Risk Sensitive Dynamic Asset Allocation

by

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Abstract: This paper develops a continuous time portfolio optimization model where the mean returns of individual securities or asset categories are explicitly affected by underlying economic factors such as dividend yields, a firm's ROE, interest rates, and unemployment rates. The factors are Gaussian processes, and the drift coefficients for the securities are affine functions of these factors. We employ methods of risk sensitive control theory, thereby using an infinite horizon objective that is natural and features the long run growth rate, the asymptotic variance, and a single risk aversion parameter. Even with constraints on the admissible trading strategies, it is shown that the optimal trading strategy has a simple characterization in terms of the factor levels. For particular factor levels, the optimal trading positions can be obtained by solving a quadratic program. The optimal objective value, as a function of the risk aversion parameter, is shown to be the solution of a partial differential equation. A simple asset allocation example, featuring a Vasicek-type interest rate which affects a stock index and also serves as a second investment opportunity, illustrates how factors which are commonly used for forecasting returns can be explicitly incorporated in a portfolio optimization model.

1 Introduction

The application of mathematics in the world's financial markets has had some great successes, as exemplified by the crucial role played by mathematics in the derivatives industry. On the other hand, and with the exception of standard statistical methods, the application of mathematics to portfolio management problems has been a disappointment. Portfolio management is a natural situation for optimization models, and yet practitioners rarely use any such models.

A lack of practitioner familiarity with optimization models does not seem to be an explanation for this, because a wide variety of models, ranging from the simple yet robust Markowitz single period model to highly sophisticated continuous time models, is readily available. A possible explanation is a lack of a realistic model that are simple enough to permit calculation of practical optimal solutions, but this also seems unlikely. A much more likely explanation is the statistical difficulty in estimating all the parameters used by these models. In particular, it seems to be very difficult to make good estimates of the mean returns (or their continuous time counterparts). For example, solving the Markowitz model with real data usually leads to foolish solutions where you put all your money in only a few securities. You can add a number of artificial constraints (e.g., never put more than 3% of your capital in a single security), but then the *ad hoc* constraints will probably have a greater bearing on the solution than the underlying economic objective.

Whatever the explanation, the fact remains that practitioners rarely use optimization models. Instead, they channel their energy into making statistical forecasts of security prices, that is, into making statistical estimates of mean returns. In the case of individual stocks, these forecasts are developed from factors such as price-earnings ratios, dividend yields, accounting ratios such as the return on equity, and so forth. In the case of asset categories, these forecasts are based on similar measures as well as macroeconomic factors such as inflation, unemployment and interest rates. In the end, practitioners take their forecasts and, using little more in the way of tools than common sense, reach their final decisions about how to allocate their funds among the securities or asset categories [1].

Meanwhile, most optimization models developed for portfolio management tend to have the opposite orientation. The parameters related to mean returns, volatilities, and correlations are the starting point of the analysis, whereas for the practitioners the estimated mean returns will come near the end of the analysis. There is no explicit attention given to the uncertainty about these parameter estimates or to their origins. Instead, the emphasis is on characterizing the optimal trading strategies under a variety of assumptions about the investor's preferences and the underlying security prices. Furthermore, and this is especially true for continuous time models with constraints and a practical number of securities or asset categories, these dynamic models tend to be intractable from the computational standpoint. Whether one takes a PDE approach or uses a more modern approach using convex optimization theory and risk neutral probability measures (see [12], [21], [22]), there is little hope for using existing optimization models to solve practical portfolio problems that will be of interest to practitioners.

[1]Note for future consideration that an implicit feature of the practitioner's "philosophy" is the notion that randomness of the underlying factors will lead to mean return estimates which also evolve in a random manner.

The purpose of this paper is to develop a new kind of portfolio optimization model that is more compatible with financial practice. The key idea is to explicitly incorporate in the model stochastic factors such as accounting ratios, dividend yields, and macroeconomic measures. These underlying factors will explicitly affect the mean returns of the securities or asset categories, thereby resulting in a complex model which captures the dependence of security prices on the underlying factors. While this approach does not circumvent the statistical difficulties of making good estimates, it sheds better light on this problem, because the variables which are used to forecast returns now reside within the model. Both the estimation and optimization parts of the portfolio management process can be addressed in an integrated fashion.

Since our aim is to develop a model that has good potential for practical use, it is necessary that we pay attention to three considerations: realism, practicalities, and computational tractability. By "realism" we mean that the securities and factors are modeled in a realistic way. This will be accomplished with a continuous time approach, using standard stochastic calculus models of security prices and factors. By "practicalities" we mean that the model must be capable of handling a moderate number of securities, say several dozen or more. Moreover, the model must be able to handle at least several factors per security. And it is desirable to include constraints on the trading strategy, such as short sales restrictions, borrowing restrictions, and upper bounds on the positions in individual securities.

The third desired feature of our model, "computational tractability," requires some discussion. The common objective in the literature for portfolio optimization models is to maximize expected utility of capital at the end of a finite planning horizon. The resulting optimal trading strategies will usually be time-dependent for a general utility function, in which case the computational difficulties may be great. The alternative, adoption of an infinite horizon optimization objective, offers the possibility of stationary policies being optimal and thus of less severe computational difficulties. In addition, an infinite horizon optimization objective model frequently serves as a good approximation to practical decision making situations with finite but relatively long planning horizons. However, the choice of infinite horizon criteria is quite limited. In fact, the only such criterion that has been widely studied for portfolio management purposes is that of maximizing the portfolio's long-run expected growth rate (i.e., the log utility). This criterion is not conservative enough for most investors, and investors vary widely in regard to their attitudes about risk. What is needed is an infinite horizon criterion which depends upon one or more risk aversion parameters.

For a candidate criterion, let $V(t)$ denote the time-t value of a portfolio and consider the objective of maximizing the quantity

$$\liminf_{t\to\infty}(1/\gamma)t^{-1}ln\mathrm{E}(V(t))^{\gamma}, \quad \gamma<1\ ,\ \gamma\neq 0.$$

This was used by [11] and [5] to study a classical portfolio problem under a drawdown constraint. Note that letting $\gamma\to 0$ this becomes, in the limit, the same as the objective of maximizing the portfolio's long-run expected growth rate (the Kelly criterion), whereas for $\gamma>0$ it is not clear how to meaningfully interpret this criterion, although it resembles expected utility with an isoelastic or power utility function.

However, suppose this expression is rewritten as

$$J_\theta := \liminf_{t\to\infty}(-2/\theta)t^{-1}ln\mathrm{E}e^{-(\theta/2)lnV(t)},$$

where $\theta > -2, \theta \neq 0$. Substituting $C(t) = lnV(t)$ enables one to establish a connection with the recently developed literature on ***risk sensitive optimal control*** (e.g., see Whittle [23]), where $C(t)$ plays the role of a cumulative cost. This means that if we adopt, as we shall, the objective of maximizing J_θ, then many of the techniques that have recently been developed for risk sensitive control can potentially be applied to our portfolio management problem.

Moreover, as is well understood in the risk sensitive control literature, a Taylor series expansion of J_θ about $\theta = 0$ yields

$$-\frac{2}{\theta} ln\mathrm{E}e^{-\frac{\theta}{2} lnV(t)} = \mathrm{E}lnV(t) - \frac{\theta}{4}\mathrm{var}(lnV(t)) + O(\theta^2). \tag{1.1}$$

Hence J_θ can be interpreted as the long-run expected growth rate minus a penalty term, with an error that is proportional to θ^2. Furthermore, the penalty term is proportional to the ***asymptotic variance***, a quantity that was studied by [15] in the case of a conventional, multivariate geometric Brownian motion model of securities. The penalty term is also proportional to θ, so θ should be interpreted as a ***risk sensitivity parameter*** or ***risk aversion parameter***, with $\theta > 0$ and $\theta < 0$ corresponding to risk averse and risk seeking investors, respectively. The special case of $\theta = 0$ will thus be referred to as the ***risk neutral case***. In this case the criterion J_0 is the classical Kelly criterion, that is

$$J_0 = \liminf_{t\to\infty} t^{-1}\mathrm{E}lnV(t).$$

Note that J_θ has the form of the large-deviations type functional for the capital process $V(t)$. Consequently, in the risk averse case maximizing J_θ protects an investor interested in maximizing the expected growth rate of the capital against large deviations of the actually realized rate from the expectations.

Some insight into our risk sensitive criterion can be obtained by considering the case where the process $V(t)$ is a simple geometric Brownian motion with parameters μ and σ. A simple calculation gives

$$J_\theta = \mu - \frac{1}{2}\sigma^2 - \frac{\theta}{4}\sigma^2,$$

so the approximation mentioned above is, in this case, exact.

This paper is not the first to apply a risk sensitive optimality criterion to a financial problem. Lefebre and Montulet [17] studied a model for a firm's optimal mix between liquid and non-liquid assets; the calculus of variations approach was used to derive an explicit expression for the optimal division. Fleming [7] used risk sensitive methods to obtain two kinds of asymptotic results. In the first he considered a conventional, finite horizon portfolio model and studied certain limits as the coefficient of risk aversion diverges to infinity. In the second he studied the long-term growth rate for conventional modles with transaction costs and HARA utility functions.

In summary, in this paper we will develop a portfolio optimization model that is highlighted by two features. First, stochastic, economic factors such as dividend yields, accounting ratios, and macroeconomic variables which can be used for estimating mean returns of securities and asset categories will be explicitly incorporated in the model. Second, the objective will be to maximize the risk sensitive criterion J_θ that was introduced above. A precise formulation of our model as well as the main results will all be found in Section 2. Various preliminary and auxiliary results are located in Section 3, while Section 4 has the principal arguments and proofs of our main results.

Sections 2 - 4 are all for the case where the risk aversion parameter $\theta > 0$. As already mentioned, the case $\theta = 0$ is of considerable importance, because it corresponds to log utility and the objective of maximizing the portfolio's long run expected growth rate. This is the subject of Section 5. As will be seen, when our model has no factors it collapses to a well-studied, complete model, but when factors are included our model is incomplete and our expected growth rate maximization results are new.

Section 6 is devoted to a special case of a simple asset allocation model featuring a Vasicek type spot interest rate and a stock index that is affected by the level of interest rates. The investor must decide how to divide up his or her capital among the two asset categories. Not only does this example illustrate the main ideas of Sections 2 and 5, but it should also be of independent interest to financial economists. In fact, it was motivated by and resembles somewhat the model studied by [4].

In Section 7 we provide some final remarks and some thoughts about future research.

Our paper concludes with two Appendices. In Appendix 1 we prove lemma 5.1. In Appendix 2 we provide some additional, interesting insight into the nature of risk-sensitive optimality in the context of dynamic asset management.

2 Formulation of the problem and the main results

We will consider a market consisting of $m \geq 2$ securities and $n \geq 1$ factors. The set of the securities may include stocks, bonds, cash and derivative securities, as in [4] for example. The set of factors may include dividend yields, price-earning ratios, short-term interest rates, the rate of inflation, etc., as in [20] for example.

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbf{P})$ be the underlying probability space. Denoting by $S_i(t)$ the price of the i-th security and by $X_j(t)$ the level of the j-th factor at time t, we consider the following market model for the dynamics of the security prices and factors:

$$\frac{dS_i(t)}{S_i(t)} = (a + AX(t))dt + \sum_{k=1}^{m+n} \sigma_{ik} dW_k(t), \quad S_i(0) = s_i, \quad i = 1, 2, \cdots, m \tag{2.1}$$

$$dX(t) = (b + BX(t))dt + \Lambda dW(t), \quad X(0) = x, \tag{2.2}$$

where $W(t)$ is a R^{m+n} valued standard Brownian motion process with components $W_k(t)$, $X(t)$ is the R^n valued factor process with components $X_j(t)$, and the market parameters a, A, $\Sigma := [\sigma_{ij}]$, b, B, $\Lambda := [\lambda_{ij}]$ are matrices of appropriate dimensions. It is well known that a unique, strong solution exists for (2.1), (2.2), and that the processes $S_i(t)$ are positive with probability 1 (see e. g. [14], chapter 5).

