A Risk Sensitive Intertemporal CAPM, With Application to Fixed Income Management *

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Abstract

This paper presents an application of risk sensitive control theory in financial decision making. A variation of Merton's continuous-time intertemporal capital asset pricing model is developed where the infinite horizon objective is to maximize the portfolio's risk adjusted growth rate. The resulting model is tractable and thus provides economic insight about optimal trading strategies as well as the fact that the strategy of 100% cash is not necessarily the least risky one. For fixed income applications we utilize the concept of rolling-horizon bonds, which are stochastic process models of certain mutual funds of zero coupon bonds. We show by numerical example that the optimal proportion of one's wealth to hold in an asset is given by a simple affine function of economic factors such as interest rates of various maturities.

Keywords: risk sensitive control, optimal portfolios, fixed income management, intertemporal capital asset pricing model

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Contents

1 Introduction

Beginning with the pioneering work by Merton [30], [31], [32] and continuing through the recent books by Karatzas and Shreve [26] and Korn [28], some very sophisticated stochastic control methods have been applied to portfolio management. But most of these applications have been concerned with, at least implicitly, the management of equities. In spite of an abundance of well known mathematical interest rate models, exemplified by the classical models of Vasicek [41], Duffie and Kan [20] and Heath, Jarrow, and Morton [23], one can find very few applications of modern control theory to fixed income management.

There are at least two possible explanations for this deficiency in the literature. First, most fixed income assets have finite lives, so they cannot be modeled by simple stochastic processes such as the infinitely-lived geometric Brownian motions that are commonly used for equities. Moreover, the maturing of bonds before the planning horizon causes modeling difficulties by forcing discontinuities in the trading strategy. Another possible explanation for the limited number of control theory applications to fixed income management is that, unlike equities, the risk and return characteristics of fixed income assets explicitly depend upon underlying exogenous factors, namely, levels of interest rates. Consequently, sensible trading strategies will need to be explicit functions of these interest rates, and so these interest rate factors need to be explicitly incorporated in any mathematical model.

Of course a stochastic control model having a general form suitable for fixed income management was formulated more than 25 years ago by Merton [31]. His famous intertemporal capital asset pricing model (ICAPM) features both assets and the exogenous factors which affect them, all of which are modeled as diffusion processes. For the problem of maximizing expected utility of consumption and/or terminal wealth, he then derived and provided economic interpretations of the corresponding Hamilton-Jacobi-Bellman equation. It is straightforward, in principle, to adapt his model to fixed income management, namely, by letting the factors be interest rates in accordance with a desired interest rate model and then letting the assets be securities like zero coupon, discount bonds. Unfortunately, this approach is unlikely to be successful because, except for a very few special cases, the ICAPM is intractable.

In one successful special case Kim and Omberg [27] derived explicit solutions of the ICAPM where the interest rate is constant and the Sharpe ratio of the only available risky asset follows an Ornstein-Uhlenbeck process. Canestrelli [13] and Canestrelli and Pontini [14] obtained explicit solutions where the short interest rate is described by an Ornstein-Uhlenbeck process and there is also a single risky asset. In all these cases the only fixed income asset is the bank account which earns interest at the short, locally riskless rate.

There have even been a few studies of the ICAPM where one or more of the risky assets are taken to be fixed income securities. Brennan and Schwartz [11] and Brennan, Schwartz, and Lagnado [10] used numerical methods to solve the Hamilton-Jacobi-Bellman partial differential equation for the optimal trading strategy, but they were hard pressed to derive a solution even though there were only three factors and a similar number of assets. Explicit results on the application of the ICAPM to fixed income management were recently obtained by Bajeux-Besnainou, Jordan, and Portait [4] and Sørensen [39]. Apparently working independently, they studied the same special case. They both assumed there is one factor, an Ornstein-Uhlenbeck process for the short, locally riskless interest rate, and there are three assets. The first is a bank account where money grows according to the short rate. The second is the zero coupon bond which matures at a fixed planning horizon T. With the Brownian motion driving the short rate being the sole Brownian motion affecting the term structure of interest rates, the zero coupon bond thus evolves according to the well-known model due to Vasicek [41]. The third asset is a stock which evolves according to a process that is a geometric Brownian motion except that its appreciation rate equals the short interest rate plus a constant; the Brownian motion driving this process is correlated with the one driving the interest rate term structure.

Both studies focus on the problem of maximizing expected utility of wealth at a finite planning horizon T, where the utility function is of the form $u(w) = w^{1-\gamma}/(1-\gamma)$ and $\gamma > 0$ is the parameter of constant relative risk aversion. For $\gamma = 1$ one actually has as a special limiting case $u(w) = \ln(w)$, giving rise to what is sometimes called the growth optimal or numeraire portfolio. Note that increasing values of the parameter γ correspond to increasing levels of risk aversion for the investor.

Using the risk neutral computational approach introduced by Pliska [35], [36], both studies derived the same general form for the optimal trading strategy: at every point in time hold the fraction $1/\gamma$ of one's wealth in the growth optimal portfolio and invest the rest in the zero coupon bond. This makes some intuitive sense, because the zero coupon bond is the riskless asset for this problem, and the bigger the value of γ, the bigger the proportion in this asset. Unfortunately, however, their solutions leave unresolved a troublesome issue. Although their proportions in the stock are constants, their proportions in the zero coupon bond are deterministic functions of time which are unbounded in every neighborhood of the planning horizon T. This difficulty is associated with the fact that the zero coupon bond's volatility converges to zero as time approaches maturity.

Research in mathematical control theory has progressed to the point where it is likely that soon there will be explicit solutions of more general versions of Merton's ICAPM. This will be very important, both for fixed income management as well as many other kinds of consumption/investment problems. However, in this paper we pursue a variation of the ICAPM where explicit, general solutions can be obtained right now. Retaining the same diffusion process model for assets and factors, our approach calls for changing from Merton's finite horizon expected utility objective to the infinite horizon objective of maximizing the risk adjusted growth rate. As explained in our earlier work [5], [6], [7], [8] and more fully in Section 3 of this paper, this criterion can be viewed as being analogous to the classical Markowitz single-period approach except that instead of trading off single-period criteria we are trading off the long run growth rate (which by itself is maximized by the growth optimal portfolio) versus the average volatility of the portfolio. Our risk adjusted growth rate objective emerges naturally from the application of recent mathematical results on risk sensitive control theory. The principal benefit of this objective is that it is an infinite horizon criterion and therefore, as with most control problems in general, the ICAPM is more tractable than if a finite horizon criterion is used.

Our initial work in this direction developed a model with Gaussian factors and with assets having constant volatilities and appreciation rates that are affine functions of the factor levels. The theoretical foundations of this model are found in [5], while its applications to asset allocation problems are explored in [8]. But while this model fully allows for there to be correlations between asset returns and movements of the factor levels, it has a critical shortcoming: the partial correlations between asset returns and movements of the factor levels must be zero, that is, the residuals of the asset returns must be independent of the residuals of the factors. While this assumption is reasonable for applications where the only interest rate asset is the bank account, it is unacceptable for models which include both interest rate factors and additional fixed income assets.

In this paper we develop a rather general model featuring our risk sensitive criterion and, as in the ICAPM, exogenous factors which explicitly affect the diffusion process dynamics of the assets. The details and main results are provided in Section 2 and 4; proofs are relegated to an Appendix. A special case of our model was recently studied by Fleming and Sheu [22]. They obtained sub-optimal investment strategies for small values of the risk-sensitivity parameter.

Not surprisingly, the form of our optimal trading strategies closely resembles the form of Merton's. With trading strategies expressed as vectors of the proportions of portfolio value held in the respective assets, there are two terms. The first is the usual myopic one which is porportional to the asset excess returns multiplied by the inverse of the asset covariance matrix. The second is what Merton called the hedging term, a term that necessarily involves the correlations between the asset residuals and the factor residuals. Its economic interpretation is unclear from Merton's set-up, however, because its sign and other properties can only be determined from an explicit solution of the corresponding Hamilton-Jacobi-Bellman equation. But in what is a significant contribution of this paper, we use our risk sensitive approach to provide a better understanding of the hedging term. In particular, by studying a special case (see Section 5) we show that the risk of a portfolio, as measured by what we call the asymptotic variance, is affected not only by the usual volatilities (i.e., by the asset diffusion coefficients) but also by correlations between the residuals of the assets and the residuals of the factors, acting in conjunction with stochastic appreciation rates (i.e., local mean returns). This correlation-dependent effect corresponding to an individual asset can either increase or decrease the asymptotic variance of the whole portfolio, and so the hedging term will accordingly adjust the optimal proportion coming from the myopic term up or down.

Our conclusions about the hedging term are consistent with Lo and Wang ([29]) as well as Campbell and Viceira ([12]). The former point out that in many cases the conventional statistical measures (such as would correspond to our asymptotic variance) of standard deviations of asset returns fail to give unbiased estimates of asset volatilities (i.e., diffusion coefficients). In our paper we extend their observations by showing that such statistical discrepancies can be attributed to correlations between asset residuals and factor residuals. The latter researchers applied a discrete time optimal portfolio model to fixed income assets and obtained a similar two-term structure for the optimal trading strategies. In particular, they obtained the same result as ourselves: as investor risk aversion increases the myopic term disappears but the intertemporal hedging component of the optimal component in a risky asset can remain non-zero. We argue that this phenomenon is due to the fact that holding 100% cash is not necessarily the least risky strategy.

In Section 6 we turn to the application of our theory to fixed income management. By utilizing the concept of rolling-horizon bonds, a concept introduced by Rutkowski [37], we are able to incorporate fixed income assets having infinite lives. These assets can be viewed, roughly, as mutual funds of zero coupon bonds, all of which mature at about the same fixed distance in the future; these bonds are rolled over in a self financing manner so this same fixed distance is preserved through the course of time. Moreover, they are closely related to Constant Maturity Treasury (CMT) rates, Constant Maturity Swap rates, and what Rutkowski [37] calls *sliding bonds*. We provide explicit theoretical results in Section 7 for a model having three kinds of assets: the bank account, rolling horizon bonds, and (possibly) equities; all the factors are various interest rates. We also use twenty years of monthly data to provide a simple numerical example. The form of the optimal strategy is in accordance with economic intuition and is shown to depend significantly on the correlations between the residuals.

2 Formulation of the Optimal Risk Sensitive Asset Management Problem

In this section we formulate a general optimal dynamic asset management problem with risk sensitive optimality criterion. We consider a market consisting of $n \geq 2$ securities and $m+1$, $m \geq 0$, economic factors. The set of securities may include stocks, certain fixed income assets, cash and derivative securities, as in Brennan, Schwartz and Lagnado [10] and Bajeux-Besnainou, Jordan, and Portait [4] for example. The set of factors may include dividend yields, price-earning ratios, short-term interest rate, yields on various bonds, the rate of inflation, etc., as in Pesaran and Timmermann [33], [34] for example.

Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, \mathbf{P})$ be the underlying probability space. Denoting by $S_i(t)$ the price of the i-th security and by $X_j(t)$ the level of the j-th factor at time t, we consider the following market model for the dynamics of the security prices and factors:

$$\frac{dS_i(t)}{S_i(t)} = (a + AX(t))_i dt + \sum_{k=0}^{N} \sigma_{ik} dW_k(t), \quad S_i(0) = s_i, \quad i = 1, 2, \cdots, n \tag{1}$$

$$dX(t) = (b + BX(t))dt + \Lambda dW(t), \quad X(0) = x, \tag{2}$$

where $W(t) = (W_0(t), W_1(t), \ldots, W_N)'$ is a R^{N+1} valued standard Brownian motion process, $X(t) = (X_0(t), X_1(t), \ldots, X_m)'$ is the R^{m+1} valued factor process, the market parameters $a,\ A,\ \Sigma := [\sigma_{ij}],\ b,\ B,\ \Lambda := [\Lambda_{ij}]$ are matrices of appropriate dimensions, and $(a + Ax)_i$ denotes the $i - th$ component of the vector $a + Ax$. It is well known that a unique, strong solution exists for (1), (2), and that the processes $S_i(t)$ are positive with probability 1 (see e. g. [25], chapter 5).

Let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$, where $S(t) = (S_1(t), S_2(t), \ldots, S_n(t))$ is the security price process. Let $h(t)$ denote an R^n valued investment process (or strategy) whose components $h_i(t),\ i = 1, 2, \ldots, n$, represent the time-$t$ proportions of wealth in the corresponding assets.

Definition 2.1 An investment process $h(t)$ is **admissible** if the following conditions are satisfied:

(i) $\sum_{i=1}^{n} h_i(t) = 1$,

(ii) $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0, t]) \otimes \mathcal{G}_t\}_{t\geq 0}$,

(iii) for every $\theta > 0$ there exists a probability measure $\mathbf{P}^{h,\theta}$ on $(\Omega, \mathcal{F})$ so that

$$\frac{d\mathbf{P}^{h(\cdot),\theta}}{d\mathbf{P}}\Big|_{\mathcal{F}_t} = \eta_t(h(\cdot), \theta),$$

where

$$\eta_t(h(\cdot), \theta) = exp\{-\frac{\theta^2}{8} \int_0^t \|h'(s)\Sigma\|^2 ds - \frac{\theta}{2} \int_0^t h'(s)\Sigma dW_s\},$$

(iv) $\limsup_{t\to\infty} t^{-1} \ln \mathbf{E}^{h(\cdot),\theta} exp\{\left(\frac{\theta}{2}\right)(X_t' K_1(\theta) X_t + K_2(\theta) X_t\} \leq 0,$ where $K_1(\theta)$ and $K_2(\theta)$ are matrices defined later in the text, and $\mathbf{E}^{h(\cdot),\theta}$ denotes expectation under $\mathbf{P}^{h(\cdot),\theta}$.

