HEDGING OF DEFAULTABLE CLAIMS

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Introduction

The goal of these lectures is to present a survey of recent developments in the practically important and challenging area of hedging credit risk. In a companion work, Bielecki et al. (2004a), we presented techniques and results related to the valuation of defaultable claims. It should be emphasized that in most existing papers on credit risk, the risk-neutral valuation of defaultable claims is not supported by any other argument than the desire to produce an arbitrage-free model of default-free and defaultable assets. Here, we focus on the possibility of a perfect replication of defaultable claims and, if the latter is not feasible, on various approaches to hedging in an incomplete setting.

These lecture notes are organized as follows. Chapter 1 is devoted to methods and results related to the replication of defaultable claims within the reduced-form approach (also known as the intensity-based approach). Let us mention that the replication of defaultable claims in the so-called structural approach, which was initiated by Merton (1973) and Black and Cox (1976), is entirely different (and rather standard), since the value of the firm is usually postulated to be a tradeable underlying asset. Since we work within the reduced-form framework, we focus on the possibility of an exact replication of a given defaultable claim through a trading strategy based on default-free and defaultable securities.

First, we shall analyze (following, in particular, Vaillant (2001)) various classes of self-financing trading strategies based on default-free and defaultable primary assets. Subsequently, we present various applications of general results to financial models with default-free and defaultable primary assets are given. We develop a systematic approach to replication of a generic defaultable claim, and we provide closed-form expressions for prices and replicating strategies for several typical defaultable claims. Finally, we present a few examples of replicating strategies for particular credit derivatives. In the last section, we present, by means of an example, the PDE approach to the valuation and hedging of defaultable claims within the framework of a complete model.

In Chapter 2, we formulate a new paradigm for pricing and hedging financial risks in incomplete markets, rooted in the classical Markowitz mean-variance portfolio selection principle and first examined within the context of credit risk by Bielecki and Jeanblanc (2003). We consider an investor who is interested in dynamic selection of her portfolio, so that the expected value of her wealth at the end of the pre-selected planning horizon is no less then some floor value, and so that the associated risk, as measured by the variance of the wealth at the end of the planning horizon, is minimized. If the perfect replication is not possible, then the determination of a price that the investor is willing to pay for the opportunity, will become subject to the investor's overall attitude towards trading. In case of our investor, the bid price and the corresponding hedging strategy is to be determined in accordance with the mean-variance paradigm.

The optimization techniques used in Chapter 2 are based on the mean-variance portfolio selection in continuous time. To the best of our knowledge, Zhou and Li (2000) were the first to use the embedding technique and linear-quadratic (LQ) optimal control theory to solve the continuous-time mean-variance problem with assets having deterministic diffusion coefficients. Their approach was subsequently developed in various directions by, among others, Li et al. (2001), Lim and Zhou (2002), Zhou and Yin (2002), and Bielecki et al. (2004b). For an excellent survey of most of these results, the interested reader is referred to Zhou (2003).

In the final chapter, we present a few alternative ways of pricing defaultable claims in the situation when perfect hedging is not possible. We study the indifference pricing approach, that was initiated by Hodges and Neuberger (1989). This method leads us to solving portfolio optimization problems in an incomplete market model, and we shall use the dynamic programming approach. In particular, we compare the indifference prices obtained using strategies adapted to the reference filtration to the indifference prices obtained using strategies based on the enlarged filtration, which encompasses also the observation of the default time. We also solve portfolio optimization problems for the case of the exponential utility; our method relies here on the ideas of Rouge and El Karoui (2000) and Musiela and Zariphopoulou (2004). Next, we study a particular indifference price based on the quadratic criterion; it will be referred to as the quadratic hedging price. In a default-free setting, a similar study was done by Kohlmann and Zhou (2000). Finally, we present a solution to a specific optimization problem, using the duality approach for exponential utilities.

