\mathbb{G}}| a_u^{\mathbb{G}} du\Big) < \infty,$$

where $a^{\mathbb{G}} \geq 0$ is defined in 2.4.

A2. The process $b^{\mathbb{G}}a^{\mathbb{G}}$ admits an \mathbb{F} -optional projection.

A3. The process $M^{\mathbb{G}}$ defined in (2.2) is a true \mathbb{G} -martingale.

Remark 3.2. In view of assumption A3, the process ${}^{o,\mathbb{F}}M^{\mathbb{G}}$ is a true \mathbb{F} -martingale as well. If the process $M^{\mathbb{G}}$ were a \mathbb{G} -local-martingale but not a true \mathbb{G} -martingale, then ${}^{o,\mathbb{F}}M^{\mathbb{G}}$ might not necessarily be an \mathbb{F} -local-martingale. See [Str77, section 2] and [FP11].

We will need the following two technical results.

Lemma 3.3. Suppose that A is an \mathbb{F} -adapted process with prelocally integrable variation, H is a process admitting an \mathbb{F} -optional projection, and such that $H \cdot A$ has an integrable variation. Then

$$M = {}^{o,\mathbb{F}}(H \cdot A) - ({}^{o,\mathbb{F}}H) \cdot A$$

is a uniformly integrable \mathbb{F} -martingale.

Proof. Applying [HWY92, Corollary 5.31.(2)] to the process $H \cdot A$, we conclude that the process

$$M = {}^{o,\mathbb{F}}(H \cdot A) - (H \cdot A)^{o,\mathbb{F}}$$

is a uniformly integrable martingale. Now, by [HWY92, Theorem 5.25] and the remark following this theorem, we have

$$(H \cdot A)^{o,\mathbb{F}} = ({}^{o,\mathbb{F}}H) \cdot A \,,$$

which finishes the proof.

Lemma 3.4. Let A1 and A2 be satisfied. Then, ${}^{o,\mathbb{F}}B^{\mathbb{G}}$ and $\int_0^{\cdot} {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du$ exist and the process M^B given as

$$M_t^B = {}^{o,\mathbb{F}}B_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du, \qquad t \ge 0,$$
(3.2)

is an \mathbb{F} -martingale. Moreover, if \mathbb{F} is \mathbb{P} -immersed in \mathbb{G} ,¹ then M^B is a null process.

Proof. Since, by assumption A1, the process $B_t^{\mathbb{G}} = \int_0^t H_s ds$, where $H_s = a_s^{\mathbb{G}} b_s^{\mathbb{G}}$, is prelocally integrable and, by assumption A2, H has an optional projection, we may apply [HWY92, Theorem 5.25] and conclude that ${}^{o,\mathbb{F}}B^{\mathbb{G}}$ and $\int_0^{\cdot} {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du$ exist.

Now, fix T > 0 and let

$$L_t = a_t^{\mathbb{G}} b_t^{\mathbb{G}} \mathbb{1}_{\{t \le T\}}, \quad t \ge 0$$

Then, invoking A1, A2 and applying Lemma 3.3 with $A_t = t$ we conclude that ${}^{o,\mathbb{F}}(L \cdot A)$ and ${}^{o,\mathbb{F}}L \cdot A$ exist and the process

$$N_t := {}^{o,\mathbb{F}} (L \cdot A)_t - ({}^{o,\mathbb{F}} L \cdot A)_t, \quad t \ge 0,$$
(3.3)

is a uniformly integrable \mathbb{F} -martingale. Now note that $L \cdot A = (L \cdot A)^T = (H \cdot A)^T$ so, by [HWY92, Theorem 5.7], we have for $t \in [0, T]$

$${}^{o,\mathbb{F}}(L\cdot A)_t = {}^{o,\mathbb{F}}((H\cdot A)^T)_t = {}^{o,\mathbb{F}}(H\cdot A)_t.$$

$$(3.4)$$

Using the definition of L and applying again [HWY92, Theorem 5.7] we have

$${}^{o,\mathbb{F}}L = {}^{o,\mathbb{F}}(L1_{[0,T]}) = 1_{[0,T]}{}^{o,\mathbb{F}}L = 1_{[0,T]}{}^{o,\mathbb{F}}(L^T) = 1_{[0,T]}{}^{o,\mathbb{F}}(H^T) = 1_{[0,T]}{}^{o,\mathbb{F}}H,$$

which implies that

$${}^{o,\mathbb{F}}L\cdot A = (1_{[0,T]}{}^{o,\mathbb{F}}H)\cdot A = ({}^{o,\mathbb{F}}H\cdot A)^T.$$
(3.5)

Using (3.4) and (3.5) we see that the martingale N defined by (3.3) can be written on [0, T] as

$$N_t = {}^{o,\mathbb{F}}(H \cdot A)_t - ({}^{o,\mathbb{F}}H \cdot A)_t = {}^{o,\mathbb{F}}B_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du, \qquad t \in [0,T].$$

Since T was arbitrary, this proves that the process given by (3.2) is an \mathbb{F} -martingale.

Finally, we will now prove that if \mathbb{F} is \mathbb{P} -immersed in \mathbb{G} , then the martingale M^B is a null process. Indeed, for any $t \geq 0$, we have

where the third equality is a consequence of immersion of $\mathbb F$ in $\mathbb G$.

The proof of the lemma is complete.

¹ We recall that \mathbb{F} is \mathbb{P} -immersed in \mathbb{G} if any (\mathbb{F}, \mathbb{P}) -martingale is a (\mathbb{G}, \mathbb{P}) -martingale.

Remark 3.5. It is important to note that Lemma 3.4 is true regardless whether the process X is adapted with respect to \mathbb{F} or not. Special versions of this lemma are known in the filtering theory. See for example Lemma 8.4 in [LS01], or the proof of Theorem 8.11 in [RW00].

The next theorem is the main result in this section.

Theorem 3.6. Assume A1-A3. Then, the \mathbb{F} -characteristic triple of X is given as

$$B^{\mathbb{F}} = \int_0^{\cdot} o_{\cdot} \mathbb{F}(b^{\mathbb{G}} a^{\mathbb{G}})_s ds, \quad C^{\mathbb{F}} = C^{\mathbb{G}}, \quad \nu^{\mathbb{F}}(dt, dx) = \left(K_t^{\mathbb{G}}(dx) a_t^{\mathbb{G}} dt\right)^{p, \mathbb{F}}.$$

Proof. Let us consider the process $X(\chi)$ given by (2.1). As observed earlier, $X(\chi)$ is a G-special semimartingale with unique canonical decomposition (2.2). Since $X(\chi)$ is \mathbb{F} -adapted, it is also an \mathbb{F} -special semimartingale, with unique canonical decomposition, say

$$X(\chi) = B^{\mathbb{F}} + M^{\mathbb{F}},\tag{3.6}$$

where $B^{\mathbb{F}}$ is \mathbb{F} -predictable process of finite variation and $M^{\mathbb{F}}$ is an \mathbb{F} -local martingale. The process $B^{\mathbb{F}}$ is the first characteristic in the \mathbb{F} -characteristic triple of X.

Our first goal is to provide a formula for $B^{\mathbb{F}}$ in terms of the \mathbb{G} -characteristics of X. Towards this end we first observe that from Lemma 3.4 if follows that ${}^{o,\mathbb{F}}B^{\mathbb{G}}$ exists. Recall that by assumption A3 the process $M^{\mathbb{G}}$ showing in (2.2) is a \mathbb{G} -martingale. Since for any bounded \mathbb{F} -stopping time $\tau \leq T$, using the fact that τ is a \mathbb{G} -stopping time and Doob's optional stopping theorem, we have

$$\mathbb{E}M_{\tau}^{\mathbb{G}} = \mathbb{E}M_0^{\mathbb{G}} < \infty.$$

Thus $M^{\mathbb{G}}$ is σ -integrable with respect to \mathcal{F}_{τ} for every bounded \mathbb{F} -stopping time τ , so its optional projection ${}^{o,\mathbb{F}}M^{\mathbb{G}}$ exists (see [HWY92, Theorem 5.1]). From this, from (2.2) and from the linearity of the optional projection we conclude that the optional projection ${}^{o,\mathbb{F}}X(\chi)$ exists, and is given as

$${}^{o,\mathbb{F}}X(\chi) = {}^{o,\mathbb{F}}M^{\mathbb{G}} + {}^{o,\mathbb{F}}B^{\mathbb{G}}.$$
(3.7)

Since $X(\chi)$ is \mathbb{F} -adapted we have

$${}^{o,\mathbb{F}}X(\chi) = X(\chi). \tag{3.8}$$

Combining (3.7) and (3.8) we obtain

$$X(\chi) = {}^{o,\mathbb{F}}M^{\mathbb{G}} + {}^{o,\mathbb{F}}B^{\mathbb{G}}.$$

Thus, since the process ${}^{o,\mathbb{F}}B_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du$ is an \mathbb{F} -martingale (by Lemma 3.4 again), and since, in view of A3, the process ${}^{o,\mathbb{F}}M^{\mathbb{G}}$ is an \mathbb{F} -martingale, we see that

$$X_t(\chi) = M_t^{\mathbb{F}} + \int_0^t o(\mathbb{F}(b^{\mathbb{G}}a^{\mathbb{G}})) ds,$$

where $M_t^{\mathbb{F}} = {}^{o,\mathbb{F}}M_t^{\mathbb{G}} + {}^{o,\mathbb{F}}B_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(b^{\mathbb{G}}a^{\mathbb{G}})_u du$. Thus, by uniqueness of the decomposition (3.6) of the special \mathbb{F} -semimartingale X, we conclude that

$$B_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}} (b^{\mathbb{G}} a^{\mathbb{G}})_s ds.$$

The second formula, $C^{\mathbb{F}} = C^{\mathbb{G}}$, follows from [Jac79, Remark 9.20].

It remains to derive a formula for $\nu^{\mathbb{F}}$. Towards this end, we recall that $\nu^{\mathbb{G}} = \mu^{p,\mathbb{G}}$ is a $\widetilde{\mathcal{P}}_{\mathbb{G}}$ -predictably σ -integrable random measure (i.e., using the notation (2.5), $M_{\nu^{\mathbb{G}}}$ is σ -finite on $\widetilde{\mathcal{P}}_{\mathbb{G}}$, such that

$$M_{\nu^{\mathbb{G}}}|_{\widetilde{\mathcal{P}}_{\mathbb{G}}} = M_{\mu}|_{\widetilde{\mathcal{P}}_{\mathbb{G}}}.$$

Thus since $\widetilde{\mathcal{P}}_{\mathbb{F}} \subset \widetilde{\mathcal{P}}_{\mathbb{G}}$ we have

$$M_{\nu^{\mathbb{G}}}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}} = M_{\mu}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}}.$$
(3.9)

Since M_{μ} is σ -finite on $\widetilde{\mathcal{P}}_{\mathbb{F}}$ (see the proof of Theorem 11.15 [HWY92], with $\widetilde{\mathcal{P}}$ there replaced by $\widetilde{\mathcal{P}}_{\mathbb{F}}$), the above implies that $M_{\nu^{\mathbb{G}}}$ is also σ -finite on $\widetilde{\mathcal{P}}_{\mathbb{F}}$. So $\nu^{\mathbb{G}}$ is \mathbb{F} -predictably σ -integrable. Thus it has the \mathbb{F} -dual predictable projection $(\nu^{\mathbb{G}})^{p,\mathbb{F}}$ which is characterized by

$$M_{(\nu^{\mathbb{G}})^{p,\mathbb{F}}}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}} = M_{\nu^{\mathbb{G}}}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}}.$$
(3.10)

This and (3.9) imply that

$$M_{(\nu^{\mathbb{G}})^{p,\mathbb{F}}}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}} = M_{\mu}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}}.$$

So, by the uniqueness of dual predictable projections we have $(\nu^{\mathbb{G}})^{p,\mathbb{F}} = \nu^{\mathbb{F}}$. The proof is complete.

