Portfolio Optimization with a Defaultable Security

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Abstract. In this paper we derive a closed-form solution for a representative investor who optimally allocates her wealth among the following securities: a credit-risky asset, a default-free bank account, and a stock. Although the inclusion of a credit-related financial product in the portfolio selection is more realistic, no closed-form solutions to date are given in the literature when a recovery value is considered in the event of a default. While most authors have assumed some recovery scheme in their initial model set up, they do not address the portfolio problem with a recovery when a default actually occurs. Given the tractability of the recovery of market value, we solved the optimal portfolio problem for the representative investor whose utility function is a Constant Relative Risk Aversion utility function.

We find that the investor will allocate larger fraction of wealth to the defaultable security as long as the default-event risk is priced. These results are very intuitive and reasonable since it indicates that if the default risk premium is not priced properly the investor purchases less defaultable securities.

Keywords: Portfolio optimization, defaultable security, credit risk, recovery of market value

1. Introduction

In today's financial markets, high yield corporate bonds are increasingly attractive to investors. Compared to stocks and default-free bonds, corporate bonds provide different risk-return profiles to investors and are often sought after because of their high yields. Although rare, corporate bond defaults do occur. One of the largest defaults in the history of a defaultable security was the default by Enron. Kmart also filed a bankruptcy, which led to significant losses to investors. As a result, optimal portfolio problems with defaultable securities have become an important area of research. In spite of several major contributions to the theory of optimal portfolio selection only a handful of papers (e.g. Korn and Kraft (2003), Walder (2002), Hou and Jin (2002), and Hou (2003)) consider the case where one or more of the securities in the portfolio are subject to default.[1]

Walder (2002) considers the optimal portfolio problem by assuming that the investor can invest in a treasury bond and a portfolio of corporate zero-coupon bonds.[2] In his model default is modeled using the intensity based approach. To obtain tractable results he makes the rather strong assumption of zero recovery in the case of a default. Hou and Jin (2002) and Hou (2003) address the optimal portfolio problem of the investor by giving the investor the ability to allocate her wealth among a stock, a default-free bank account, and a credit-risky financial instrument. They also employ the intensity based approach. Their approach, however, differs from Walder's as they assume that in the case of default the investor receives a fraction of the market value of the debt just prior to the default, commonly referred to as the recovery of market value (RMV). A key weakness of the approach by both Walder (2002) and Hou and Jin (2002) and Hou (2003) is that they do not model explicitly in their optimization problem optimal allocation of the recovered amount in default using the conditional diversification assumption of Jarrow, Lando, and Yu (2005) (hereafter JLY).[3]

JLY demonstrate that if the market for defaultable bonds consists of infinitely many bonds in which defaults are conditionally independent, the market does not value the risk associated with the default event. JLY refer to the asymptotic disappearance of the jump-risk premium as "conditional diversification." However, if the market for defaultable bonds is thin, then the concept of conditional diversification does not hold. In this case there would be a significant compensation for jump risk. The validity of conditional diversification is an empirical issue. To date, research by Driessen (2005) has shown that jump risk is significantly priced in the spreads of credit default swaps (CDS).

With growing empirical evidence supporting the presence of jump-risk premia in the defaultable security market, we provide a model for optimal asset allocation where the agent allocates her wealth in response to changing jump-risk premia. This is key innovation of our paper. By explicitly modeling the jump-risk premium in the dynamics of a defaultable bond we are able to provide results which were otherwise missing from the literature.

In our analysis the intensity based approach to credit risk modeling along with the assumption of RMV is fused with the theory of stochastic optimal control. We derived the optimal amount of wealth allocated to a defaultable security as well as to a stock and to a default-free money market account. The following relationships between the optimal amount invested in the defaultable security and the default-risk premium are obtained: (i) for a default-risk premium greater than one, the investor will optimally invest a positive amount in the defaultable bond, and (ii) for a risk premium equal to one, the investor will optimally invest nothing in the defaultable bond. A default-risk premium greater than one can be interpreted as the market fairly pricing the default risk in the defaultable bond. On the other hand, a default-risk premium equal to one, can be interpreted as the market not pricing the default risk in the defaultable bond.

The rest of the paper is organized as follows: Section 2 presents the dynamics of a defaultable bond, a stock as well as a savings account under two equivalent probability measures $\mathbb{Q}$ and $\mathbb{P}$, which will be used for the portfolio optimization in Section 3.[4] Implications of the analytic result for asset allocation are discussed in Section 4. Section 5 carries out sensitivity analysis by adopting benchmark parameter values from the literature. Section 6 summarizes the results and concludes the paper.