Let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$, where $S(t) = (S_1(t), S_2(t), \ldots, S_m(t))$ is the security price process. Let $h(t)$ denote an R^n valued investment process whose components are $h_i(t)$, $i = 1, 2, \ldots, m$.

Definition 2.1 *An investment process $h(t)$ is* **admissible** *if the following conditions are satisfied:*

(i) $h(t)$ takes values in a given measurable subset χ of R^m, and $\sum_{i=1}^m h_i(t) = 1$,

(ii) $h(t)$ is measurable, $\mathcal{G}_t$-adapted,

(iii) $\mathbf{P}[\int_0^t h'(s)h(s)\, ds < \infty] = 1$, for all finite $t \geq 0$.
The class of admissible investment strategies will be denoted by $\mathcal{H}$.

Let now $h(t)$ be an admissible investment process. Then there exists a unique, strong, and almost surely positive solution $V(t)$ to the following equation:

$$dV(t) = \sum_{i=1}^{m} h_i(t)V(t)[\mu_i(X(t))dt + \sum_{k=1}^{m+n} \sigma_{ik} dW_k(t)], \quad V(0) = v > 0\ , \tag{2.3}$$

where $\mu_i(x)$ is the i-th coordinate of the vector $a + Ax$ for $x \in R^n$. The process $V(t)$ represents the investor's capital at time t, and $h_i(t)$ represents the proportion of capital that is invested in security i, so that $h_i(t)V(t)/S_i(t)$ represents the number of shares invested in security i, just as in, for example, Section 3 of [13].

In this paper we shall investigate the following family of risk sensitized optimal investment problems, labeled as $(\mathbf{P}_\theta)$:

for $\theta \in (0, \infty)$, maximize the risk sensitized expected growth rate

$$J_\theta(v, x; h(\cdot)) := \liminf_{t\to\infty} (-2/\theta)t^{-1} ln\ \mathbf{E}^{h(\cdot)}\ [e^{-(\theta/2) ln V(t)} | V(0) = v, X(0) = x] \tag{2.4}$$

over the class of all admissible investment processes $h(\cdot)$,

subject to (2.2) and (2.3),

where $\mathbf{E}$ is the expectation with respect to $\mathbf{P}$. The notation $\mathbf{E}^{h(\cdot)}$ emphasizes that the expectation is evaluated for process $V(t)$ generated by (2.3) under the investment strategy $h(t)$.

Remark 2.1 *As mentioned in the introduction, the positive value of the risk sensitivity parameter θ corresponds to a risk averse investor. The techniques used in this paper can also be used to study problems $(\mathbf{P}_\theta)$ for negative values of θ, corresponding to risk seeking investors. The risk neutral case, for $\theta = 0$, will be studied in section 5 as the limit of the risk averse situation when the risk sensitivity parameter θ goes to zero.*

Before we can present the main results contributed by this paper, we need to introduce the following notation, for $\theta \geq 0$ and $x \in R^n$:

$$K_\theta(x) := \inf_{h\in\chi,\ 1'h=1} [(1/2)(\theta/2\ + 1)h'\Sigma\Sigma'h\ -\ h'(a + Ax)]. \tag{2.5}$$

We also need to introduce the following assumptions:

Assumption (A1) *The investment constraint set χ satisfies one of the following two conditions:*

(a) $\chi = R^n$, or

(b) $\chi = \{h \in R^n : h_{1i} \leq h_i \leq h_{2i},\ i = 1, 2, \ldots, m\}$, where $h_{1i} < h_{2i}$ are finite constants.

Assumption (A2) *For $\theta > 0$,*

$$\lim_{\|x\|\to\infty} K_\theta(x) = -\infty.$$

Assumption (A3) *The matrix $\Lambda\Lambda'$ is positive definite.*

Remark 2.2

(i) Note that if $\Sigma\Sigma'$ is positive definite, then assumption (A2) is implied by assumption (A1)(a).

(ii) These assumptions are sufficient for the results below to be true, but, as will be seen for the example considered in section 6, Assumption (A2) is not necessary, in general.

Theorems 1 and 2 below contain the main results of this paper.

Theorem 2.1 *Assume (A1)-(A3). Fix $\theta > 0$.*

Let $H_\theta(x)$ denote a minimizing selector in (2.5), that is,

$$K_\theta(x) = (1/2)(\theta/2\ +1)H_\theta(x)'\Sigma\Sigma' H_\theta(x)\ -\ H_\theta(x)'(a + Ax).$$

Then the investment process

$$h_\theta(t) := H_\theta(X(t)) \tag{2.6}$$

is optimal sure in the sense of Foldes (see [10]). That is, letting for each $\tau \geq 0$

$$J_\theta^\tau(v, x; h(\cdot)) := (-2/\theta) ln \mathbf{E}^{h(\cdot)}[V^{-\theta/2}(\tau)|V(0) = v, X(0) = x],$$

we have

$$J_\theta^\tau(v, x; h(\cdot)) \leq J_\theta^\tau(v, x; h_\theta(\cdot)) \tag{2.7}$$

for all admissible strategies $h(\cdot)$, $v > 0$, $x \in R^n$, and all $\tau \geq 0$.

Corollary 2.1 *The investment process $h_\theta(t)$ is optimal for problem (P_θ), that is*

$$J_\theta(v, x; h(\cdot)) \leq J_\theta(v, x; h_\theta(\cdot))$$

holds for all $h(\cdot) \in \mathcal{H}$, $v > 0$, $x \in R^n$.

Theorem 2.2 *Assume (A1)-(A3), fix $\theta > 0$, and consider problem P_θ. Let $h_\theta(t)$ be as in theorem 2.1. Then*

(a) For all $v > 0$ and $x \in R^n$ we have

$$J_\theta(v, x; h_\theta(\cdot)) = \lim_{t\to\infty}(-2/\theta)t^{-1} ln\ \mathbf{E}^{h_\theta(\cdot)}\ [e^{-(\theta/2) ln V(t)}|V(0) = v, X(0) = x] =: \rho(\theta).$$

(b) The constant $\rho(\theta)$ in (a) is the unique non-negative constant, which is a part of the solution $(\rho(\theta), v(x;\theta))$ to the following equation:

$$
\begin{aligned}
\rho &= (b+Bx)'grad_x v(x) \\
&- (\theta/4)\sum_{i,j=1}^{n}\frac{\partial v(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk} + (1/2)\sum_{i,j=1}^{n}\frac{\partial^2 v(x)}{\partial x_i \partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk} \\
&- K_\theta(x), \\
& v(x)\in C^2(R^n), \quad \lim_{\|x\|\to\infty} v(x) = \infty \\
& \rho = const. \qquad (2.8)
\end{aligned}
$$

The key point of the first equality in (a) is, of course, that the optimal objective value is given by an ordinary lim rather than the lim inf as in (2.4). The key point of the second equality in (a) is that the optimal objective value does not depend on either the initial amount of the investor's capital (v) or on the initial values of the underlying economic factors (x), although it depends, of course, on the investor's attitude towards risk (encoded in the value of θ). The key point of (b) is that the optimal objective value is characterized in terms of the equation (2.8). It is important to observe that for the problem studied in this paper the part $v(x;\theta)$ of the solution to (2.8) is a classical, smooth solution of a pde. This is mainly because the diffusion term in the factor equation (2.2) is non-degenerate. For more general formulations, viscosity solutions may have to be considered. The example in Section 6 illustrates how to solve the system (2.8).

3 Auxiliary results

In this section we shall formulate several technical results that will be needed later. Let $K(x)$ be a real valued function on R^n. Throughout this section it will be assumed that $K(x)$ has the following properties:

Assumption (B1)

(a) $K(x) \le 0$,

(b) $lim_{\|x\|\to\infty} K(x) = -\infty$,

(c) $|K(x)| \le c(1+\|x\|^2)$, *where c is a positive constant,*

(d) Let $K_i \subset R^n$, $i = 1,2,\ldots,I$, *I finite, be disjoint, open sets, such that* $\bigcup_{i=1}^{I} \bar{K}_i = R^n$, *where* $\bar{K}_i$ *is the closure of* K_i. *Then* $K(x)$ *is smooth on each of the* K_i*'s,*

(e) $K(x)$ *is locally Lipschitz on* R^n.

We begin by considering the following Cauchy problem:

$$
\begin{aligned}
\frac{\partial f(t,x)}{\partial t} &= (1/2)\sum_{i,j=1}^{n}\frac{\partial^2 f(t,x)}{\partial x_i \partial x_j}\sum_{k=1}^{m+n}\lambda_{ik}\lambda_{jk} \\
&+ (b+Bx)'grad_r f(t,r) + K(x)f(x,t), \\
f(0,x) &= 1, \qquad (3.1)
\end{aligned}
$$

for $t \in (0,T]$, $T < \infty$, and $x \in R^n$. For the convenience of the reader we proceed with a construction of the classical solution to (3.1), closely following [16] chapter IV. Let

$$K(x,\xi,t,\tau) := (b+Bx)' grad_x Z_0(x-\xi,\xi,t,\tau) + K(x) Z_0(x-\xi,\xi,t,\tau), \quad (x,\xi) \in R^n, \ 0 \le \tau < t \le T. \tag{3.2}$$

where $Z_0(x-\xi,\xi,t,\tau)$ is defined as in (IV.11.2) of [16], that is:

$$Z_0(x-\xi,\xi,t,\tau) = \frac{1}{[4\pi(t-\tau)]^{(\frac{n}{2})}((1/2) det \Lambda\Lambda')^{(1/2)}} exp\Big(- \frac{1}{4(t-\tau)} \sum_{i,j=1}^{n} \Lambda^{(i,j)}(x_i-\xi_i)(x_j-\xi_j) \Big),$$

for $(x,\xi) \in R^n$, $0 \le \tau < t \le T$, and where $\Lambda^{(i,j)}$ is the (i,j)'th element of the matrix inverse to $\Lambda\Lambda'$.

The following estimate easily follows from the above definition and the condition (B1)(c):

$$|K(x,\xi,t,\tau)| \le c_1(t-\tau)^{-\frac{n+1+\alpha}{2}} exp\{ - c_2 \frac{\|x-\xi\|^2}{t-\tau} \}, \tag{3.3}$$

for some positive constants c_1, c_2 independent of (x,ξ,t,τ), and $\alpha \in (0,1)$. This is the estimate (IV.11.17) in [16], and it implies that (compare (IV.11.25) in [16])

$$|K_m(x,\xi,t,\tau)| \le c_1^m (\pi/c_2)^{\frac{n(m-1)}{2}} \frac{\Gamma^m(\alpha/2)}{\Gamma(m\alpha/2)} (t-\tau)^{-\frac{m\alpha-n-2}{2}} \ exp\{ - c_2 \frac{\|x-\xi\|^2}{t-\tau} \}, \tag{3.4}$$

for $m \ge 1$, where

$$\begin{aligned} K_1(x,\xi,t,\tau) &= K(x,\xi,t,\tau) \\ K_m(x,\xi,t,\tau) &= \int_\tau^t d\lambda \int_{R^n} K(x,y,t,\lambda) K_{m-1}(y,\xi,\lambda,\tau) dy, \quad m \ge 2. \end{aligned}$$

Owing to (3.4) we can now define (as in (IV.11.23) of [16])

$$Q(x,\xi,t,\tau) := \sum_{m=1}^{\infty} (-1)^m K_m(x,\xi,t,\tau) \tag{3.5}$$

and derive the estimate (compare (IV.11.26) of [16])

$$|Q(x,\xi,t,\tau)| \le c_1(t-\tau)^{-\frac{n+1+\alpha}{2}} exp\{ - c_2 \frac{\|x-\xi\|^2}{t-\tau} \}. \tag{3.6}$$

Let now $Z(x,\xi,t,\tau)$ be given as in (11.13) of [16] for $x,\xi \in R^n$ and $0 \le \tau < t \le T$, that is:

$$Z(x,\xi,t,\tau) = Z_0(x-\xi,\xi,t,\tau) + \int_\tau^t d\lambda \int_{R^n} Z_0(x-y,y,t,\tau) Q(y,\xi,\lambda,\tau) dy. \tag{3.7}$$

Define

$$g(t,x) := \int_{R^n} Z(x,\xi,t,0)d\xi, \qquad (t,x) \in [0,T] \times R^n. \tag{3.8}$$

Using the above estimates and the formulas (11.8), (11.9) on page 358 of [16], it follows that

$$g \in C^{1,2}((0,T),R^n) \cap C([0,T],R^n). \tag{3.9}$$

and

$$g \in C^{1+\alpha/2,2+\alpha}((0,T),B_r) \cap C([0,T],R^n), \tag{3.10}$$

where $B_r = \{x \in R^n : \|x\| < r\}$, and $r > 0$ is arbitrary. In addition, we also have (see the discussion in section IV.14 of [16]) that g is a solution to the Cauchy problem (3.1). Finally, using the Feynmann-Kac formula (see e. g. [14]) we obtain the following stochastic representation for g:

$$g(t,x) = \mathbf{E}[e^{\int_0^t K(X(s))ds} | X(0) = x], \tag{3.11}$$

where $X(t)$ is our factor process. Since every smooth solution for the Cauchy problem (3.1) has the above representation, then g is the unique solution to the problem (3.1). In view of the conditions (B1)(a),(c), the representation (3.11) implies the following estimates for g:

$$0 < g(t,x) \leq 1, \quad (t,x) \in [0,T] \times R^n, \tag{3.12}$$

$$\frac{\partial g(t,x)}{\partial t} \leq 0, \quad (t,x) \in (0,T) \times R^n. \tag{3.13}$$

Let us now fix $\theta > 0$ and define

$$u_{\theta,T}(t,x) := -(2/\theta) ln\ g(t,x), \quad (t,x) \in [0,T] \times R^n. \tag{3.14}$$

The following lemma summarizes properties of $u_{\theta,T}$ that we will need.