Remark 3.7. Let us note that we also have

$$(\nu^{\mathbb{G}})^{p,\mathbb{F}} = ((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}.$$

Indeed, by analogous reasoning as in the proof of [HWY92, Theorem 11.8] we can prove that the random measure $\nu^{\mathbb{G}}$ admits an \mathbb{F} -dual optional projection if and only if it is \mathbb{F} -optionally σ -integrable. Now, recall that $M_{\nu^{\mathbb{G}}}$ is σ -finite on $\widetilde{\mathcal{P}}_{\mathbb{F}}$. This and the fact that $\widetilde{\mathcal{P}}_{\mathbb{F}} \subset \widetilde{\mathcal{O}}_{\mathbb{F}}$ imply that $M_{\nu^{\mathbb{G}}}$ is also σ -finite on $\widetilde{\mathcal{O}}_{\mathbb{F}}$, so $\nu^{\mathbb{G}}$ is \mathbb{F} -optionally σ -integrable. Thus there exists $(\nu^{\mathbb{G}})^{o,\mathbb{F}}$ – the \mathbb{F} -dual optional projection of $\nu^{\mathbb{G}}$, i.e., the unique \mathbb{F} -optional measure which is \mathbb{F} -optionally σ -integrable such that

$$M_{\nu^{\mathbb{G}}}|_{\widetilde{\mathcal{O}}_{\mathbb{F}}} = M_{(\nu^{\mathbb{G}})^{o,\mathbb{F}}}|_{\widetilde{\mathcal{O}}_{\mathbb{F}}}.$$

$$M_{\nu^{\mathbb{G}}}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}} = M_{(\nu^{\mathbb{G}})^{o,\mathbb{F}}}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}}.$$
(3.11)

Hence we have

Since
$$M_{\nu^{\mathbb{G}}}$$
 is σ -finite on $\widetilde{\mathcal{P}}_{\mathbb{F}}$, so is $M_{(\nu^{\mathbb{G}})^{o,\mathbb{F}}}$. Therefore, invoking again [HWY92, Theorem 11.8], we conclude that there exists the \mathbb{F} -predictable projection of $(\nu^{\mathbb{G}})^{o,\mathbb{F}}$, i.e. $((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}$, for which we have

$$M_{((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}} = M_{(\nu^{\mathbb{G}})^{o,\mathbb{F}}}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}}$$

From the latter equality and from (3.9) and (3.11) we deduce that

$$M_{((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}} = M_{\mu}|_{\widetilde{\mathcal{P}}_{\mathbb{F}}}.$$

By uniqueness of the \mathbb{F} -dual predictable projection of μ we finally obtain

$$\nu^{\mathbb{F}} = \mu^{p,\mathbb{F}} = ((\nu^{\mathbb{G}})^{o,\mathbb{F}})^{p,\mathbb{F}}.$$

The case of immersion between \mathbb{F} and \mathbb{G} .

We briefly discuss here the case when \mathbb{F} is \mathbb{P} -immersed in \mathbb{G} . We will show that $(B^{\mathbb{F}}, C^{\mathbb{F}}, \nu^{\mathbb{F}}) = (B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$. Towards this end, let us consider the process $X(\chi)$ defined by (2.2).

Clearly, the process $X(\chi)$ has bounded jumps and is both G-adapted and F-adapted. Thus, it is a special semimartingale in both filtrations, and hence it has the canonical decompositions

$$X(\chi) = M^{\mathbb{F}} + B^{\mathbb{F}} = M^{\mathbb{G}} + B^{\mathbb{G}}.$$

Since, by immersion, $M^{\mathbb{F}}$ is a \mathbb{G} -martingale and, obviously, $B^{\mathbb{F}}$ is \mathbb{G} -predictable, one has that $M^{\mathbb{G}} = M^{\mathbb{F}}$ and $B^{\mathbb{G}} = B^{\mathbb{F}}$ (by uniqueness of canonical \mathbb{G} -decomposition of \check{X}).

The fact that $C^{\mathbb{F}} = C^{\mathbb{G}}$ follows, again, from [Jac79, Remark 9.20, p.288].

Finally, we verify that $\nu^{\mathbb{G}} = \nu^{\mathbb{F}}$. Note that, for any positive real measurable function g, the process $g * \mu - g * \nu^{\mathbb{F}}$ is an \mathbb{F} -local martingale and hence, by immersion, a \mathbb{G} -local martingale. This implies, by uniqueness of the compensator and by the fact that $\nu^{\mathbb{F}}$ is \mathbb{G} -predictable, that $\nu^{\mathbb{F}} = \nu^{\mathbb{G}}$.

In conclusion, we have

Proposition 3.8. Assume that \mathbb{F} is \mathbb{P} -immersed in \mathbb{G} . Then,

$$(B^{\mathbb{F}}, C^{\mathbb{F}}, \nu^{\mathbb{F}}) = (B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}}).$$

4 Study of Problem B: The Case of X not adapted to \mathbb{F}

In this section we consider the case where X is a \mathbb{G} -special semimartingale, but it is not adapted to \mathbb{F} . Therefore, we shall study here the \mathbb{F} -optional projection ${}^{o,\mathbb{F}}X$ of X and its semimartingale characteristics. In particular, in Theorem 4.2 we provide sufficient conditions on X under which the \mathbb{F} -optional projection ${}^{o,\mathbb{F}}X$ exists and is an \mathbb{F} -special semimartingale.

We have the following canonical decompositions of X (see [HWY92, Corollary 11.26] or [JS03, Corollary II 2.38]):

$$X_t = X_0 + X_t^{c,\mathbb{G}} + \int_0^t \int_{\mathbb{R}} x(\mu^{\mathbb{G}}(ds, dx) - \nu^{\mathbb{G}}(ds, dx)) + \widehat{B}_t^{\mathbb{G}} = X_0 + \widehat{M}_t^{\mathbb{G}} + \widehat{B}_t^{\mathbb{G}}, \qquad (4.1)$$

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where $\widehat{B}^{\mathbb{G}}$, called the modified first characteristic, and the \mathbb{G} local martingale $\widehat{M}^{\mathbb{G}}$ are given by

$$\widehat{B}_{t}^{\mathbb{G}} = B_{t}^{\mathbb{G}} + \int_{0}^{t} \int_{|x|>1} x\nu^{\mathbb{G}}(ds, dx), \qquad t \ge 0,$$

$$\widehat{M}_{t}^{\mathbb{G}} = X_{t}^{c,\mathbb{G}} + \int_{0}^{t} \int_{\mathbb{R}} x(\mu^{\mathbb{G}}(ds, dx) - \nu^{\mathbb{G}}(ds, dx))$$

$$(4.2)$$

$$= M_t^{\mathbb{G}} + \int_0^t \int_{|x|>1} x(\mu^{\mathbb{G}}(ds, dx) - \nu^{\mathbb{G}}(ds, dx)), \quad t \ge 0.$$

Moreover if ${}^{o,\mathbb{F}}X$ exists and is an \mathbb{F} -special semimartingale, then

$${}^{o,\mathbb{F}}X_t = X_0 + ({}^{o,\mathbb{F}}X)_t^{c,\mathbb{F}} + \int_0^t \int_{\mathbb{R}} x(\mu^{\mathbb{F}}(ds, dx) - \nu^{\mathbb{F}}(ds, dx)) + B_t^{\mathbb{F}} = X_0 + \widehat{M}_t^{\mathbb{F}} + \widehat{B}_t^{\mathbb{F}}.$$
 (4.3)

Note that $\mu^{\mathbb{F}}$ is the random measure of jumps of ${}^{o,\mathbb{F}}X$ and $\nu^{\mathbb{F}}$ denotes its \mathbb{F} -compensator whereas in (4.1) $\mu^{\mathbb{G}}$ denotes random measure of jumps of X and $\nu^{\mathbb{G}}$ its \mathbb{G} -compensator. We also note that the first \mathbb{F} -characteristic $B^{\mathbb{F}}$ can be computed from the modified one (i.e. $\widehat{B}^{\mathbb{F}}$) and $\nu^{\mathbb{F}}$ by means of a counterpart of the formula (4.2) i.e.

$$\widehat{B}_t^{\mathbb{F}} = B_t^{\mathbb{F}} + \int_0^t \int_{|x|>1} x\nu^{\mathbb{F}}(ds, dx).$$
(4.4)

In this section we will work under the following additional standing assumptions: $\widehat{A1}$. For every $t \ge 0$ we have

$$\mathbb{E}\Big(\int_0^t |\widehat{b}_u^{\mathbb{G}}| a_u^{\mathbb{G}} du\Big) < \infty,$$

where $a^{\mathbb{G}} \ge 0$ is defined in 2.4, and

$$\widehat{b}_t^{\mathbb{G}} = b_t^{\mathbb{G}} + \int_{|x|>1} x K_t^{\mathbb{G}}(dx), \quad t \ge 0.$$
(4.5)

 $\widehat{A2}$. The process $\widehat{b}^{\mathbb{G}}a^{\mathbb{G}}$ admits an \mathbb{F} -optional projection.

 $\widehat{A3}$. $\widehat{M}^{\mathbb{G}}$ is a square integrable martingale.²

B1. There exists a square integrable \mathbb{F} -martingale Z such that the predictable representation property holds for (\mathbb{F}, Z) : any square integrable \mathbb{F} -martingale M admits a representation $M_t = M_0 + \int_0^t \psi_u \, dZ_u, t \ge 0$, with an \mathbb{F} -predictable process ψ .

B2. The \mathbb{F} -martingale Z is a \mathbb{G} -martingale.

B3. \mathcal{G}_0 is trivial (so \mathcal{F}_0 is also trivial).

B4. The predictable projection ${}^{p,\mathbb{F}}\left(\frac{d\langle \widehat{M}^{\mathbb{G}},Z\rangle_t^{\mathbb{G}}}{d\langle Z\rangle_t^{\mathbb{G}}}\right)$ exists for each $t \ge 0$.

²Note that in Assumption A3 we postulated that $M^{\mathbb{G}}$ is a martingale, but not necessarily a square integrable martingale.

Remark 4.1. Note that, under B1 and B2, the immersion property holds between $\mathbb F$ and $\mathbb G.$ Thus, we have

$$\langle Z \rangle = \langle Z \rangle^{\mathbb{G}},$$

where, we recall, $\langle Z \rangle$ denotes the F-predictable quadratic variation process of Z. Consequently, $[Z] - \langle Z \rangle$ is a G-martingale.

In order to proceed, we denote by $\lambda_{\langle Z \rangle}$ a measure on $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ defined by

$$\lambda_{\langle Z \rangle}(A) = \mathbb{E}\Big(\int_{[0,\infty[} \mathbb{1}_A(\cdot, s) d\langle Z \rangle_s(\cdot)\Big), \quad \text{for } A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$$

In particular, for any $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable function $f(\omega, s)$ we have

$$\int_{\Omega \times \mathbb{R}_+} f(\omega, s) d\lambda_{\langle Z \rangle}(\omega, s) = \mathbb{E} \Big(\int_{[0, \infty[} f_s d\langle Z \rangle_s \Big),$$

where $f_s(\omega) := f(\omega, s)$, if the expression on the right hand side is well defined. The above relation gives definition of integrability with respect to the measure $\lambda_{\langle Z \rangle}$.