2. Dynamics of Financial Securities

Let us first assume that there exists a fixed probability space $(\Omega, \mathbb{G}, \mathbb{Q})$, which is endowed with the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ where $\mathcal{G}_t$ is often called an enlarged filtration and is given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. The filtration $\mathcal{F}_t$ is assumed to be generated by the Wiener process which drives the change of stock price and the filtration $\mathcal{H}_t$ is driven by a Poisson process that is used to denote the arrival of defaults. In the probability space $\mathbb{Q}$ denotes the martingale probability measure (or risk neutral measure), which is assumed to be equivalent to the some real-world probability measure which we denote by $\mathbb{P}$.[5]

Now we are ready to define a default process.

Definition 1. *A nondecreasing right continuous process which makes discrete jumps at a random time τ is called default process. We denote a default process by $H_t = 1_{\{\tau \leq t\}}$. Where 1 represents the indicator function which takes the value of one if there is a jump and zero otherwise and $H_{u-} := \lim_{s\uparrow u} H_s$.*

The default process H is assumed to be a Poisson process with a constant intensity h.

Definition 2. *The default process $H = (H_t)_{t\geq 0}$ is said to be a Poisson process if*

- $H_0 = 0$
- $(H_t)_{t\geq 0}$ *has independent increments*

– *The number of events in any interval of length t has a Poisson distribution with mean ht. That is, for all s, $t \geq 0$,*

$$Prob\{H_{t+s} - H_s = n\} = e^{-ht}\frac{(ht)^n}{n!},\ n \geq 0.$$

Here, h represents the intensity of the Poisson process, which measures the arrival rate of a default. The mean arrival rate ht is called the compensator for the Poisson process. We now define a martingale (compensated) Poisson process.

Definition 3. *The martingale default process is thus given by the following equation:*

$$M_t := H_t - \int_0^t (1 - H_{u-})hdu.$$

The stochastic differential equation (SDE) of compensated Poisson process defined above is $dM_t = dH_t - h(1 - H_{t-})dt$. We extensively utilize this SDE to derive the dynamics of the defaultable bond.

2.1. Dynamics of a Defaultable Bond Price Process, a Stock Price, and a Saving Account under $\mathbb{Q}$

We assume that there exists a defaultable zero coupon bond with a maturity date T_1. The value of the defaultable bond after default is assumed to be zero. In the case of default the investor recovers a fraction of the market value of the defaultable bond just prior to default. The loss rate is denoted by ζ which is assumed to be constant and take value between zero and one.[6] The arrival rate of the default $h^{\mathbb{Q}}$ of the Poisson process is also assumed to be constant.

We will now define the price process for a defaultable bond under the assumption of constant interest rate and intensity under the risk neutral measure $\mathbb{Q}$ as follows:

$$p(t, T_1) = 1_{\{\tau > t\}} e^{-(r+\delta)(T_1 - t)} + 1_{\{\tau \leq t\}}(1 - \zeta)e^{-(r+\delta)(T_1 - \tau)} e^{r(t-\tau)}. \tag{1}$$

We refer to $p(t, T_1)$ as being a fictitious security because it is not really a traded security. It actually consists of two components, which are mutually exclusive, i.e., the component of the left hand side is the actual defaultable bond price given that default has yet to occur, whereas the component of the right hand side represents the value (or recovered amount) of an already defaulted bond. By accounting for the second component we are implicitly modeling the difference between the bond price if there is a full recovery and fractional recovery at default. This is referred to by Driessen (2005) among others as the jump risk event. Accounting for the impact of recovery at default will lead us to accurately specify the dynamics of the defaultable bond, that is, we will be able to account for the jump risk premium in the expected return of the defaultable bond. The explicit inclusion of the recovery is the first innovation of this paper.

$\mathbb{Q}$ dynamics of the price process of the defaultable bond is derived in the following lemma.

Lemma 1. *The $\mathbb{Q}$ dynamics for the price process of the defaultable bond at time t is represented in terms of the compensated jump process $M^{\mathbb{Q}}$, which is $\mathbb{Q}$ martingale process, such as:*

$$dp(t, T_1) = rp(t, T_1)dt - \zeta e^{-(r+\delta)(T_1-t)} dM_t^{\mathbb{Q}}. \tag{2}$$

Proof (of Lemma 1). Use Itô's lemma on equation (1).

Note that under our specification of the price process of the defaultable bond, the expected return of the defaultable bond in equation (2) is the risk free short rate of interest under the risk neutral measure $\mathbb{Q}$.

Now we consider that the agent can invest money in the bank at the constant short rate of interest r, i.e. she has access to the risk-free asset B_t with

$$dB_t = rB_t dt. \tag{3}$$

The investor can also invest in a risky stock with price process S_t, the dynamics under $\mathbb{Q}$ is given by the following diffusion equation:

$$dS_t = S_t(rdt + \sigma dW_t^{\mathbb{Q}}) \tag{4}$$

where σ instantaneous volatility of the equity return process is assumed to be constant. Note that the expected return under $\mathbb{Q}$ for each security is the risk free interest rate. This is considered to be arbitrage free condition, which is assumed to be satisfied whenever there exists a martingale probability measure (see Harrison and Pliska (1983)).