Lemma 3.1 *Assume (A2) and (B1). Then the function $u_{\theta,T}$ defined by (3.14) enjoys the following properties:*

(a) $u_{\theta,T} \geq 0$,

(b) $\frac{\partial u_{\theta,T}}{\partial t} \geq 0$,

(c) $u_{\theta,T}$ is the only, non-negative, classical solution of

$$\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= (b+Bx)' grad_x u(t,x) \\
&+ (1/2)[(-\theta/2) \sum_{i,j=1}^{n} \frac{\partial u(t,x)}{\partial x_i} \frac{\partial u(t,x)}{\partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk} + \sum_{i,j=1}^{n} \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk}] \\
&- (2/\theta)K(x), \\
u(0,x) &= 0,
\end{aligned} \tag{3.15}$$

for $(t,x) \in (0,T] \times R^n$.

(d) $u_{\theta,T}(t,x)$ is three times continuously differentiable in x on $\cup_{i=1}^{I} K_i$.

Proof. Properties (a), (b), and (c) are obvious consequences of (3.12)-(3.14) and the fact that $g(t,x)$ is the unique solution to (3.1). The property (d) follows since $K(x)$ is smooth on $\cup_{i=1}^{I} K_i$. ∎

In view of the above lemma and the fact that $grad_x u_{\theta,T}$ and $\frac{\partial u_{\theta,T}}{\partial t}$ are continuous, we can now apply the argument used in the proof of Lemma 1.5 of Nagai [19] in order to obtain the following important estimate (compare (1.28) in [19]):

$$t\left(\|grad_x u_{\theta,T}\|^2 + \gamma \frac{\partial u_{\theta,T}}{\partial t} \right) \leq t K_{r,\gamma} + L_{r,\gamma}, \quad on \quad (0,T] \times B_r, \tag{3.16}$$

where γ, $K_{r,\gamma}$, and $L_{r,\gamma}$ are some positive constants that are independent of t and T.

We want to extend the solution $u_{\theta,T}(t,x)$ from $[0,T] \times R^n$ to $[0,\infty) \times R^n$. Towards this end, following the argument of the section 1.4 in Nagai [19] with $u_R(t,x) = u_1(t,x) = u_2(t,x) = u_{\theta,T}(t,x)$ on $[0,T] \times \bar{B}_R$ (using the notation of [19]), we arrive at the following result (compare Theorem 1.1 in [19]):

Lemma 3.2 *Assume (A2) and (B1). Then,*

(a) The equation

$$\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= (b+Bx)' grad_x u(t,x) \\
&+ (1/2)[(-\theta/2) \sum_{i,j=1}^{n} \frac{\partial u(t,x)}{\partial x_i} \frac{\partial u(t,x)}{\partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk} + \sum_{i,j=1}^{n} \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk}], \\
&- (2/\theta) K(x), \quad (t,x) \in (0,\infty) \times R^n, \\
u(0,x) &= 0, \quad x \in R^n,
\end{aligned} \tag{3.17}$$

has a non-negative solution $u_\theta \in C^{1,2}((0,\infty), R^n) \cap C([0,\infty), R^n)$. u_θ *is an extension of* $u_{\theta,T}$ *to* $[0,\infty) \times R^n$,

(b) $\frac{\partial u_\theta}{\partial t} \geq 0$,

(c)

$$t\left(\|grad_x u_\theta\|^2 + \gamma \frac{\partial u_\theta}{\partial t} \right) \leq t K_{r,\gamma} + L_{r,\gamma}, \quad on \quad (0,\infty) \times B_r, \tag{3.18}$$

for some positive constants γ, $K_{r,\gamma}$ *and* $L_{r,\gamma}$ *that are independent of* t.

We see now that Theorem 3.4 of Nagai [19] applies in our context. We state the version of this theorem, appropriate for the situation considered here, as

Lemma 3.3 *Assume (A2) and (B1). Then*

(a) As $t \to \infty$

the function $u_\theta(t,x) - u_\theta(t,0)$ *converges to a function* $\nu_\theta(x)$ *in* $W^1_{2,loc}$ *and uniformly on each compact subset of* R^n,

the function $\frac{\partial u_\theta(t,x)}{\partial t}$ *converges to a constant* ρ_θ, *(b) The pair $(\nu_\theta, \rho_\theta)$ is the unique solution to equation (ν_θ is unique up to an additive constant)*

$$\begin{aligned}
\rho &= (b+Bx)'grad_x\nu(x) \\
&- (\theta/4)\sum_{i,j=1}^{n}\frac{\partial\nu(x)}{\partial x_i}\frac{\partial\nu(x)}{\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk} + (1/2)\sum_{i,j=1}^{n}\frac{\partial^2\nu(x)}{\partial x_i\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk} \\
&- (2/\theta)K(x),
\end{aligned}$$

$$\nu(x)\in C^2(R^n),\ \lim_{\|x\|\to\infty}\nu(x)=\infty$$
$$\rho = const. \tag{3.19}$$

Since $\frac{\partial u_\theta(t,x)}{\partial t}$ converges to a constant ρ_θ, we thus have an obvious

Corollary 3.1

$$\lim_{t\to\infty}\frac{u_\theta(t,x)}{t}\equiv\rho_\theta. \tag{3.20}$$

4 Proofs of the main results

In this section we verify validity of the results stated in section 2. Assumptions (A1)-(A3) are supposed to hold throughout the section.

Fix $\theta>0$ and consider the following Bellman-Hamilton-Jacobi equation

$$\begin{aligned}
0 &= \inf_{h\in\chi}\Big[L^h\phi(t,x,v)\Big], \\
\phi(0,x,v) &= v^{-(\theta/2)},
\end{aligned}\tag{4.1}$$

for $t>0,\ x\in R^n,\ v>0$, where

$$\begin{aligned}
L^h\phi(t,x,v) &:= -\frac{\partial\phi(t,x,v)}{\partial t}+\frac{\partial\phi(t,x,v)}{\partial v}h'(a+Ax)v+(b+Bx)'grad_x\phi(t,x,v) \\
&+ (1/2)\frac{\partial^2\phi(t,x,v)}{\partial^2 v}h'\Sigma\Sigma'hv^2 \\
&+ (1/2)\sum_{i,j=1}^{n}\frac{\partial^2\phi(t,x,v)}{\partial_i^x\partial_j^x}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}.
\end{aligned}$$

We shall seek a solution to the equation (4.1) in the form

$$\Phi(t,x,v;\theta)=v^{-(\theta/2)}e^{-(\theta/2)U(t,x;\theta)}, \tag{4.2}$$

for some suitable function $U(t,x;\theta)$. To this end, let us consider first the following two equations

$$
\begin{aligned}
\frac{\partial U(t,x)}{\partial t} &= (b+Bx)' grad_x U(t,x) \\
&+ (1/2)[(-\theta/2)\sum_{i,j=1}^{n} \frac{\partial U(t,x)}{\partial x_i}\frac{\partial U(t,x)}{\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk} + \sum_{i,j=1}^{n}\frac{\partial^2 U(t,x)}{\partial x_i \partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}], \\
&- K_\theta(x), \quad (t,x) \in (0,\infty)\times R^n , \\
U(0,x) &= 0, \quad x \in R^n, \qquad (4.3)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \bar{U}(t,x)}{\partial t} &= (b+Bx)' grad_x \bar{U}(t,x) \\
&+ (1/2)[(-\theta/2)\sum_{i,j=1}^{n} \frac{\partial \bar{U}(t,x)}{\partial x_i}\frac{\partial \bar{U}(t,x)}{\partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk} + \sum_{i,j=1}^{n}\frac{\partial^2 \bar{U}(t,x)}{\partial x_i \partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk}], \\
&- \bar{K}_\theta(x), \quad (t,x) \in (0,\infty)\times R^n , \\
\bar{U}(0,x) &= 0, \quad x \in R^n \qquad (4.4)
\end{aligned}
$$

where

$$\bar{K}_\theta(x) = K_\theta(x) - K_\theta$$

and $K_\theta(x)$ and K_θ are defined in (2.5) and (2.9), respectively. We now have the following

Proposition 4.1 *The constant K_θ is finite, and the function $\bar{K}_\theta(x)$ satisfies assumption (B1).*

Proof. Invoking the results from section 5.5 of Bank et al. [1] we conclude that $K_\theta(x)$ satisfies conditions (c) and (d) of the assumption (B1). In view of this and the assumption (A2), we see that the constant K_θ is finite, and that $\bar{K}_\theta(x)$ satisfies conditions $(a)-(d)$ of the assumption (B1). Condition (e) is satisfied in view of the problem (11)(a) in [8]. ∎

From the above proposition and from lemma 3.2 it follows that there exists a non-negative classical solution $\bar{U}(t,x;\theta)$ to equation (4.4). Now, by letting

$$U(t,x;\theta) = \bar{U}(t,x;\theta) - tK_\theta, \ (t,x) \in [0,\infty)\times R^n \qquad (4.5)$$

we see that $U(t,x;\theta)$ is a classical solution to equation (4.3). We thus have the following

Proposition 4.2 *Let $\Phi(t,x,v;\theta)$ be as in (4.2), with $U(t,x;\theta)$ as in (4.5). Then $\Phi(t,x,v;\theta)$ is a classical solution to the Bellman-Hamilton-Jacobi equation (4.1).*

Proof. The result follows by direct inspection. ∎

We proceed with the statement and proof of a verification result, from which theorem 2.1 and corollary 2.1 follow immediately:

Proposition 4.3 *Let $\Phi_\theta(t,x,v)$ denote any classical solution to (4.1). For each $h(\cdot) \in \mathcal{H}$ we have*

$$\Phi_\theta(t,x,v) \le \mathbf{E}^{h(\cdot)}[V(t)^{-(\theta/2)}|V(0)=v, X(0)=x], \quad t \ge 0,\ (x,v) \in R^n \times (0,\infty), \tag{4.6}$$

For $h_\theta(\cdot)$ defined in (2.6) we have

$$\Phi_\theta(t,x,v) = \mathbf{E}^{h_\theta(\cdot)}[V(t)^{-(\theta/2)}|V(0)=v, X(0)=x], \quad t \ge 0,\ (x,v) \in R^n \times (0,\infty), \tag{4.7}$$

Proof. The results are clearly true for $t=0$.