The class of admissible investment strategies will be denoted by $\mathcal{H}$. □

Let now $h(t)$ be an admissible investment process. Then there exists a unique, strong, and almost surely positive solution $V(t)$ to the following equation:

$$dV(t) = \sum_{i=1}^{m} h_i(t)V(t)\big[\mu_i(X(t))dt + \sum_{k=0}^{N} \sigma_{ik} dW_k(t)\big], \quad V(0) = v > 0\ , \tag{3}$$

where $\mu_i(x)$ is the i-th coordinate of the vector $a + Ax$ for $x \in R^{m+1}$. The process $V(t)$ represents the investor's capital at time t, and $h_i(t)$ represents the proportion of capital that is invested in security i, so that $h_i(t)V(t)/S_i(t)$ represents the number of shares invested in security i, just as in, for example, Section 3 of [24].

In this paper we consider the following family of risk sensitized optimal investment problems, labeled as $\mathcal{P}_\theta$:

for $\theta \in (0, \infty)$, maximize the risk sensitized expected growth rate

$$J_\theta(v, x; h(\cdot)) := \liminf_{t\to\infty} (-2/\theta)t^{-1} ln\ \mathbf{E}^{h(\cdot)}\ [e^{-(\theta/2) ln V(t)} | V(0) = v, X(0) = x] \tag{4}$$

over the class of all admissible investment processes $h(\cdot)$, subject to (1) and (2),

where $\mathbf{E}^{h(\cdot)}$ is the expectation with respect to $\mathbf{P}$. The notation $\mathbf{E}^{h(\cdot)}$ emphasizes that the expectation is evaluated for process $V(t)$ generated by (3) under the investment strategy $h(t)$.

3 Discussion of the Risk-Sensitive Optimality Criterion

In this section we shall explain and discuss some important aspects of the risk-sensitive optimality criterion. We do this by drawing connections between it and various concepts in economics, probability theory, and statistics. Although we restrict attention throughout the rest of the paper to the case where the risk aversion parameter $\theta > 0$, in this section we often allow θ to be non-positive.

3.1 Origins of risk sensitivity and its relation to the Arrow-Pratt measure of absolute risk aversion

In order to motivate our risk sensitive optimality criterion, which applies to an infinite horizon, dynamic setting, consider a simple one-period gamble and a Von Neumann-Morgenstern type utility function denoted by U. The function U is a real-valued function on $(0, \infty)$ or $(-\infty, \infty)$ assumed in this subsection to be strictly increasing and strictly concave. Given these assumptions about U we know that there exists an inverse function to U, denoted as U^{-1}.

Suppose now that we are given an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a real-valued random variable C such that both C and the random variable $U(C)$ are integrable [with respect to the probability measure $\mathbf{P}$]. Then Jensen's inequality tells us that

$$\mathbf{E}(U(C)) \le U(\mathbf{E}(C)), \tag{5}$$

where $\mathbf{E}$ denotes expectation with respect to $\mathbf{P}$. Thus as Whittle ([42], page 6) observes, the variability of the random variable C is penalized by a concave utility function. Or in rough terms it can be said that

$$\mathbf{E}(U(C)) = U\big(m + variability\big) \le U(m), \tag{6}$$

where we introduced the expected value $m := \mathbf{E}(C)$. Note that the value $\tilde{m} := m + variability$ here can be viewed as the certainty equivalent for the random variable C relative to the function U (see for example Alexander and Francis [1], page 21).

One may interpret this statement in a more meaningful way if one imposes suitable regularity and integrability conditions on the function U and the random variable C, because then one has [compare Whittle [42], page 6, or Samperi, [38], section 4.8]:

$$U^{-1}\big(\mathbf{E}(U(C))\big) = m - \frac{1}{2} l_U(m) Var(C) + \ldots, \tag{7}$$

where the " ... " terms involve the higher order central moments of C and the higher order derivatives of U evaluated at $m = \mathbf{E}(C)$. The expression $l_U(m)$ here is defined by

$$l_U(c) := -\frac{U''(c)}{U'(c)},$$

and so in (7) it represents a value of the Arrow-Pratt function of absolute risk-aversion (see for example Alexander and Francis [1], Section 2.8). The terms $-\frac{1}{2} l_U(m) Var(C) + \ldots$ in (7) account for the variability of C. Note that combining (6) with (7) gives the following expression for the certainty equivalent:

$$\tilde{m} = m - \frac{1}{2} l_U(m) Var(C) + \ldots. \tag{8}$$

A *risk sensitive utility function* is of the form

$$U(c;\theta) = -\frac{2}{\theta} e^{-\frac{\theta}{2} c}, \tag{9}$$

where $\theta > 0$ is a fixed parameter. One easily calculates the Arrow-Pratt measure in this case: $l_U(c) = \frac{\theta}{2} > 0$ for every real c. This means that the investor is risk averse and that the strength of the investor's risk aversion is entirely encoded in the value of the *risk sensitivity parameter* θ. For the risk sensitive utility function U we have that $U^{-1}(u;\theta) = -\frac{2}{\theta}\ln(-\theta u/2)$. Thus (7) becomes

$$\delta_\theta(C) := -\frac{2}{\theta} \ln \mathbf{E}(e^{-\frac{\theta}{2} C}) = \mathbf{E}(C) - \frac{\theta}{4} Var(C) + \ldots \quad . \tag{10}$$

We introduced the function δ_θ because using it our risk sensitive optimality criterion (4) can be written simply as

$$J_\theta(v, x; h(\cdot)) := \liminf_{t\to\infty} t^{-1} \delta_\theta(\ln V(t)) \tag{11}$$

for the appropriate expectation operator $\mathbf{E}$. Thus (10) gives an interpretation of a portion of our risk sensitive optimality criterion: ignoring the higher order terms it equals the expected value of $\ln V(t)$ minus the penalty term $\frac{\theta}{4} Var(\ln V(t))$. This again shows that the bigger the value of the risk aversion parameter θ, the bigger the investor's worry about the variance of time-t capital.

It turns out that there is justification for ignoring the higher order terms. For the rest of this subsection we shall allow θ to vary over the entire real line. Moreover, we shall fix the random variable C and consider δ as a function of θ. This function is well defined for all $\theta \neq 0$. And since

$$\lim_{x\to 0} \Big(-\frac{2}{x}\Big) \ln \mathbf{E}(e^{-\frac{x}{2} C}) = \mathbf{E}(C), \tag{12}$$

we can naturally extend the definition of this function by taking $\delta_0(C) = \mathbf{E}(C)$. Applying the Taylor expansion of δ_θ around $\theta = 0$ we thus see that

$$\delta_\theta = \mathbf{E}(C) - \frac{\theta}{4} Var(C) + O(\theta^2). \tag{13}$$

This provides another interpretation of (10) and, in particular, of the " ... " terms in (10). Moreover, this shows that $\theta < 0$ corresponds to a risk seeking investor and that $\theta = 0$ corresponds to an investor who does not care one way or another about risk.

Next, letting $C = \ln V(t)$ we obtain from (13)

$$\delta_\theta(\ln V(t)) = \mathbf{E}(\ln V(t)) - \frac{\theta}{4} Var(\ln V(t)) + O(\theta^2, t), \tag{14}$$

where the last term on the right hand side of (14) may now depend on t. Suppose that the following limits exist: our measure of performance $\lim_{t\to\infty} t^{-1}\delta_\theta(\ln V(t))$, the portfolio's *expected growth rate* $\lim_{t\to\infty} t^{-1}\mathbf{E}(\ln V(t))$, and the portfolio's *asymptotic variance* $\lim_{t\to\infty} t^{-1}Var(\ln V(t))$. Then it follows from (14) that the limit $\lim_{t\to\infty} t^{-1}O(\theta^2, t)$ also exists. In many situations this last limit is equal to zero or, at least, a function of θ that goes to zero as θ goes to zero. This will be the case in the example discussed in Section 5. So supposing that $\lim_{t\to\infty} t^{-1}O(\theta^2, t) = 0$ and combining this with (11) and (14) we obtain

$$J_\theta(v, x; h(\cdot)) = \lim_{t\to\infty} t^{-1}\mathbf{E}(\ln V(t)) - \frac{\theta}{4} \lim_{t\to\infty} t^{-1}Var(\ln V(t)). \tag{15}$$

This relation, representing the risk sensitive growth rate of the portfolio $J_\theta(v, x; h(\cdot))$ in terms of the portfolio's expected growth rate $\lim_{t\to\infty} t^{-1}\mathbf{E}(\ln V(t))$ and the portfolio's asymptotic variance $\lim_{t\to\infty} t^{-1}Var(\ln V(t))$, will be used in Section 5.

3.2 Investor Risk Aversion and the Parameter θ

The preceding calculations involving the function δ_θ suggest that, at the cost of losing concavity, the risk sensitive utility function $U(c;\theta)$ can be extended to non-positive values of θ as follows:

$$U(c;\theta) = \begin{cases} -\frac{2}{\theta} e^{-\frac{\theta}{2}c}, & \text{if } \theta \neq 0 \\ c, & \text{if } \theta = 0\ . \end{cases} \tag{16}$$

We can use this to obtain an additional interpretation of our risk sensitive optimality criterion (4), for all real numbers θ. Recall that for any admissible trading strategy $h(\cdot)$ the wealth process V_t defined in (3) is (strictly) positive with probability one. For the functions $U(c;\theta)$ we thus obtain the following rates:

$$\liminf_{t\to\infty} t^{-1}U^{-1}\Big(\mathbf{E}^{h(\cdot)}\Big[U(\ln V_t;\theta)\Big];\theta\Big) = \begin{cases} \liminf_{t\to\infty} (-2/\theta) t^{-1} \ln \mathbf{E}^{h(\cdot)}\Big(e^{-\frac{\theta}{2}\ln V_t}\Big), & \text{if } \theta \neq 0 \\ \liminf_{t\to\infty} t^{-1}\mathbf{E}^{h(\cdot)}(\ln V_t), & \text{if } \theta = 0\ . \end{cases} \tag{17}$$

In cases where $\theta \neq 0$ we call these rates the *risk sensitive growth rates* or the *risk adjusted growth rates* of the portfolio. Cases with $\theta < 0$ correspond to a risk-seeking investor and cases with $\theta > 0$ correspond to a risk-averse investor. In the case $\theta = 0$ we call the respective rate the *risk-null* growth rate. This is the well known Kelly criterion.

3.3 Cumulant Functions, Geometric Brownian Motion, and the Risk Sensitive Frontier

A *moment generating function* of the random variable C is the function

$$M_C(t) := \mathbf{E}(e^{tC}),$$

where t is a real-valued variable [the domain of the function M_C may not be the entire real line though]. Observe that $M_C(t)$ is strictly positive for these values of t for which it is well defined. Thus we may define the corresponding *cumulant generating function*

$$K_C(t) := \ln M_C(t). \tag{18}$$

It can be shown (see e.g. Billingsley [9], page 144) that the cumulant generating function can be expanded into a series [at least in some neighborhood of $t = 0$] of the form

$$K_C(t) = \sum_{i=1}^{\infty} \frac{k_i}{i!} t^i, \tag{19}$$

where the coefficients k_i are called the cumulants (or semi-invariants) of C. In particular we have

$$k_1 = \mathbf{E}(C), \quad k_2 = Var(C). \tag{20}$$

Thus letting $t = -\frac{\theta}{2}$ we obtain from (19) and (20)

$$\ln \mathbf{E}(e^{-\frac{\theta}{2}C}) = K_C(-\frac{\theta}{2}) = -\frac{\theta}{2}\Big(\mathbf{E}(C) - \frac{\theta}{4}Var(C)\Big) + \sum_{i=3}^{\infty} \frac{k_i}{i!}\Big(-\frac{\theta}{2}\Big)^i. \tag{21}$$

In particular, if C has a Gaussian distribution with mean μ and variance σ^2 then the cumulants of order 3 and higher are zero. In this case (21) takes the form

$$\ln \mathbf{E}(e^{-\frac{\theta}{2}C}) = K_C(-\frac{\theta}{2}) = -\frac{\theta}{2}\Big(\mu - \frac{\theta}{4}\sigma^2\Big). \tag{22}$$

Now suppose the wealth process V_t is a simple geometric Brownian motion (which happens if the entire portfolio is held in a single asset following model (1) with the constant appreciation rate μ and the constant volatility σ)

$$V_t = V_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t}, \quad t \geq 0, \tag{23}$$

where W_t is a standard, real-valued Brownian motion. It follows from (22) that for every $\theta \neq 0$ the risk sensitive growth rate of such a portfolio is

$$\lim_{t\to\infty} (-2/\theta)t^{-1} \ln \mathbf{E}\Big(e^{-\frac{\theta}{2}\ln V_t}\Big) = \lim_{t\to\infty} (-2/\theta)t^{-1} K_{\ln V_t}(-\frac{\theta}{2}) = \mu - \frac{1}{2}\sigma^2 - \frac{\theta}{4}\sigma^2. \tag{24}$$

More generally, if the assets are described by a multivariate geometric Brownian motion with constant appreciation rates and constant volatilities and if the investor rebalances continuously to constant proportions $h(t) = h_{\rm const}$, then the wealth process $V(t)$ will still be a simple geometric Brownian motion and we will still have

$$\lim_{t\to\infty}(-2/\theta)t^{-1}\ln\mathbf{E}^{h_{\rm const}}\left(e^{-\frac{\theta}{2}\ln V_t}\right) = \lim_{t\to\infty}(-2/\theta)t^{-1}K_{\ln V_t}(-\frac{\theta}{2}) = \mu_{h_{\rm const}} - \frac{1}{2}\sigma^2_{h_{\rm const}} - \frac{\theta}{4}\sigma^2_{h_{\rm const}}, \tag{25}$$

where $\mu_{h_{\rm const}}$ and $\sigma_{h_{\rm const}}$ are the appreciation rate and volatility of the portfolio, respectively. This observation leads to the concepts of the *(asymptotic) risk sensitive frontier* and the *risk sensitive feasible region*, concepts which are analogous to the efficient frontier and the feasible region of the single period Markowitz model. Corresponding to any constant proportion trading strategy are a measure analogous to the mean return, namely, the growth rate $\mu_{h_{\rm const}} - \frac{1}{2}\sigma^2_{h_{\rm const}}$, and a measure analogous to the standard deviation of the return, namely, the *asymptotic variance* $\sigma^2_{h_{\rm const}}$. The set of all such pairs corresponding to all constant proportion trading strategies is the risk sensitive feasible region. The portion of the boundary of this region where the growth rate is maximized for a fixed level of asymptotic variance is the risk sensitive frontier. Viewed in this way, for the problem of maximizing the growth rate subject to a fixed asymptotic variance, the parameter θ plays the role of a Lagrange multiplier.