2.1.1. *Dynamics of the Price Process under* $\mathbb{P}$

We change the measure from the risk neutral probability measure $\mathbb{Q}$ to the real world probability measure $\mathbb{P}$. This is necessary since the investor optimizes her utility under the real world probability measure $\mathbb{P}$.

The following Girsanov's theorem (see for example Klebaner (1998)) is used to change of measures. We state the theorem without proof.

Lemma 2. *A probability* $\mathbb{P}$ *is equivalent to* $\mathbb{Q}$ *on* $\mathcal{G}$ *if and only if there exists progressively measurable, R-valued process* β *and a predictable,* $R_+ :=]0,\infty[$*-valued process* $\triangle > 0$ *such that*

1. $\mathbb{E}_{\mathbb{P}}(L(T^*)) = 1$, *where*

$$\begin{aligned} L(t) &:= L_1(t)L_2(t) \\ L_1(t) &:= exp\left\{ \int_0^t \beta_u dW_u^{\mathbb{Q}} - 1/2 \int_0^t \beta_u^2 du \right\} \\ L_2(t) &:= exp\left\{ \int_0^t ln(\triangle_u) dH_u - h^{\mathbb{Q}} \int_0^{t\wedge\tau} [\triangle_u - 1] du \right\}, \forall t \in [0, T^*] \end{aligned}$$

2. $\frac{d\mathbb{P}}{d\mathbb{Q}} = L(t)$

Moreover, the process $W^{\mathbb{P}} = W_t^{\mathbb{Q}} - \int_0^t \beta_u du$ *is a* $\mathbb{G}$*-Brownian motion under* $\mathbb{P}$ *and the process* $M_t^{\mathbb{P}} = H_t - h^{\mathbb{Q}} \int_0^t \triangle_u (1 - H_u) du$ *is a* $\mathbb{G}$*-martingale under the physical measure,* $\mathbb{P}$.

The existence of the risk neutral measure implies that there are no arbitrage opportunities.

By applying Lemma 2 we obtain the following $\mathbb{P}$ dynamics of the price process for the defaultable bond.

Lemma 3. *For positive interest rate, the dynamics of defaultable bond price is represented as the exponential form of SDE as follows.*

$$\begin{aligned} dp(t,T_1) &= (1-H_t)(r+\delta-\delta\triangle)p(t,T_1)dt + H_t r p(t,T_1)dt - (1-H_{t-})\zeta p(t,T_1)dM_t^{\mathbb{P}} \\ &= p(t-,T_1)\Big[rdt + (1-H_t)\delta(1-\triangle)dt - (1-H_{t-})\zeta dM_t^{\mathbb{P}}\Big] \end{aligned} \tag{5}$$

where we use

$$\begin{cases} p(t,T_1) = e^{-(r+\delta)(T_1-t)} & \text{if } \tau > t \\ p(t,T_1) = (1-\zeta)e^{-(r+\delta)(T_1-\tau)}e^{r(t-\tau)} & \text{if } \tau \le t \end{cases}$$

The instantaneous expected return of the defaultable bond, given in equation (5), can be seen to consist of two components similar to what Yu (2002) argues (see equation (15) in Yu (2002). The first component is the return on an otherwise identical default-free bond. The second is the difference between the risk neutral credit spread and the real world credit spread provided that default has not already occurred by time t. By using a diversification argument similar to that of the original APT, JLY (2005) showed asymptotic equivalence between the risk neutral intensity $h^{\mathbb{Q}}$ and the real world intensity $h^{\mathbb{P}}$ (see Proposition 1 in Jarrow et al. (2005)). The implication of JLY's argument is that the second component will disappear. In order to see this clearly we explicitly express the difference of the credit spreads in terms of the intensities under $\mathbb{Q}$ and $\mathbb{P}$ such as:

$$\delta - \delta\triangle = h^{\mathbb{Q}}\zeta - h^{\mathbb{Q}}\zeta\triangle = h^{\mathbb{Q}}\zeta - h^{\mathbb{P}}\zeta = (\frac{h^{\mathbb{Q}}}{h^{\mathbb{P}}} - 1)h^{\mathbb{P}}\zeta$$

where $\delta^{\mathbb{P}} = h^{\mathbb{P}}\zeta$ is the credit spread under the real world probability measure. Therefore, if $h^{\mathbb{Q}}$ is equal to $h^{\mathbb{P}}$ then the second term in equation (5) vanishes provided that default has not already occurred by time t. In such a case, the investor requires no compensation for bearing the risk of the default event.