The following theorem presents computation of the \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$.

Theorem 4.2. Let X be a special G-semimartingale with G-characteristic triple $(B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$. Assume that $\widehat{A1} - \widehat{A3}$ and B1 - B4 are satisfied. Then the optional projection ${}^{o,\mathbb{F}}X$ exists, it is an \mathbb{F} -special semimartingale and its \mathbb{F} -characteristics are

$$B^{\mathbb{F}}_{\cdot} = \int_{0}^{\cdot} {}^{o,\mathbb{F}} \left(b^{\mathbb{G}}_{s} a^{\mathbb{G}}_{s} + \int_{|x|>1} x K^{\mathbb{G}}_{s}(dx) a^{\mathbb{G}}_{s} \right) ds - \int_{0}^{\cdot} \int_{|x|>1} x \nu^{\mathbb{F}}(ds, dx), \tag{4.6}$$

$$C^{\mathbb{F}}_{\cdot} = \int_0^{\cdot} h_s^2 d\langle Z^c \rangle_s, \qquad (4.7)$$

$$\nu^{\mathbb{F}}(A, dt) = \int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}}(h_t x) \nu^{Z, \mathbb{F}}(dx, dt), \qquad A \in \mathcal{B}(\mathbb{R}),$$
(4.8)

where $\nu^{Z,\mathbb{F}}$ is the \mathbb{F} -compensator of the jump measure of Z and

$$h_{t} = {}^{p,\mathbb{F}} \left(\frac{d\langle \widehat{M}^{\mathbb{G}}, Z \rangle_{t}^{\mathbb{G}}}{d\langle Z \rangle_{t}^{\mathbb{G}}} \right) = \mathbb{E} \left(\frac{d\langle \widehat{M}^{\mathbb{G}}, Z \rangle_{t}^{\mathbb{G}}}{d\langle Z \rangle_{t}^{\mathbb{G}}} \middle| \mathcal{F}_{t-} \right) = \mathbb{E} \left(\frac{d\langle \widehat{M}^{\mathbb{G}}, Z \rangle_{t}^{\mathbb{G}}}{d\langle Z \rangle_{t}} \middle| \mathcal{F}_{t-} \right) \qquad \lambda_{\langle Z \rangle} - a.e.$$

$$(4.9)$$

Proof. Step 1. First we will show that ${}^{o,\mathbb{F}}X$ exists and is an \mathbb{F} -special semimartingale. Towards this end it suffices to show that for every \mathbb{F} -stopping time τ random variable $X_{\tau}\mathbb{1}_{\tau<\infty}$ is σ -integrable with respect to \mathcal{F}_{τ} . Let us take \mathbb{F} stopping time τ and consider increasing sequence of sets $A_n := \{\tau \land n = \tau\} \in \mathcal{F}_{\tau \land n}$. Note that by (2.4) we have

$$\widehat{B}_t^{\mathbb{G}} = \int_0^t \widehat{b}_u^{\mathbb{G}} a_u^{\mathbb{G}} du.$$

Using above formula, $\widehat{A1}$ and $\widehat{A3}$, i.e. \mathbb{G} -martingale property of $\widehat{M}^{\mathbb{G}}$, we obtain

$$\mathbb{E}(X_{\tau}\mathbb{1}_{\tau<\infty}\mathbb{1}_{A_{n}}) = \mathbb{E}(X_{\tau\wedge n}\mathbb{1}_{A_{n}}) = \mathbb{E}\left((X_{0} + \widehat{M}_{\tau\wedge n}^{\mathbb{G}} + \widehat{B}_{\tau\wedge n}^{\mathbb{G}})\mathbb{1}_{A_{n}}\right)$$
$$= \mathbb{E}\left(X_{0}\mathbb{1}_{A_{n}} + \mathbb{E}(\widehat{M}_{n}^{\mathbb{G}}\mathbb{1}_{A_{n}}|\mathcal{G}_{\tau\wedge n}) + \mathbb{1}_{A_{n}}\int_{0}^{\tau\wedge n}\widehat{b}_{u}^{\mathbb{G}}a_{u}^{\mathbb{G}}du\right)$$
$$= \mathbb{E}\left(X_{0}\mathbb{1}_{A_{n}} + \widehat{M}_{n}^{\mathbb{G}}\mathbb{1}_{A_{n}} + \mathbb{1}_{A_{n}}\int_{0}^{\tau\wedge n}\widehat{b}_{u}^{\mathbb{G}}a_{u}^{\mathbb{G}}du\right)$$
$$\leq \mathbb{E}\left(X_{0}\mathbb{1}_{A_{n}} + |\widehat{M}_{n}^{\mathbb{G}}|\mathbb{1}_{A_{n}} + \mathbb{1}_{A_{n}}\int_{0}^{n}|\widehat{b}_{u}^{\mathbb{G}}|a_{u}^{\mathbb{G}}du\right) < \infty.$$

Thus using [HWY92, Theorem 5.1] we conclude that ${}^{o,\mathbb{F}}X$ exists. Note that as a by-product of the above estimate we also get that X_0 , $\widehat{M}^{\mathbb{G}}$ and $\widehat{B}^{\mathbb{G}}$ are σ -integrable with respect to \mathcal{F}_{τ} for every \mathbb{F} -stopping time τ and hence using again [HWY92, Theorem 5.1] we conclude that ${}^{o,\mathbb{F}}\widehat{M}^{\mathbb{G}}$ and ${}^{o,\mathbb{F}}\widehat{B}^{\mathbb{G}}$ exist. By linearity of \mathbb{F} -optional projections and assumptions B3 and $\widehat{A2}$, we may now write

$${}^{o,\mathbb{F}}X_t = {}^{o,\mathbb{F}}X_0 + {}^{o,\mathbb{F}}\widehat{M}_t^{\mathbb{G}} + {}^{o,\mathbb{F}}\widehat{B}_t^{\mathbb{G}}$$

$$= X_0 + {}^{o,\mathbb{F}}\widehat{M}_t^{\mathbb{G}} + {}^{o,\mathbb{F}}\widehat{B}_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(\widehat{b}^{\mathbb{G}}a^{\mathbb{G}})_u du + \int_0^t {}^{o,\mathbb{F}}(\widehat{b}^{\mathbb{G}}a^{\mathbb{G}})_u du,$$
(4.10)

The process $\widehat{M}_t^B = {}^{o,\mathbb{F}}\widehat{B}_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(\widehat{b}^{\mathbb{G}}a^{\mathbb{G}})_u du$ is an \mathbb{F} -martingale (see Lemma 3.4 and Remark 3.5). Invoking assumptions B1 and B2, which, in fact, imply the immersion between \mathbb{F} and \mathbb{G} , and recalling Lemma 3.4 again we see that this process is null. Hence we conclude that

$${}^{o,\mathbb{F}}X_t = X_0 + {}^{o,\mathbb{F}}\widehat{M}_t^{\mathbb{G}} + \int_0^t {}^{o,\mathbb{F}}(\widehat{b}^{\mathbb{G}}a^{\mathbb{G}})_s ds$$

$$(4.11)$$

The process ${}^{o,\mathbb{F}}\widehat{M}^{\mathbb{G}}$ is an \mathbb{F} -martingale, since for an arbitrary bounded \mathbb{F} -stopping time τ we have

$$\mathbb{E}({}^{o,\mathbb{F}}\widehat{M}^{\mathbb{G}}_{\tau}) = \mathbb{E}(\mathbb{E}(\widehat{M}^{\mathbb{G}}_{\tau}|\mathcal{F}_{\tau})) = \mathbb{E}(\widehat{M}^{\mathbb{G}}_{\tau}) = \mathbb{E}(\widehat{M}^{\mathbb{G}}_{0}) = 0$$

Moreover, the process $\int_0^{\cdot o,\mathbb{F}}(\widehat{b}^{\mathbb{G}}a^{\mathbb{G}})_s ds$ is an \mathbb{F} -predictable process with finite variation. From this and from (4.11) we deduce that the process ${}^{o,\mathbb{F}}X$ is an \mathbb{F} -special semimartingale. Hence, using again (4.11), by uniqueness of canonical decomposition of ${}^{o,\mathbb{F}}X_t = X_0 + \widehat{M}_t^{\mathbb{F}} + \widehat{B}_t^{\mathbb{F}}$, we have

$$\widehat{M}_t^{\mathbb{F}} = {}^{o,\mathbb{F}} \widehat{M}_t^{\mathbb{G}}, \qquad \widehat{B}_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}} (\widehat{b}^{\mathbb{G}} a^{\mathbb{G}})_s ds.$$
(4.12)

Step 2. Now we will compute the \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$.

The formulae (4.12), (4.5) and (4.4) imply that the first characteristic of ${}^{o,\mathbb{F}}X$, that is $B^{\mathbb{F}}$, is given by (4.6).

Now, since $\widehat{M}^{\mathbb{G}}$ is square integrable then, invoking the Jensen inequality, we conclude that $\widehat{M}^{\mathbb{F}} = {}^{o,\mathbb{F}} \widehat{M}^{\mathbb{G}}$ is square integrable. Next, invoking the predictable representation property we see that there exists an \mathbb{F} -predictable process h such that $\mathbb{E} \int_0^t h_s^2 d[Z]_s < \infty$ and

$$\widehat{M}_t^{\mathbb{F}} = \int_0^t h_s dZ_s. \tag{4.13}$$

Now we will compute remaining characteristics of ${}^{o,\mathbb{F}}X$ in terms of the process h and in the following step we will compute h. The continuous martingale part of ${}^{o,\mathbb{F}}X$ is thus given as $\int_0^t h_s dZ_s^c$, where Z^c is the continuous part of the \mathbb{F} -martingale Z, so that

$$C_t^{\mathbb{F}} = \int_0^t h_s^2 d\langle Z^c \rangle_s.$$

Here, Z^c being continuous, $\langle Z^c \rangle = \langle Z^c \rangle^{\mathbb{G}}$. To complete this step of the proof we need to justify (4.8). This formula is a consequence of the fact that $\Delta^{o,\mathbb{F}}X_t = h_t\Delta Z_t$, which entails that the jump measure of ${}^{o,\mathbb{F}}X$ is the image of the jump measure of Z under the mapping $(t, x) \to (t, xh_t \mathbb{1}_{\{h_t \neq 0\}})$, and thus the \mathbb{F} -compensator of ${}^{o,\mathbb{F}}X$ is the image of the \mathbb{F} -compensator of Z.