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Chapter 1

Replication of Defaultable Claims

The goal of this chapter is the present some methods and results related to the replication of defaultable claims within the *reduced-form approach* (also known as the *intensity-based approach*). In contrast to some other related works, in which this issue was addressed by invoking a suitable version of the martingale representation theorem (see, for instance, Bélanger et al. (2001) or Blanchet-Scalliet and Jeanblanc (2004)), we analyze here the possibility of a perfect replication of a given defaultable claim through a trading strategy based on default-free and defaultable securities. Therefore, the important issue of the choice of primary assets that are used to replicate a defaultable claim (e.g., a vulnerable option or a credit derivative) is emphasized. Let us stress that replication of defaultable claims within the structural approach to credit risk is rather standard, since in this approach the default time is, typically, a predictable stopping time with respect to the filtration generated by prices of primary assets.

By contrast, in the intensity-based approach, the default time is not a stopping time with respect to the filtration generated by prices of default-free primary assets, and it is a totally inaccessible stopping time with respect to the enlarged filtration, that is, the filtration generated by the prices of primary assets and the jump process associated with the random moment of default.

Our research in the chapter was motivated, in particular, by the paper by Vaillant (2001). Other related works include: Wong (1998), Arvanitis and Laurent (1999), Greenfield (2000), Lukas (2001), Collin-Dufresne and Hugonnier (2002) and Jamshidian (2002).

For a more exhaustive presentation of the mathematical theory of credit risk, we refer to the monographs by Cossin and Pirotte (2000), Arvanitis and Gregory (2001), Bielecki and Rutkowski (2002), Duffie and Singleton (2003), or Schönbucher (2003).

The chapter is organized as follows. Section 1.1 is devoted to a brief description of the basic concepts that are used in what follows. In Section 1.2, we formally introduce the definition of a generic defaultable claim (X, Z, C, τ) and we examine the basic features of its ex-dividend price and pre-default value. The well-known valuation results for defaultable claims are also provided. In the next section, we analyze (following, in particular, Vaillant (2001)) various classes of self-financing trading strategies based on default-free and defaultable primary assets.

Section 1.4 deals with applications of results obtained in the preceding section to financial models with default-free and defaultable primary assets. We develop a systematic approach to replication of a generic defaultable claim, and we provide closed-form expressions for prices and replicating strategies for several typical defaultable claims. A few examples of replicating strategies for particular credit derivatives are presented.

Finally, in the last section, we examine the PDE approach to the valuation and hedging of defaultable claims.

1.1 Preliminaries

In this section, we introduce the basic notions that will be used in what follows. First, we introduce a default-free market model; for the sake of concreteness we focus on default-free zero-coupon bonds. Subsequently, we shall examine the concept of a random time associated with a prespecified hazard process.

1.1.1 Default-Free Market

Consider an economy in continuous time, with the time parameter $t \in \mathbb{R}_+$. We are given a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P}^*)$ endowed with a d-dimensional standard Brownian motion W^*. It is convenient to assume that $\mathbb{F}$ is the $\mathbb{P}^*$-augmented and right-continuous version of the natural filtration generated by W^*. As we shall see in what follows, the filtration $\mathbb{F}$ will also play an important role of a *reference filtration* for the intensity of default event. Let us recall that any (local) martingale with respect to a Brownian filtration $\mathbb{F}$ is continuous; this well-known property will be of frequent use in what follows.

In the first step, we introduce an arbitrage-free default-free market. In this market, we have the following primary assets:

- A *money market account* B satisfying
$$dB_t = r_t B_t \, dt, \quad B_0 = 1,$$
or, equivalently,
$$B_t = \exp\left(\int_0^t r_u \, du\right),$$
where r is an $\mathbb{F}$-progressively measurable stochastic process. Thus, B is an $\mathbb{F}$-adapted, continuous, and strictly positive process of finite variation.