In a situation, however, where the risk inherent in the actual default event cannot be diversified away, one has to consider the second component explicitly. Through simple parameterization of the default event risk we can show the difference of the credit spreads in terms of the default event risk premium, namely,

$$\delta - \delta\triangle = (\frac{h^{\mathbb{Q}}}{h^{\mathbb{P}}} - 1)h^{\mathbb{P}}\zeta = (\frac{1}{\triangle} - 1)\delta^{\mathbb{P}},$$

where $\frac{1}{\triangle} = \frac{h^{\mathbb{Q}}}{h^{\mathbb{P}}}$. Thus the compensation for the default event risk premium becomes $(\frac{1}{\triangle} - 1)$. Since an investor's risk aversion toward the default event implies a $\frac{1}{\triangle}$ greater than one, ignoring the second component will lead to an underestimation of the expected return on the defaultable bond. Thus $\frac{1}{\triangle}$ is called a default event risk premium.

Utilizing Lemma 2 we can also express the dynamics of stock price under $\mathbb{P}$ as follows.

$$dS_t = S_t(\mu dt + \sigma dW^{\mathbb{P}}) \tag{6}$$

The stock price grows at a instantaneous rate of μ and has the instantaneous volatility of σ under the real world probability measure $\mathbb{P}$. Note that we change the drift term of the dynamics of stock price, but do not change the volatility of it through the change of risk neutral probability measure to real world probability measure. On the other hand, the dynamics of a savings account has the same representation under both measure $\mathbb{Q}$ and $\mathbb{P}$ since it is non-stochastic.

The pricing dynamics of each security under the real world probability measure $\mathbb{P}$ such as equation (3), (5), and (6) are utilized in the next section.

3. Optimal Portfolio Problems: Constant Relative Risk Aversion (CRRA) Utility Function

We use stochastic optimal control theory to find the optimal allocations among a savings account, a stock (or stock index), and a defaultable bond for the investor whose objective is to maximize her conditional expected utility of the terminal wealth over the investment horizon $[0, T]$. We assume throughout that $T < T_1$, where T_1 is the time of maturity of the defaultable bond. We consider that short rate of interest is constant and non-negative.

3.1. Admissible Strategies and Wealth Processes

To start, let ϕ_t^s be the number of shares of stocks the investor longs ($\phi_t^s \geq 0$) or shorts ($\phi_t^s \leq 0$) at time t. Similarly, ϕ_t^p and ϕ_t^B are the number of shares of the defaultable bond and a savings account she buys or sells at time t for all $t \in [0, T]$. Then the process $\phi = (\phi^s, \phi^p, \phi^B)$ is called a portfolio process. Now we define $V_t(\phi)$ as the wealth of the portfolio process $\phi = (\phi^s, \phi^p, \phi^B)$ at time t, that is

$$V_t(\phi) = \phi_t^s S_t + \phi_t^p p(t, T_1) + \phi_t^B B_t, \ \forall\, t \in [0, T].$$

The wealth of the portfolio process ϕ starting with $v > 0$ at time t will be denoted as $V_t^v(\phi)$.

Let $(\pi_t^s, \pi_t^p, \pi_t^B)$ be now defined as

$$\begin{aligned} \pi_t^s &:= \frac{\phi_t^s S_t}{V_{t-}(\phi)}, \\ \pi_t^p &:= \frac{\phi_t^p p(t-, T_1)}{V_{t-}(\phi)}, \\ \pi_t^B &:= \frac{\phi_t^B B_t}{V_{t-}(\phi)}. \end{aligned} \tag{7}$$

Note that $\pi_t = (\pi_t^s, \pi_t^p, \pi_t^B)$ represents the fraction of the wealth invested in each asset at time t and is called *fractional strategy* or simply *strategy* in the sequel. For all $t \in [0, T]$ we shall consider the following wealth equation:

$$\begin{cases} dV_t(\pi) = V_{t-}(\pi)\Big[\Big[\pi_t^s\mu + \pi_t^p\big(\tilde{H}_t\delta(1-\triangle)+r\big) + (1-\pi_t^s-\pi_t^p)r\Big]dt + \pi_t^s\sigma dW_t^{\mathbb{P}} - \pi_t^p\zeta dM_t^{\mathbb{P}}\Big] \\ V_0(\pi) = v \end{cases} \tag{8}$$

where $\tilde{H}_t = 1 - H_t$, $\tilde{H}_{t-}dM_t^{\mathbb{P}} = dM_t^{\mathbb{P}}$, and $h^{\mathbb{P}} = h^{\mathbb{Q}}\triangle$.

3.2. Optimization Problems

We consider the following joint dynamics of the wealth equation $V_t(\pi)$ and for the factor equation H_t for $t \in [0,T]$,

$$\begin{cases} dV_t(\pi) = V_{t-}(\pi)\Big[\Big[\pi_t^s\mu + \pi_t^p\big(\tilde{H}_t\delta(1-\triangle)+r\big) + (1-\pi_t^s-\pi_t^p)r\Big]dt + \pi_t^s\sigma dW_t^{\mathbb{P}} - \pi_t^p\zeta dM_t^{\mathbb{P}}\Big] \\ dH_t = h^{\mathbb{P}}(1-H_t)dt + dM_t^{\mathbb{P}} \end{cases} \tag{9}$$

with $V_0(\pi) = v, H_0 = 0$. We use $h^{\mathbb{P}} = h^{\mathbb{Q}}\triangle$ and $\delta = h^{\mathbb{Q}}\zeta$.