Step 3. We will now compute h. Towards this end, we fix $t \ge 0$ and we observe using (4.12) that for any bounded \mathcal{F}_t -measurable random variable γ we have

$$\mathbb{E}(\gamma \widehat{M}_t^{\mathbb{G}}) = \mathbb{E}(\gamma \widehat{M}_t^{\mathbb{F}}).$$
(4.14)

By using integration by parts formula we may write the left-hand side of (4.14) as

$$\mathbb{E}(\gamma \widehat{M}_t^{\mathbb{G}}) = \mathbb{E}\Big(\int_0^t \widehat{\gamma}_{s-} d\widehat{M}_s^{\mathbb{G}} + \int_0^t \widehat{M}_{s-}^{\mathbb{G}} k_s dZ_s + [\widehat{\gamma}, \widehat{M}^{\mathbb{G}}]_t\Big), \tag{4.15}$$

where $(\widehat{\gamma}_s)_{s \in [0,t]}$ is the bounded martingale defined by $\widehat{\gamma}_s := \mathbb{E}(\gamma | \mathcal{F}_s)$ which admits the representation

$$\widehat{\gamma}_s = \mathbb{E}(\gamma) + \int_0^s k_u dZ_u, \quad s \in [0, t],$$
(4.16)

from which k is obtained. In view of assumption $\widehat{A3}$, we know that the process $\widehat{M}^{\mathbb{G}}$ admits a Kunita-Watanabe decomposition of the form

$$\widehat{M}_t^{\mathbb{G}} = \widehat{M}_0^{\mathbb{G}} + \int_0^t H_s dZ_s + \widehat{O}_t^{\perp}, \qquad (4.17)$$

where \widehat{O}^{\perp} is a square integrable \mathbb{G} -martingale orthogonal to Z satisfying $\widehat{O}_0^{\perp} = 0$, and H is a \mathbb{G} -predictable process such that $\int_0^{\cdot} H_s dZ_s$ is a square integrable \mathbb{G} -martingale (see e.g. [Sch01]). Hence, since Z is assumed to be square integrable, we have

$$H_t = \frac{d\langle \widehat{M}^{\mathbb{G}}, Z \rangle_t^{\mathbb{G}}}{d\langle Z \rangle_t^{\mathbb{G}}}.$$
(4.18)

Now, let us note that from the representation (4.16) of $\hat{\gamma}$ as stochastic integral with respect to Z and from (4.17) we may write

$$[\widehat{\gamma}, \widehat{M}^{\mathbb{G}}]_t = \int_0^t k_s d[Z, \widehat{M}^{\mathbb{G}}]_s = \int_0^t k_s \Big(H_s d[Z]_s + d[Z, \widehat{O}^{\perp}]_s \Big).$$

Using this we obtain from (4.15)

$$\mathbb{E}(\gamma \widehat{M}_t^{\mathbb{G}}) = \mathbb{E}\Big(\int_0^t \widehat{\gamma}_{s-} d\widehat{M}_s^{\mathbb{G}} + \int_0^t \widehat{M}_{s-}^{\mathbb{G}} k_s dZ_s + \int_0^t k_s H_s d[Z]_s + \int_0^t k_s d[Z, \widehat{O}^{\perp}]_s\Big).$$
(4.19)

Now we prove that the stochastic integrals $\int_0^{\cdot} \widehat{\gamma}_{s-} d\widehat{M}_s^{\mathbb{G}}$, $\int_0^{\cdot} \widehat{M}_{s-}^{\mathbb{G}} k_s dZ_s$ and $\int_0^{\cdot} k_s d[Z, \widehat{O}^{\perp}]_s$ in (4.19) are \mathbb{G} -martingales on [0, t]. The first stochastic integral, i.e., $\int_0^{\cdot} \widehat{\gamma}_{s-} d\widehat{M}_s^{\mathbb{G}}$, is a \mathbb{G} -martingale since $(\widehat{\gamma}_{s-})_{s \in [0,t]}$ is a bounded predictable process. Next, we prove that the local martingale

$$\int_0^u \widehat{M}_{s-}^{\mathbb{G}} k_s dZ_s, \qquad u \in [0, t], \tag{4.20}$$

is a G-martingale. Using [CE15, Lemma 16.2.5], [CE15, Theorem 16.2.6] and the Èmery inequality for BMO ([CE15, Theorem A.8.15.]) applied to local martingales (with p = 1) we obtain

$$\mathbb{E}(\widehat{M}_{-}^{\mathbb{G}} \cdot (k \cdot Z))_{t}^{*} \leq C \mathbb{E}(\widehat{M}^{\mathbb{G}})_{t}^{*} \| (k \cdot Z) \|_{BMO}.$$

From the Doob maximal inequality, we obtain that $\mathbb{E}(\widehat{M}^{\mathbb{G}})_t^* < \infty$. Next, since $\widehat{\gamma}$ is bounded, using [CE15, Remark A.8.3.] we see that $||(k \cdot Z)||_{BMO} < \infty$. Therefore the local martingale given by (4.20) is a martingale. Finally, we consider

$$\int_{0}^{u} k_{s} d[Z, \widehat{O}^{\perp}]_{s}, \qquad u \in [0, t].$$
(4.21)

Using Kunita-Watanabe's inequality and Cauchy-Schwartz's inequality we obtain that

$$\mathbb{E}(k \cdot [Z, \widehat{O}^{\perp}])_t^* \le \mathbb{E} \int_0^t |k_s| |d[Z, \widehat{O}^{\perp}]_s| \le \left(\mathbb{E} \int_0^t k_s^2 d[Z]_s\right)^{1/2} \left(\mathbb{E}[\widehat{O}^{\perp}]_t\right)^{1/2} < +\infty.$$

So the process given by (4.21) is a martingale. Consequently, since the processes $\int_0^{\cdot} \widehat{\gamma}_{s-} d\widehat{M}_s^{\mathbb{G}}$, $\int_0^{\cdot} \widehat{M}_{s-}^{\mathbb{G}} k_s dZ_s$ and $\int_0^{\cdot} k_s d[Z, \widehat{O}^{\perp}]_s$ in (4.19) are \mathbb{G} -martingales on [0, t], the left hand side of (4.14) takes the form

$$\mathbb{E}(\gamma \widehat{M}_t^{\mathbb{G}}) = \mathbb{E}\Big(\int_0^t k_s H_s d[Z]_s\Big).$$
(4.22)

Now we deal with the right-hand side of (4.14). Invoking (4.12) and (4.16), and using integration by parts formula, we may write the right-hand side of (4.14) as

$$\mathbb{E}(\gamma \widehat{M}_t^{\mathbb{F}}) = \mathbb{E}(\gamma^{o,\mathbb{F}} \widehat{M}_t^{\mathbb{G}}) = \mathbb{E}\Big(\int_0^t \widehat{\gamma}_{s-} d^{o,\mathbb{F}} \widehat{M}_s^{\mathbb{G}} + \int_0^t {}^{o,\mathbb{F}} \widehat{M}_{s-}^{\mathbb{G}} k_s dZ_s + [\widehat{\gamma},{}^{o,\mathbb{F}} \widehat{M}^{\mathbb{G}}]_t\Big).$$
(4.23)

Next, let us note that from (4.12), (4.16) and (4.13) we get

$$[\widehat{\gamma}, {}^{o,\mathbb{F}}\widehat{M}^{\mathbb{G}}]_t = \int_0^t k_s d[Z, \widehat{M}^{\mathbb{F}}]_s = \int_0^t k_s h_s d[Z]_s.$$

Using this and (4.23) we obtain

$$\mathbb{E}(\gamma \widehat{M}_t^{\mathbb{F}}) = \mathbb{E}\Big(\int_0^t \widehat{\gamma}_{s-} d\widehat{M}_s^{\mathbb{F}} + \int_0^t \widehat{M}_{s-}^{\mathbb{F}} k_s dZ_s + \int_0^t k_s h_s d[Z]_s\Big).$$
(4.24)

Applying reasoning analogous to the one that led to (4.22), and invoking (4.24) we conclude that

$$\mathbb{E}(\gamma \widehat{M}_t^{\mathbb{F}}) = \mathbb{E}\Big(\int_0^t k_s h_s d[Z]_s\Big).$$
(4.25)

Putting together (4.14), (4.22) and (4.25), we see that (4.14) is equivalent to

$$\mathbb{E}\left(\int_0^t H_s k_s d[Z]_s\right) = \mathbb{E}\left(\int_0^t h_s k_s d[Z]_s\right)$$
(4.26)

for any k which is \mathbb{F} -predictable and such that $\int_0^t k_s dZ_s$ is bounded.

We will now show that (4.26) extends to any \mathbb{F} -predictable and bounded k, a result that we will need in what follows. Towards this end let us take an arbitrary predictable and bounded k and define a square integrable random variable ψ by

$$\psi := \int_0^t k_s dZ_s.$$

The random variable ψ is a (point-wise) limit of the sequence $\psi_n := \psi \wedge n$ of bounded random variables and hence $\mathbb{E}\psi_n \to \mathbb{E}\psi = 0$. Moreover, for each *n* we have the predictable representation $\psi_n = \mathbb{E}(\psi_n) + \int_0^t k_s^n dZ_s$, and thus

$$\mathbb{E}\Big(\int_0^t (k_s^n - k_s)^2 d[Z]_s\Big) = \mathbb{E}\Big(\int_0^t k_s^n dZ_s - \int_0^t k_s dZ_s\Big)^2 = \mathbb{E}((\psi_n - \psi - \mathbb{E}(\psi_n))^2)$$
$$\leq 2\mathbb{E}((\psi_n - \psi)^2) + 2\left(\mathbb{E}(\psi_n)\right)^2 \underset{n \to \infty}{\longrightarrow} 0.$$

Using this and the Kunita-Watanabe inequality we obtain

$$\mathbb{E}\left(\int_0^t |H_s(k_s^n - k_s)|d[Z]_s\right) \le \left(\mathbb{E}\left(\int_0^t |H_s|^2 d[Z]_s\right)\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_0^t |k_s^n - k_s|^2 d[Z]_s\right)\right)^{\frac{1}{2}} \underset{n \to \infty}{\longrightarrow} 0$$

and

$$\mathbb{E}\left(\int_0^t |h_s(k_s^n - k_s)|d[Z]_s\right) \le \left(\mathbb{E}\left(\int_0^t |h_s|^2 d[Z]_s\right)\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_0^t |k_s^n - k_s|^2 d[Z]_s\right)\right)^{\frac{1}{2}} \underset{n \to \infty}{\longrightarrow} 0.$$

Using these two facts and (4.26) for k^n , we can pass to the limit in (4.26) and obtain that (4.26) holds for any bounded k.

Recall that $[Z] - \langle Z \rangle$ is a G-martingale (cf. Remark 4.1). Thus, using the Kunita-Watanabe inequality we obtain that $(\int_0^{\cdot} H_s k_s d([Z]_s - \langle Z \rangle_s))$ is a G-martingale and hence

$$\mathbb{E}\left(\int_0^t H_s k_s d[Z]_s\right) = \mathbb{E}\left(\int_0^t H_s k_s d\langle Z \rangle_s\right).$$

Similarly

$$\mathbb{E}\left(\int_0^t h_s k_s d[Z]_s\right) = \mathbb{E}\left(\int_0^t h_s k_s d\left\langle Z\right\rangle_s\right).$$

Hence and from (4.26), we have

$$\mathbb{E}\left(\int_0^t (H_s - h_s) k_s d\left\langle Z \right\rangle_s\right) = 0.$$

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This implies, by using assumption B.4, [HWY92, Theorem 5.16] and the remark right below it, that

$$\mathbb{E}\left(\int_{0}^{t} ({}^{p,\mathbb{F}}H_{s} - h_{s})k_{s}d\left\langle Z\right\rangle_{s}\right) = 0$$

for any bounded \mathbb{F} -predictable k such that $(H_t - h_t)k_t \geq 0$ for $t \geq 0$. Hence, we have

$$\int_{\Omega \times \mathbb{R}_+} \mathbb{1}_{(0,t]}(s) ({}^{p,\mathbb{F}}H_s(\omega) - h_s(\omega)) k_s(\omega) d\lambda_{\langle Z \rangle}(\omega,s) = 0$$

for any bounded \mathbb{F} -predictable k such that $(H_t - h_t)k_t \ge 0$ for $t \ge 0$. Since, by convention, semi-martingales are right continuous, then the measure $\lambda_{\langle Z \rangle}$ does not charge any set of the form $B \times \{0\}$. Consequently, we conclude that

$$\int_{\Omega \times \mathbb{R}_+} \mathbb{1}_{[0,t]}(s) ({}^{p,\mathbb{F}}H_s(\omega) - h_s(\omega)) k_s(\omega) d\lambda_{\langle Z \rangle}(\omega,s) = 0$$

for any bounded \mathbb{F} -predictable k such that $(H_t - h_t)k_t \ge 0$ for $t \ge 0$.