Fix $t>0$ and a strategy $h(\cdot) \in \mathcal{H}$. Applying Ito formula to $\Psi_\theta(s,x,v) = \Phi_\theta(t-s,x,v)$ for $0 \le s \le t$, we get for each sufficiently small $\epsilon > 0$ the following equality

$$\begin{aligned}&\mathbf{E}^{h(\cdot)}[\Phi_\theta(\epsilon, X(t-\epsilon), V(t-\epsilon))|V(0)=v, X(0)=x] - \Phi_\theta(t,x,v) = \\ &\mathbf{E}^{h(\cdot)}[\int_\epsilon^t L^{h(\cdot)}\Phi_\theta(r, X(t-r), V(t-r))dr|V(0)=v, X(0)=x]\end{aligned} \tag{4.8}$$

for all $x \in R^n$ and $v>0$, where $L^{h(\cdot)}\Phi_\theta(s,X(s),V(s))$ is defined similarly as $L^h\Phi_\theta(s,X(s),V(s))$ with $h(s)$ substituting for h.

It follows from (4.1) that the expression on the right hand side of (4.8) is non-negative. Thus, letting ϵ go to zero, we obtain (4.6).

It follows from the results of section 5.5 in Bank et al. [1] that $H_\theta(x)$ (defined in Theorem 2.1) is a piece-wise affine function on R^n. Thus $h_\theta(\cdot)$ is an admissible strategy, and the conclusion (4.7) follows since the right hand side of (4.8) is equal to zero for $h(\cdot) \equiv h_\theta(\cdot)$. ∎

We are ready now to prove theorem 2.1.

Proof. of Theorem 2.1

Let $\Phi(t,x,v;\theta)$ be as in (4.2), with $U(t,x;\theta)$ as in (4.5). Then it follows from propositions 4.2 and 4.3 that $\Phi(t,x,v;\theta)$ is the unique solution to the Bellman-Hamilton-Jacobi equation (4.1), and that it satisfies (4.6) and (4.7). This implies (2.7). ∎

It remains to demonstrate theorem 2.2.

Proof. of Theorem 2.2 As in the proof of theorem 2.1 we first observe that $\Phi(t,x,v;\theta)$ is the unique solution to (4.1). Thus $\bar{U}(t,x.v;\theta)$ is the unique non-negative solution to (4.4), and we have

$$-(2/\theta)ln\Phi(t,x,v;\theta) = lnv + \bar{U}(t,x;\theta) - tK_\theta, \tag{4.9}$$

for $(t,x,v) \in [0,\infty) \times R^n \times (0,\infty)$. Applying lemma 3.3 and corollary 3.1 to the equations (2.8) and (4.4) we conclude that

$$\lim_{t\to\infty} \frac{\bar{U}(t,x;\theta)}{t} = \rho(\theta) + K_\theta. \tag{4.10}$$

The conclusions of theorem 2.2 follow now from (4.9) and (4.10) since $\Phi(t,x,v;\theta)$ satisfies (4.6) and (4.7). ∎

5 Risk neutral problem ($\theta = 0$)

In this section we are going to study limit when $\theta \downarrow 0$ of the problems $\mathbf{P}_\theta$. This leads to consideration of the classical problem of maximizing the portfolio's expected growth rate, or the growth rate of the log-utility function (see e.g.[12]). We label this problem as $\mathbf{P}_0$, and formulate as follows:

maximize the expected growth rate

$$J_0(v, x; h(\cdot)) := \liminf_{t\to\infty} t^{-1}\, \mathbf{E}^{h(\cdot)}\, [lnV(t)|V(0) = v, X(0) = x] \tag{5.1}$$

over the class of all admissible investment processes $h(\cdot)$,

subject to (2.2) and (2.3).

We intend to establish a relationship between the *risk neutral problem* $\mathbf{P}_0$ and the *risk sensitive problems* $\mathbf{P}_\theta, \theta > 0$. Towards this end let us consider the following equation

$$\begin{aligned}
\rho(0) &= (b + Bx)' grad_x v_0(x) + (1/2) \sum_{i,j=1}^{n} \frac{\partial^2 v_0(x)}{\partial x_i \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk} \\
&- K_0(x), \\
& v_0(x) \in C^2(R^n), \quad \lim_{\|x\|\to\infty} v_0(x) = \infty \\
& \rho(0) = const,
\end{aligned} \tag{5.2}$$

and introduce the following assumption

Assumption C1

$$\lim_{\|x\|\to\infty} K_0(x) = -\infty.$$

We now have the following two results,

Lemma 5.1 *Assume (A1), (A3) and (C1). Then there exists a solution pair $(\rho(0), v_0)$ to the equation (5.2).*

Proof. See Appendix 1. ∎

Proposition 5.1 *Let $H_0(x)$ be a minimizing selector on the right hand side of (5.2). Define a strategy $h_0(\cdot)$ as in (2.6) with 0 replacing θ. If $(\rho(0), v_0)$ is a solution to (5.2) then we have*

$$J_0(v, x; h_0(\cdot)) = \rho(0) - \liminf_{t\to\infty} t^{-1}\, \mathbf{E}^{h_0(\cdot)}\, [v_0(X(t))|V(0) = v, X(0) = x]. \tag{5.3}$$

Proof. It follows from the results of section 5.5 in Bank et al. [1] that $h_0(\cdot)$ is an admissible strategy. Applying Ito formula to $v_0(x)$ we obtain

$$\begin{aligned} v_0(X(t)) - v_0(v) &= \int_0^t [(b+Bx)' grad_x v_0(X(s)) + (1/2)\sum_{i,j=1}^{n} \frac{\partial^2 v_0(X(s))}{\partial x_i \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk}] ds \\ &- \int_0^t grad_x' v_0(X(s)) \Lambda dW(s) \\ &= t\rho(0) + \int_0^t [(1/2) h_0'(X(s)) \Sigma\Sigma' h_0(X(s)) - h_0'(X(s))(a + AX(s))] ds \\ &- \int_0^t grad_x' v_0(X(s)) \Lambda dW(s), \quad t \geq 0. \end{aligned} \tag{5.4}$$

Under the strategy $h_0(\cdot)$ the capital process $V(t)$ is the geometric Brownian motion (see e.g. [14], p. 361)

$$\begin{aligned} V(t) &= v \cdot exp\Big\{ - \int_0^t [(1/2) h_0'(X(s)) \Sigma\Sigma' h_0(X(s)) - h_0'(X(s))(a + AX(s))] ds \\ &+ \int_0^t h_0'(X(s)) \Sigma dW(s) \Big\}, \quad t \geq 0. \end{aligned}$$

Therefore, we obtain from (5.4)

$$\begin{aligned} & \mathbf{E}^{h_0(\cdot)} \, [v_0(X(t)) | V(0) = v, X(0) = x] - v_0(v) \\ =\; & t\rho(0) + lnv - \mathbf{E}^{h_0(\cdot)} \, [lnV(X(t)) | V(0) = v, X(0) = x], \; t \geq 0, \end{aligned} \tag{5.5}$$

which proves (5.2). ∎

Before we state our next result we need to introduce

Assumption C2 *For each solution $(\rho(0), v_0)$ of the equation (5.2), and each $h(\cdot) \in \mathcal{H}$ we assume that*

$$\liminf_{t\to\infty} (1/t) \mathbf{E}^{h(\cdot)} \, [v_0(X(t)) | V(0) = v, X(0) = x] = 0.$$

In addition, we assume that lim inf *can be replaced with* lim *in this equality when $h(\cdot) \equiv h_\theta(\cdot)$.*

Remark 5.1 *Assumption C2 will most likely be satisfied in case the factor process $X(t)$ is stable (B is negative definite). In the next section we shall study an example of a market situation for which this assumption holds.*

Proposition 5.2 *Let $(\rho(0), v_0)$ be a solution of (5.2). Assume C2. Then, the following conclusions hold:*

(a) The strategy $h_0(\cdot)$ is optimal for $\mathbf{P}_0$, and

$$J_0(v, x; h_0(\cdot)) = \lim_{t\to\infty} t^{-1} \, \mathbf{E}^{h_0(\cdot)} \, [lnV(t) | V(0) = v, X(0) = x] = \rho(0). \tag{5.6}$$

(b) The constant $\rho(0)$ is unique.

Proof. Fix an arbitrary admissible strategy $h(\cdot)$. Applying Ito formula to v_0 and using (5.2) we get

$$\begin{aligned} v_0(X(t)) - v_0(v) &= \int_0^t [(b+Bx)' grad_x v_0(X(s)) + (1/2)\sum_{i,j=1}^{n} \frac{\partial^2 v_0(X(s))}{\partial x_i \partial x_j} \sum_{k=1}^{n+m} \lambda_{ik}\lambda_{jk}]ds \\ &- \int_0^t grad_x' v_0(X(s))\Lambda dW(s) \\ &\leq t\rho(0) + \int_0^t [(1/2)h'(X(s))\Sigma\Sigma' h(X(s)) - h'(X(s))(a + AX(s))]ds \\ &- \int_0^t grad_x' v_0(X(s))\Lambda dW(s), \quad t \geq 0. \end{aligned} \tag{5.7}$$

Thus we conclude that

$$\begin{aligned} &\mathbf{E}^{h(\cdot)}\ [v_0(X(t))|V(0) = v, X(0) = x] - v_0(x) \\ \leq\ & t\rho(0) - \mathbf{E}^{h(\cdot)}\ [lnV(X(t))|V(0) = v, X(0) = x],\ t \geq 0. \end{aligned} \tag{5.8}$$

This proves (a) in view of the assumption C2 and the results of proposition 5.1, since $h(\cdot)$ was selected arbitrarily. Uniqueness of $\rho(0)$ follows from its stochastic representation in (5.6). ∎

Remark 5.2 *(a) Note that assumption C2 is not needed to prove optimality of* $h_0(\cdot)$ *in proposition 5.2, only that the* lim inf *is actually an ordinary* lim *equal to* $\rho(0)$.
(b) In case when $\Sigma\Sigma'$ *is positive definite the optimal strategy* $h_0(\cdot)$ *agrees with the one derived by Karatzas (see (9.19) in [12]).*

In order to relate the risk sensitive problems $\mathbf{P}_\theta$, $\theta > 0$ and the risk-neutral problem $\mathbf{P}_0$ we need to introduce the following assumption:

Assumption C3 *The constants* $\rho(\theta)$, $\theta > 0$, *in the solutions to equation (2.8) converge to the constant* $\rho(0)$ *in the solution to equation (5.2) when* θ *converges to zero.*

Remark 5.3 *Assumption C3 is quite natural. It is satisfied for the example considered in section 6. We believe that it is satisfied for equations (2.8) and (5.2) in the general framework of this paper, but we are unable to verify this at this time.*

We now have the following proposition, which says that the optimal objective values for problems (P_θ) converge to the optimal objective value for the risk neutral problem P_0 when the risk-aversion parameter decays to zero:

Proposition 5.3 *Assume Assume (A1)-(A3) and (C1)-(C3). Then,*

$$\begin{aligned} &\lim_{\theta\downarrow 0} \max_{h(\cdot)\in\mathcal{H}} \left[\lim_{t\to\infty} (-2/\theta) t^{-1} ln\ \mathbf{E}^{h(\cdot)}\ [e^{-(\theta/2) lnV(t)}|V(0) = v, X(0) = x] \right] \\ &= \max_{h(\cdot)\in\mathcal{H}} \left[\lim_{t\to\infty} t^{-1}\ \mathbf{E}^{h(\cdot)}\ [lnV(t)|V(0) = v, X(0) = x] \right]. \end{aligned} \tag{5.9}$$

Proof. It follows from corollary 2.1 and theorem 2.1 that the left hand side of (5.9) is equal to the $\lim_{\theta\downarrow 0}\rho(\theta)$. Proposition 5.2 implies that the right hand side of (5.9) is equal to $\rho(0)$. This proves the result in view of the assumption C3. ∎

The following result characterizes the portfolio expected growth rate corresponding to the optimal investment strategy for the risk aversion level $\theta > 0$ and will be used in section 6.