Let $\mathcal{U}$ be an CRRA utility function satisfying[7]

$$\mathcal{U}(x) = \frac{x^\gamma}{\gamma} \quad for\ 0 < \gamma < 1, \quad x \geq 0. \tag{10}$$

We shall study the optimization problems in case of CRRA utility function such as:

$$\mathcal{J}(t,v,z) = \sup_{(\pi^s,\pi^p)\in\Pi_t(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\Big[\frac{V_T^\gamma(\pi)}{\gamma}\Big| V_t(\pi) = v, H_t = z\Big] \tag{11}$$

for all $(t,v,z) \in (0,T)\times(0,\infty)\times\{0,1\}$ over all admissible strategies π.[8]

Let us define the following process.

$\pi_t^{s*} = \frac{(\mu-r)}{\sigma^2(1-\gamma)}, \quad t \in [0,T]$

$$\pi_t^{p*} = \begin{cases} \frac{1}{\zeta}\Bigg(1 - \Bigg(\big(\frac{1}{\triangle}\big)^{\frac{1}{1-\gamma}} e^{-(\frac{h^{\mathbb{P}}}{1-\gamma} - \frac{h^{\mathbb{P}}}{\triangle}\frac{\gamma}{1-\gamma})(T-t)} + \big(e^{-(\frac{h^{\mathbb{P}}}{1-\gamma} - \frac{h^{\mathbb{P}}}{\triangle}\frac{\gamma}{1-\gamma})(T-t)} - e^{-\frac{\gamma^2 r}{1-\gamma}(T-t)}\big)h^{\mathbb{P}}\frac{1-\gamma}{h^{\mathbb{P}}(\gamma-\triangle)+\gamma^2\triangle r}\Bigg)^{-1}\Bigg), \\ \quad for \quad 0 \leq t < (\tau \wedge T) \\ 0, \quad for \quad t \in [(\tau\wedge T), T] \end{cases}$$

$$\pi_t^{B*} = \begin{cases} 1 - \pi_t^{s*} - \pi_t^{p*}, & 0 \leq t < (\tau \wedge T) \\ 1 - \pi_t^{s*}, & t \in [(\tau \wedge T), T] \end{cases}$$

Then the process π^* defined above is an optimal control.

Now we are ready to state the optimality theorem.

Theorem 4 (Optimal Trading Strategies). *The process π^* defined above is an optimal control.*

Note that the optimal trading strategies are independent of the wealth since the CRRA utility is a homogeneous function of wealth.

Next we derive the optimal value function $\mathcal{J}(\cdot,\cdot,\cdot)$. This is often called the indirect utility function since it measures the highest attainable expected utility the investor can derive from her current wealth in the current state of the world.

Theorem 5 (Indirect Utility Function). *Given the CRRA utility function, we obtain the optimal value function, $\mathcal{J}(t, v, z)$*

$$\mathcal{J}(t,v,z) = \begin{cases} \Big(f(t)\Big)^{1-\gamma} \times \Big(\frac{v^\gamma}{\gamma}\Big), & z=0 \\ e^{\big(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}+\gamma r\big)(T-t)} \times \Big(\frac{v^\gamma}{\gamma}\Big), & z=1 \end{cases}$$

where

$$f(t) = e^{\big(\frac{1}{2}\frac{\gamma}{(1-\gamma)^2}\frac{(\mu-r)^2}{\sigma^2}+\frac{\gamma r}{1-\gamma}\big)(T-t)} \times$$
$$\Big(e^{\big(-\frac{h^{\mathbb{P}}}{1-\gamma}+\frac{h^{\mathbb{P}}}{\triangle}(\frac{\gamma}{1-\gamma})\big)(T-t)} + (e^{-(\frac{h^{\mathbb{P}}}{1-\gamma}-\frac{h^{\mathbb{P}}}{\triangle}\frac{\gamma}{1-\gamma})(T-t)} - e^{-\frac{\gamma^2 r}{1-\gamma}(T-t)})h^{\mathbb{P}}(\frac{1}{\triangle})^{\frac{1}{\gamma-1}}\frac{1-\gamma}{h^{\mathbb{P}}(\gamma-\triangle)+\gamma^2\triangle r}\Big).$$

Note that $\mathcal{J}_v(t,v,z) > 0$ and $\mathcal{J}_{vv}(t,v,z) < 0$. This indicates that $\mathcal{J}(t,v,z)$ is an increasing and concave function of initial wealth v.