Of course when assets are not geometric Brownian motions the constant proportion strategies will not necessarily be optimal, and so strictly speaking this geometric interpretation does not hold in general. However, in very general situations this geometric picture will be valid in an approximate way. Corresponding to each trading strategy will be a growth rate and an asymptotic variance, the latter being, in general, $\liminf_{t\to\infty} t^{-1}Var(\ln V(t))$. Thus corresponding to a problem there will be a risk sensitive feasible region and a risk sensitive frontier. Moreover, the solution of a risk sensitive optimization problem will provide a solution that corresponds approximately to some point on the risk sensitive frontier. The accuracy of this approximation hinges on "the higher order terms" in (10). In other words, subject to the higher order terms and possibly some regularity conditions, the risk sensitive optimality criterion seeks to maximize a portfolio's growth rate subject to a constraint on its asymptotic variance.

3.4 Large Deviations, Donsker-Varadhan Rate and Risk-Sensitivity

Once again let us consider the wealth process V_t as a simple geometric Brownian motion

$$V_t = V_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W_t}, \quad t \geq 0, \tag{26}$$

where W_t is a standard, real-valued Brownian motion. The law of large numbers for a standard Brownian motion implies that

$$\lim_{t\to\infty}\frac{1}{t}\ln V_t = \mu - \frac{1}{2}\sigma^2, \quad \mathbf{P} - \text{a.s.} \tag{27}$$

Typically, the process $t^{-1}\ln V_t$ will oscillate around the limiting value $\mu - \frac{1}{2}\sigma^2$. The limiting probabilities of "large deviations" of the process $t^{-1}\ln V_t$ from the limiting value $\mu - \frac{1}{2}\sigma^2$ are controlled by the so called *(good) rate function* $I(x)$, (see e.g. [16], [19], [17]). There is an interesting connection between the risk sensitive limit (24) and the rate function $I(x)$ given by the Donsker-Varadhan variational formula (Donsker and Varadhan [18]; see Dembo and Zeitouni [16] or Ellis and Dupuis [19] for the most recent monographs incorporating the Donsker-Varadhan theory):

$$\lim_{t\to\infty} t^{-1}\ln \mathbf{E}\Big(e^{-\frac{\theta}{2}\ln V_t}\Big) = \inf_{x\in\mathbf{R}}\big\{-\frac{\theta}{2}x - I(x)\big\}. \tag{28}$$

The left hand side of (28) is sometimes referred to as the Donsker-Varadhan rate. Using duality relationships (see e.g. [16], Theorem 4.5.10) one can easily compute the rate function $I(x)$ in this geometric Brownian motion case:

$$I(x) = \frac{1}{2\sigma^2}\Big(x-\mu+\frac{\sigma^2}{2}\Big)^2.$$

3.5 Entropy Games and Risk-Sensitivity

The observations made in this sub-section follow from the Donsker-Varadhan theory. We denote by $\mathcal{P}(\mathbf{R})$ the set of probability measures on the real line $\mathbf{R}$, and we fix a measure $P\in\mathcal{P}(\mathbf{R})$. The (relative) entropy $H(\cdot|P)$ is a function defined on $\mathcal{P}(\mathbf{R})$ as (see e.g. [19], page 32)

$$H(Q|P) = \begin{cases} \int_{\mathbf{R}}\big(\ln\frac{dQ}{dP}\big)dQ, & \text{if } Q \text{ is absolutely continuous w.r.t. } P\\ +\infty, & \text{otherwise.}\end{cases}$$

Suppose now that ψ is a bounded, measurable function on $\mathbf{R}$, and let us fix a time horizon $T>0$. In the context of the market model of Section 2 let us denote by $\mathbf{P}_T^{h(\cdot)}$ the probability measure generated on $\mathbf{R}$ by the random variable V_T under the admissible investment strategy $h(\cdot)$. It follows from the Donsker-Varadhan theory that

$$-\ln\mathbf{E}^{h(\cdot)}\Big(e^{-\frac{\theta}{2}\psi(V_T)}\Big) = \inf_{Q\in\mathcal{P}(\mathbf{R})}\{H(Q|P_T^{h(\cdot)}) - \frac{\theta}{2}\mathbf{E}_Q\psi(V_T)\}, \tag{29}$$

where $\mathbf{E}_Q$ denotes expectation with respect to the measure Q. Thus if one is interested in maximizing the risk sensitive criterion $-\frac{\theta}{2}\ln\mathbf{E}^{h(\cdot)}\Big(e^{-\frac{\theta}{2}\psi(V_T)}\Big)$ over the class of all admissible strategies, then one arrives at the following entropy game:

$$\sup_{h(\cdot)\in\mathcal{H}}\Big(-\frac{\theta}{2}\Big)\ln\mathbf{E}^{h(\cdot)}\Big(e^{-\frac{\theta}{2}\psi(V_T)}\Big)$$

$$=\sup_{h(\cdot)\in\mathcal{H}}\Big\{\frac{\theta}{2}\inf_{Q\in\mathcal{P}(\mathbf{R})}\big\{H(Q|P_T^{h(\cdot)}) - \frac{\theta}{2}\mathbf{E}_Q\psi(V_T)\big\}\Big\}. \tag{30}$$

This result is of potential importance due to duality theory. In particular, there may be circumstances where the dual problem, that is, the entropy game, is easier to solve than the original, primal problem. Unfortunately, we do not know at this time whether an analogous characterization will hold for linear or other unbounded, measurable functions ψ. This is a challenging problem for future research. Interesting developments in this direction are presented in Delbaen et al. [15].

3.6 Coherent Risk Measures and Risk-Sensitivity

Artzner et al. ([2],[3]) defined and analyzed coherent measures of risk. One of the conclusions of their analysis was that VaR (Value-at-Risk) is not a coherent measure of risk. This is because VaR violates the sub-additivity axiom (see e.g. Artzner et al. [3]).

The risk sensitive criterion has an appropriately weighted measure of risk built into it. Its main objective is to allow for finding an optimal compromise between the investor's desire of maximizing the growth rate of her/his portfolio and minimizing the probabilities of the portfolio's deviations from the expected maximal growth rate. It turns out that the risk sensitive criterion is a *coherent optimization criterion*. It satisfies four dynamic coherence axioms which are counter-parts of the four Artzner et al. coherence axioms defined for risk measures. To see this, it is convenient to make use of the function δ_θ that was introduced in (10). In particular, we shall investigate

$$\delta_\theta(\ln V) = (-\frac{2}{\theta})\ln \mathbf{E}\Big(e^{(-\frac{\theta}{2})\ln V}\Big),$$

where V is any positive random variable and $\theta > 0$.

It can be easily verified that $\delta_\theta(lnV)$ satisfies the four axioms below. For their economic interpretations the reader is invited to compare them with the corresponding coherence axioms presented in Artzner et al. [3], keeping in mind that here a bigger value of the risk measure $\delta_\theta(\ln V)$ is "better."

<u>Axiom 1</u> (logarithmic super-additivity and sub-additivity)

$$\delta_\theta(\ln V_1 + \ln V_2) \geq \delta_\theta(\ln V_1) + \delta_\theta(\ln V_2),$$

if V_1 and V_2 are non-negatively correlated,

$$\delta_\theta(\ln V_1 + \ln V_2) \leq \delta_\theta(\ln V_1) + \delta_\theta(\ln V_2),$$

if V_1 and V_2 are non-positively correlated,

<u>Axiom 2</u> (logarithmic homogeneity)

$$\delta_\theta(t \ln V) = t\delta_{t\theta}(\ln V),$$

<u>Axiom 3</u> (monotonicity)

$$\delta_\theta(\ln V_1) \geq \delta_\theta(\ln V_2) \quad iff \quad V_1 \geq V_2,$$

<u>Axiom 4</u> (logarithmic risk-free condition)

$$\delta_\theta(\ln V + nr) = \delta_\theta(\ln V) + nr,$$

where r is the (constant) risk-free interest rate and n is any integer.

The above "logarithmic" form of the axioms is appropriate due to the fact that portfolios grow at some exponential rate in a continuous time framework.

3.7 Infinite Horizon Criterion vs Finite Horizon Criterion

Bielecki and Pliska [5] considered problem (4) under the assumption that $\Sigma\Lambda' = 0$. They showed that the investment strategy which is optimal for this infinite time horizon is also *universally optimal with respect to the planning horizon*. This means that the same investment strategy maximizes exponential utility from terminal wealth at any terminal date T. In other words, investors with a specific finite planning horizon will also optimally benefit from investing according to this same strategy provided the market is driven according to the model of Bielecki and Pliska [5]. Such a strong result is not generally true. However, there are good reasons to believe that the optimal investment strategy for problem (4), derived later in this paper, will also perform well for relatively long, but finite, time horizons. This sub-optimality issue will require further studies though.

4 The Main Mathematical Results

We make the following standing assumptions with regard to the problem (4):

Assumption (A1) *The spectrum of the matrix B is contained in the left half plane [i.e. the matrix B is stable].*

Remark 4.1 We observe here that Assumption (A1) is not a necessary assumption for the results below to hold. Nevertheless, we make this assumption as it results in the factor process to be an ergodic Gaussian process. In economic terms this means that the factor process mean reverts to its ergodic mean. □

Assumption (A2) *(a) The matrix $\Sigma\Sigma'$ is positive definite; (b) The matrix $\Lambda\Lambda'$ is positive definite.*

Remark 4.2 It is worth emphasizing at this point that we do not assume independence of random perturbations driving the price and the factor dynamics. That is, we do not assume that $\Sigma\Lambda' = 0$. Such an assumption was made in [5]. □

Let us consider now the following continuous algebraic Riccati equation (CARE)

$$K'R_1(\theta)K + K'R_2(\theta) + R_2(\theta)'K + R_3(\theta) = 0, \tag{31}$$

where

$$R_1(\theta) := \frac{\theta^2}{2}\Big(\frac{\theta}{2}+1\Big)^{-1}\Lambda\Sigma'\Psi\Sigma\Lambda' - \theta\Lambda\Lambda',$$

$$R_2(\theta) := -\frac{\theta}{2}\Big(\frac{\theta}{2}+1\Big)^{-1}\Lambda\Sigma'\Psi A + B,$$

$$R_3(\theta) := (1/2)\Big(\frac{\theta}{2}+1\Big)^{-1}A'\Psi A,$$

and

$$\Psi := (\Sigma\Sigma')^{-1} - \Gamma,$$

$$\Gamma := \frac{(\Sigma\Sigma')^{-1}\mathbf{1}\mathbf{1}'(\Sigma\Sigma')^{-1}}{\mathbf{1}'(\Sigma\Sigma')^{-1}\mathbf{1}},$$

$$\mathbf{1} = (1, 1, \ldots, 1)'.$$

In order to formulate our main results we need one more assumption:

Assumption (A3) *For every $\theta > 0$ the equation (31) admits a unique, positive semi-definite solution, say $K_1(\theta)$. Moreover, the matrix $G(\theta)$ defined below is stable:*

$$G(\theta) := R_2(\theta) + R_1(\theta)K_1(\theta). \tag{32}$$

Remark 4.3 Standard controllability and observability conditions imposed on the parameters of problems $\mathcal{P}_\theta$ will guarantee that Assumption (A3) is satisfied. As a matter of fact under these conditions stability of the matrix $G(\theta)$ is implied by positive semi-definiteness of the matrix $K_1(\theta)$. See Wonham [43], for example. □

In view of the Assumption (A3), for every $\theta > 0$ we may define a matrix $K_2(\theta)$ as follows:

$$K_2(\theta) := \Big(G(\theta)'\Big)^{-1}\bigg\{\Big(A - \theta\Sigma\Lambda'K_1(\theta)\Big)'\Big[\frac{(\Sigma\Sigma')^{-1}\mathbf{1}}{\mathbf{1}'(\Sigma\Sigma')^{-1}\mathbf{1}} - (\frac{\theta}{2}+1\big)^{-1}\Psi a\Big] - 2K_1(\theta)b'\bigg\}. \tag{33}$$

Theorems 2.1 and 2.2 below contain the main results of this paper. Theorem 2.1 characterizes optimal, risk-sensitive investment strategies. Theorem 2.2 characterizes the optimal value of the objective criterion.

Theorem 4.1 *Assume (A1),(A2) and (A3). Fix $\theta > 0$ and consider the process $h^\theta(t)$ defined as*

$$h^\theta(t) := (\frac{\theta}{2}+1\big)^{-1}(\Sigma\Sigma')^{-1}\Big[\beta(X_t) + \lambda(X_t)\mathbf{1}\Big], \tag{34}$$

where

$$\beta(x) := \big(A - \theta\Sigma\Lambda'K_1(\theta)\big)x + a - \frac{\theta}{2}\Sigma\Lambda'K_2(\theta), \tag{35}$$

$$\lambda(x) := \frac{(\frac{\theta}{2}+1\big) - \mathbf{1}'(\Sigma\Sigma')^{-1}\beta(x)}{\mathbf{1}'(\Sigma\Sigma')^{-1}\mathbf{1}}, \tag{36}$$

the matrix $K_1(\theta)$ is as in Assumption (A3), and the corresponding matrix $K_2(\theta)$ is defined in (33). Then for θ sufficiently small the investment process $h_\theta(t)$ is optimal for problem (P_θ), that is,

$$J_\theta(v, x; h(\cdot)) \leq J_\theta(v, x; h_\theta(\cdot))$$

holds for all $h(\cdot) \in \mathcal{H}$, $v > 0$, $x \in R^{m+1}$.