Lemma 5.2 *Assume (A3). Fix $\theta > 0$. Let $H_\theta(x)$ be as in theorem 2.1 and assume that*

$$\lim_{\|x\|\to\infty}[(1/2)H_\theta(x)'\Sigma\Sigma' H_\theta(x) \; - \; H_\theta(x)'(a+Ax)] = -\infty. \tag{5.10}$$

Consider the equation

$$\begin{aligned}
\rho_\theta &= (b+Bx)'grad_x v_{\theta,0}(x) + (1/2)\sum_{i,j=1}^{n}\frac{\partial^2 v_{\theta,0}(x)}{\partial x_i \partial x_j}\sum_{k=1}^{n+m}\lambda_{ik}\lambda_{jk} \\
&- [(1/2)H_\theta(x)'\Sigma\Sigma' H_\theta(x) \; - \; H_\theta(x)'(a+Ax)], \\
& v_{\theta,0}(x) \in C^2(R^n), \lim_{\|x\|\to\infty} v_{\theta,0}(x) = \infty \\
& \rho_\theta(0) = const.
\end{aligned} \tag{5.11}$$

Then, there exist solution $(\rho_\theta, v_{\theta,0})$ to the above equation, the constant ρ_θ, is unique, and we have

$$J_0(v, x; h_\theta(\cdot)) = \rho_\theta. \tag{5.12}$$

for all $(v,x) \in (0,\infty)\times R^n$, where $h_\theta(\cdot)$ is defined as in (2.6).

Proof. The proof is analogous to the proofs of proposition 5.2 and therefore is omitted. ∎

We shall see in section 6 that the condition (5.10) is sufficient but not necessary.

6 Example: Asset Allocation With Vasicek Interest Rates

In this section we present a simple example which not only illustrates the ideas developed in the preceding sections, but also is of independent interest in its own right. We study a model of an economy where the mean returns of the stock market are affected by the level of interest rates. Consider a single risky asset, say a stock index, that is governed by the SDE

$$\frac{dS_1(t)}{S_1(t)} = (\mu_1 + \mu_2 r(t))dt + \sigma dW_1(t), \quad S_1(0) = s,$$

where the spot interest rate $r(\cdot)$ is governed by the classical "Vasicek" process

$$dr(t) = (b_1 + b_2 r(t))dt + \lambda dW_2(t), \quad r(0) = r > 0.$$

Here μ_1, μ_2, b_1, b_2, σ, and λ are fixed, scalar parameters, to be estimated, while W_1 and W_2 are two independent Brownian motions. We assume $b_1 > 0$ and $b_2 < 0$ in all that follows.

The investor can take a long or short position in the stock index as well as borrow or lend money, with continuous compounding, at the prevailing interest rate. It is therefore convenient to follow the common approach and introduce the "bank account" process S_2, where

$$\frac{dS_2(t)}{S_2(t)} = r(t)dt.$$

Thus $S_2(t)$ represents the time-t value of a savings account when $S_2(0) = 1$ dollar is deposited at time-0. This enables us to formulate the investor's problem as in the preceding sections, for there are $m = 2$ securities S_1 and S_2, there is $n = 1$ factor $X = r$, and we can set $b = b_1$, $B = b_2$, $a = (\mu_1, 0)'$, $A = (\mu_2, 1)', \Lambda = (0, 0, \lambda)'$, and

$$\Sigma = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With only two assets it is convenient to describe the investor's trading strategy in terms of the scalar valued function $H_\theta(r)$, which is interpreted as the proportion of capital invested in the stock index, leaving the proportion $1 - H_\theta(r)$ invested in the bank account. We suppose for simplicity that there are no special restrictions (e.g., short sales constraints, borrowing restrictions, etc.) on the investor's trading strategy, so the investment constraint set χ is taken to be the whole real line.

In view of Theorem 2.1 and Corollary 2.1, the optimal trading strategy is easy to work out. With (see (2.5))

$$K_\theta(r) = \inf_{h \in R} [(1/2)(\theta/2 + 1)(h, 1 - h)\Sigma\Sigma'(h, 1 - h)' - (h, 1 - h)(a + Ar)],$$

it follows that the optimal trading strategy is $h_\theta(t) = [\tilde{h}_\theta(t), 1 - \tilde{h}_\theta(t)]'$ where $\tilde{h}_\theta(t) = H_\theta(r(t))$, and where

$$H_\theta(r) = \frac{\mu_1 + \mu_2 r - r}{(\theta/2 + 1)\sigma^2},$$

in which case

$$K_\theta(r) = -r - \frac{(\mu_1 + \mu_2 r - r)^2}{(\theta + 2)\sigma^2}.$$

It is interesting to note the obvious similarity between this optimal strategy and the well known results (see [18] and [12]) for the case of conventional complete models of securities markets and power utility functions. In particular, when $\mu_2 = 0$, so the mean returns of the stock market are independent of the interest rates, the expressions for the trading strategies are identical. Another special case of interest is when $\mu_2 = 1$, so that the "market risk premium" $(\mu_1 + \mu_2 r - r)/\sigma$ is constant. Here the results are somewhat boring, in that H_θ is constant with respect to r and $K_\theta(r)$ is linear in r.

More interesting is the study of $\rho(\theta)$, our measure of performance under the optimal trading strategy (see Theorem 2.2). In view of (2.8) this is obtained as part of the solution (ρ, v) of the equation

$$\rho = \frac{1}{2}\lambda^2 v''(r) + (b_1 + b_2 r)v'(r) - (\theta/4)\lambda^2(v'(r))^2 - K_\theta(r), \tag{6.1}$$

where v is a unique (up to a constant) function satisfying $\lim_{|r|\to\infty} v(r) = \infty$. To solve this, we conjecture that a solution is obtained with v having the quadratic form

$$v(r) = \alpha r^2 + \beta r + \gamma$$

for suitable constants α, β, and γ. Substituting this and the expression for $K_\theta(r)$ into (6.1) and then collecting terms, we see that the quadratic terms cancel out if and only if

$$\lambda^2\theta\alpha^2 - 2b_2\alpha - \frac{(\mu_2 - 1)^2}{(\theta+2)\sigma^2} = 0.$$

This quadratic equation in α has two roots, one of which is positive, while the other is negative. However, the requirement that $\lim_{|r|\to\infty} v(r) = \infty$ is satisfied only for the positive root, so recalling our assumption that $b_2 < 0$ it follows that for the value of α we should take (for future purposes it is convenient to denote the dependence on θ and λ)

$$\alpha(\theta,\lambda) = \frac{b_2 + \sqrt{b_2^2 + \theta\lambda^2(\mu_2-1)^2/[(\theta+2)\sigma^2]}}{\lambda^2\theta}. \tag{6.2}$$

The linear terms on the right hand side of (6.1) cancel if and only if the value of β is

$$\beta(\theta,\lambda) = \frac{1 + 2\mu_1(\mu_2-1)/[(\theta+2)\sigma^2] + 2b_1\alpha(\theta)}{\sqrt{b_2^2 + \theta\lambda^2(\mu_2-1)^2/[(\theta+2)\sigma^2]}}. \tag{6.3}$$

Thus (6.1) does indeed have a solution with v as indicated; this solution is unique up to the constant γ, the value of which does not matter. The value of $\rho(\theta,\lambda)$ will then equal the remaining terms on the right hand side of (6.1), that is,

$$\rho(\theta,\lambda) = \lambda^2\alpha(\theta) + b_1\beta(\theta) - \frac{\lambda^2\theta[\beta(\theta)]^2}{4} + \frac{\mu_1^2}{(\theta+2)\sigma^2}. \tag{6.4}$$

Remark 6.1 *Note that the above results are valid also in the case when $\mu_2 = 1$. The assumption (A2) is not satisfied in this case since $\lim_{r\to-\infty} K_\theta(r) = \infty$.*

It is interesting to consider the risk neutral case, because here $\rho(0)$ will turn out to be the long-run expected growth rate under the strategy that is optimal when $\theta = 0$. Using L'Hospital's rule we compute the limits

$$\alpha(0,\lambda) = \lim_{\theta\downarrow 0}\alpha(\theta,\lambda) = -\frac{(\mu_2-1)^2}{4b_2\sigma^2},$$

$$\beta(0,\lambda) = \lim_{\theta\downarrow 0}\beta(\theta,\lambda) = \frac{b_1(\mu_2-1)^2}{2b_2^2\sigma^2} - \frac{1}{b_2} - \frac{\mu_1(\mu_2-1)}{b_2\sigma^2},$$

and

$$\rho(0,\lambda) = \lim_{\theta \downarrow 0} \rho(\theta,\lambda) = -\frac{b_1}{b_2} + \frac{[\mu_1 - (b_1/b_2)(\mu_2 - 1)]^2}{2\sigma^2} - \frac{\lambda^2(\mu_2 - 1)^2}{4b_2\sigma^2}. \tag{6.5}$$

Note that each of the three terms is non-negative. The Vasicek interest rate has a limiting distribution with a mean equal to the so-called "mean reversion" level $-b_1/b_2$, which is the first term. The second term equals the contribution to the long-run expected growth rate due to trading in the stock index, assuming the interest rate is the constant mean reversion level. The third term equals the contribution to the long-run expected growth rate due to the volatility of the interest rate.

Another quantity of interest is the long-run expected growth rate which results from using the strategy $h_\theta(t)$ that is optimal for a particular value of θ, a quantity that will be denoted by $\rho_\theta(\lambda)$. Of course, $\rho_0(\lambda) = \rho(0,\lambda)$, which is given by (6.5), whereas for $\theta > 0$ we use lemma 5.2 and obtain the quantity $\rho_\theta(\lambda)$ by solving for the constant ρ and the function v such that $\lim_{|r|\to\infty} v(r) = \infty$ and

$$\begin{aligned} \rho &= \frac{1}{2}\lambda^2 v''(r) + (b_1 + b_2 r)v'(r) - \Big[(1/2)(H_\theta(r), 1 - H_\theta(r))\Sigma\Sigma(H_\theta(r), 1 - H_\theta(r))' \\ &- (H_\theta(r), 1 - H_\theta(r))(a + Ar)\Big]. \end{aligned} \tag{6.6}$$

We solve (6.6) in exactly the same way as (6.1), obtaining

$$\rho_\theta(\lambda) = -\frac{b_1}{b_2} + \frac{2(\theta+1)}{(\theta+2)^2\sigma^2}\left[[\mu_1 - (b_1/b_2)(\mu_2 - 1)]^2 - \frac{\lambda^2(\mu_2 - 1)^2}{2b_2}\right]. \tag{6.7}$$

Note that the second and third terms, respectively, of (6.5) and (6.7) differ by the factor $4(\theta + 1)/(\theta+2)^2$. This factor is strictly less than one for all $\theta > 0$, so $\rho(0,\lambda) > \rho_\theta(\lambda)$ for all $\theta > 0$. Thus the optimal expected growth rate when $\theta = 0$ is greater than when θ is positive, as anticipated.

Remark 6.2 *At this point we want to emphasize one more time that the main advantage of the risk-sensitive approach to dynamic asset allocation over the classical log-utility approach is that the risk-sensitive approach provides an optimal compromise between maximization of the capital expected growth rate and controlling the investment risk, given the investor's attitude towards risk encoded in the value of* θ. *Even though the long-run expected growth rate of the capital under* $h_0(\cdot)$ *is greater than under* $h_\theta(\cdot)$, *if* $\theta > 0$, *the asymptotic risk of investment decreases with the increasing values of* θ *(see the discussion below, as well as our numerical results that conclude this section). In Appendix 2 we demonstrate that the asymptotic ratio of "up-side chance" to "down-side risk" is maximal for some* θ *positive, which provides yet another justification of the superiority of the risk-sensitive approach over the log-utility one.*

Still another quantity of interest is $(4/\theta)[\rho_\theta - \rho(\theta)]$ which, by equation (1.1) can be interpreted as an estimate of the asymptotic variance of $lnV(t)$ under the strategy that is optimal for the particular value of θ. In general, this is a messy formula when expressed in terms of the original data; no simplifications seem possible. However, there is interest in computing $\rho'(0,\lambda) := \frac{\partial^+\rho(\theta,\lambda)}{\partial\theta}|_{\theta=0}$, because when $\theta = 0$ the asymptotic variance under the optimal trading strategy will be $\lim_{\theta\downarrow 0}(4/\theta)[\rho_\theta - \rho(\theta)] = -4\rho'(0,\lambda)$. After lengthy, tedious calculations using L'Hospital's rule and so forth, we obtained

$$\begin{aligned}\rho'(0,\lambda) &= -\frac{[\mu_1-(b_1/b_2)(\mu_2-1)]^2}{4\sigma^2}\\ &+ \frac{\lambda^2(\mu_2-1)^2}{32b_2^3\sigma^4}[\lambda^2(\mu_2-1)^2+4b_2^2\sigma^2] - \frac{\lambda^2}{4b_2^2\sigma^4}[\sigma^2+\mu_1(\mu_2-1)-(b_1/b_2)(\mu_2-1)^2]^2.\end{aligned}$$

Note that each of the three terms is non-positive, as desired.