Now, note that $H_s - h_s > 0$ if and only if ${}^{p,\mathbb{F}}H_s - h_s > 0$. Thus, first taking

$$k_s = \mathbb{1}_{\{(H_s - h_s) > 0\}} = \mathbb{1}_{\{(p, \mathbb{F}_{H_s} - h_s) > 0\}}$$

and then taking

$$k_s = -\mathbb{1}_{\{(H_s - h_s) < 0\}} = -\mathbb{1}_{\{(p, \mathbb{F}_{H_s} - h_s) < 0\}}$$

in the above, we obtain

$$h_s = {}^{p,\mathbb{F}}H_s \qquad \lambda_{\langle Z \rangle} - \text{ a.e. on } \Omega \times [0,t].$$
(4.27)

This, together with formula (4.18) gives (4.9). The proof is complete.

5 Examples

Examples 5.1–5.4 below illustrate the results in the case when X is \mathbb{F} -adapted. In what follows the natural filtration of any process A is denoted by \mathbb{F}^A .

Example 5.1. Consider two one-point càdlàg processes Y^1 and Y^2 on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $Y = (Y^1, Y^2)$. That is, Y^i (i = 1, 2) starts from 0 at time t = 0 and jumps to 1 at some random time. Thus, Y can be identified with a pair of positive random variables T_1 and T_2 given by $T_i := \inf\{t > 0 : Y_t^i = 1\}, i = 1, 2$. In other words, $Y_t^i = \mathbb{1}_{\{T_i \leq t\}}, i = 1, 2$. We assume that, under \mathbb{P} , the probability distribution of (T_1, T_2) admits a density function f(u, v) which is continuous in both variables.

Now, let $X = Y^1$, $\mathbb{F} = \mathbb{F}^X$ and $\mathbb{G} = \mathbb{F}^Y$. Clearly, X is a special \mathbb{G} -semimartingale and a special \mathbb{F} -semimartingale on $(\Omega, \mathcal{F}, \mathbb{P})$.

The \mathbb{G} -characteristics of X are $(B^{\mathbb{G}}, 0, \nu^{\mathbb{G}})$, where

$$B_t^{\mathbb{G}} = \int_0^t \kappa_s ds, \quad \nu^{\mathbb{G}}(ds, dx) = \delta_1(dx)\kappa_s ds,$$

 δ_1 is the Dirac measure at 1, and κ is given by (this result follows, for example, by application

of [LB95, Theorem 4.1.11])

$$\kappa_s = \frac{\int_s^\infty f(s,v) \, dv}{\int_s^\infty \int_s^\infty f(u,v) \, du \, dv} \mathbb{1}_{\{s \le T_1 \land T_2\}} + \frac{f(s,T_2)}{\int_s^\infty f(u,T_2) \, du} \mathbb{1}_{\{T_2 < s \le T_1\}}, \quad s \ge 0.$$

Thus, according to Theorem 3.6, the \mathbb{F} -characteristics of X are $(B^{\mathbb{F}}, 0, \nu^{\mathbb{F}})$, where

$$B_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}}(\kappa)_s ds, \quad \nu^{\mathbb{F}} = (\nu^{\mathbb{G}})^{p,\mathbb{F}}.$$

Now, we will provide explicit formulae for $B^{\mathbb{F}}$ and $\nu^{\mathbb{F}}$; for the latter, we only need to compute $\nu^{\mathbb{F}}(dt, \{1\})$. It can be easily shown that these computations boil down to computing the \mathbb{F} -optional projection of the process κ . Indeed, for an arbitrary \mathbb{F} -predictable, bounded function W on $\Omega \times \mathbb{R}$ we have

$$\mathbb{E}\Big(\int_{\mathbb{R}_+\times\mathbb{R}} W(s,x)\nu^{\mathbb{G}}(ds,dx)\Big) = \mathbb{E}\Big(\int_{\mathbb{R}_+} W(s,1)\kappa_s ds\Big) = \mathbb{E}\Big(\int_{\mathbb{R}_+}{}^{p,\mathbb{F}}(W(\cdot,1)\kappa_{\cdot})_s ds\Big)$$
$$= \mathbb{E}\Big(\int_{\mathbb{R}_+}{}^{p,\mathbb{F}}(\kappa)_s W(s,1) ds\Big) = \mathbb{E}\Big(\int_{\mathbb{R}_+\times\mathbb{R}}{}^{p,\mathbb{F}}(\kappa)_s W(s,x)\delta_1(dx) ds\Big),$$

where $p,\mathbb{F}(\kappa)$ denotes the \mathbb{F} -predictable projection of κ . Next, we note that the measure ρ defined as

$$\rho(dt, dx) := {}^{p, \mathbb{F}}(\kappa)_t \delta_1(dx) dt$$

is \mathbb{F} -predictable, and thus, due to uniqueness of the dual predictable projections, we have $\rho = (\nu^{\mathbb{G}})^{p,\mathbb{F}}$, and so $\nu^{\mathbb{F}} = \delta_1(dx)^{p,\mathbb{F}}(\kappa)$. Finally, we note that, in view of the continuity assumptions on f and that fact that κ admits two jumps only, we have

$$\mathbb{E}\Big(\int_{\mathbb{R}_+\times\mathbb{R}}{}^{p,\mathbb{F}}(\kappa)_s W(s,x)\delta_1(dx)ds\Big) = \mathbb{E}\Big(\int_{\mathbb{R}_+\times\mathbb{R}}{}^{o,\mathbb{F}}(\kappa)_s W(s,x)\delta_1(dx)ds\Big),$$

where ${}^{o,\mathbb{F}}(\kappa)$ denotes the \mathbb{F} -optional projection of κ . Using the key lemma (see e.g. [AJ17, Lemma 2.9]) we obtain

$${}^{o,\mathbb{F}}(\kappa)_{s} = \mathbb{E}\left(\frac{\int_{s}^{\infty} f(s,v) \, dv}{\int_{s}^{\infty} \int_{s}^{\infty} f(u,v) \, du \, dv} \mathbb{1}_{\{s \le T_{1} \land T_{2}\}} + \frac{f(s,T_{2})}{\int_{s}^{\infty} f(u,T_{2}) \, du} \mathbb{1}_{\{T_{2} < s \le T_{1}\}} \middle| \mathcal{F}_{s}\right)$$

$$= \frac{\int_{0}^{\infty} f(s,v) \, dv}{\int_{s}^{\infty} \int_{0}^{\infty} f(u,v) \, du \, dv} \mathbb{1}_{\{T_{1} > s\}}.$$

Consequently,

$$B_t^{\mathbb{F}} = \int_0^t \frac{\int_0^\infty f(s, v) \, dv}{\int_s^\infty \int_0^\infty f(u, v) \, du \, dv} \mathbb{1}_{\{T_1 > s\}} ds$$

and $\nu^{\mathbb{F}}((0,t],\{1\})$ is given as

$$\nu^{\mathbb{F}}((0,t],\{1\}) = \int_0^t \frac{\int_0^\infty f(s,v) \, dv}{\int_s^\infty \int_0^\infty f(u,v) \, du \, dv} \mathbb{1}_{\{T_1 > s\}} ds.$$

We note that the last result agrees with the classical computation of intensity of T_1 in its own filtration, which is given as $\lambda_s^1 = \frac{f^1(s)}{1-F^1(s)}$ with $F^1(s) = \mathbb{P}(T_1 \leq s)$ and $f^1(s) = \frac{\partial F^1(s)}{\partial s}$. \Box

Example 5.2. Let X be a real-valued process on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying

$$dX_t = m_t dt + \sum_{j=1}^2 \sigma_t^j dW_t^j + dM_t, \quad t \ge 0,$$

where W^j s are independent standard Brownian motions (SBMs), and $M_t = \int_0^t \int_{\mathbb{R}} x(\mu(ds, dx) - \nu(ds, dx))$ is a pure jump martingale, with absolutely continuous compensating part, say $\nu(dx, dt) = \eta(t, dx)dt$. We assume that M is independent of W^j s. The coefficients m and $\sigma^j > 0$, j = 1, 2 are adapted to $\mathbb{G} := \mathbb{F}^{W^1, W^2, M}$ and bounded.

Let $\mathbb{F} = \mathbb{F}^X$. Since M and $\sigma^1 \cdot W^1 + \sigma^2 \cdot W^2$ are true \mathbb{G} -martingales, then X is a special semimartingale in \mathbb{G} and thus in \mathbb{F} .

The \mathbb{G} -characteristics of X are

$$B_t^{\mathbb{G}} = \int_0^t m_s ds, \quad C_t^{\mathbb{G}} = \int_0^t ((\sigma_s^1)^2 + (\sigma_s^2)^2) ds, \quad \nu^{\mathbb{G}}(dx, dt) = \eta(t, dx) dt.$$

Now, in view of Theorem 3.6, we conclude that the \mathbb{F} -characteristics of X are

$$B_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}}(m)_s ds, \quad C_t^{\mathbb{F}} = \int_0^t ((\sigma_s^1)^2 + (\sigma_s^2)^2) ds, \quad \nu^{\mathbb{F}}(dx, dt) = (\eta(t, dx) dt)^{p,\mathbb{F}}.$$

Example 5.3. In this example we consider time homogeneous Poisson process with values in \mathbb{R}^2 . There is a one-to-one correspondence between any time homogeneous Poisson process with values in \mathbb{R}^2 , say $N = (N^1, N^2)$, and a homogeneous Poisson measure, say μ , on $E := \{0, 1\}^2 \setminus \{(0, 0)\}$.³ See for instance discussion in [BJVV08].

Let $\mathbb{G} = \mathbb{F}^{N}$, and let ν denote the \mathbb{G} -dual predictable projection of μ . The measure ν is a measure on a finite set, so it is uniquely determined by its values on the atoms in E. Therefore the Poisson process $N = (N^1, N^2)$ is uniquely determined by

$$\nu(dt, \{1, 0\}) = \lambda_{10} dt, \quad \nu(dt, \{0, 1\}) = \lambda_{01} dt, \quad \nu(dt, \{1, 1\}) = \lambda_{11} dt \tag{5.1}$$

for some positive constants λ_{10} , λ_{01} and λ_{11} . Clearly, the Poisson process $N = (N^1, N^2)$ is a G-special semimartingale, and the G-characteristic triple of N is $(B, 0, \nu)$, where

$$B_t = \begin{bmatrix} (\lambda_{10} + \lambda_{00})t \\ (\lambda_{01} + \lambda_{00})t \end{bmatrix}.$$

Let $X = N^1$. Then, X is a G-special semimartingale, and the G-characteristic triple of X is $(B^{\mathbb{G}}, 0, \nu^{\mathbb{G}})$, where

$$\nu^{\mathbb{G}}(dt, \{1\}) = \nu(dt, \{(1,0)\}) + \nu(dt, \{(1,1)\}) = \lambda_{10}dt + \lambda_{11}dt, \qquad \nu^{\mathbb{G}}(dt, \{0\}) = 0,$$

and $B_t^{\mathbb{G}} = (\lambda_{10} + \lambda_{00})t$.