- *Default-free zero-coupon bonds* with prices

$$B(t,T) = B_t \, \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} \,|\, \mathcal{F}_t), \quad \forall \, t \leq T,$$

where T is the bond's maturity date. Since the filtration $\mathbb{F}$ is generated by a Brownian motion, for any maturity date $T > 0$ we have

$$dB(t,T) = B(t,T)\big(r_t \, dt + b(t,T) \, dW_t^*\big)$$

for some $\mathbb{F}$-predictable, $\mathbb{R}^d$-valued process $b(t,T)$, referred to as the *bond's volatility.*

For the sake of expositional simplicity, we shall postulate throughout that the default-free term structure model is complete. The probability $\mathbb{P}^*$ is thus the unique martingale measure for the default-free market model. This assumption is not essential, however. Notice that all price processes introduced above are continuous $\mathbb{F}$-semimartingales.

Remarks. The bond was chosen as a convenient and practically important example of a tradeable financial asset. We shall be illustrating our theoretical derivations with examples in which the bond market will play a prominent role. Most results can be easily applied to other classes of financial models, such as: models of equity markets, futures markets, or currency markets, as well as to models of LIBORs and swap rates.

1.1.2 Random Time

Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, termed a *random time* (it will be later referred to as a *default time*). We introduce the jump process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and we denote by $\mathbb{H}$ the filtration generated by this process.

Hazard process. We now assume that some *reference filtration* $\mathbb{F}$ such that $\mathcal{F}_t \subseteq \mathcal{G}$ is given. We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as to the *full filtration*: it includes the observations of default event. It is clear that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time (but not necessarily an $\mathbb{F}$-stopping time). The concept of the hazard process of a random time τ is closely related to the process F_t which is defined as follows:

$$F_t = \mathbb{Q}^*\{\tau \leq t \,|\, \mathcal{F}_t\}, \quad \forall \, t \in \mathbb{R}_+.$$

Let us denote $G_t = 1 - F_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\}$ and let us assume that $G_t > 0$ for every $t \in \mathbb{R}_+$ (hence, we exclude the case where τ is an $\mathbb{F}$-stopping time). Then the process $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, given by the formula

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t, \quad \forall \, t \in \mathbb{R}_+,$$

is termed the *hazard process* of a random time τ with respect to the reference filtration $\mathbb{F}$, or briefly the $\mathbb{F}$-*hazard process* of τ.

Notice that $\Gamma_\infty = \infty$ and Γ is an $\mathbb{F}$-submartingale, in general. We shall only consider the case when Γ is an increasing process (for a construction of a random time associated with a given hazard process Γ, see Section 1.1.2). This indeed is not a serious compromise to generality. We refer to Blanchet-Scalliet and Jeanblanc (2004) for a discussion regarding completeness of the underlying financial market and properties of the process Γ. They show that if the underlying financial market is complete then the so-called (H) hypothesis is satisfied and, as a consequence, the process Γ is indeed increasing.

Remarks. The simplifying assumption that $\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} > 0$ for every $t \in \mathbb{R}_+$ can be relaxed. First, if we fix a maturity date T, it suffices to postulate that $\mathbb{Q}^*\{\tau > T \,|\, \mathcal{F}_T\} > 0$. Second, if we have $\mathbb{Q}^*\{\tau \le T\} = 1$, so that the default time is bounded by some $U = \operatorname{ess\,sup} \tau \le T$, then it suffices to postulate that $\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} > 0$ for every $t \in [0, U)$ and to examine separately the event $\{\tau = U\}$. For a general approach to hazard processes, the interested reader is referred to Bélanger et al. (2001).

Deterministic intensity. The study of a simple case when the reference filtration $\mathbb{F}$ is trivial (or when a random time τ is independent of the filtration $\mathbb{F}$, and thus the hazard process is deterministic) may be instructive. Assume that τ is such that the cumulative distribution function $F(t) = \mathbb{Q}^*\{\tau \le t\}$ is an absolutely continuous function, that is,

$$F(t) = \int_0^t f(u)\, du$$

for some density function $f : \mathbb{R}_+ \to \mathbb{R}_+$. Then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)\, du}, \quad \forall\, t \in \mathbb{R}_+,$$

where (recall that we postulated that $G(t) = 1 - F(t) > 0$)

$$\gamma(t) = \frac{f(t)}{1 - F(t)}\,, \quad \forall\, t \in \mathbb{R}_+.$$

The function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ is non-negative and satisfies $\int_0^\infty \gamma(u)\, du = \infty$. It is called the *intensity function* of τ (or the *hazard rate*). It can be checked by direct calculations that the process $H_t - \int_0^{t\wedge\tau} \gamma(u)\, du$ is an $\mathbb{H}$-martingale.