Theorem 4.2 *Assume (A1),(A2) and (A3), fix $\theta > 0$ sufficiently small, and consider the problem P_θ. Let $h_\theta(t)$ be as in Theorem 2.1. Then*

(a) For all $v > 0$ and $x \in R^{m+1}$ we have

$$J_\theta(v, x; h_\theta(\cdot)) = \lim_{t\to\infty}(-2/\theta)t^{-1}\ln\ \mathbf{E}^{h_\theta(\cdot)}\ [e^{-(\theta/2)\ln V(t)}|V(0) = v, X(0) = x] =: \rho(\theta).$$

(b) The constant $\rho(\theta)$ in (a) is given by the formula

$$\rho(\theta) = b'K_2(\theta) + (1/2)\Big[(-\theta/2)K_2(\theta)'\Lambda\Lambda'K_2(\theta) + 2tr\Lambda'K_1(\theta)\Lambda\Big]$$

$$+(1/2)\left(\frac{\theta}{2}+1\right)^{-1}\Big(a-(\theta/2)\Sigma\Lambda' K_2(\theta)\Big)'\Psi\Big(a-(\theta/2)\Sigma\Lambda' K_2(\theta)\Big)$$
$$-\frac{\mathbf{1}'(\Sigma\Sigma')^{-1}}{\mathbf{1}'(\Sigma\Sigma')^{-1}\mathbf{1}}\big(a-(\theta/2)\Sigma\Lambda' K_2(\theta)\big)+\frac{(1/2)\left(\frac{\theta}{2}+1\right)}{\mathbf{1}'(\Sigma\Sigma')^{-1}\mathbf{1}}, \tag{37}$$

where the matrix $K_1(\theta)$ is as in Assumption (A3) and the corresponding matrix $K_2(\theta)$ is defined in (33).

Remark 4.4 The key point of the first equality in (a) is, of course, that the optimal objective value is given by an ordinary lim rather than the lim inf as in (4). The key point of the second equality in (a) is that the optimal objective value does not depend on either the initial amount of the investor's capital (v) or on the initial values of the underlying economic factors (x), although it depends, of course, on the investor's attitude towards risk (encoded in the value of θ). □

Remark 4.5 It appears that the results of Theorems 2.1 and 2.2 are stronger than the results of section 7 in Fleming and Sheu [22], where only suboptimal investment strategies are discussed. □

Remark 4.6 Even though we require in Theorems 2.1 and 2.2 that the parameter θ is sufficiently small [the case that we can prove at the moment], we conjecture that the theorems remain true for all θ positive. □

Remark 4.7 Theorems 2.1 and 2.2 are demonstrated in the Appendix. □

4.1 The case where one of the assets is "risk free"

Let us consider the market (1) and (2) provided with an additional instantaneously risk-free asset, say $S_0(t)$, whose dynamics are given as

$$\frac{dS_0(t)}{S_0(t)}=(a_0+A_0X(t))dt, \quad S_0(0)=s_0. \tag{38}$$

Let $h(t)$ denote an R^{n+1}-valued investment process or strategy whose components are $h_i(t)$, $i=0,1,2,\ldots,n$, and which satisfies conditions analogous to the ones specified in definition 2.1. It is convenient to partition $h(t)$ as

$$h(t)=(h_0(t),\tilde{h}(t)).$$

Define

$$\tilde{A}:=A-\mathbf{1}A_0, \quad \tilde{a}:=a-\mathbf{1}a_0, \quad \tilde{\Psi}:=(\Sigma\Sigma')^{-1}.$$

In order to characterize the optimal investment strategy $h^\theta(t)$ in this case we need to consider the CARE (31) and the formulas (32) and (33) with A, a, and Ψ replaced with $\tilde{A}$, $\tilde{a}$ and $\tilde{\Psi}$, respectively. Then we have that an optimal investment strategy is given as

$$h^\theta(t)=(h_0^\theta(t),\tilde{h}^\theta(t)), \tag{39}$$

where

$$\tilde{h}^\theta(t)=(\theta/2+1)^{-1}\tilde{\Psi}\big[\big(\tilde{A}-\theta\Sigma\Lambda'\tilde{K}_1(\theta)\big)X_t+\tilde{a}-(\theta/2)\Sigma\Lambda'\tilde{K}_2(\theta)\big], \tag{40}$$

$$h_0^\theta(t) = 1 - \mathbf{1}'\tilde{h}^\theta(t), \tag{41}$$

$\tilde{K}_1(\theta)$ is the unique solution to (31) with A and Ψ replaced with $\tilde{A}$ and $\tilde{\Psi}$, respectively, and

$$\tilde{K}_2(\theta) = -\Big(\tilde{G}(\theta)'\Big)^{-1}\Big((\theta/2)(\theta/2+1)^{-1}\big[\tilde{A} - \theta\Sigma\Lambda'\tilde{K}_1(\theta)\big]'\tilde{\Psi}\tilde{a} + A_0' + 2\tilde{K}_1(\theta)'b\Big). \tag{42}$$

In the above formula the matrix $\tilde{G}(\theta)$ is defined just as $G(\theta)$ with A and Ψ replaced with $\tilde{A}$ and $\tilde{\Psi}$, respectively.

Finally, in this case we have

$$\rho(\theta) = (1/2)(\theta/2+1)^{-1}\Big(\tilde{a} - (\theta/2)\Sigma\Lambda'\tilde{K}_2(\theta)\Big)'\tilde{\Psi}\Big(\tilde{a} - (\theta/2)\Sigma\Lambda'\tilde{K}_2(\theta)\Big)$$

$$+(1/2)\Big[-(\theta/2)\tilde{K}_2(\theta)'\Lambda\Lambda'\tilde{K}_2(\theta) + 2\mathrm{tr}\Lambda'\tilde{K}_1(\theta)\Lambda\Big] + b'\tilde{K}_2(\theta). \tag{43}$$

In section 7 we shall apply the general results in this section to a fixed income management problem where there are a money market account, some fixed income assets, and possibly some equities. But first we provide an explicit example.

5 Explicit Results for a One-Factor Example

In this section we investigate a special case where there is a single economic factor and thus the continuous algebraic Riccati equation (31) is an ordinary quadratic equation in a single variable. The explicit solutions of this equation are used to obtain explicit formulas for the optimal trading strategy and related quantities, thereby providing useful economic insight.

The scalar-valued factor X is governed by the SDE

$$dX(t) = (b + BX(t))dt + \lambda_1 dW_1 + \lambda_2 dW_2, \tag{44}$$

where W_1 and W_2 are independent Brownian motions. There are a single risky asset S_1 with

$$dS_1/S_1 = (a + AX(t))dt + \sigma dW_1 \tag{45}$$

and an instantaneously risk-free asset S_0 (i.e., a bank account) with

$$dS_0/S_0 = (a_0 + A_0X(t))dt. \tag{46}$$

The quantities $b, B, \lambda_1, \lambda_2, a, A, \sigma, a_0,$ and A_0 are all scalar parameters with $B < 0$, $\sigma > 0$, and $\lambda_2 > 0$.

The CARE (31) in this case becomes (after substituting $\tilde{A} = A - A_0$ and $\tilde{\Psi} = \sigma^{-2}$)

$$\sigma^2\Big(2\theta\lambda_1^2 + \theta(\theta+2)\lambda_2^2\Big)K^2 + \Big(2\theta\lambda_1\sigma(A - A_0) - 2\sigma^2(\theta+2)B\Big)K - (A - A_0)^2 = 0. \tag{47}$$

To investigate the roots we first look at the discriminant

$$\Delta := \Big[2\theta\lambda_1\sigma(A-A_0) - 2\sigma^2(\theta+2)B\Big]^2 + 4\sigma^2(A-A_0)^2\Big[2\theta\lambda_1^2 + \theta(\theta+2)\lambda_2^2\Big]. \tag{48}$$

Both terms are nonnegative, so both roots of (47) will be real. Moreover, one root will be negative whereas the other will be nonnegative. In view of Assumption (A3) and Theorem 4.1 the latter is the one we want, so we take

$$\tilde{K}_1(\theta) = \frac{[\sigma^2(\theta+2)B - \theta\lambda_1\sigma(A-A_0)] + \sqrt{\Delta/4}}{\sigma^2[2\theta\lambda_1^2 + \theta(\theta+2)\lambda_2^2]}. \tag{49}$$

Note that $\tilde{K}_1(\theta) = 0$ if and only if $A = A_0$.

In view of (32) and (42) we have

$$\tilde{G}(\theta) = -\frac{\sqrt{\Delta/4}}{(\theta+2)\sigma} \tag{50}$$

and

$$\tilde{K}_2(\theta) = \frac{(\theta+2)\sigma}{\sqrt{\Delta/4}}\Big[\Big(\frac{\theta(a-a_0)}{(\theta+2)\sigma^2}\Big)[A - A_0 - \theta\lambda_1\sigma\tilde{K}_1(\theta)] + A_0 + 2\tilde{K}_1(\theta)b\Big]. \tag{51}$$

Note that $\tilde{G}(\theta)$ is stable, as desired. If $A = A_0$ then $\tilde{K}_1(\theta) = 0$, but in general, the sign of $\tilde{K}_2(\theta)$ is unclear.

In view of (40) the optimal proportion in the risky asset is given by

$$\tilde{h}^\theta(t) = \frac{2(a + AX(t) - [a_0 + A_0X(t)])}{(\theta+2)\sigma^2} - \frac{\theta}{\theta+2}\Big(\frac{\lambda_1}{\sigma}\Big)\Big[\tilde{K}_2(\theta) + 2\tilde{K}_1(\theta)X(t)\Big]. \tag{52}$$

The first term is the classical myopic term; it is proportional to the inverse of the covariance matrix multiplied by the risky asset's local excess return. The second term is the so-called hedging term. Since the sign of $\tilde{K}_2(\theta)$ is unclear, so is the sign of the hedging term. However, $\tilde{K}_1(\theta) > 0$ (unless $A = A_0$), so the hedging term causes a "rotation" of the optimal strategy from what it would be if there were only the myopic term. In particular, if $\lambda_1 > 0$ (so the residuals of the factor and the risky asset are positively correlated), then the rotation is clockwise because the hedging term is negative for all large values of $X(t)$ and positive for all small values of $X(t)$. Similarly, the rotation is counter-clockwise if the residuals are negatively correlated.

If $A = A_0$ (so the excess return of the risky asset is constant), then $\tilde{K}_1(\theta) = 0, \tilde{K}_2(\theta) = A/(\sigma|B|)$, and

$$\tilde{h}^\theta(t) = \frac{2}{\theta+2}\Big(\frac{a-a_0}{\sigma^2}\Big) - \frac{\theta}{\theta+2}\Big(\frac{\lambda_1 A}{\sigma^2|B|}\Big). \tag{53}$$

Hence the hedging term can still be present, but if so it will be constant with respect to the level $X(t)$ of the factor. Note that the sign of the hedging term will be negative if and only if $\lambda_1 A > 0$. For instance, if higher levels of the factor increase the risky asset's appreciation rate (i.e., if $A > 0$), then positively correlated residuals cause the optimal proportion in the risky asset to be smaller than what it would be if there were only the myopic term.

To summarize matters at this point, the preceding calculations of $\tilde{h}^\theta(t)$ make it clear that the presence of the hedging term is related to interactions between the residuals combined with the factor's effects on the assets via the appreciation rates. In particular, if the residuals are correlated, then the hedging term disappears if and only if $A = A_0 = 0$. The hedging term is present even when $A = 0$ (so the risky asset is ordinary geometric Brownian motion) provided both λ_1 and A_0 are non-zero (for instance, when $a_0 = 0$ and $A_0 = 1$, in which case the factor is precisely a Vasicek interest rate). However, it remains unclear from the economic standpoint why and how these interactions affect the optimal trading strategy the way they do.

A clue is provided by the observation that the optimal proportion in the risky asset does not necessarily converge to zero as the investor's risk aversion parameter increases to infinity. The myopic term in (52) certainly converges to zero as $\theta \to \infty$, and a little algebra reveals that $\tilde{K}_1(\theta)$ has the same limit. However, some additional algebra shows that $\tilde{K}_2(\theta)$ and thus the hedging term in (52) can have a non-zero limit as $\theta \to \infty$. Hence for the very risk averse investor it may be optimal to invest at least some capital in the risky asset, rebalancing continuously so as to maintain a constant proportion (since $\tilde{K}_1(\theta) = 0$ for large θ, this investor's optimal proportion does not depend upon the level of the factor).

This result is disconcerting, because the conventional wisdom says that the trading strategy of holding 100% cash is the least risky one. Campbell and Viceira [12] reported a similar result for their discrete time asset management problem. They argued that the departure from conventional wisdom in their specific situation should be explained by the fact that their particular "risky" asset is actually less risky than holding cash in their stochastic interest rate economy. While their explanation is not wrong, we offer a much more general explanation that is not tied to a specific economic interpretation of the risky asset. We contend that when risk is measured by the asymptotic variance, and when the residuals are not correlated, then the risk of the portfolio might not be minimized by the strategy which holds 100% of the capital in cash.