4. Implications for Asset Allocation

We investigate the implications of the optimal strategies.

4.1. Optimal Strategy for a Stock

The optimal strategy for the stock is constant irrespective of pre-default or post-default. The reason is that the stock has no correlation with the defaultable bond, which means there is no need to hedge for the default. Thus the optimal strategy for the stock is invariant to the default event risk.

Let us decompose the optimal strategy π^{s*} for the stock as follows.

$$\pi_t^{s*} = \Big(-\frac{J_v}{vJ_{vv}}\Big) \times \Big(\frac{\mu - r}{\sigma} \times \frac{1}{\sigma}\Big) = \Big(-\frac{1}{\gamma - 1}\Big) \times \Big(\frac{\mu - r}{\sigma^2}\Big). \tag{12}$$

The optimal strategy π^{s*} for the stock is interpreted as the product of the relative risk tolerance $-\frac{J_v}{vJ_{vv}}$ (i.e. the inverse of the relative risk aversion) of the indirect utility function and market price of equity risk (or Sharpe ratio of a stock) $\frac{\mu-r}{\sigma}$ normalized by the volatility of the stock return σ. Thus the demand for equity contains only so-called "myopic" term despite the fact that the investment opportunity set is stochastic in this economy.[9]

As usual, the myopic demand for the stock decreases as the risk aversion coefficient γ is close to zero. This is reasonable since γ close to zero indicates that the relative risk aversion $(1-\gamma)$ of the investor whose utility function is CRRA is close to one, which means the investor is risk-averse. We note that the optimal strategy π^{s*} for the stock is independent of the investment horizon, but linearly dependent on the interest rate.

4.2. Optimal Demand for the Defaultable Bond

It is clear from Lemma 3 that the drift and volatility of the return of the defaultable bond changes discretely at default. That is, the stochastically evolving default process H, which is the only risk source of randomness for the defaultable bond, is captured in the drift and volatility of the return of the defaultable bond. We can interpret the change that occurs at default as a change in the investment opportunity set of the investor. In the default literature it is commonly assumed that the investor knows whether the default event has occurred or not. Thus we can simply analyze the optimal defaultable bond investment on the pre and post default space.[10]

The pre-default optimal strategy for defaultable bond is an increasing function of the default event risk premium. That is, the investor purchases more defaultable bonds when the price of the default event is greater. Another result is that the greater the interest rate the lower the investment is in the defaultable bond, holding other factors constant. Intuitively we can interpret this result as saying that when there is no default event risk premium and interest rate equals zero the investor allocates zero amount of wealth to the defaultable bond. On the other hand, if interest rate is positive the investor takes a short position in the defaultable bond whenever there is no default event risk premium, ceteris paribus. That is, the investor disinvests in the defaultable security whenever the interest rate is positive and the default risk premium is equal to one. For positive interest rate and a risk premium greater than one the investor always allocates a positive amount of wealth in the defaultable bond.

From the analytical result it is clear that the pre-default optimal strategy of the defaultable bond depends on the investment planning horizon. The investor allocates more when she has longer planing horizon. It can be easily shown that the probability of default in our framework is greater for longer planning horizon. Nevertheless, we find that the investor allocates greater amount of wealth the longer the amount of time remaining. Although this result at first appears to be perverse it can be explained in terms of the default risk premium. Recall from Section 2 that if default follows a Poisson process then the pre-default price of the defaultable bond is given by $p(t,T_1) = e^{-(r+\delta)(T_1-t)}$ where $T < T_1$. We can see that as time t approaches to T, both the investment horizon $T-t$ and the time to maturity of the defaultable bond $T_1 - t$ tend to zero. This leads to an increase in the price of the defaultable bond providing no default has yet to occur. In other words, the defaultable bond is getting relatively cheap as the investment horizon increases. Since the default event risk premium $\frac{h^{\mathbb{Q}}}{h^{\mathbb{P}}}$ is assumed to be the same throughout the planning horizon and the time to maturity then the only risk the investor faces with regards to her return in the defaultable bond is the default itself. Thus the investor would naturally invest greater amount at longer planning horizon since for given probability of default the bond naturally appreciates in value as it achieves par value at maturity providing there is no default. Following the buy low sell high rule, the investor should invest more in the defaultable bond when its price is low, that is, for longer maturity date, all else equal.

4.3. Optimal Strategy for a Savings Account

The optimal strategy for a savings account is calculated by $\pi_t^{B*} = 1 - \pi_t^{s*} - \pi_t^{p*}$ for $t \in [0, \tau \wedge T)$. Upon default, the value of corporate bond goes down to zero, and the investor recoups only the recovery. Since the defaultable bond disappears after default, it is enough to find the optimal investment for the savings account, which satisfies $\pi_t^{s*} + \pi_t^{B*} = 1$ for all $t \in [\tau \wedge T, T]$.