Stochastic intensity. Assume that the hazard process Γ is absolutely continuous with respect to the Lebesgue measure (and therefore an increasing process), so that there exists a process γ such that $\Gamma_t = \int_0^t \gamma_u\, du$ for every $t \in \mathbb{R}_+$. Then the $\mathbb{F}$-predictable version of γ is called the *stochastic intensity* of τ with respect to $\mathbb{F}$, or simply the $\mathbb{F}$-intensity of τ. In terms of the stochastic intensity, the conditional probability of the default event $\{t < \tau \le T\}$, given the full information $\mathcal{G}_t$ available at time t, equals

$$\mathbb{Q}^*\{t < \tau \le T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(1 - e^{-\int_t^T \gamma_u\, du} \,\Big|\, \mathcal{F}_t\Big).$$

Thus

$$\mathbb{Q}^*\{\tau > T \,|\, \mathcal{G}_t\} = \mathbb{1}_{\{\tau>t\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(e^{-\int_t^T \gamma_u\, du} \,\Big|\, \mathcal{F}_t\Big).$$

It can be shown (see, for instance, Jeanblanc and Rutkowski (2002) or Bielecki and Rutkowski (2004)) that the process

$$H_t - \Gamma_{\tau\wedge t} = H_t - \int_0^{\tau\wedge t} \gamma_u\, du = \int_0^t (1-H_u)\gamma_u\, du, \quad \forall\, t \in \mathbb{R}_+,$$

is a (purely discontinuous) $\mathbb{G}$-martingale

Construction of a Random Time

We shall now briefly describe the most commonly used construction of a random time associated with a given hazard process Γ. It should be stressed that the random time obtained through this particular method – which will be called the *canonical construction* in what follows – has certain specific features that are not necessarily shared by all random times with a given $\mathbb{F}$-hazard process Γ. We start by assuming that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process Γ defined on a filtered probability space $(\widetilde{\Omega}, \mathbb{F}, \mathbb{P}^*)$. As usual, we assume that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$. In many instances, the hazard process Γ is given by the equality

$$\Gamma_t = \int_0^t \gamma_u\, du, \quad \forall\, t \in \mathbb{R}_+,$$

for some non-negative, $\mathbb{F}$-predictable, stochastic intensity γ. To construct a random time τ such that Γ is the $\mathbb{F}$-hazard process of τ, we need to enlarge the underlying probability space $\widetilde{\Omega}$. This also means that Γ is not the $\mathbb{F}$-hazard process of τ under $\mathbb{P}^*$, but rather the $\mathbb{F}$-hazard process of τ under a suitable extension $\mathbb{Q}^*$ of the probability measure $\mathbb{P}^*$. Let ξ be a random variable defined on some probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{Q}})$, uniformly distributed on the interval $[0,1]$ under $\widehat{\mathbb{Q}}$. We consider the product space $\Omega = \widetilde{\Omega} \times \widehat{\Omega}$, endowed with the product σ-field $\mathcal{G} = \mathcal{F}_\infty \otimes \widehat{\mathcal{F}}$ and the product probability measure $\mathbb{Q}^* = \mathbb{P}^* \otimes \widehat{\mathbb{Q}}$. The latter equality means that for arbitrary events $A \in \mathcal{F}_\infty$ and $B \in \widehat{\mathcal{F}}$ we have $\mathbb{Q}^*\{A \times B\} = \mathbb{P}^*\{A\}\widehat{\mathbb{Q}}\{B\}$. We define the random time $\tau : \Omega \to \mathbb{R}_+$ by setting