In order to understand this better we shall now compute, as a function of h which is the fixed proportion of capital held in the risky asset (the balance is held in cash, and there is continuous rebalancing), the portfolio's asymptotic variance, namely

$$\lim_{t\to\infty} t^{-1} Var\Big(\ln V(t)\Big). \tag{54}$$

In view of equation (15) in Section 3, this can be derived from a computation of the portfolio's risk sensitive growth rate [which exists as an ordinary limit rather than the lim inf in this case]

$$J_\theta^h := \lim_{t\to\infty} \Big(-\frac{2}{\theta}\Big) t^{-1} ln\ \mathbf{E}\ [e^{-(\theta/2) ln V(t)}]. \tag{55}$$

It follows from the Appendix that J_θ^h can be computed from the variation of the Bellman-Hamilton-Jacobi equation for this asset management problem that is obtained by fixing the proportion in the risky asset equal to the constant h. This means that we need to solve (with h fixed) the differential equation

$$\begin{aligned} \rho = (b+Bx)\phi'(x) - \frac{\theta}{4}(\lambda_1^2+\lambda_2^2)\Big(\phi'(x)\Big)^2 + (\lambda_1^2+\lambda_2^2)\phi''(x) \\ -\frac{1}{2}\Big(\frac{\theta}{2}+1\Big)\sigma^2 h^2 + a_0(1-h) + ah + A_0(1-h)x + Ahx - \frac{\theta}{2}\sigma\lambda_1\phi'(x)h \end{aligned} \tag{56}$$

for a scalar constant ρ (which will turn out to be equal to J_θ^h) and a convex (or linear) quadratic function ϕ. This is straightforward. Taking $\phi'(x) = cx + d$ for some scalar constants c and d it easily follows that we must have $c = 0, d = -[A_0(1-h) + Ah]/B,$ and

$$\rho = J_\theta^h = [a_0 + A_0(-b/B)](1-h) + [a + A(-b/B)]h - \frac{1}{2}\sigma^2 h^2$$
$$-\frac{\theta}{4}\Big[\sigma^2 h^2 + (\lambda_1^2+\lambda_2^2)\Big(\frac{A_0(1-h)+Ah}{B}\Big)^2 + 2\sigma\lambda_1\Big(\frac{A_0(1-h)+Ah}{B}\Big)h\Big]. \tag{57}$$

In view of equation (15) in Section 3, the portion of this expression that is constant with respect to θ is precisely equal to the portfolio's long-run growth rate. This makes intuitive sense: with the factor X having a long-run mean value of $-b/B$, the portfolio's average appreciation rate is $\mu := [a_0 + A_0(-b/B)](1-h) + [a + A(-b/B)]h$, in which case the long run growth rate is $\mu - \sigma^2 h^2/2$.

It also follows from equation (15) in Section 3 that the asymptotic variance of the portfolio is

$$\sigma^2 h^2 + (\lambda_1^2+\lambda_2^2)\Big(\frac{A_0(1-h)+Ah}{B}\Big)^2 - 2\sigma\lambda_1\Big(\frac{A_0(1-h)+Ah}{B}\Big)h. \tag{58}$$

Notice that this is a quadratic (in fact, convex) function of h. A little differential calculus shows that the minimum is not necessarily located at $h = 0$. In fact, this minimizing value can be non-zero even if the residuals are uncorrelated (i.e., $\lambda_1 = 0$). When $h = 0$ expression (58) gives the asymptotic variance for the strategy of holding 100% cash, namely, $(\lambda_1^2 + \lambda_2^2)A_0^2/B^2$. Hence holding cash is risky if the short interest rate depends upon the level of the factor, and this risk can sometimes be reduced by putting some capital in the risky asset.

Additional insight is obtained by looking at the asymptotic variance of the risky asset. By taking $h = 1$ in expression (58) this is seen to be

$$\sigma^2 + (\lambda_1^2+\lambda_2^2)\Big(\frac{A}{B}\Big)^2 - 2\sigma\lambda_1 A/B. \tag{59}$$

Each of these three terms has a direct interpretation. The first is the contribution to the asymptotic variance made by the usual volatility of the asset. The second is the contribution due to predictable yet random variations in the asset's appreciation rate caused by fluctuations of the factor. To understand this better it is useful to consider how one would simulate the asset with a discrete time approximation. Successive returns will vary not only due to the volatility σ but also due to random variations in the period-by-period **mean** returns. The presence of the second term means that the local variance of the asset's return will be stochastic even though the classical volatility σ is constant. This all is consistent with Lo and Wang [29] who observed that with a stochastic appreciation rate the variance of an asset's t-period return will not necessarily equal $\sigma^2 t$, as it does for simple geometric Brownian motion.

But note that this second term does not have an explicit counterpart in the earlier expressions for $\tilde{h}^\theta(t)$, the optimal proportion in the risky asset. In particular, the increase in the asymptotic variance associated with this term does not automatically lead to a reduction in the optimal position in the risky asset, due to the higher risk. This is because this contribution to the asymptotic variance is predictable, corresponding to the observed level of the factor X. Moreover, this contribution can be fully hedged, as reflected by the variation with respect to the factor level X of the myopic term in the expressions for $\tilde{h}^\theta(t)$.

The third term in the expression for the asymptotic variance of the risky asset provides some useful insight about the hedging term in the formulas for $\tilde{h}^\theta(t)$. When $\lambda_1 A \neq 0$ there is an additional contribution to the asymptotic variance, a contribution that can be positive or negative. Apparently the dynamics of the stochastic system are such that a correlation of the residuals combined with a factor-dependent appreciation rate will affect the asset's asymptotic variance. Furthermore, unlike what we saw for the second term, the third term's contribution to the asymptotic variance is not fully predictable and cannot be fully hedged. Compare, for instance, the third term in the expression for the asymptotic variance with the hedging term in expression (53). If $\lambda_1 A > 0$ then (since $B < 0$) the third term will **increase** the asymptotic variance of the risky asset. Intuitively, then, it will be optimal to **decrease** the proportion in the risky asset. This is borne out by the hedging term in (53) for the case when $A = A_0$, for it will have a negative sign when $\lambda_1 A > 0$. The analogous but opposite situation pertains when $\lambda_1 A < 0$.

Additional evidence for the hedging term's role as an adjustment for asset risk can be seen by looking at formula (52) for the optimal proportion in the risky asset. A simple calculation shows that the hedging term disappears as the risk aversion parameter θ decreases to zero. In view of Section 3 this limiting situation corresponds to the Kelly criterion where the investor's objective is to simply maximize the portfolio's growth rate without regard to risk. For the risk-null investor the portfolio's asymptotic variance and thus its component due to correlated residuals are irrelevant, so this investor's optimal strategy will have no hedging term, even if the residuals are correlated and the factor level affects the asset's appreciation rate.

In conclusion, this example suggests that the presence of the hedging term in our general risk sensitive asset management model is due to contributions to asset asymptotic variance caused by correlations between asset and factor residuals operating in conjunction with factor-dependent asset appreciation rates. This term can be non-zero for a very risk averse investor because holding 100% cash is not necessarily the minimum-risk trading strategy. The local variance of asset returns can be stochastic even though the classical volatilities in our model are constants. The hedging term arises because it is appropriate for the optimal proportions to be based on asymptotic variances rather than solely on classical volatilities. We conjecture that similar conclusions should be made about the hedging term in very general versions of Merton's ICAPM.

6 Rolling-Horizon Bonds and Gaussian Yield-Factor Models

In the finance literature one can find studies where the authors assume infinitely-lived geometric Brownian motion models of fixed income securities, but this choice of model is never justified. In particular, no arguments can be found which start with a specific interest rate model together with a specific fixed income asset and then derive the stochastic dynamics of this asset's price process, showing this process to be a geometric Brownian motion. A purpose of this section is to make such an argument.

The appropriate asset is called a *rolling-horizon bond*, a concept introduced by Rutkowski [37]. To understand this it is convenient to assume the so-called Heath-Jarrow-Morton framework for interest rate models, namely, that for any maturity date T the instantaneous forward rate $f(\cdot, T)$ satisfies the stochastic differential equation

$$df(t,T) = \alpha(t,T)dt + \nu(t,T) \cdot dW_t, \tag{60}$$

where $\alpha(\cdot, T)$ and $\nu(\cdot, T)$ are adapted stochastic processes, W is a standard (possibly vector valued) Brownian motion defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, \mathbf{P})$, and $\mathbf{P}$ is the actual, real-world probability measure. Letting $P(t,T)$ denote the time-t price of a zero-coupon (i.e., pure discount) bond that pays one unit of cash at time T, it is well known (see Heath *et al.* [23]) that under a martingale measure $\mathbf{P}^*$ the dynamics of P are given by

$$dP(t,T) = P(t,T)(r_t dt + b(t,T) \cdot dW_t^*), \tag{61}$$

where W^* is a standard (possibly vector valued) Brownian motion under $\mathbf{P}^*$, $r_t := f(t,t)$ is the short-term, locally riskless interest rate, and $b(t,T) := -\int_t^T \nu(t,u)du$ is the bond's volatility.

Now consider a self-financing trading strategy in which the total wealth is continuously reinvested in zero-coupon bonds which mature exactly T time units in the future. This strategy is a bit abstract, but it can be viewed as the limit as $\delta \to 0$ of the self-financing strategy which every δ time units rolls over all the funds from the bonds that mature in $T-\delta$ time units into bonds that mature in T time units. The wealth process denoted $U(\cdot, T)$ of such a strategy is called a rolling horizon bond. As reported by Rutkowski [37], the dynamics of $U(\cdot, T)$ under the martingale measure $\mathbf{P}^*$ are

$$dU(t,T) = U(t,T)(r_t dt + b(t, t+T) \cdot dW_t^*). \tag{62}$$

Meanwhile, under the actual probability measure $\mathbf{P}$ the rolling-horizon bond process will have the same volatility $b(t, t+T)$, while its drift will depend in a simple way upon the market price of risk (i.e., the Girsanov transformation relating $\mathbf{P}$ and $\mathbf{P}^*$). In particular, and this is the case that will be studied in this paper, for the Gaussian HJM model, where the diffusion coefficient ν in the forward rate SDE is nonrandom, the volatility $b(t, t+T)$ of the rolling-horizon bond is also a deterministic function of time. Hence the rolling-horizon bond can be a geometric Brownian motion whose appreciation rate depends upon the market price of risk.

We now turn to the interest rate model that is the subject of this paper. We fix k times $T_1 < T_2 < \ldots < T_k$, where $T_1 > 0$, and consider the yields $Y_t^1, Y_t^2, .., Y_t^k$ of the corresponding *sliding bonds* (this is Rutkowski's [37] terminology) $P(t, t+T_i), i = 1, 2, .., k$. In other words,

$$Y_t^i = -\frac{1}{T_i} \ln P(t, t+T_i), i = 1, 2, .., k. \tag{63}$$

There is also the usual short-term, locally riskless interest rate r_t, the yield corresponding to a zero coupon bond having a maturity of zero, if you will. These $k+1$ interest rates will be the factors affecting and thus characterizing our interest rate model. We assume they satisfy, under the actual, real-world probability measure $\mathbf{P}$, the system of stochastic differential equations:

$$dr_t = (b_0 + B_{00} r_t + B_{01} Y_t^1 + .. + B_{0k} Y_t^k)dt + \sum_{j=0}^{N} \lambda_{0j} dW_j(t), \tag{64}$$

$$dY_t^i = (b_i + B_{i0} r_t + B_{i1} Y_t^1 + .. + B_{ik} Y_t^k)dt + \sum_{j=0}^{N} \lambda_{ij} dW_j(t), \quad i = 1, 2, .., k. \tag{65}$$

Here b_i, B_{ij}, and λ_{il} are all fixed constants for all $i, j = 0, 1, .., k$, $l = 0, 1, \ldots, N$ and $W = (W_0, W_1, .., W_N)$ is a standard $(N+1)$-dimensional Brownian motion under $\mathbf{P}$, as before. Note that the $(k+1)$-dimensional factor process $(r, Y^1, .., Y^k)$ is Markovian with drift coefficients that are affine functions of the factor levels and with constant diffusion coefficients.

We now make an important

Assumption (A4). *The market price of risk is an affine function of the $k+1$ interest rate factors.*

The market price of risk, of course, provides the connection via Girsanov between the real-world probability measure $\mathbf{P}$ and the martingale measure $\mathbf{P}^*$. Therefore, as a consequence of our assumption (or "equivalently," if you prefer), under $\mathbf{P}^*$ the dynamics of the $k+1$ interest rate factors will have exactly the same form as under $\mathbf{P}$. In other words, under $\mathbf{P}^*$ the SDE's describing the interest rate dynamics will have drift coefficients that are affine functions of the interest rate levels and diffusion coefficients that are constants.

We thus have specified what is a special case of the affine yield-factor model developed by Duffie and Kan [20]. Accordingly, there exist deterministic, real-valued functions $\alpha(\cdot), \beta_0(\cdot), .., \beta_k(\cdot)$ on $[0, \infty)$ such that the price of any sliding bond is given by

$$P(t, t+T) = exp\{\alpha(T) + \beta_0(T)r_t + \beta_1(T)Y_t^1 + ... + \beta_k(T)Y_t^k\}, \quad \forall t \geq 0. \tag{66}$$

These $k+1$ deterministic functions can be computed from the parameters in equations (64) and (65) together with the market price of risk parameters, but doing so will not be necessary for our purposes. Instead, it suffices to observe that by taking $T = T_i$ for $i = 1, 2, .., k$ and using equation (63) we must have (see also (5.2) of Duffie and Kan [20])

$$\alpha(T_i) = \beta_j(T_i) = 0, \ j \neq i; \ \beta_i(T_i) = -T_i, \quad i, j = 1, .., k. \tag{67}$$

We shall use this to derive the real-world dynamics of the rolling-horizon bonds. Applying Ito's formula to the above expression for the sliding bond we obtain the following real-world dynamics:

$$dP(t, t+T)/P(t, t+T) = (\ldots)dt + \beta(T)\Lambda \cdot dW(t), \tag{68}$$

where $\beta(T) := (\beta_0(T), \beta_1(T), \ldots, \beta_k(T))$ is a row vector and $\Lambda := (\lambda_{ij})$ is the $(k+1) \times (N+1)$ matrix of diffusion coefficients (as in (64), (65) as well as the corresponding risk neutral SDE's for the interest rates); the drift coefficient here does not matter, so we omit the details.

In particular, when $T = T_i$ for any $i = 1, \ldots, k$ we can apply (67) and thus obtain

$$dP(t, t+T_i)/P(t, t+T_i) = (\ldots)dt - T_i\lambda_{i0}dW_0(t) - \ldots - T_i\lambda_{iN}dW_N(t). \tag{69}$$

As explained by Rutkowski [37], it thus follows that our bond price volatilities $b(t, T)$ satisfy

$$b(t, t+T_i) = -T_i(\lambda_{i0}, \lambda_{i1}, .., \lambda_{iN}), \quad i = 1, .., k. \tag{70}$$

Hence substituting this in (62) and using again our assumption that the market price of risk is an affine function of the $k+1$ interest rate factors, we obtain the following characterization of the real-world dynamics of the k rolling-horizon bonds corresponding to the k specified maturities and interest rate factors:

$$dU(t, T_i)/U(t, T_i) = (\alpha_i + \mathcal{A}_{i0}r_t + \mathcal{A}_{i1}Y_t^1 + ... + \mathcal{A}_{ik}Y_t^k)dt - T_i\sum_{j=0}^{N}\lambda_{ij}dW_t^j, \ i = 1..., k. \tag{71}$$

Here α_i and $\mathcal{A}_{ij}$ for $i = 1, .., k$ and $j = 0, 1, .., k$ are all fixed constants that can be computed from the market price of risk and other parameters (recall that each $U(t, T_i)$ is an asset price which under the martingale measure has drift coefficient $r_t U(t, T_i)$ and the same diffusion coefficient as under the real-world probability measure). Alternatively, and this is more convenient for implementation purposes, one can estimate these drift parameters directly, ignoring the market price of risk parameters.