Our various calculations can be reconciled with classical results (e.g., see [18] and [12]) by considering various limits as the data parameter $\lambda \to 0$. This is because in the long-run when $\lambda = 0$ the interest rate is essentially equal to the constant mean reverting value $-b_1/b_2$, in which case the drift coefficient in the SDE for the stock index is the constant $\mu_1 - \mu_2 b_1/b_2$. Hence, for instance, we have

$$\lim_{\lambda\to 0}\alpha(\theta,\lambda) = -\frac{(\mu_2-1)^2}{2(\theta+2)b_2\sigma^2},$$

$$\lim_{\lambda\to 0}\beta(\theta,\lambda) = -\frac{1}{b_2} - \frac{2\mu_1(\mu_2-1)-(b_1/b_2)(\mu_2-1)^2}{b_2(\theta+2)\sigma^2},$$

$$\lim_{\lambda\to 0}\rho(\theta,\lambda) = -\frac{b_1}{b_2} + \frac{[\mu_1-(b_1/b_2)(\mu_2-1)]^2}{(\theta+2)\sigma^2},$$

and

$$\lim_{\lambda\to 0}\rho_\theta(\lambda) = -\frac{b_1}{b_2} + \frac{2(\theta+1)[\mu_1-(b_1/b_2)(\mu_2-1)]^2}{(\theta+2)^2\sigma^2}.$$

We conclude this section with some numerical calculations that are intended to generate some economic intuition about our asset allocation problem. Throughout we envision time units in years and set $b_1 = 0.05$ and $b_2 = -1$, so the mean reverting interest rate is 5% per annum. We also set $\mu_1 = 0.1 + (b_1/b_2)\mu_2$ so that the stock index's drift coefficient is always 0.1 whenever the interest rate is at the mean reverting level. Finally, the volatility parameter for the stock index is always taken to be $\sigma = 0.2$. Thus if the interest rate is fixed at the mean reverting level, then the stock index evolves like ordinary geometric Brownian motion and has a long run expected growth rate equal of 8% per annum and an asymptotic variance equal to 0.04.

This leaves two unspecified parameters: λ and μ_2. For Figures 1 - 3 we fix $\lambda = 0.02$ and consider the effect of the interest rate sensitivity parameter μ_2.

Figure 1 shows three graphs of the function $\rho(\theta)$ corresponding to, from top to bottom, respectively, $\mu_2 = -1$, $\mu_2 = 0$, and $\mu_2 = 1$. The numerical values are expressed as percentages; the value of θ varies between 0 and 6.0. Although the function $\rho(\theta)$ involves the factor $(\mu_2 - 1)$ raised to the first power, it turns out for our chosen parameters that the value of $\rho(\theta)$ when $\mu_2 = 1 + \delta$ is not much different when $\mu_2 = 1 - \delta$, for all $\delta > 0$ and $\theta > 0$. Hence, roughly speaking, the greater the sensitivity of the stock index risk premium $(\mu_1 + \mu_2 r - r)/\sigma$ to the interest rate, the greater the optimal objective value $\rho(\theta)$.

Figure 2 shows three graphs of the function ρ_θ corresponding to, from top to bottom, respectively, $\mu_2 = -1$, $\mu_2 = 0$, and $\mu_2 = 1$. It is interesting to compare these values with 8%, the long run expected growth rate of the stock index itself when the interest rate is fixed at the mean reverting level. Note that μ_2 enters the equation for ρ_θ only as part of the factor $(\mu_2 - 1)^2$.

Figure 3 shows three graphs of the estimated asymptotic variance corresponding to, from top to bottom, respectively, $\mu_2 = -1$, $\mu_2 = 0$, and $\mu_2 = 1$. Plotted is the quantity $(4/\theta)[\rho_\theta - \rho(\theta)]$, with ρ_θ and $\rho(\theta)]$ expressed as percentages. It is interesting to compare these values with 4.0, the asymptotic variance for the stock index itself when the interest rate is fixed at the mean reverting level. As with ρ_θ, the estimated asymptotic variances are more sensitive to the value of $|\mu_2 - 1|$ than to the value of $|\mu_2|$ itself.

For Figures 4 - 6 we fix $\mu_2 = 0$ and study the effect of the interest rate volatility parameter λ. Figures 4 - 6 show graphs of $\rho(\theta), \rho_\theta$, and the estimated asymptotic variances, respectively. Each figure shows three graphs, corresponding to three different values of λ : 0, 0.02, and 0.04. In all three figures, the bigger the value of λ, the bigger the value of the corresponding function. Hence it seems that the greater the volatility of the interest rate, the greater the investment opportunities, although these opportunities will be accompanied by greater volatilities.

7 Final remarks and future research

In the literature, the risk-sensitive control problem is frequently converted to an equivalent zero-sum game problem (e.g., see Fleming and McEneaney [9] and the references therein). We did not follow the game theoretic approach in this paper, but perhaps it will be interesting for the reader to consider how one can approach our problem using a zero-sum game formulation. We shall briefly present one possible formulation, and the reader is referred to [9] for comparisons.

The main observation is that the Hamilton-Jacobi-Bellman equations (4.3) and (2.8) can be rewritten as Isaacs equations corresponding to some zero-sum stochastic differential games. Let us consider the equation (4.3) first, and rewrite its version with t restricted to the interval $[0, T]$ as

$$\begin{aligned}
0 &= -\frac{\partial U(t,x,\theta)}{\partial t} + (b + Bx)' grad_x U(t,x,\theta) + (1/2) \sum_{i,j=1}^{n} \frac{\partial^2 U(t,x,\theta)}{\partial x_i \partial x_j} \sum_{k=1}^{m+n} \lambda_{ik}\lambda_{jk} \\
&- \sup_{k \in R^n} \inf_{(h \in \chi, 1'h=1)} \Big[grad_x' U(t,x,\theta) k - (1/\theta) k'(\Lambda\Lambda')^{-1} k \\
&+ (1/2)(\theta/2+1) h'\Sigma\Sigma' h - h'(a + Ax) \Big], (t,x) \in (0,T] \times R^n \\
U(0,x) &= 0, \quad x \in R^n. \qquad (7.1)
\end{aligned}$$

This equation is an optimality equation corresponding to the zero-sum stochastic differential game

$$\begin{aligned}
&\inf_{k(\cdot)\in\mathcal{K}} \sup_{h(\cdot)\in\mathcal{H}} \mathbf{E}\Big[\int_0^T \Big\{ h'(s)(a + A\zeta(s)) - (\theta/2+1)(1/2) h'(s)\Sigma\Sigma' h(s) \\
&+ (1/\theta) k'(s)(\Lambda\Lambda')^{-1} k(s) \Big\} ds | \zeta(0) = x \Big], \qquad (7.2)
\end{aligned}$$

subject to,

$$d\zeta(t) = [b + B\zeta(t) + k(t)]dt + \Lambda dW(t), \quad \zeta(0) = x \in R^n . \tag{7.3}$$

where $\mathcal{K}$ is some suitably defined set of admissible controls $k(\cdot)$ for the minimizing player.

Similarly, equation (2.8) can be rewritten as

$$\begin{aligned} \rho &= (b + Bx)' grad_x v(x) + (1/2) \sum_{i,j=1}^{n} \frac{\partial^2 v(x)}{\partial x_i \partial x_j} \sum_{k=1}^{m+n} \lambda_{ik}\lambda_{jk} \\ &- \sup_{k \in R^n} \inf_{(h \in \chi, 1'h=1)} \Big[grad_x' v(x) k - (1/\theta) k' (\Lambda\Lambda')^{-1} k \\ &+ (1/2)(\theta/2 + 1) h' \Sigma\Sigma' h - h'(a + Ax) \Big] \\ & v(x) \in C(R^n), \lim_{\|x\| \to \infty} v(x) = \infty \\ & \rho = const. \end{aligned} \tag{7.4}$$

This is an optimality equation corresponding to a zero-sum stochatic differential game of the form

$$\begin{aligned} &\inf_{k(\cdot)\in\mathcal{K}} \sup_{h(\cdot)\in\mathcal{H}} \liminf (T^{-1}) \mathbf{E}\Big[\int_0^T \Big\{ h'(s)(a + A\zeta(s)) - (\theta/2 + 1)(1/2) h'(s)\Sigma\Sigma' h(s) \\ &+ (1/\theta) k'(s)(\Lambda\Lambda')^{-1} k(s) \Big\} ds | \zeta(0) = x \Big], \end{aligned} \tag{7.5}$$

subject to (7.3).

The game theoretic perspective on risk sensitive control problems is also closely related to so-called *robust* or H^∞ control principles (e.g., see [23], [9] for discussions).

As for future research, it is important to study risk sensitive investment problems with partial information. Typically, the values of the market parameters a, A, , Σ, b, B and Λ are not known to an investor. Therefore, the optimal risk sensitive investment theory developed in this paper cannot be directly applied in the practice of dynamic asset management. What are the alternatives then? One possibility is that the investor obtains initial estimates of the market parameters based on historical time series, and then holds onto these estimates throughout the entire future investment horizon. A potentially better approach for the investor would be to *adaptively* select her or his investment decisions based on currently available market information and the optimal decision strategies (perhaps the ones developed in this paper). This means that the estimates of market parameters are updated as time goes by and new market information is acquired, and subsequently the updated estimates are used instead of the "true" values of those market parameters in the formulas for optimal risk sensitive investment rules.

It is assumed that security prices and the values of the factors can be accurately observed by the investor. It is also assumed that the market volatility parameter matrix Σ is known with a reasonably good accuracy. Therefore, as it is clear from (2.5), the investor needs to update primarily her or his estimates regarding the immediate return parameters a and A in order to compute updated versions of the risk sensitive investment rule $H_\theta(x)$.

Let $H_\theta(x; a, A)$ denote the rule computed from (2.5) (this notation is used in order to emphasize its dependence on the parameters a and A). Let also $\tilde{h}_\theta(t)$ denote the adaptive investment process defined by

$$\tilde{h}_\theta(t) = H_\theta(X(t); a(t), A(t)), \tag{7.6}$$

where $a(t)$ and $A(t)$ are estimates of a and A based on market information available through time t, and $X(t)$ is our factor process.

Let $\bar{X}(t) = [1,\ X(t)]'$, and let $\mathcal{R}_i(t)$ denote the i-th return process corresponding to (2.1), that is

$$\mathcal{R}_i(t) = \int_0^t \frac{1}{S_i(u)} dS_i(u), \quad i = 1, 2, \ldots, m\ . \tag{7.7}$$

In a future paper we shall investigate adaptive risk sensitive investment strategies based on the following estimation scheme:

$$[a(t),\ A(t)]' = \Phi^{-1}(t) M(t)\ , \tag{7.8}$$

where

$$\begin{aligned} \Phi(t) &= \int_0^t \bar{X}(s)\bar{X}'(s)ds + I_{n+1}\ , \\ I_{n+1} & \quad \text{is the } (n+1) \times (n+1) \text{ identity matrix}\ , \\ M(t) &= \int_0^t \bar{X}(s) d\mathcal{R}'(s)\ , \\ \mathcal{R}'(t) &= [\mathcal{R}_1(t), \ldots, \mathcal{R}_m(t)]. \end{aligned}$$

Mathematical questions that arise here include:

- Asymptotic study of the estimation scheme (7.9). For example, we would like to know whether the estimates (7.9) are strongly consistent (i.e. whether they almost surely converge to $[a\ A]$ as $t \to \infty$).
- Asymptotic study of the adaptive investment process (7.6) corresponding to (7.8). We would like to know whether this process is optimal for (P_θ) , that is whether

  $$J_\theta(v, x; h_\theta(\cdot)) = J_\theta(v, r; \tilde{h}_\theta(\cdot))\ .$$

  This is not an unreasonable expectation due to the time averaging in (2.4).
- Analysis of discretization and computational schemes for (7.8). We would like to effectively use financial data collected in our data bases.