$$\tau = \inf\{\, t \in \mathbb{R}_+ : e^{-\Gamma_t} \le \xi \,\} = \inf\{\, t \in \mathbb{R}_+ : \Gamma_t \ge \eta \,\},$$

where the random variable $\eta = -\ln \xi$ has a unit exponential law under $\mathbb{Q}^*$. It is not difficult to find the process $F_t = \mathbb{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}$. Indeed, since clearly $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$ and the random variable Γ_t is $\mathcal{F}_\infty$-measurable, we obtain

$$\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} = \mathbb{Q}^*\{\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_\infty\} = \widehat{\mathbb{Q}}\{\xi < e^{-x}\}_{x=\Gamma_t} = e^{-\Gamma_t}.$$

Consequently, we have

$$1 - F_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} = \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t\big) = e^{-\Gamma_t},$$

and so F is an $\mathbb{F}$-adapted, right-continuous, increasing process. It is also clear that Γ is the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}^*$. Finally, it can be checked that any $\mathbb{P}^*$-Brownian motion W^* with respect to $\mathbb{F}$ remains a Brownian motion under $\mathbb{Q}^*$ with respect to the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

1.2 Defaultable Claims

A generic defaultable claim (X, C, Z, τ) with maturity date T consists of:

- The *default time* τ specifying the random time of default and thus also the default events $\{\tau \le t\}$ for every $t \in [0, T]$. It is always assumed that τ is strictly positive with probability 1.
- The *promised payoff* X, which represents the random payoff received by the owner of the claim at time T, if there was no default prior to or at time T. The actual payoff at time T associated with X thus equals $X \mathbb{1}_{\{\tau > T\}}$.
- The finite variation process C representing the *promised dividends* – that is, the stream of (continuous or discrete) random cash flows received by the owner of the claim prior to default or up to time T, whichever comes first. We assume that $C_T - C_{T-} = 0$.
- The *recovery process* Z, which specifies the recovery payoff Z_τ received by the owner of a claim at time of default, provided that the default occurs prior to or at maturity date T.

It is convenient to introduce the *dividend process* D, which represents all cash flows associated with a defaultable claim (X, C, Z, τ). Formally, the dividend process D is defined through the formula

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty)}(t) + \int_{(0,t]} (1 - H_u)\, dC_u + \int_{(0,t]} Z_u \, dH_u,$$

where both integrals are (stochastic) Stieltjes integrals.

Definition 1.1 The *ex-dividend price process* U of a defaultable claim of the form (X, C, Z, τ) which settles at time T is given as

$$U_t = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{(t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \quad \forall \, t \in [0, T),$$

where $\mathbb{Q}^*$ is the *spot martingale measure* and B is the savings account. In addition, at maturity date we set $U_T = U_T(X) + U_T(Z) = X \mathbb{1}_{\{\tau > T\}} + Z_T \mathbb{1}_{\{\tau = T\}}$
.

Observe that $U_t = U_t(X) + U_t(Z) + U_t(C)$, where the meaning of $U_t(X), U_t(Z)$ and $U_t(C)$ is clear. Recall also that the filtration $\mathbb{G}$ models the full information, that is, the observations of the default-free market and of the default event.

1.2.1 Default Time

We assume from now on that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process Γ on $(\Omega, \mathbb{F}, \mathbb{P}^*)$ with $\Gamma_\infty = \infty$. The default time τ and the probability measure $\mathbb{Q}^*$ are constructed as in Section 1.1.2. The probability $\mathbb{Q}^*$ will play the role of the *martingale probability* for the defaultable market. It is essential to observe that:

- The Wiener process W^* is also a Wiener process with respect to $\mathbb{G}$ under the probability measure $\mathbb{Q}^*$.
- We have $\mathbb{Q}^*_{|\mathcal{F}_t} = \mathbb{P}^*_{|\mathcal{F}_t}$ for every $t \in [0, T]$.