In the following section we incorporate this Gaussian yield-factor model and the corresponding rolling-horizon bonds as part of the general risk sensitive asset management model that was developed in Section 2.

7 Optimal Dynamic Risk-Sensitive Management of Cash, Equities and Bonds

In this section we consider a specification of the general model of Section 2.1. Specifically, we consider the case where [in the notation of Sections 2 and 3]:

$$X(t) = (r_t, Y_t^1, \ldots, Y_t^k)'. \tag{72}$$

That is to say, the economic factors considered are the short rate r_t and the yields Y_t^i. As the tradable assets we take the bank account $S_0(t)$ whose dynamics are given as

$$\frac{dS_0(t)}{S_0(t)} = r_t dt, \quad S_0(0) = s_0, \tag{73}$$

$\bar{m}$ stocks and stock indices $S_i(t)$, $i = 1, \ldots, \bar{n}$ whose dynamics are given by (1), and k rolling bonds $S_{\bar{n}+j}(t) := U(t, T_j)$, $j = 1, \ldots k$ whose dynamics are given by (71).

Thus, we are considering the model of section 2.1 with the following parameterization:

$$n = \bar{n} + k, \quad m = k, \tag{74}$$

$$a_0 = 0, \quad A_0 = (1, 0, \ldots, 0)_{1\times(1+k)}, \tag{75}$$

$$a_{\bar{n}+j} = \alpha_j, \quad A_{\bar{n}+j,l} = \mathcal{A}_{j,l}, \quad j, l = 1, \ldots, k, \tag{76}$$

$$\Sigma_{i,l} = \sigma_{i,l}, \quad i = 1, \ldots, \bar{n},\ l = 0, 1, \ldots, N, \tag{77}$$

$$\Sigma_{i,l} = -T_{i-\bar{n}}\lambda_{i-\bar{n},l}, \quad i = \bar{n}+1, \ldots, \bar{n}+k, \quad l = 0, 1, \ldots, N, \tag{78}$$

$$\Lambda_{i,l} = \lambda_{i,l}, \quad i = 0, 1, \ldots, k, \quad l = 0, 1, \ldots, N, \tag{79}$$

Remark 7.1 For example, letting $\bar{n} = k = 1$ and $N = 3$ implies

$$n = 2, \quad m = 1 \tag{80}$$

$$W(t) = (W_0(t), W_1(t), W_2(t), W_3(t))', \tag{81}$$

$$S_2(t) = U(t, T_1), \quad X_0(t) = r_t, \quad X_1(t) = Y_t^1, \tag{82}$$

$$A_0 = (1,0), \quad a = \begin{pmatrix} a_1 \\ \alpha_1 \end{pmatrix}, \quad A = \begin{pmatrix} A_{1,0} & A_{1,1} \\ \mathcal{A}_{1,0} & \mathcal{A}_{1,1} \end{pmatrix}, \tag{83}$$

$$\Sigma = \begin{pmatrix} \sigma_{1,0} & \sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} \\ -T_1\lambda_{1,0} & -T_1\lambda_{1,1} & -T_1\lambda_{1,2} & -T_1\lambda_{1,3} \end{pmatrix}, \tag{84}$$

$$b = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}, \quad B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}, \tag{85}$$

$$\Lambda = \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} & \lambda_{0,2} & \lambda_{0,3} \\ \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} \end{pmatrix}. \tag{86}$$

On the other hand, letting $\bar{n} = 1$, $k = 2$ and $N = 2$ implies

$$n = 3, \quad m = 2 \tag{87}$$

$$W(t) = (W_0(t), W_1(t), W_2(t))', \tag{88}$$

$$S_2(t) = U(t,T_1), \quad S_3(t) = U(t,T_2), \quad X_0(t) = r_t, \quad X_1(t) = Y_t^1, \quad X_2(t) = Y_t^2 \tag{89}$$

$$A_0 = (1,0,0), \quad a = \begin{pmatrix} a_1 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_{1,0} & A_{1,1} & A_{1,2} \\ \mathcal{A}_{1,0} & \mathcal{A}_{1,1} & \mathcal{A}_{1,2} \\ \mathcal{A}_{2,0} & \mathcal{A}_{2,1} & \mathcal{A}_{2,2} \end{pmatrix}, \tag{90}$$

$$\Sigma = \begin{pmatrix} \sigma_{1,0} & \sigma_{1,1} & \sigma_{1,2} \\ -T_1\lambda_{1,0} & -T_1\lambda_{1,1} & -T_1\lambda_{1,2} \\ -T_2\lambda_{2,0} & -T_2\lambda_{2,1} & -T_2\lambda_{2,2} \end{pmatrix}, \tag{91}$$

$$b = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{pmatrix}, \tag{92}$$

$$\Lambda = \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} & \lambda_{0,2} \\ \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} \end{pmatrix}. \tag{93}$$

□

Formulas of section 4.1 can now be applied to compute optimal investment strategies and the optimal value of the objective criterion for Problem (4) specified to the present setting. We shall now illustrate this with a numerical example where there is $\tilde{n} = 1$ stock index, namely, the S&P 500, and $m = k = 1$ rolling horizon bond, namely, the one corresponding to a maturity of $T_1 = 20$ years. Hence there are two factors: the short rate $X_0(t) = r_t$ and the 20-year (U.S. Treasury) zero coupon yield $X_1(t) = Y_t^1$. This system involves three Brownian motions, so $N = 2$.

We estimated the model parameters using monthly S&P and interest rate data from October 1979 to September 1999. From this we computed $P_i(t)$, the time-t price of the zero coupon bond that expires in i months, and so for the return of the rolling horizon bond we took $\ln(P_{239}(t+1)/P_{240}(t))$. We then regressed the asset returns against the levels of the two factors to obtain

$$a = \begin{pmatrix} 0 \\ 0.2532 \\ -0.3690 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ -0.6064 & -0.5640 \\ -6.2848 & 10.5624 \end{pmatrix}. \tag{94}$$

Note from the second row of A that low interest rates were bullish for the stock market.

For the second step we regressed the monthly changes in interest rates against the beginning-of-month levels of interest rates to obtain

$$b = \begin{pmatrix} 0.0122 \\ 0.0192 \end{pmatrix}, \quad B = \begin{pmatrix} -0.4468 & 0.1691 \\ 0.4058 & -0.5579 \end{pmatrix}. \tag{95}$$

Note that B is indeed a stable matrix, with mean reverting levels of the short rate and 20-year rate being 5.56% and 7.49%, respectively.

For the Λ matrix we computed the 2×2 covariance matrix using the residuals from the regressions in step 2. We then computed the corresponding Cholesky matrix, thereby giving (since we have three Brownian motions)

$$\Lambda = \begin{pmatrix} 0.0210 & 0 & 0 \\ 0.0127 & 0.0104 & 0 \end{pmatrix}. \tag{96}$$

For the row of the volatility matrix Σ corresponding to the rolling horizon bond we used $\sigma_{2,0} = -T_1 \lambda_{1,0} = -0.2540$ and so forth. Incidentally, this gives an overall volatility for the rolling horizon bond of 32.83%, not far above the measured value of 26.24%. For the row of Σ corresponding to the stock index we have from the residuals of the S&P returns that its overall volatility is 15.00%. We also computed the correlation between the S&P residuals and the short rate residualts (respectively, the 20-year yield residuals) to be approximately -1% (resp. -2.5%). It follows that

$$\Sigma = \begin{pmatrix} 0 & 0 & 0 \\ -0.001255 & -0.004489 & 0.140633 \\ -0.2540 & -0.2080 & 0 \end{pmatrix}. \tag{97}$$

These correlations are very close to zero and probably not statistically significant, but as will be seen below the presence of these non-zero correlations has a substantial effect on the optimal trading strategy. In any event, since this is a theoretical paper our aim is not to conduct a definitive empirical analysis of the U.S. asset markets. Rather, our aim here is to produce an illustrative example having plausible if not realistic parameter values.

We used MATLAB to solve the continuous algebraic Riccati equation and compute the quantities of interest. For example, with $\theta = 2$ we obtained

$$K_1(\theta) = \begin{pmatrix} 242.5919 & -144.9067 \\ -144.9067 & 328.2342 \end{pmatrix} \tag{98}$$

and

$$\tilde{h}^\theta(t) = \begin{pmatrix} 6.2883 \\ -2.0752 \end{pmatrix} + \begin{pmatrix} -42.8551 & -11.1381 \\ -36.6818 & 44.2618 \end{pmatrix} \begin{pmatrix} r_t \\ Y_t^1 \end{pmatrix}. \tag{99}$$

In particular, with the factors at their mean reverting levels of $r_t = 5.56\%$ and $Y_t^1 = 7.49\%$ this calls for a highly leveraged position of 307% of one's wealth in the stock index, a short position equal to 80% of one's wealth in the rolling horizon bond, and the borrowing of an amount of cash equal to 127% of one's wealth. These extreme positions are consistent with an aggressive investor having a small risk aversion parameter such as $\theta = 2$.

With the risk aversion parameter fixed at $\theta = 10$, Figure 1 shows a graph of the optimal proportion in the rolling horizon bond versus the two interest rate factors. As one should expect, the lower the 20-year yield, the lower this proportion, whereas the lower the short rate, the higher this proportion. With $\theta = 10$ again, Figure 2 shows the analogous graph for the optimal proportion in the stock index. There is a slight tendency for this proportion to decrease as the 20-year interest rate increases, and there is a stronger tendency for this proportion to be decreasing with respect to the short rate.

In order to get a better sense about the contribution of the hedging term to the optimal proportion in the stock index, we revised the stock's volatilities so that its residuals are uncorrelated with the interest rate residuals and yet its overall volatility of 15.0% is maintained. We then recomputed the optimal proportion in the stock index for $\theta = 10$. Figure 3 shows a graph of the difference of proportions, the original values minus the corresponding revised values. The differences range from -10% to +20% over the range of interest rates shown. These differences are surprisingly large, given the correlations of the residuals that are so close to zero.

With the short rate fixed at 5% and the 20-year yield fixed at 6.15%, Figure 4 shows a graph of the optimal stock and rolling horizon bond proportions versus the risk aversion parameter θ. It is interesting to note that the bond proportion is negative for all values of θ. Although no investor will take a long position in the rolling horizon bond when $\theta = 10$ in this economy, long positions in the underlying zero coupon bonds could still be attractive. Notice also that, in accordance with the discussion in Section 5, a very risk averse investor will want to take a short position in the rolling horizon bond.

8 Appendix: Proofs of The Main Mathematical Results

We fix $\theta > 0$. Also, without any loss of generality, we may and we do assume that $V(0) = v = 1$.

Step 1: The BHJ Equation. We begin with writing down the *Bellman-Hamilton-Jacobi* equation corresponding to the problem $\mathcal{P}_\theta$:

$$\rho = \Phi_x'(x)Bx + b'\Phi_x(x) + \frac{1}{2}\Big[-\Big(\frac{\theta}{2}\Big)\Phi_x'(x)\Lambda\Lambda'\Phi_x(x) + tr\Lambda'\Phi_{xx}(x)\Lambda\Big]$$

$$-\inf_{\{h\in\mathbf{R}^n,\,\mathbf{1}'h=1\}}\Big[\frac{1}{2}\Big(\frac{\theta}{2}+1\Big)h'\Sigma\Sigma'h - h'\big(Ax+a\big) + \frac{\theta}{2}h'\Sigma\Lambda'\Phi_x(x)\Big], \quad \forall x\in\mathbf{R}^{m+1}. \tag{100}$$

The equation (100) needs to be solved for the pair (ρ, Φ) where ρ is a real constant, and $\Phi : \mathbf{R}^{m+1} \to \mathbf{R}$ is a function. We denote the gradient and the Hessian of Φ by Φ_x and Φ_{xx} respectively.

Step 2: A Solution of the BHJ Equation. Consider a function Φ^θ of the quadratic form

$$\Phi^\theta(x) := x'K_1(\theta)x + K_2'(\theta)x, \quad \forall x\in\mathbf{R}^{m+1}, \tag{101}$$

where the matrix $K_1(\theta)$ is as in Assumption (A3) and the corresponding matrix $K_2(\theta)$ is defined in (33). Consider also the constant $\rho(\theta)$ defined in (37). Direct calculations verify that the pair $(\rho(\theta), \Phi^\theta)$ satisfies the equation (100).

Step 3: Minimal Selector in the BHJ Equation. It is easy to verify that the function

$$H^\theta(x) := (\frac{\theta}{2}+1)^{-1}(\Sigma\Sigma')^{-1}\Big[\beta(x) + \lambda(x)\mathbf{1}\Big], \tag{102}$$

where $\beta(x)$ and $\gamma(x)$ are given by the formulas (35) and (36), is the minimal selector in the BHJ equation (100).