In our study of the problems described in this task we intend to use some of the ideas developed in [3], [2], and [24] [2].

Note that the estimation scheme (7.8) is a very simple one. This, combined with (2.5) should lead to development of practically feasible algorithms for a real-time dynamic asset management, as postulated in the Introduction.

8 Appendix 1: Proof of Lemma 5.1

9 Appendix 2: Up-side Chance and Down-side Risk

Let us consider a special case of the price and factor model considered in section 6. Specifically, let us assume the following

- $\mu_2 = 0$,
- $b_1 = b_2 = \lambda = 0$, and
- $\mu_1 > r$.

Observe that we have assumed a constant spot rate (that is, $r(t) = r$ for all t) and a constant rate of return (μ_1) on the stock index. We thus have a very conventional, two-asset model. Our assumptions are imposed in order to simplify the following development of a new interpretation of the risk sensitive optimality criterion. In particular, we will analyze dependence on $\theta \geq 0$ of the following quantity,

$$R(\theta) := \lim_{t\to\infty}(1/t)ln\frac{\mathbf{P}^{h_\theta(\cdot)}((1/t)lnV(t) > \rho(0))}{\mathbf{P}^{h_\theta(\cdot)}((1/t)lnV(t) < r)}. \tag{9.1}$$

To interpret this quantity, using the strong law of large numbers for Brownian motion process it can easily be shown that, under the parametrization considered here,

$$\lim_{t\to\infty}(1/t)\mathbf{E}^{h_\theta(\cdot)}[lnV(t)|V(0) = v] = \lim_{t\to\infty}(1/t)lnV(t) = \rho(\theta), \quad \mathbf{P}^{h_\theta(\cdot)} \ a.s., \tag{9.2}$$

for all $v \in (0, \infty)$ and $\theta \geq 0$. In particular, $\rho(0) = (1/2)\frac{(\mu_1 - r)^2}{\sigma^2} + r$ is the maximal (expected) growth rate of the investor's portfolio. Thus the quantity $R(\theta)$ above can be interpreted as the asymptotic, logarithmic ratio of the chance that the actual growth rate of the investor's portfolio under the strategy $h_\theta(\cdot) = [\tilde{h}_\theta(\cdot), 1 - \tilde{h}_\theta(\cdot)]'$ will exceed the maximal limit, to the risk that the growth rate will fall below the spot rate.

It is interesting to see which value of $\theta \geq 0$ maximizes $R(\theta)$. Toward this end let us first note that in the current situation we have

$$\tilde{h}(t) = H_\theta(r) := \frac{\mu_1 - r}{\sigma^2}$$

[2]Recent developments obtained by the *Kansas Adaptive Control Group* (see their web page: *http://www.math.ukans.edu/ksacg/*) will also be helpful.

for all $t \geq 0$. Thus, under $h_\theta(\cdot)$, we have

$$(1/t)lnV(t) = \mu_1 H_\theta(r) + (1 - H_\theta(r))r - (1/2)H_\theta^2(r)\sigma^2 + H_\theta(r)\frac{W_1(t)}{t}. \tag{9.3}$$

Consequently (using (1.1.4) of Deuschel and Stroock [6]) we obtain

$$R(\theta) = \inf\{\frac{x^2}{2} : x \in \Lambda(\theta)\} - \inf\{\frac{x^2}{2} : x \in \Gamma(\theta)\}, \tag{9.4}$$

where $\Lambda(\theta) := (\infty, -\frac{1}{2}(\mu_1 - r)\frac{\theta+1}{\frac{\theta}{2}+1})$ and $\Gamma(\theta) := (\frac{1}{8}\frac{\theta^2(\mu_1-r)}{\frac{\theta}{2}+1}, \infty)$. Thus

$$R(\theta) = (1/4)\frac{(\mu_1 - r)^2}{(\frac{\theta}{2}+1)^2}\Big[(\theta+1)^2 - \frac{\theta^4}{16}\Big]. \tag{9.5}$$

Finally we see that $R(\theta)$ is maximized over $[0, \infty)$ by $\theta^* = 2$. This means that the ratio of *up-side chance* to *down-side risk* is maximized if the investor maximizes the growth of $-(1/t)ln\mathbf{E}\frac{1}{V(t)}$!!!

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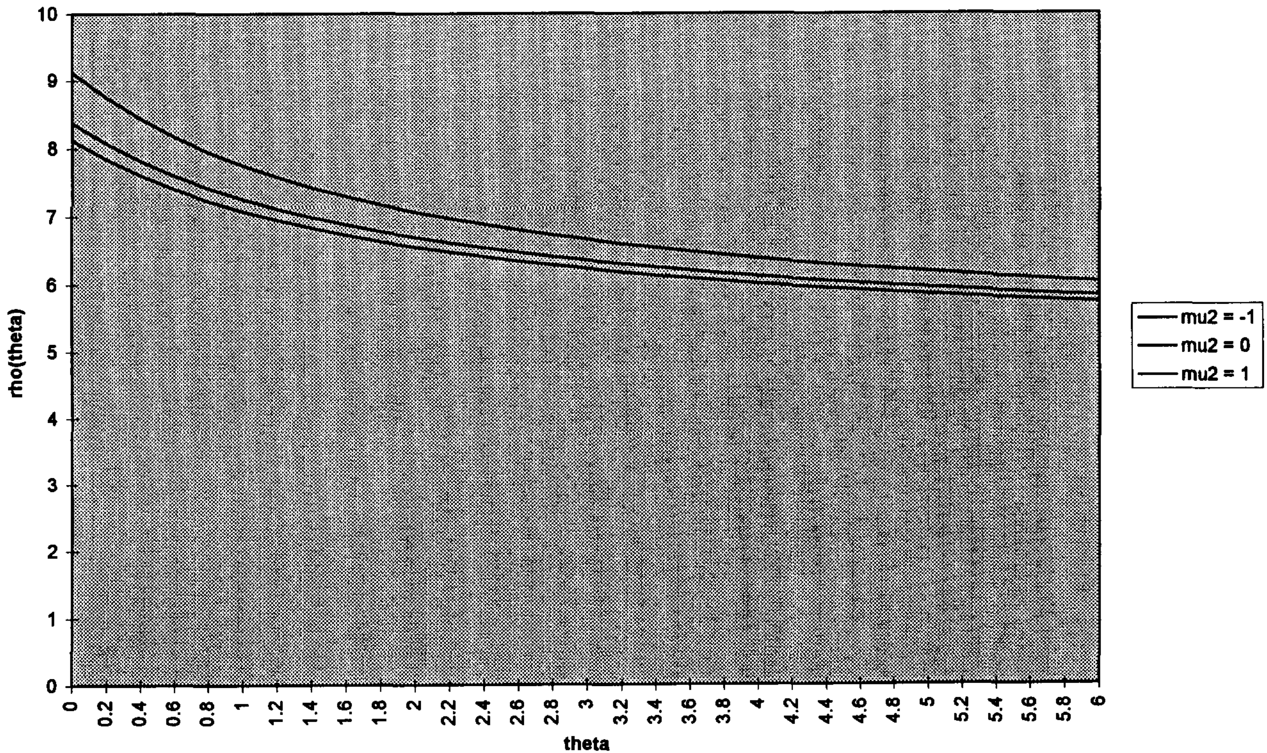
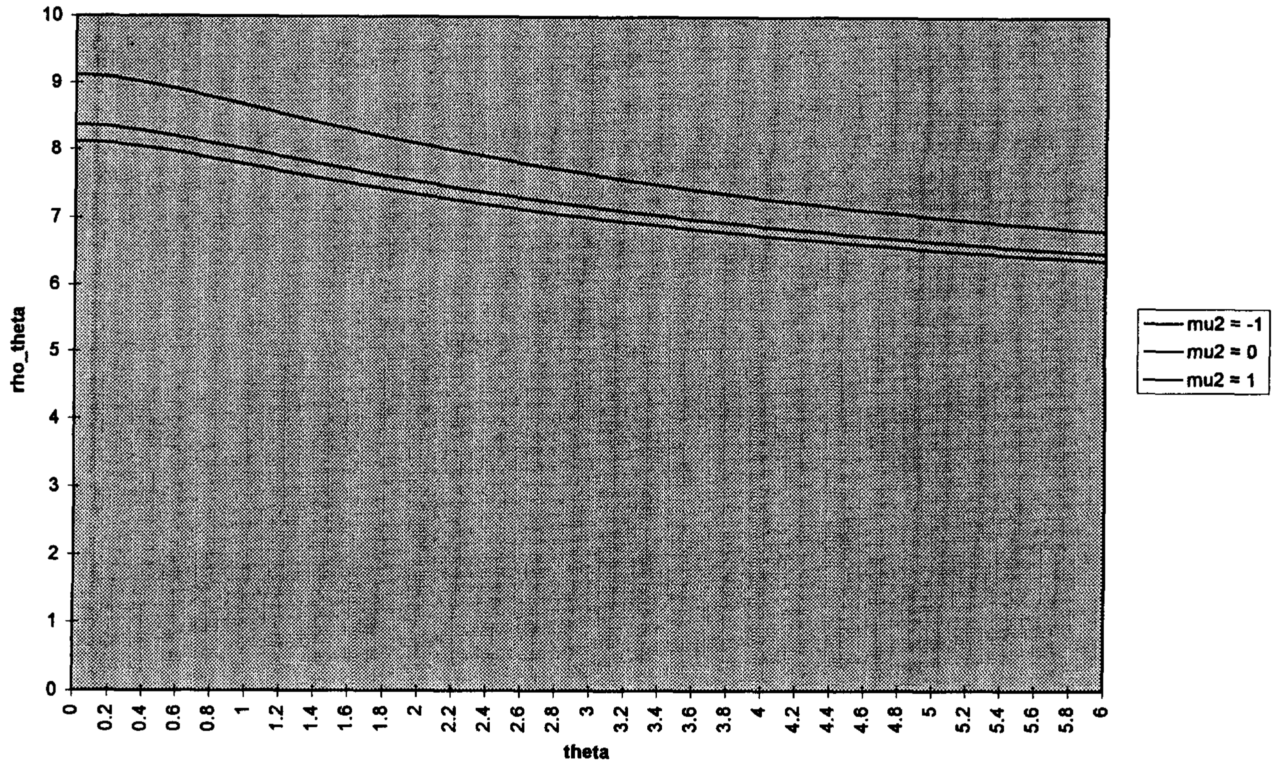
Figure 1



rho(theta)
10
9
8
7
6
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4
4.2
4.4
4.6
4.8
5
5.2
5.4
5.6
5.8
6
theta
mu2 = -1
mu2 = 0
mu2 = 1

Figure 2


10
9
8
7
6
5
4
3
2
1
0
rho_theta
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
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3
3.2
3.4
3.6
3.8
4
4.2
4.4
4.6
4.8
5
5.2
5.4
5.6
5.8
6
theta
mu2 = -1
mu2 = 0
mu2 = 1