If the hazard process Γ admits the integral representation $\Gamma_t = \int_0^t \gamma_u \, du$ then the process γ is called the (stochastic) *intensity of default* with respect to the reference filtration $\mathbb{F}$.

1.2.2 Risk-Neutral Valuation

We shall now present the well-known valuation formulae for defaultable claims within the reduced-form setup (see, e.g., Lando (1998), Schönbucher (1998), Bielecki and Rutkowski (2004) or Bielecki et al. (2004a)).

Terminal payoff. The valuation of the terminal payoff is based on the following classic result.

Lemma 1.1 *For any $\mathcal{G}$-measurable, integrable random variable X and any $t \leq T$ we have*

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau > T\}} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau > T\}} X \,|\, \mathcal{F}_t)}{\mathbb{Q}^*(\tau > t \,|\, \mathcal{F}_t)}.$$

If, in addition, X is $\mathcal{F}_T$-measurable then

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{\tau > T\}} X \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \, \mathbb{E}_{\mathbb{Q}^*}(e^{\Gamma_t - \Gamma_T} X \,|\, \mathcal{F}_t).$$

Let X be an $\mathcal{F}_T$-measurable random variable representing the promised payoff at maturity date T. We consider a defaultable claim of the form $\mathbb{1}_{\{\tau > T\}} X$ with zero recovery in case of default (i.e., we set $Z = C = 0$). Using the definition of the ex-dividend price of a defaultable claim, we get the following *risk-neutral valuation formula*

$$U_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} \mathbb{1}_{\{\tau > T\}} X \,|\, \mathcal{G}_t)$$

which holds for any $t < T$. The next result is a straightforward consequence of Lemma 1.1.

Proposition 1.1 *The price of the promised payoff X satisfies for $t \in [0, T]$*

$$U_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} X \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \widetilde{U}_t(X), \tag{1.1}$$

where we define

$$\widetilde{U}_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} e^{\Gamma_t - \Gamma_T} X \,|\, \mathcal{F}_t) = \widehat{B}_t \, \mathbb{E}_{\mathbb{Q}^*}(\widehat{B}_T^{-1} X \,|\, \mathcal{F}_t),$$

where the risk-adjusted savings account $\widehat{B}_t$ equals $\widehat{B}_t = B_t e^{\Gamma_t}$. If, in addition, the default time admits the intensity process γ then

$$\widehat{B}_t = \exp\left(\int_0^t (r_u + \gamma_u) \, du \right).$$

The process $\widetilde{U}_t(X)$ represents the *pre-default value* at time t of the promised payoff X. Notice that $\widetilde{U}_T(X) = X$ and the process $\widetilde{U}_t(X)/\widehat{B}_t$, $t \in [0,T]$, is a continuous $\mathbb{F}$-martingale (thus, the process $\widetilde{U}(X)$ is a continuous $\mathbb{F}$-semimartingale).

Remark. The valuation formula (1.1), as well as the concept of pre-default value, should be supported by replication arguments. To this end, we need first to construct a suitable model of a defaultable market. In fact, if we wish to use formula (1.1), we need to know the joint law of all random variables involved, and this appears to be a non-trivial issue, in general.

Recovery payoff. The following result appears to be useful in the valuation of the recovery payoff Z_τ which occurs at time τ. The process $\widetilde{U}(Z)$ introduced below represents the pre-default value of the recovery payoff.

For the proof of Proposition 1.2 we refer, for instance, to Bielecki and Rutkowski (2004) (see Propositions 5.1.1 and 8.2.1 therein).