Step 4: Representations of the Wealth Process $V(\cdot)$. Here we provide some representations of $V(\cdot)^{-\theta/2}$, where $V(\cdot)$ is the wealth process (3). It easily follows [using the Itó formula] that

$$\Big(V(t)\Big)^{-\theta/2} = \exp\Big\{ (-\theta/2) \int_0^t h'(s)(AX_t + a)ds + \Big(\theta/4\Big)\big(\theta/2+1\big) \int_0^t \|h'(s)\Sigma\|^2 ds - \Big(\theta^2/8\Big) \int_0^t \|h'(s)\Sigma\|^2 ds - \Big(\theta/2\Big) \int_0^t h'(s)\Sigma dW_s \Big\}, \tag{103}$$

where $\|\cdot\|$ denotes the Euclidean norm. Now, using the above representation (103), with $h(t)$ substituted with $h^\theta(t) = H^\theta(X_t)$, and invoking the BHJ equation (100) combined with the Itó formula, we conclude that

$$\Big(V(t)\Big)^{-\theta/2} = \exp\Big\{ (-\theta/2)\rho(\theta)t + (\theta/2)\Phi^\theta(X_t) - (\theta/2)\Phi^\theta(X_0) - \Big(\theta^2/8\Big) \int_0^t \|\eta_s\|^2 ds - \Big(\theta/2\Big) \int_0^t \eta_s dW_s \Big\}, \tag{104}$$

where $\eta_t := \Big(\Phi_x^\theta(X_t)\Big)'\Lambda + \Big(h^\theta(t)\Big)'\Sigma$.

Step 5: Girsanov Change of Measure. Let now $\mathbf{P}^\theta$ be a probability measure on $(\Omega,\mathcal{F})$ such that

$$\frac{d\mathbf{P}^\theta}{d\mathbf{P}}\Big|_{\mathcal{F}_t} = \xi_t, \quad \forall t \geq 0, \tag{105}$$

where

$$\xi_t := \exp\Big\{ - \Big(\theta^2/8\Big) \int_0^t \|\eta_s\|^2 ds - \Big(\theta/2\Big) \int_0^t \eta_s dW_s \Big\}.$$

Remark 8.1 It follows from the Lemma (4.2) in Stroock [40] that a sufficient condition for existence of such a measure $\mathbf{P}^\theta$ is that the process ξ is a martingale on $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, \mathbf{P})$, a condition which is satisfied here. The restrictions of measures $\mathbf{P}$ and $\mathbf{P}^\theta$ to each $\sigma-$field $\mathcal{F}_t$, $t \geq 0$, are mutually equivalent. However, the measures $\mathbf{P}$ and $\mathbf{P}^\theta$ may be mutually singular on $(\Omega,\mathcal{F})$. Thus, completions of $(\Omega,\mathcal{F})$ w.r.t. each of the measures $\mathbf{P}$ and $\mathbf{P}^\theta$ will differ in general, which implies that the "almost sure" statements w.r.t. to the measure $\mathbf{P}$ may not be equivalent to the "almost sure" statements w.r.t. to the measure $\mathbf{P}^\theta$.

Step 6: Dynamics of the Factor Process Under the Measure $\mathbf{P}^\theta$. It follows from the Lemma (4.3) in Stroock [40] that the process $W_t^\theta := W_t + \Big(\theta/2\Big)\int_0^t \eta_s' ds$ is a standard Brownian motion under the measure $\mathbf{P}^\theta$. A straightforward algebra yields the following representation for the dynamics of the factor process X under the measure $\mathbf{P}^\theta$:

$$dX_t = G(\theta)X_t dt + g(\theta)dt + \Lambda dW_t^\theta \tag{106}$$

where $G(\theta)$ is the matrix defined in the Assumption (A3), and $g(\theta)$ is a remainder term that only depends on θ [and on the other parameters of the problem $\mathcal{P}_\theta$.] Recall that the matrix $G(\theta)$ is stable. Thus, under the measure $\mathbf{P}^\theta$ the process X is a strongly ergodic Gaussian diffusion. This means in particular that if $F(x)$ is any function which is integrable w.r.t. the invariant measure for X [under the measure $\mathbf{P}^\theta$] then we have

$$\lim_{t\to\infty} \mathbf{E}_{\mathbf{P}^\theta}\Big[F(X_t)\big|X_0 = x\Big] = \int F d\mu^\theta, \quad \forall x \in \mathbf{R}^{m+1}, \tag{107}$$

where by μ^θ we denote the corresponding invariant measure. The invariant measure μ^θ is in fact a Gaussian measure on $\mathbf{R}^{m+1}$ with the covariance operator $C(\theta)$ given as

$$C(\theta) = \int_0^\infty e^{tG(\theta)}\Lambda\Lambda' e^{tG'(\theta)}dt. \tag{108}$$

Observe that the matrix $C(\theta)$ is positive definite and that it solves the Lyapunov equation

$$G(\theta)C(\theta) + C(\theta)G'(\theta) + \Lambda\Lambda' = 0. \tag{109}$$

Step 7: Representation of the Risk Sensitive Criterion Under the Measure $\mathbf{P}^\theta$. It follows from (104) and (105) that for every $x \in \mathbf{R}^{m+1}$ we have [recall that we have assumed that $V(0) = 1$]

$$\begin{aligned} &\liminf_{t\to\infty}(-2/\theta)t^{-1}\ln\ \mathbf{E}^{h^\theta(\cdot)}\ [e^{-(\theta/2)\ln V(t)}|X(0) = x] \\ &\le \rho(\theta) + \liminf_{t\to\infty}(-2/\theta)t^{-1}\ln\ \mathbf{E}_{\mathbf{P}^\theta}\ [e^{(\theta/2)\Phi^\theta(X_t)}|X(0) = x], \end{aligned} \tag{110}$$

with equality holding if the limit rather than lim inf exists on the right hand side [in which case the lim inf on the left hand side may be replaced by the limit]. Similar considerations combined with the condition (iv) of Definition 2.1 imply that for every admissible investment strategy $h(\cdot)$ we have

$$\begin{aligned} &\liminf_{t\to\infty}(-2/\theta)t^{-1}\ln\ \mathbf{E}^{h(\cdot)}\ [e^{-(\theta/2)\ln V(t)}|X(0) = x] \\ &\le \rho(\theta) + \liminf_{t\to\infty}(-2/\theta)t^{-1}\ln\ \mathbf{E}^{h(\cdot),\theta}\ [e^{(\theta/2)\Phi^\theta(X_t)}|X(0) = x] \le \rho(\theta). \end{aligned} \tag{111}$$

In order to conclude the proof of Theorems 2.1 and 2.2 it is thus sufficient to verify that the second term on the right hand side of (110) is zero for θ sufficiently small. This will be done in the remaining step.

Step 8: Conclusion of the Proof. It follows from (109) that the matrix $C^{-1}(\theta)$ satisfies the reciprocal equation

$$G(\theta)C^{-1}(\theta) + C^{-1}(\theta)G'(\theta) + C^{-1}(\theta)\Lambda\Lambda'C^{-1}(\theta) = 0. \tag{112}$$

Substituting for $G(\theta)$ we thus obtain

$$C^{-1}(\theta)R_2(\theta) + C^{-1}(\theta)R_1(\theta)K_1(\theta) + R_2'(\theta)C^{-1}(\theta) + K_1(\theta)R_1(\theta)C^{-1}(\theta) + C^{-1}(\theta)\Lambda\Lambda'C^{-1}(\theta) = 0. \tag{113}$$

Consequently,

$$\begin{aligned} &\big(C^{-1}(\theta) - \theta K_1(\theta)\big)R_2(\theta) + \big(C^{-1}(\theta) - \theta K_1(\theta)\big)R_1(\theta)K_1(\theta) \\ &+R_2'(\theta)\big(C^{-1}(\theta) - \theta K_1(\theta)\big) + R_1(\theta)K_1(\theta)\big(C^{-1}(\theta) - \theta K_1(\theta)\big) \end{aligned}$$

$$+\theta\Big[K_1(\theta)R_2(\theta)+K_1(\theta)R_1(\theta)K_1(\theta)+R_2'(\theta)K_1(\theta)+R_3(\theta)\Big]$$
$$+\theta K_1(\theta)R_1(\theta)K_1(\theta)-\theta R_3(\theta)+C^{-1}(\theta)\Lambda\Lambda' C^{-1}(\theta)=0, \tag{114}$$

so that

$$(C^{-1}(\theta)-\theta K_1(\theta))G(\theta)+G'(\theta)(C^{-1}(\theta)-\theta K_1(\theta))$$
$$+\theta K_1(\theta)R_1(\theta)K_1(\theta)-\theta R_3(\theta)+C^{-1}(\theta)\Lambda\Lambda' C^{-1}(\theta)=0. \tag{115}$$

Now, observe that for θ sufficiently close to 0 the matrix $\theta K_1(\theta)R_1(\theta)K_1(\theta)-\theta R_3(\theta)+C^{-1}(\theta)\Lambda\Lambda' C^{-1}(\theta)$ is positive definite. This implies that the matrix $(C^{-1}(\theta)-\theta K_1(\theta))$, is positive definite, and thus the matrix $(\theta K_1(\theta)-C^{-1}(\theta))$ is negative definite. Consequently, the function $F(x)=e^{\left(\theta/2\right)\Phi^\theta(x)}$ is integrable w.r.t. μ^θ which implies that

$$\liminf_{t\to\infty}(-2/\theta)t^{-1}\ln\ \mathbf{E}_{\mathbf{P}^\theta}\ [e^{(\theta/2)\Phi^\theta(X_t)}|X(0)=x]=0. \tag{116}$$

This completes the proof of Theorems 2.1 and 2.2. □

References

[1] Alexander G.J. and Francis, J.C., (1986), "Portfolio Analysis. Third Edition,", Prentice-Hall, Englewood Cliffs, NJ.

[2] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D., (1997), "Thinking Coherently," *RISK*, vol. 10, no. 11, pp. 68-71.

[3] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D., (1999), "Coherent Measures of Risk," *Math. Finance*, vol. 9, no. 3, pp. 203-228.

[4] Bajeux-Besnainou, I., Jordan, J.V., and Portait, R., (1999), "Dynamic Allocation of Stocks, Bonds and Cash for Long Horizons," *working paper.*

[5] Bielecki, T.R., and Pliska, S.R., (1999), "Risk Sensitive Dynamic Asset Management," *Appl. Math. Optim.*, vol. 39., pp. 337-360.

[6] Bielecki, T. R., Hernandez-Hernandez, D., and Pliska S.R., (1999), "Risk sensitive control of finite state Markov chains in discrete time, with applications to portfolio management," *Math. Meth. Oper. Res.*, vol. 50, pp. 167-188.

[7] Bielecki, T. R., and Pliska, S.R., (2000), "Risk-sensitive dynamic asset management in the presence of transaction costs," *Finance and Stochastics*, vol. 4, pp. 1-33.

[8] Bielecki, T. R., Pliska, S. R., and M. Sherris (2000), "Risk Sensitive Asset Allocation," *Journal of Economic Dynamics and Control*, vol. 24, pp. 1145-1177.

[9] Billingsley, P., (1986), "Probability and measure. Second Edition," John Wiley & Sons, New York, NY.

[10] Brennan, M.J., Schwartz, E.S., and Lagnado, R. (1997) "Strategic asset allocation," *J. Economic Dynamics and Control*, vol. 21, pp. 1377-1403.

[11] Brennan, M. J., and Schwartz, E. S., (1996), "The use of treasury bill futures in strategic asset allocation programs," In *World-Wide Asset and Liability Modeling*, J.M. Mulvey and W.T. Ziemba, eds. Cambridge University Press.

[12] Campbell, J. Y., and Viceira, L. M., (1997), "Who should buy long-term bonds?" Harvard University working paper.

[13] Canestrelli, E., (1998), "Applications of Multidimensional Stochastic Processes to Financial Investments," *preprint*, Dipartimento di Matematicas Applicata ed Informatica, Universita "Ca' Foscari" di Venezia.

[14] Canestrelli, E., and Pontini, S., (1998), "Inquires on the Applications of Multidimensional Stochastic Processes to Financial Investments," *preprint*, Dipartimento di Matematica Applicata ed Informatica, Universita "Ca' Foscari" di Venezia.

[15] Delbaen, F., Grandits, P., Rheinländer, T., Samperi, D., Schweitzer, M. and Striker. C., (2000), "Exponential Hedging and Entropic Penalties," *preprint*

[16] Dembo, A. and Zeitouni, O., (1998), "Large Deviations Techniques and Applications. Second Edition," Springer, New York, NY.

[17] Deuschel, J.D. and Stroock, D.W., (1989), "Large Deviations," Academic Press, Boston, MA.

[18] Donsker, M.D. and Varadhan, S.R.S., "Asymptotic evaluation of certain Markov processes expectations for large time, I, II, III" *Comm. Pure Appl. Math.*, vol. 28, (1975), pp. 1-45, 279-301; vol. 29 (1976), pp. 389-461.

[19] Dupuis, P. and Ellis, R.S., (1997), "A Weak Convergence Approach to the Theory of Large Deviations," John Wiley & Sons, New York, NY.

[20] Duffie, D., and Kan, R., (1996), "A yield-factor model of interest rates," *Math. Finance*, vol. 6, pp. 379-406.

[21] Fleming, W. H., and Sheu, S. J., (1998), "Optimal long term growth rate of expected utility of wealth," *Ann. Appl. Probab*, vol. 9, pp. 871-903.

[22] Fleming, W. H., and Sheu, S. J., (2000), "Risk Sensitive Control and an Optimal Investment Model," *Mathematical Finance*, vol. 10, pp. 197-213.

[23] Heath, D., Jarrow, R., and Morton, A., (1992), "Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation," *Econometrica*, vol. 60, pp. 77-105.

[24] Karatzas I., and Kou S.G., (1996), "On the pricing of contingent claims under constraints," *Annals of Applied Probability*, vol. 6, pp. 321-369.

[25] Karatzas I., and Shreve S. E., (1988), "Brownian Motion and Stochastic Calculus," Springer-Verlag, New York.

[26] Karatzas, I., and Shreve, S. E., (1998), "Methods of Mathematical Finance," Springer-Verlag, New York.

[27] Kim, T.S., and Omberg, E., (1996), "Dynamic Non-myopic Portfolio Behavior," *The Review of Financial Studies*, vol. 9, pp. 141-161.

[28] Korn, R., (1997), "Optimal Portfolios. Stochastic Models for Optimal Investment and Risk Management in Continuous Time," World Scientific, Singapore.

[29] Lo, A. W., and Wang, J., (1995), "Implementing option pricing models when asset returns are predictable," *J. Finance*, vol. 50, pp. 87-130.