Figure 3

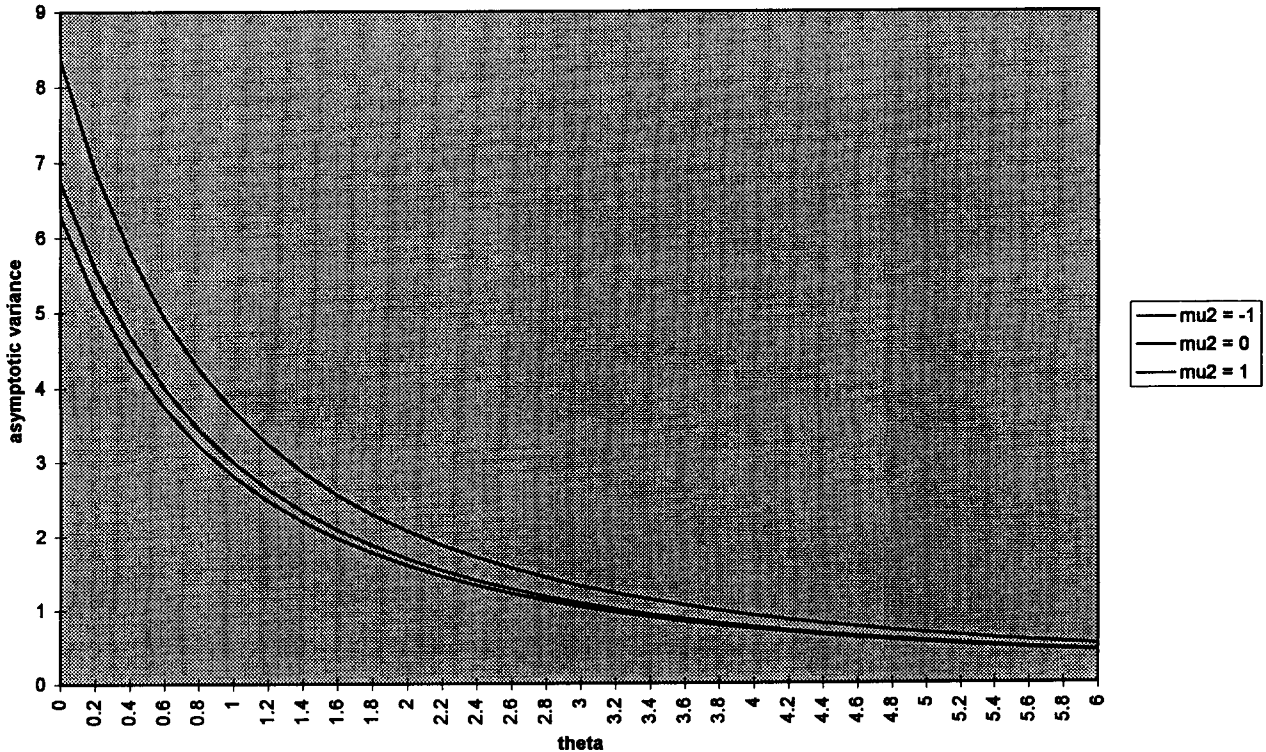

asymptotic variance
0
1
2
3
4
5
6
7
8
9
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2 2.4 2.6 2.8 3 3.2 3.4 3.6 3.8 4 4.2 4.4 4.6 4.8 5 5.2 5.4 5.6 5.8 6
theta
mu2 = -1
mu2 = 0
mu2 = 1

Figure 4

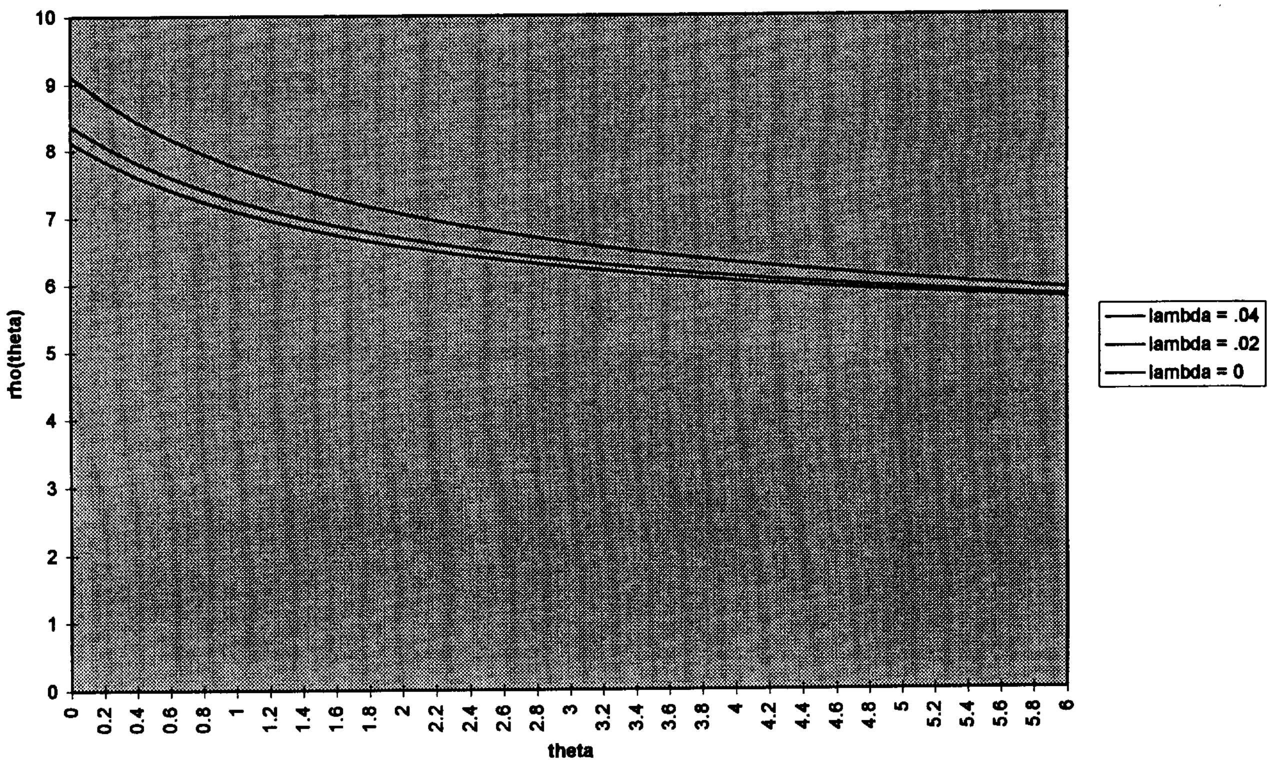
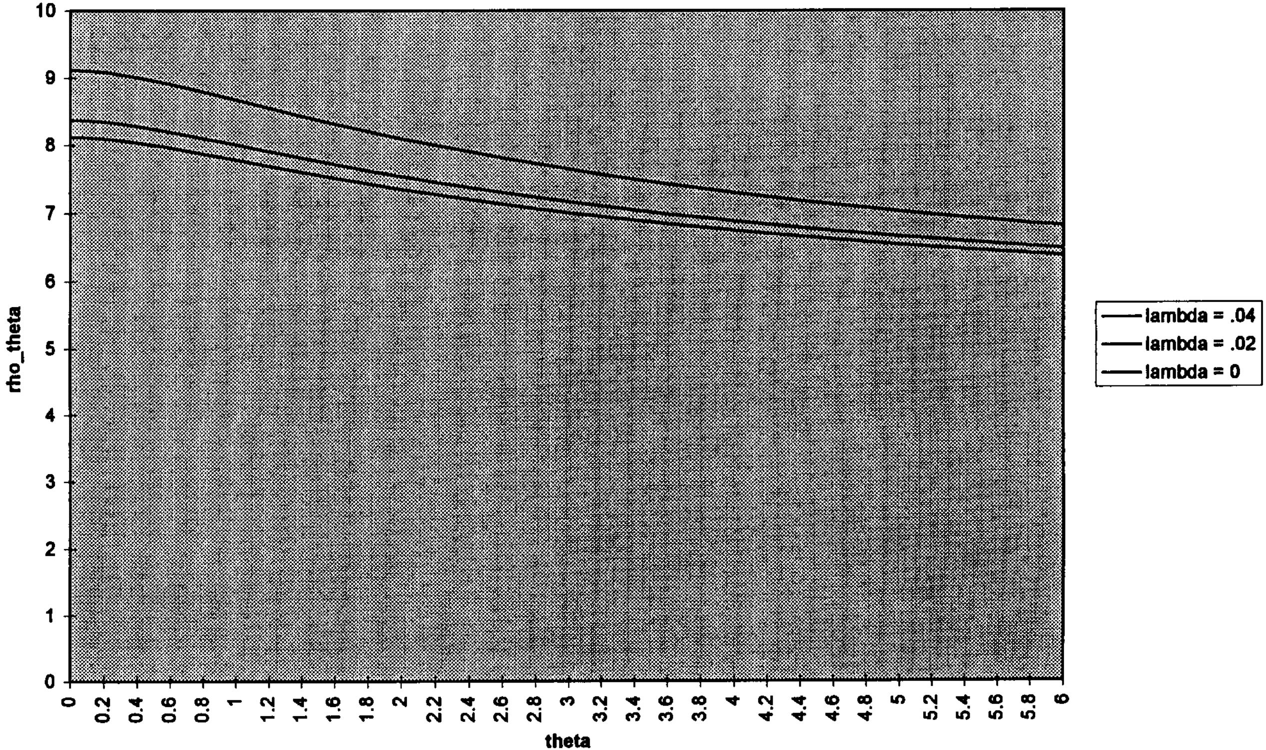
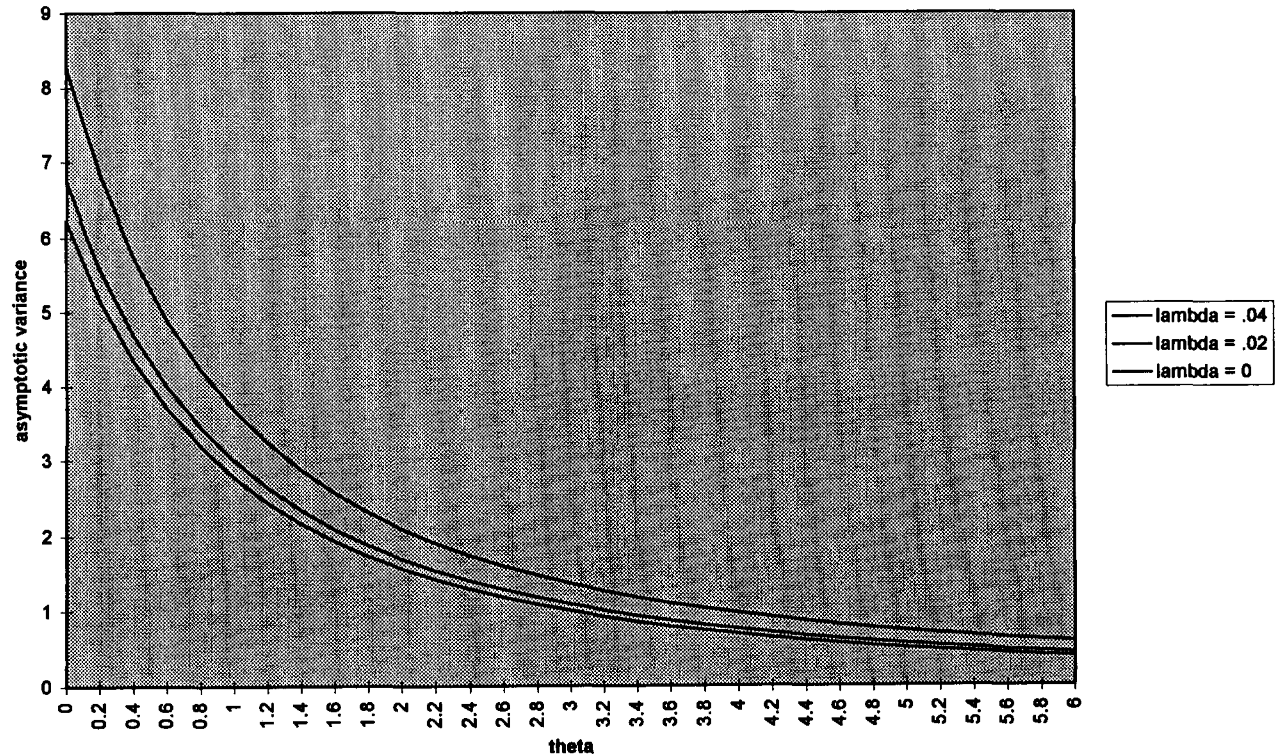


Figure 5



10
9
8
7
6
5
4
3
2
1
0
rho_theta
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2 2.4 2.6 2.8 3 3.2 3.4 3.6 3.8 4 4.2 4.4 4.6 4.8 5 5.2 5.4 5.6 5.8 6
theta
lambda = .04
lambda = .02
lambda = 0

Figure 6



9
8
7
6
5
4
3
2
1
0
asymptotic variance
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4
4.2
4.4
4.6
4.8
5
5.2
5.4
5.6
5.8
6
theta
lambda = .04
lambda = .02
lambda = 0

CONTRIBUTED PAPERS