5. Sensitivity Analysis

In this section we analyze the behavior of the optimal strategy of the defaultable bond in particular. Since the optimal strategy of the defaultable security is a function of the default risk premium,

Table I. Parameter Definition and values
This table summarizes parameters and values used in this paper. Parameters of default risk premium and loss rates for three industries are taken from Duffie et al. (2004), Hou (2003), and Driessen (2005).

Symbol	*Definition*	*Values*
$\frac{h^{\mathbb{Q}}}{h^{\mathbb{P}}} = \frac{1}{\triangle}$	default risk premium	2.53 for Oil and Gas
		5.50 for BE
		5.59 for Healthcare
$h^{\mathbb{Q}}$	risk neutral default intensity	0.013
$T - t$	investment time horizon remaining	1 (year)
γ	risk aversion parameter	0.01 for nearly risk neutral
		0.5 for risk averse
		0.9 for more risk averse
ζ	write-down rate	0.52 for Oil and Gas
		0.62 for BE
		0.673 for Healthcare
		1.00 for zero recovery
		0.00 for full recovery
μ	instantaneous rate of return of stock	0.06765 for $r = 0$
σ	instantaneous volatility of stock's return	0.15
r	risk free short rate of interest	0

the loss rate, the time to maturity, the risk aversion, the default intensity, and the interest rate. We quantify the optimal investment for the defaultable bond by adopting parameter values from the literature. We take the estimates of risk premia from Duffie et al. (2004) for three industries such as Oil and Gas, Broadcasting and Entertainment (BE), and Health-care. Hou (2003) and Driessen (2005) also provide parameter values similar to Duffie et al. (2004). Table I summarizes the parameters' values.

Since a higher default risk premium induces a positive amount of investment in the defaultable security, we are not surprised to find a positive relationship between optimal investment in the defaultable security and the default risk premium. Figure 1 shows that the investor buys more defaultable security as the default risk premium increases. Note that the optimal investment in the defaultable bond is increasing with a decreasing rate when the default risk premia increase.

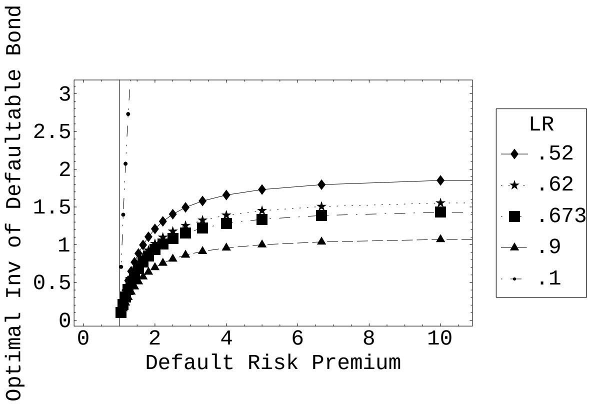

Figure 1. Optimal investment of a defaultable bond versus the default risk premium when the loss rate (LR) is 0.1 0.52 0.62 0.673 0.9 given $\gamma = 0.3$

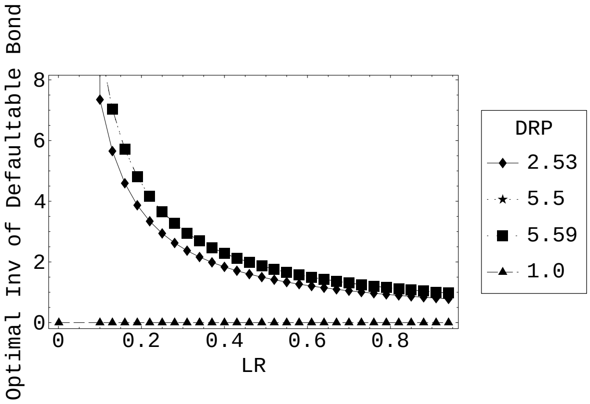

Figure 2. Optimal investment of a defaultable bond versus the loss rate (LR) when default risk premium (DRP) is 1.0, 2.53, 5.5, 5.59 given $\gamma = 0.3$

In Figure 1, we also observe that if the loss rate is small then the investor responds more to the increase in the default risk premium. As the loss rate increases, however, the investor still enjoys the high risk premium, but with much care. The concave shape of optimal investment of the defaultable security is getting flatter as ζ increases. When the loss rate (recovery rate) is very low (high), the marginal increase of the optimal strategy of the defaultable bond is huge. Thus the investor enjoys the priced default event risk premium since she can recover a lot even in the case of default. Note that if the default risk premium is equal to one the investor does not invest in the defaultable security, which is shown as the horizontal line.

Next we analyze the relationship between the loss rate and the optimal strategy. Since the higher loss rates mean less recovery values the optimal investment in the defaultable security shows a decaying pattern. Figure 2 indicates that when the default risk premium is high the investor decreases the investment in the defaultable security, but still invests more compared to the case of low default risk premium as the loss rate increases.