[30] Merton, R. C., (1971), "Optimum consumption and portfolio rules in a continuous time model," *J. Econ. Th.* vol. 3, pp. 373-413.

[31] Merton, R. C., (1973), "An intertemporal capital asset pricing model," *Econometrica* vol. 41, pp. 866-887.

[32] Merton, R. C., (1990), "Continuous-Time Finance," Basil Blackwell, Cambridge.

[33] Pesaran M. H., and Timmermann A., (1995), "Predictability of stock returns: robustness and economic significance," *J. Finance*, vol. 50, pp. 1201-1228.

[34] Pesaran M. H., and Timmermann A., (1998), "A recursive modelling approach to predicting UK stock returns," *working paper*, Trinity College, Cambridge.

[35] Pliska, S. R., (1982), "A discrete time stochastic decision model," in *Advances in Filtering and Optimal Stochastic Control*, eds. W. H. Fleming and L. G. Gorostiza, Lecture Notes in Control and Information Sciences 42, Springer Verlag, New York, pp. 290-304.

[36] Pliska, S. R., (1986), "A stochastic calculus model of continuous trading: optimal portfolios," *Math. Operations Research*, vol. 11, pp. 371-384.

[37] Rutkowski, M., (1997), "Self-financing Trading Strategies for Sliding, Rolling-horizon, and Consol Bonds," *Mathematical Finance* vol. 9, pp. 361-385.

[38] Samperi, D., (1998), "Inverse Problems, Model Selection and Entropy in Derivative Security Pricing," *Ph.D. Thesis*, New York University.

[39] Sørensen, C., (1999), "Dynamic Asset Allocation and Fixed Income Management," *Journal of Financial and Quantitative Analysis*, forthcoming.

[40] Stroock, D.W., (1987), "Lectures on Stochastic Analysis: Diffusion Theory," Cambridge University Press, Cambridge.

[41] Vasicek, O. A., (1977), "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics* vol. 5, pp. 177-188.

[42] Whittle P., (1990), "Risk Sensitive Optimal Control," John Wiley, New York.

[43] Wonham, W.M., (1979), "Linear Multivariable Control: a Geometric Approach," Springer-Verlag, New York.

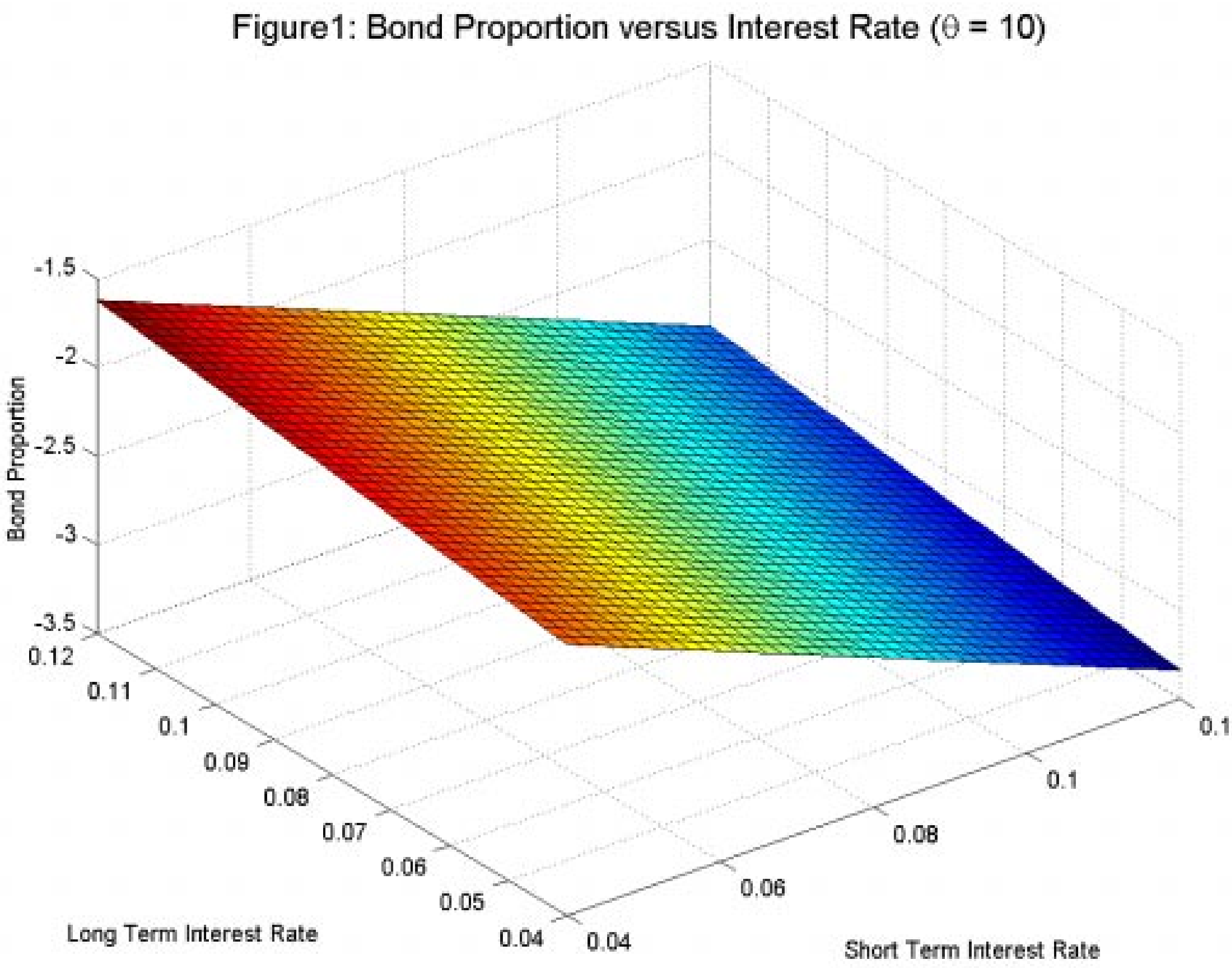
Figure1: Bond Proportion versus Interest Rate (θ = 10)
Bond Proportion
-1.5
-2
-2.5
-3
-3.5
0.12
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
Long Term Interest Rate
0.04
0.06
0.08
0.1
0.12
Short Term Interest Rate

Figure 2: Stock Proportion versus Interest Rate (θ = 10)

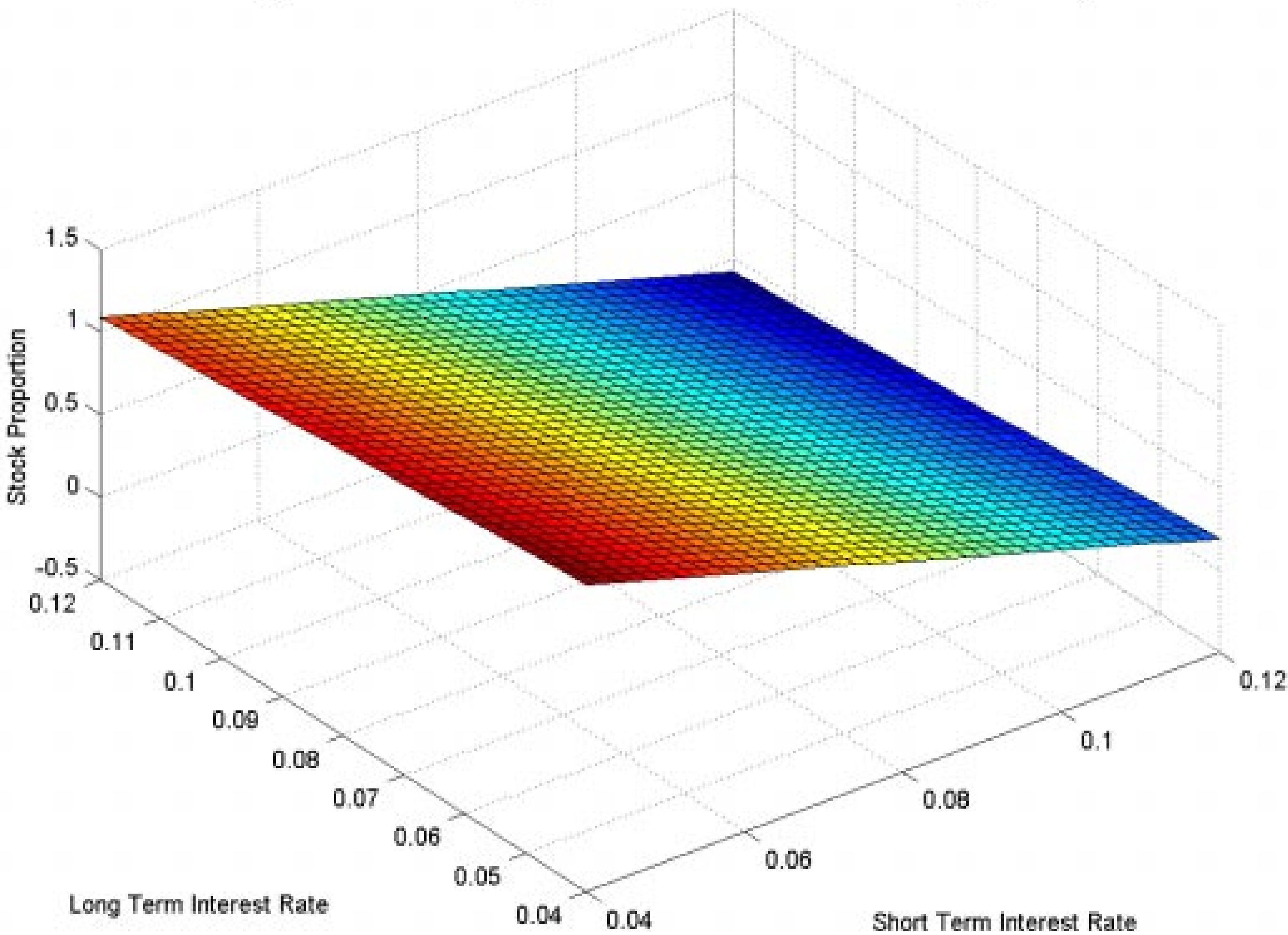

Figure 3: Stock Proportion Difference versus Interest Rate (θ = 10)

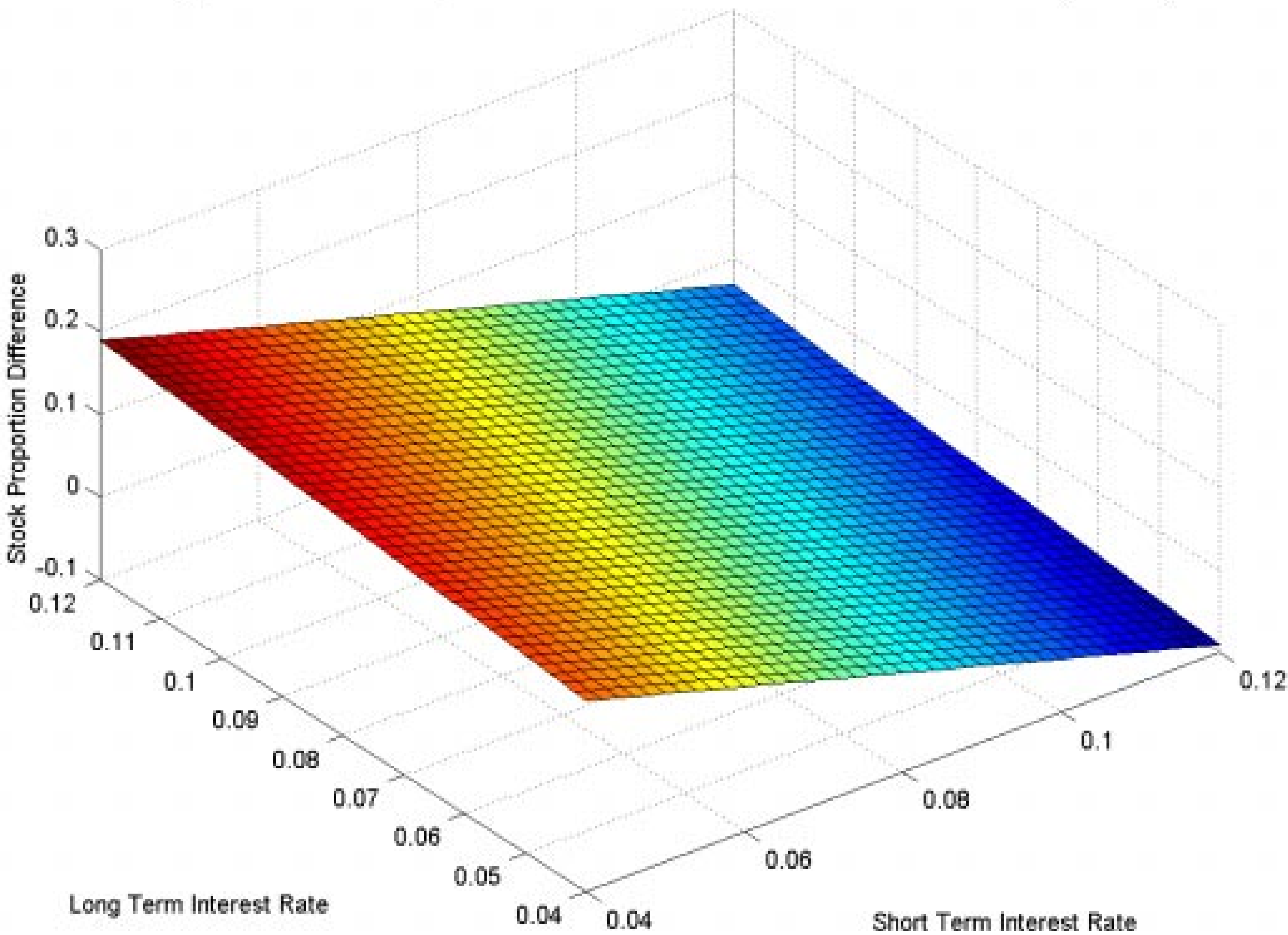

Figure 4: Bond and Stock Proportion versus θ