Now, let us set $\mathbb{F} = \mathbb{F}^X$. In view of Proposition 3.1 we have

$$(B^{\mathbb{F}}, 0, \nu^{\mathbb{F}}) = (B^{\mathbb{G}}, 0, \nu^{\mathbb{G}}).$$

 $^{^{3}}$ We refer to [JS03] for the definition of the Poisson measure.

Example 5.4. Let $Y = (Y^1, Y^2)^{\top}$ be given as the strong solution of the SDE

$$dY_t = m(Y_t)dt + \Sigma(Y_t)dW_t, \quad Y(0) = (1,1)^{\top},$$
(5.2)

where $W = (W_1, W_2)^{\top}$ is a two dimensional SBM process on $(\Omega, \mathcal{F}, \mathbb{P})$, and where

$$m(y^1, y^2) = (m_1(y^1, y^2), m_2(y^1, y^2))^{\top}, \quad \Sigma(y^1, y^2) = \begin{pmatrix} \sigma_{11}(y^1, y^2) & \sigma_{12}(y^1, y^2) \\ \sigma_{21}(y^1, y^2) & \sigma_{22}(y^1, y^2) \end{pmatrix}$$

are bounded. Next, let us set $\mathbb{G} = \mathbb{F}^Y$, $X = Y^1$ and $\mathbb{F} = \mathbb{F}^X$. Hence

$$dX_t = m_1(X_t, Y_t^2)dt + \sigma_{1,1}(X_t, Y_t^2)dW_t^1 + \sigma_{1,2}(X_t, Y_t^2)dW_t^2$$
(5.3)

Suppose that the function Σ satisfies the following condition

$$\sigma_{11}^2(y^1, y^2) + \sigma_{12}^2(y^1, y^2) = \sigma_1^2(y^1), \quad (y^1, y^2) \in \mathbb{R}^2,$$

for some function $\sigma_1 > 0$, and suppose that the function m_1 satisfies

$$m_1(y^1, y^2) = \mu_1(y^1) \quad (y^1, y^2) \in \mathbb{R}^2.$$

Then (5.3) takes form

$$dX_t = \mu_1(X_t)dt + \sigma_1(X_t)dZ_t, \qquad X(0) = 1,$$

where

$$Z_t = \int_0^t \frac{\sigma_{1,1}(X_s, Y_s^2)}{\sigma_1(X_s)} dW_s^1 + \int_0^t \frac{\sigma_{1,2}(X_s, Y_s^2)}{\sigma_1(X_s)} dW_s^2$$

is a G-adapted process, which is a continuous G-local martingale. Since $(Z^2(t) - t)_{t\geq 0}$ is a local martingale we obtain by Lévy's characterization theorem that Z is a standard Brownian motion in the filtration G. Thus using continuity of paths of X we conclude that X has the G-characteristic triple given as $(B^{\mathbb{G}}, C^{\mathbb{G}}, 0)$, where

$$B_t^{\mathbb{G}} = \int_0^t \mu_1(X_u) du, \quad C_t^{\mathbb{G}} = \int_0^t \sigma_1^2(X_u) du, \quad t \ge 0.$$

We will now apply Theorem 3.6 so to compute the \mathbb{F} -characteristics of X. Since X is \mathbb{F} -adapted the \mathbb{F} -characteristics of X are

$$B_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}}(\mu_1(X_u))du = \int_0^t \mu_1(X_u)du, \quad C_t^{\mathbb{F}} = C_t^{\mathbb{G}} = \int_0^t \sigma_1^2(X_u)du$$

Finally by continuity of paths

$$\nu^{\mathbb{F}}(dt, dx) = (\nu^{\mathbb{G}}(dt, dx))^{p, \mathbb{F}} = (0)^{p, \mathbb{F}} = 0.$$

So we conclude that $(B^{\mathbb{G}}, C^{\mathbb{G}}, 0) = (B^{\mathbb{F}}, C^{\mathbb{F}}, 0).$

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The remaining examples refer to the case when X is not \mathbb{F} -adapted. We begin with providing, in Remark 5.5, a sufficient condition for the \mathbb{F} -optional and the \mathbb{F} -predictable projections of a \mathbb{G} -adapted process Y to exist. We will only use this condition to deduce existence of \mathbb{F} -predictable projections though.

Remark 5.5. The \mathbb{F} -optional and \mathbb{F} -predictable projections of a \mathbb{G} -adapted process Y exist under the condition that $\mathbb{E}(Y_t^*) < \infty$ for all t > 0, where $Y_t^* = \sup_{s \le t} |Y_s|$. Indeed, taking an arbitrary \mathbb{F} stopping time and $A_n = \{\tau \le n\}, n = 1, 2, \ldots$ we have $A_n \in \mathcal{F}_{\tau-} \subseteq \mathcal{F}_{\tau}$ and

$$\mathbb{E}(Y_{\tau}\mathbb{1}_{\tau<\infty}\mathbb{1}_{A_n}) \le \mathbb{E}(Y_n^*\mathbb{1}_{\tau<\infty}\mathbb{1}_{A_n}) \le \mathbb{E}(Y_n^*) < \infty,$$

for each $n = 1, 2, \ldots$. Hence by [HWY92, Theorem 5.1] the F-optional projection of Y exists. Taking τ to be an F-predictable stopping time we have by [HWY92, Theorem 5.2] that the F-predictable projection of Y exists.

Example 5.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space supporting a Brownian motion W and an independent time inhomogeneous Poisson process N. Let \mathbb{G} be the filtration generated by W and N. Suppose that N has deterministic compensator $\nu(t) = \int_0^t \lambda(s) ds, t \in \mathbb{R}_+$, so that $(\nu(t))_{t \in \mathbb{R}_+}$ is the unique continuous deterministic function such that

$$M_t := N_t - \nu(t), \quad t \ge 0,$$

is an $(\mathbb{F}^N, \mathbb{P})$ -martingale. Additionally, suppose that λ is such that $\lim_{t\to\infty} \nu(t) = \infty$ and $\nu(t) < \infty, t \ge 0$. Finally, note that by independence of W and N under \mathbb{P} , M is also a (\mathbb{G}, \mathbb{P}) -martingale.

Let now X be a process with the following integral representation

$$X_t = X_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dW_s + \int_0^t \kappa_s \mathbb{1}_{|\kappa_s| \le 1} dM_s + \int_0^t \kappa_s \mathbb{1}_{|\kappa_s| > 1} dN_s, \tag{5.4}$$

for some \mathbb{G} -predictable processes γ and κ such that for all $t \geq 0$

$$\mathbb{E}\Big(\sup_{s\leq t}\left(|\beta_s|+|\kappa_s\lambda(s)|\mathbb{1}_{|\kappa_s|>1}\right)\Big)<\infty,\tag{5.5}$$

$$\mathbb{E}\Big(\int_0^t \left(\gamma_s^2 + \kappa_s^2 \lambda(s)\right) ds\Big) < \infty.$$
(5.6)

Note that $L_t = \int_0^t \gamma_s dW_s + \int_0^t \kappa_s \mathbb{1}_{|\kappa_s| \leq 1} dM_s$ and $A_t = \int_0^t \beta_s ds + \int_0^t \kappa_s \mathbb{1}_{|\kappa_s| > 1} dN_s$, $t \geq 0$ are a \mathbb{G} -adapted local-martingale and \mathbb{G} -adapted process with locally integrable variation, respectively. Thus for $A_t^{\mathbb{G}} = t$ the process X is a special \mathbb{G} -semimartingale with \mathbb{G} -characteristics $(B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$, where

$$B^{\mathbb{G}}_{\cdot} = \int_{0}^{\cdot} \beta_{t} dt,$$
$$C^{\mathbb{G}}_{\cdot} = \int_{0}^{\cdot} \gamma_{t}^{2} dt,$$

and

$$\nu^{\mathbb{G}}(A, dt) = \left(\int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}}(x) \delta_{\kappa_t}(dx)\right) \lambda(t) dt, \qquad A \in \mathcal{B}(\mathbb{R})$$

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In particular, we have (cf. (2.3))

$$a_t^{\mathbb{G}} = 1, \quad b_t^{\mathbb{G}} = \beta_t, \quad c_t^{\mathbb{G}} = \gamma_t^2, \quad K_t^{\mathbb{G}}(dx) = \delta_{\kappa_t}(dx) \mathbb{1}_{\kappa_t \neq 0} \lambda(t).$$

Moreover, X as a special \mathbb{G} -semimartingale has the unique canonical decomposition

$$X = X_0 + \widehat{B}^{\mathbb{G}} + \widehat{M}^{\mathbb{G}},$$

where

$$\widehat{B}_t^{\mathbb{G}} = \int_0^t (\beta_s + \mathbb{1}_{|\kappa_s| > 1} \kappa_s \lambda(s)) ds \quad \text{and} \quad \widehat{M}_t^{\mathbb{G}} = \int_0^t \gamma_s dW_s + \int_0^t \kappa_s dM_s.$$
(5.7)

Now, we will verify that for arbitrary $\mathbb{F} \subseteq \mathbb{G}$ assumptions $\widehat{A1} - \widehat{A3}$ are satisfied. Since $a^{\mathbb{G}} = 1$ and

$$\widehat{b}^{\mathbb{G}} = \beta + \mathbb{1}_{|\kappa| > 1} \kappa \lambda$$

from (5.5) it immediately follows that $\widehat{A1}$ holds. Note that (5.5) also implies σ -integrability of $\widehat{b}^{\mathbb{G}}a^{\mathbb{G}}$ with respect to \mathcal{F}_{τ} for every bounded \mathbb{F} -stopping time τ , so $\widehat{A2}$ is satisfied. Using (5.6) we see that $\widehat{M}^{\mathbb{G}}$ is a square integrable \mathbb{G} -martingale and hence $\widehat{A3}$ follows.

In what follows we will give the form of characteristics of ${}^{o,\mathbb{F}}X$ for different specifications of \mathbb{F} and Z.

a) Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\alpha : \mathbb{R}_+ \longrightarrow (0, \infty)$ be two deterministic functions, with $\int_0^t \alpha^2(s) ds < \infty$, $t \ge 0$. Let $i(t) = \mathbb{1}_{\{\phi(t)=0\}}$, and set

$$\lambda(t) = \begin{cases} \alpha^2(t)/\phi^2(t) & \text{if } \phi(t) \neq 0, \\ 0 & \text{if } \phi(t) = 0, \end{cases} \quad t \ge 0.$$

By Proposition 4 in [É89] the process V given by

$$dV_t = i(t)dW_t + \frac{\phi(t)}{\alpha(t)}(dN_t - \lambda(t)dt), \quad t \ge 0, \quad V_0 = 0,$$
(5.8)

is the unique strong solution of the following structure equation

$$d[V]_t = dt + \frac{\phi(t)}{\alpha(t)} dV_t, \quad t \ge 0, \quad V_0 = 0.$$
 (5.9)

By Proposition 3 ii) in [É89] the process V has the predictable representation property in \mathbb{F}^V .