6. Conclusion

In this paper we address the optimal portfolio problem with a defaultable security. We adopt the intensity based approach to modeling defaultable securities. In addition, we assume the recovery of market value throughout the paper. This is a key distinguishing feature, as most papers ignore the problem of a positive recovery value in the event of a default or they just simply assume zero recovery. Thus this paper improves upon the existing literature by explicitly modeling the recovery amount in the dynamics of the defaultable bond price. The theory of stochastic optimal control is utilized as opposed to the martingale approach to solve the investor's problem. We obtained closed form solutions for the optimal strategies for a default-free bank account, a stock, and a defaultable bond.

We also obtained the results that link the optimal strategy of the defaultable security to the default risk premium, the loss rate, the investment planning horizon, relative risk aversion, and the intensity. In fact, from the optimal amount invested in the defaultable security and the default-risk premium given zero interest rates we find: (1) for a default-risk premium greater than one, the investor will optimally invest a positive amount in the defaultable bond, and (2) for a default-risk premium equal to one, the investor will optimally invest nothing in the defaultable bond. A default-risk premium greater than one (equal to one) can be interpreted as the market is pricing (not pricing) the default-event risk premium in the defaultable bond, respectively. Thus we can say that if the market is pricing the default-event risk, the investor takes a long position in the defaultable security and if it is not priced the investor puts zero money in the defaultable bond.

We provide numerical examples showing that the higher the default-risk premium is, the more defaultable securities the investor buys.

In spite of the intuitive contributions of this paper we made some rather strong assumptions. For example we assumed that a intensity rate, a loss rate (or recovery rate), and a interest rate are constants. We did not address any of benefits that might be obtained from a diverified portfolio with several defaultable assets. However, our intention was to study a relatively simple, but not trivial, mathematical set-up in order not to obscure the main ideas with technicalities, which will be a part of a more realistic model. These are issues which we are addressing in the current research.

Notes

[1] Default is usually defined as a violation of the covenant of a debt contract, such as a delay of payment or outright bankruptcy.

[2] He excludes the equity market from the optimal portfolio problem and thus ignores any interaction that may occur between the equity and fixed income market (default-free and defaultable).

[3] Korn and Kraft (2003) took an alternative approach to solving the portfolio problem with a default risky security. They utilized the structural approach (Merton (1974)) to model default risk in their paper. The optimal amounts invested in each security are expressed as functions of either the elasticity or duration variable.

[4] A probability measure $\mathbb{Q}$ is said to be equivalent to another probability measure $\mathbb{P}$ if the two probabilities have the same null sets.

[5] Typically in the asset pricing literature one starts with the specification of a real world (statistical) probability measure $\mathbb{P}$ and assume there exists an equivalent probability measure $\mathbb{Q}$ which is defined as the risk neutral measure. This is necessary when one wishes to obtain the arbitrage free price of a security. In what follows we will be more concerned with the real world probability measure as our interest does not lie in the pricing of securities, but in the optimal asset allocation in these securities. Finally, we take this approach because the probability measure that is relevant to investor as it concerns taking or not taking risks is the real world probability measure. See Jarrow, Lando, and Turnbull (1997) in this area.

[6] Zero and one are both included. $\zeta = 0$ means the full recovery of market value, and $\zeta = 1$ indicates the zero recovery.

[7] Our analysis is valid when $\gamma < 0$. The utility function still exhibits concave shape, which means it is increasing with decreasing rates. When $\gamma = 0$ then the utility function is a logarithmic function. We can see this case ($\gamma = 0$) as follows: Except for a constant, the utility function $\mathcal{U}(x) = \frac{x^{\gamma}-1}{\gamma}$ is equal to the utility function specified in (**??**). The two utility function is equivalent in the sense that they generate the same optimal choices. The advantage in using

this modified form is that this function has a well-defined limit as $\gamma \longrightarrow 0$. From L'Hospital's rule we have that

$$\lim_{\gamma \longrightarrow 0} \frac{x^\gamma - 1}{\gamma} = \ln x,$$

which is the important special case of logarithmic utility. When the utility function is logarithmic the optimal portfolio found is a growth optimum portfolio. If $\gamma > 1$, however, the utility function is convex, which means the investor is risk-seeking. In the sensitivity analysis section, we analyze the impact of default risk premium on the portfolio selection by varying γ in the range of zero to one.

[8] $z = 1$ for post-default

[9] A strategy is called "myopic" if at each point in the investment planning horizon it depends only on the current values of the investment opportunity set $\{\mu(t), r(t), \sigma(t)\}$, independently of whether or not these values will change in the future. In addition, it does not depend on the planning horizon T either. (see Merton (1990)) In our set up, $\{\mu, r, \sigma\}$ are assumed to be constant.

[10] Note the post default case is simply a Merton problem and thus we spend no time analyzing this case.

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