We now take $\mathbb{F} = \mathbb{F}^V \subseteq \mathbb{G}$ and Z given as

$$Z_t = \int_0^t \alpha(s) dV_s, \ t \ge 0$$

Thus, following [JP02], we see that Z satisfies

$$dZ_t = i(t)\alpha(t)dW_t + \phi(t) (dN_t - \lambda(t)dt), \quad t \ge 0, \ Z_0 = 0.$$
(5.10)

Thus, using the fact that

 $\phi(t)dZ_t = \phi^2(t)(dN_t - \lambda(t)dt)$ and $i^2(t)\alpha^2(t) + \phi^2(t)\lambda(t) = \alpha^2(t),$

we conclude that

$$d[Z]_t = i^2(t)\alpha^2(t)dt + \phi^2(t)dN_t = \alpha^2(t)dt + \phi(t)dZ_t, \quad t \ge 0, \ Z_0 = 0.$$
(5.11)

The process Z is obviously a square integrable (\mathbb{G}, \mathbb{P})-martingale and from (5.11) we see that $\langle Z \rangle_t^{\mathbb{G}} = \int_0^t \alpha^2(s) ds$. Since $\mathbb{F} \subset \mathbb{G}$, Z is a square integrable (\mathbb{F}, \mathbb{P})-martingale. Clearly, Z has predictable representation property in \mathbb{F} since $\alpha > 0$ and V has predictable representation property in \mathbb{F}^V . Therefore for such Z we see that conditions B1-B2 are satisfied. By definition of \mathbb{G} we have that \mathcal{G}_0 is trivial, so B3 holds. Moreover, we additionally assume that, for $t \geq 0$,

$$\mathbb{E}\Big(\sup_{s\leq t}\Big(\frac{|\gamma_s|}{\alpha(s)}i(s) + (1-i(s))\Big|\frac{\kappa_s}{\phi(s)}\Big|\Big)\Big) < \infty.$$
(5.12)

From (5.10) and (5.7) we have

$$H_t := \frac{d\langle M^{\mathbb{G}}, Z \rangle_t}{d\langle Z \rangle_t} = \frac{\gamma_t i(t)\alpha(t) + \kappa_t \phi(t)\lambda(t)}{i^2(t)\alpha^2(t) + \phi^2(t)\lambda(t)} = \frac{\gamma_t}{\alpha(t)}i(t) + (1 - i(t))\frac{\kappa_t}{\phi(t)} \quad dt \otimes d\mathbb{P} \ a.e.$$

Hence, assumption (5.12) and Remark 5.5 imply that the predictable projection of the process H exists, so B4 holds.

Now, using Theorem 4.2 we obtain that ${}^{o,\mathbb{F}}X$ is a special semimartingale whose characteristics are expressed in terms of h given by (4.9). We will now proceed with computation of h. Since $d\langle Z \rangle_t = \alpha^2(t)dt$, $\alpha > 0$, we see from (4.9) that $h_t = {}^{p,\mathbb{F}}H_t$ for $t \ge 0$ outside of an evanescent set, so that $h_t = \mathbb{E}(H_t | \mathcal{F}_{t-})$ for t > 0 outside of an evanescent set. Thus we may write

$$h_t = \frac{\mathbb{E}(\gamma_t | \mathcal{F}_{t-})}{\alpha(t)} i(t) + (1 - i(t)) \frac{\mathbb{E}(\kappa_t | \mathcal{F}_{t-})}{\phi(t)}.$$

In view of Theorem 4.2 again, having the above form of h, we find the \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$. Since $d\langle Z^c \rangle_t = i^2(t)\alpha^2(t)dt$, the \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$ are

$$C^{\mathbb{F}} = \int_0^{\cdot} h_s^2 i^2(s) \alpha^2(s) ds = \int_0^{\cdot} (\mathbb{E}(\gamma_s | \mathcal{F}_{s-}))^2 i(s) ds, \ t \ge 0,$$

and, for any $A \in \mathcal{B}(\mathbb{R})$,

$$\begin{split} \nu^{\mathbb{F}}(A,dt) &= \Big(\int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}}(h_t x) \delta_{\phi(t)}(dx) \Big) \lambda(t) dt \\ &= \Big(\int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}} \big(\mathbb{E}(H_t | \mathcal{F}_{t-}) x) \delta_{\phi(t)}(dx) \Big) (1-i(t)) \lambda(t) dt \\ &= \Big(\int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}} \Big((1-i(t)) \frac{\mathbb{E}(\kappa_t | \mathcal{F}_{t-})}{\phi(t)} x \Big) \delta_{\phi(t)}(dx) \Big) (1-i(t)) \lambda(t) dt \\ &= \Big(\int_{\mathbb{R}} \mathbb{1}_{A \setminus \{0\}}(x) \delta_{\mathbb{E}(\kappa_t | \mathcal{F}_{t-})}(dx) \Big) \lambda(t) dt, \ t \ge 0, \end{split}$$

and finally the first \mathbb{F} -characteristic is given by

$$B_t^{\mathbb{F}} = \int_0^t \left({}^{o,\mathbb{F}} (b_s + \mathbb{1}_{|\kappa_s| > 1} \kappa_s \lambda(s)) - \mathbb{1}_{|\mathbb{E}(\kappa_s|\mathcal{F}_{s-})| > 1} \mathbb{E}(\kappa_s|\mathcal{F}_{s-}) \lambda(s) \right) ds, \ t \ge 0.$$

b) Now, we take $\mathbb{F} = \mathbb{F}^M \subseteq \mathbb{G}$ and Z = M. We additionally assume that

$$\mathbb{E}\Big(\sup_{s\leq t}|\kappa_s|\Big)<\infty,\ t\geq 0.$$
(5.13)

Then, proceeding in a way analogous to what is done in a) above, we compute

$$\begin{split} d\langle \widehat{M}^{\mathbb{G}}, Z \rangle_t &= d \Big\langle \int_0^{\cdot} \gamma_s dW_s + \int_0^{\cdot} \kappa_s dM_s, M \Big\rangle_t \\ &= \gamma_t d\langle W, M \rangle_t + \kappa_s d\langle M \rangle_t = \kappa_t d\langle M \rangle_t = \kappa_t d\langle Z \rangle_t, \ t \ge 0, \end{split}$$

so that

$$H_t = \frac{d\langle \widehat{M}^{\mathbb{G}}, Z \rangle_t}{d\langle Z \rangle_t} = \kappa_t \quad dt \otimes d\mathbb{P} \ a.e.$$

Hence, assumption (5.13) implies that the predictable projection of the process H exist and we conclude that B4 holds. Then, from Theorem 4.2 and the fact that $Z^c = M^c = 0$ we obtain that \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$ are given by

$$B_t^{\mathbb{F}} = \int_0^t \left({}^{o,\mathbb{F}} (\beta_s + \kappa_s \mathbb{1}_{|\kappa_s| > 1} \lambda(s)) - \mathbb{1}_{|\mathbb{E}(\kappa_s|\mathcal{F}_{s-})| > 1} \mathbb{E}(\kappa_s|\mathcal{F}_{s-}) \lambda(s) \right) ds, \quad C_t^{\mathbb{F}} = 0,$$
$$\nu^{\mathbb{F}} (dx, dt) = \delta_{\mathbb{E}(\kappa_t|\mathcal{F}_{t-})} (dx) \mathbb{1}_{\{\mathbb{E}(\kappa_t|\mathcal{F}_{-t}) \neq 0\}} \lambda(t) dt, \ t \ge 0.$$

c) Here, we take $\mathbb{F} = \mathbb{F}^W \subseteq \mathbb{G}$ and Z = W. We additionally assume that

$$\mathbb{E}\Big(\sup_{s\leq t}|\gamma_s|\Big)<\infty,\ t\geq 0.$$
(5.14)

We have

$$d\langle \widehat{M}^{\mathbb{G}}, Z \rangle_{t} = d \Big\langle \int_{0}^{t} \gamma_{s} dW_{s} + \int_{0}^{t} \kappa_{s} dM_{s}, W \Big\rangle_{t}$$
$$= \gamma_{t} d\langle W \rangle_{t} + \kappa_{t} d\langle M, W \rangle_{t} = \gamma_{t} d\langle W \rangle_{t}, \ t \ge 0.$$

Thus

$$\frac{d\langle \widehat{M}^{\mathbb{G}}, Z \rangle_t}{d\langle Z \rangle_t} = \gamma_t \quad dt \otimes d\mathbb{P} \ a.e.$$

Hence, assumption (5.14) implies that the predictable projection of the process H exist and thus assumption B4 holds. Then, applying Theorem 4.2, we conclude that \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$ are given by

$$B_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}} (\beta_s + \kappa_s \mathbb{1}_{|\kappa_s| > 1} \lambda(s)) ds, \quad C_t^{\mathbb{F}} = \int_0^t (\mathbb{E}(\gamma_s | \mathcal{F}_{s-}))^2 ds, \ t \ge 0, \quad \nu^{\mathbb{F}} \equiv 0,$$

where the third equality follows from $\nu^{Z,\mathbb{F}} = \nu^{W,\mathbb{F}} \equiv 0$.

d) Let us take $\mathbb{F} = \mathbb{F}^W \subseteq \mathbb{G}$ and Z = W. Moreover, take $\lambda(s) = \lambda > 0$, $\beta_s = \lambda$, $\kappa_s = 1$ and $\gamma_s = 0$ for $s \ge 0$, and $X_0 = 0$. Thus X = N is a Poisson process with intensity λ . Clearly X is a special G-semimartingale with G-characteristics $(B^{\mathbb{G}}, C^{\mathbb{G}}, \nu^{\mathbb{G}})$, where

$$B_t^{\mathbb{G}} = \lambda t, \quad C_t^{\mathbb{G}} = 0, \quad \nu^{\mathbb{G}}(dx, dt) = \lambda \delta_1(dx) dt,$$

Applying Theorem 4.2 we see that \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$ are given by

$$B_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}} \lambda ds = \lambda t, \quad C_t^{\mathbb{F}} = 0, \quad \nu^{\mathbb{F}} \equiv 0.$$

So a purely discontinuous special semimartingale X admits continuous optional projection ${}^{o,\mathbb{F}}\!X$. $\hfill\square$

We will now present an example where X is a continuous special \mathbb{G} -semimartingale, and ${}^{o,\mathbb{F}}X$ is a purely discontinuous special \mathbb{F} -semimartingale.

Example 5.7. Consider a standard Brownian motion W. Let X = W and take $\mathbb{G} = \mathbb{F}^X$. The \mathbb{G} -characteristics triple of W is $(0, C^{\mathbb{G}}, 0)$, where $C_t^{\mathbb{G}} = t$. In particular, we have $b^{\mathbb{G}} = 0$ and $a^{\mathbb{G}} = 1$. Next, define the filtration \mathbb{F} as

$$\mathcal{F}_t = \mathcal{F}_n^X, \quad t \in [n, n+1), \ n = 0, 1, 2, \dots$$

The optional projection of X on \mathbb{F} exists and is given as

$${}^{p,\mathbb{F}}X_t = X_n, \quad t \in [n, n+1), \ n = 0, 1, 2, \dots$$

In order to compute the \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$ we first observe that the canonical semimartingale representation of ${}^{o,\mathbb{F}}X$, with respect to the standard truncation function, is given as

$${}^{o,\mathbb{F}}X = x * \mu = (x\mathbb{1}_{|x|\leq 1}) * \nu + (x\mathbb{1}_{|x|\leq 1}) * (\mu - \nu) + (x\mathbb{1}_{|x|>1}) * \mu,$$
(5.15)

where

$$\mu(dt, dx) = \sum_{n \ge 1} \delta_{(n, X_n - X_{n-1})}(dt, dx),$$

and

$$\nu(\omega, dt, dx) = \sum_{n \ge 1} \delta_n(dt) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

From (5.15) we obtain

$$B_t^{\mathbb{F}} = \int_0^t \int_{|x| \le 1} x\nu(ds, dx) = \int_{(0,t] \times [-1,1]} x \sum_{n \ge 1} \delta_n(ds) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\right)$$
$$= \int_{(0,t]} \sum_{n \ge 1} \left(\int_{-1}^1 x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\right) \delta_n(ds) = 0.$$

Thus, the \mathbb{F} -characteristics triple of ${}^{o,\mathbb{F}}X$ is $(0,0,\nu^{\mathbb{F}})$, where

$$\nu^{\mathbb{F}} = \nu.$$

Example 5.8. Let us consider the case where \mathbb{F} is a Brownian filtration, \mathbb{G} its progressive enlargement with a strictly positive random time τ . Taking $X_t = \mathbb{1}_{\{\tau \leq t\}}, t \geq 0$ we have (cf. [AJ17]),

$$\mathbb{G} = \mathbb{F} \nabla \mathbb{F}^X,$$

where $\mathbb{F} \nabla \mathbb{F}^X$ is the smallest right-continuous filtration which contains \mathbb{F} and \mathbb{F}^X . Now, we define the Azéma supermartingale A by

$$A_t = \mathbb{P}(\tau > t | \mathcal{F}_t), \quad t \ge 0,$$

and we write its Doob-Meyer decomposition as

$$A_t = m_t - b_t, \quad t \ge 0,$$

where m is an \mathbb{F} -martingale, and b is an \mathbb{F} -predictable, increasing process which is the \mathbb{F} dual predictable projection of X. We assume that τ satisfies the following Jacod absolute continuity assumption

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \int_s^\infty \alpha_t(u) du, \quad s, t \ge 0,$$
(5.16)

where, for any $u \ge 0$, the process $\alpha_{\cdot}(u)$ is a positive continuous \mathbb{F} -martingale and the map $(\omega, t, u) \to \alpha_t(\omega; u)$ is $\widetilde{\mathcal{P}}_{\mathbb{F}}$ -measurable. Using the fact that $\int_0^\infty \alpha_t(u) du = \mathbb{P}(\tau > 0) = 1$ and $\alpha_{\cdot}(u)$ is a martingale, it is shown in Proposition 4.1 in [EKJJ10] that

$$db_t = \alpha_t(t)dt$$

and

$$m_t = \mathbb{E}\Big(\int_0^\infty \alpha_u(u)du | \mathcal{F}_t\Big) = 1 + \int_0^t \alpha_u(u)du - \int_0^t \alpha_t(u)du.$$
(5.17)

Note that in the above set-up, the process A is continuous.

The process X is a special G-semimartingale and we know (cf. Corollary 5.27 in [AJ17]) that its canonical decomposition is given as

$$X = M^{\mathbb{G}} + B^{\mathbb{G}},$$

and its \mathbb{G} -characteristics are $(B^{\mathbb{G}}, 0, \nu^{\mathbb{G}})$, where

$$B_t^{\mathbb{G}} = \int_0^t (1 - X_s) \frac{db_s}{A_s} = \int_0^t \frac{(1 - X_s)\alpha_s(s)}{A_s} ds, \quad t \ge 0$$

and

$$\nu^{\mathbb{G}}(dt, dx) = \delta_1(dx) \frac{(1 - X_{t-})\alpha_t(t)}{A_t} dt.$$

In particular, note that here we have $b_t^{\mathbb{G}} = \frac{(1-X_t)\alpha_t(t)}{A_t}$ and $a_t^{\mathbb{G}} = 1$. Now, using Lemma 3.4 and observing that ${}^{o,\mathbb{F}}X = 1 - A$ we can easily compute the first

Now, using Lemma 3.4 and observing that ${}^{o,\mathbb{F}}X = 1 - A$ we can easily compute the first \mathbb{F} -characteristic of ${}^{o,\mathbb{F}}X$,

$$B_t^{\mathbb{F}} = \int_0^t {}^{o,\mathbb{F}} \left(\frac{(1-X_s)\alpha_s(s)}{A_s} \right) ds = \int_0^t \alpha_s(s) ds.$$

Next, recalling that A is a continuous process we conclude that $\nu^{\mathbb{F}} = 0$. Moreover, we see that $C^{\mathbb{F}} = \langle m \rangle$. This completes the computation of the \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$ which are $(B^{\mathbb{F}}, \langle m \rangle, 0)$.

The next example is, in a sense, opposite to Example 5.7: here, X is a purely discontinuous special \mathbb{G} -semimartingale, and ${}^{o,\mathbb{F}}X$ is a continuous special \mathbb{F} -semimartingale. Thus, this example complements Example 5.6 d).

Example 5.9. Let \mathbb{F} be a Brownian filtration and \mathbb{G} its progressive enlargement with a strictly positive random time $\tau \in \mathcal{F}_{\infty}$ satisfying Jacod's absolute continuity assumption (5.16) with some density $\alpha_t(u)$, $t, u \geq 0$. Such a random time can be defined as $\tau := \psi(\int_0^\infty f(t)dW_t)$, where ψ is a differentiable, positive and strictly increasing function, and W is a real valued standard \mathbb{F} -Brownian motion (see [EKJJZ14]). Let \hat{X} be the compensated martingale

$$\widehat{X}_t = \mathbbm{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \frac{\alpha_s(s)}{A_s} ds, \quad t \geq 0.$$

We see that its \mathbb{G} -characteristic triple is $(0, 0, \nu^{\mathbb{G}})$ where, as in the previous example,

$$\nu^{\mathbb{G}}(dt, dx) = \delta_1(dx) \frac{\mathbbm{1}_{\{t < \tau\}} \alpha_t(t)}{A_t} dt.$$

The \mathbb{F} -optional projection of \widehat{X} , say v, is a continuous martingale, which is not constant. Indeed, note that if v were constant then $v_{\infty} = v_0 = 0$. Given that, one has $\widehat{X}_{\infty} = 1 - \int_0^\tau \frac{\alpha_s(s)}{A_s} ds \in \mathcal{F}_{\infty}$ and $v_{\infty} = \widehat{X}_{\infty}$. But since $v_{\infty} = 0$, then $\widehat{X}_{\infty} = 0$, and \widehat{X} being a martingale would be null, which it is not. This is a contradiction, showing that v is not constant. Consequently, its \mathbb{F} characteristic triple is $(0, C^{\mathbb{F}}, 0)$, with $C^{\mathbb{F}} \neq 0$.

6 Conclusion and open problems for future research

As stated in the Introduction this paper is meant to initiate a systematic study of the change of properties of semimartingales under shrinkage of filtrations and, when appropriate, under respective projections.

Given its pioneering nature the study originated here leads to numerous open problems and calls for extensions in numerous directions. Below, we indicate some such open problems and suggestions for continuation of the research presented in this paper.

The results presented in this paper use several non-trivial assumptions. A natural direction for continuation of the present work will be to try to eliminate some of these assumptions.

Recall the decomposition (4.10)

$${}^{o,\mathbb{F}}X_t = {}^{o,\mathbb{F}}X_0 + {}^{o,\mathbb{F}}\widehat{M}_t^{\mathbb{G}} + {}^{o,\mathbb{F}}\widehat{B}_t^{\mathbb{G}}$$
$$= X_0 + {}^{o,\mathbb{F}}\widehat{M}_t^{\mathbb{G}} + {}^{o,\mathbb{F}}\widehat{B}_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(\widehat{b}^{\mathbb{G}}a^{\mathbb{G}})_u du + \int_0^t {}^{o,\mathbb{F}}(\widehat{b}^{\mathbb{G}}a^{\mathbb{G}})_u du.$$

As it was shown in the proof of Theorem 4.2, if the immersion hypothesis B2 is postulated, then the martingale $\widehat{M}_t^B = {}^{o,\mathbb{F}}\widehat{B}_t^{\mathbb{G}} - \int_0^t {}^{o,\mathbb{F}}(\widehat{b}^{\mathbb{G}}a^{\mathbb{G}})_u du$ is null. Therefore it does not intervene

in the representation of the \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$. If however the martingale \widehat{M}^B is not null, then the computation of the \mathbb{F} -characteristics of ${}^{o,\mathbb{F}}X$ in terms of the \mathbb{G} -characteristics of X is much more challenging, and perhaps may not be doable.

The immersion hypothesis B2 postulated Theorem 4.2 is also heavily exploited in computation of the second \mathbb{F} -characteristic of X, that is in computation of $C^{\mathbb{F}}$. In fact, computation of $C^{\mathbb{F}}$ in terms of \mathbb{G} -canonical decomposition appears to be much more difficult, or even impossible, without the hypothesis B2, as the following reasoning shows: Assume that \mathbb{F} is a Brownian filtration generated by W, so that W enjoys the predictable representation property in \mathbb{F} . Also, take \mathbb{G} to be the progressive enlargement of \mathbb{F} by a random time τ .⁴ Assume that there exists a \mathbb{G} -predictable integrable process μ such that $W^{\mathbb{G}}$ defined for any $t \in \mathbb{R}_+$ as

$$W_t^{\mathbb{G}} = W_t + \int_0^t \mu_s ds$$

is a \mathbb{G} -martingale (hence, a \mathbb{G} -Brownian motion). Then, any \mathbb{G} -martingale X can be written as

$$X_t = X_0 + \int_0^t \psi_s dW_s^{\mathbb{G}} + M_t^{\perp}, \quad t \ge 0,$$

where ψ is a \mathbb{G} -predictable process and M^{\perp} a \mathbb{G} -martingale orthogonal to $W^{\mathbb{G}}$ (in fact, it is a purely discontinuous martingale). Moreover, one can show (using the same methodology as in [GJW19]) that ${}^{o,\mathbb{F}}X$, which is an \mathbb{F} -martingale, has the form

$${}^{o,\mathbb{F}}X_t = {}^{o,\mathbb{F}}X_0 + \int_0^t \gamma_s dW_s, \quad t \ge 0,$$

where γ satisfies $\gamma_t = \mathbb{E}(\psi_t + \mu_t X_t | \mathcal{F}_t)$. So here we have that

$$C_t^{\mathbb{F}} = \int_0^t (\mathbb{E}(\psi_s + \mu_s X_s | \mathcal{F}_s))^2 ds,$$
$$C_t^{\mathbb{G}} = \int_0^t \psi_s^2 ds.$$

Clearly, $C^{\mathbb{G}}$ alone does not suffice to compute $C^{\mathbb{F}}$, unless $\mu \equiv 0$ – i.e., \mathbb{F} is immersed in \mathbb{G} . In fact, it is not clear at all, how to compute the $C^{\mathbb{F}}$ characteristic of X in terms of the canonical decomposition and \mathbb{G} characteristics of X.

The discussion above points to an important open problem: extend, if possible, the results of Theorem 4.2 to the case when the immersion hypothesis B2 is abandoned, and extend the result of [GJW19] to the case of general continuous semi-martingales.

Another challenging problem for future research is weakening of the predictable representation property condition B1, and replacing it with the postulate of the weak predictable representation property condition for Z, that is with the postulate that every local \mathbb{F} martingale Y admits the representation

$$Y_t = Y_0 + \psi \cdot Z_t^c + \xi * \widetilde{\mu}_t^Z, \quad t \ge 0,$$

⁴See e.g. Chapter 5 in [AJ17] for the concept of the progressive enlargement of filtrations.

where ψ is an \mathbb{F} -predictable process, ξ is a $\widetilde{\mathcal{P}}_{\mathbb{F}}$ -measurable function, Z^c is the continuous martingale part of Z, and $\widetilde{\mu}^Z$ is the \mathbb{F} -compensated measure of jumps of Z.

Finally, it might be worthwhile to study the following interesting question: Suppose we have two different semimartingales X and Y with different laws but with the same characteristics in \mathbb{G} . Will their characteristics in \mathbb{F} be the same as well?

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