Proposition 1.2 *Let the hazard process Γ be continuous, and let Z be an $\mathbb{F}$-predictable bounded process. Then for every $t \in [0,T]$ we have*

$$\begin{aligned} U_t(Z) &= B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_\tau^{-1} Z_\tau \mathbb{1}_{\{t<\tau\le T\}} \,|\, \mathcal{G}_t) \\ &= \mathbb{1}_{\{\tau>t\}} B_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T Z_u B_u^{-1} e^{\Gamma_t - \Gamma_u}\, d\Gamma_u \,\Big|\, \mathcal{F}_t\Big) = \mathbb{1}_{\{\tau>t\}} \widetilde{U}_t(Z). \end{aligned}$$

where we set

$$\widetilde{U}_t(Z) = \widehat{B}_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T Z_u \widehat{B}_u^{-1} d\Gamma_u \,\Big|\, \mathcal{F}_t\Big), \quad \forall\, t \in [0,T].$$

If the default intensity γ with respect to $\mathbb{F}$ exists then we have

$$\widetilde{U}_t(Z) = \mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T Z_u e^{-\int_t^u (r_v+\gamma_v)\,dv} \gamma_u\, du \,\Big|\, \mathcal{F}_t\Big).$$

Remark. Notice that $\widetilde{U}_T(Z) = 0$ while, in general, $U_T(Z) = Z_T \mathbb{1}_{\{\tau=T\}}$ is non-zero. Note, however, that if the hazard process Γ is assumed to be continuous then we have $\mathbb{Q}^*\{\tau = T\} = 0$, and thus $\widetilde{U}_T(Z) = 0 = U_T(Z)$.

Promised dividends. To value the promised dividends C that are paid prior to default time τ we shall make use of the following result. Notice that at any date $t < T$ the process $\widetilde{U}(C)$ gives the pre-default value of future promised dividends.

Proposition 1.3 *Let the hazard process Γ be continuous, and let C be an $\mathbb{F}$-predictable, bounded process of finite variation. Then for every $t \in [0,T]$*

$$\begin{aligned} U_t(C) &= B_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,T]} B_u^{-1}(1-H_u)\, dC_u \,\Big|\, \mathcal{G}_t\Big) \\ &= \mathbb{1}_{\{\tau>t\}} B_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,T]} B_u^{-1} e^{\Gamma_t-\Gamma_u}\, dC_u \,\Big|\, \mathcal{F}_t\Big) = \mathbb{1}_{\{\tau>t\}} \widetilde{U}_t(C), \end{aligned}$$

where we define

$$\widetilde{U}_t(C) = \widehat{B}_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,T]} \widehat{B}_u^{-1} \, dC_u \, \Big| \, \mathcal{F}_t \Big), \quad \forall \, t \in [0,T].$$

If, in addition, the default time τ *admits the intensity* γ *with respect to* $\mathbb{F}$ *then*

$$\widetilde{U}_t(C) = \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{(t,T]} e^{-\int_t^u (r_v + \gamma_v)\, dv} \, dC_u \, \Big| \, \mathcal{F}_t \Big).$$

1.2.3 Defaultable Term Structure

For a defaultable discount bond with zero recovery it is natural to adopt the following definition (the superscript 0 refers to the postulated zero recovery scheme) of the price

$$D^0(t,T) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} \mathbb{1}_{\{\tau > T\}} \, | \, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \widetilde{D}^0(t,T),$$

where $\widetilde{D}^0(t,T)$ stands for the *pre-default value* of the bond, which is given by the following equality:

$$\widetilde{D}^0(t,T) = \widehat{B}_t \, \mathbb{E}_{\mathbb{Q}^*}(\widehat{B}_T^{-1} \, | \, \mathcal{F}_t).$$

Since $\mathbb{F}$ is the Brownian filtration, the process $\widetilde{D}^0(t,T)/\widehat{B}_t$ is a continuous, strictly positive, $\mathbb{F}$-martingale. Therefore, the pre-default bond price $\widetilde{D}^0(t,T)$ is a continuous, strictly positive, $\mathbb{F}$-semimartingale. In the special case, when Γ is deterministic, we have $\widetilde{D}^0(t,T) = e^{\Gamma_t - \Gamma_T} B(t,T)$.

Remark. The case zero recovery is, of course, only a particular example. For more general recovery schemes and the corresponding bond valuation results, we refer, for instance, to Section 2.2.4 in Bielecki et al. (2004a).

Let $\mathbb{Q}_T$ stand for the *forward martingale measure*, given on $(\Omega, \mathcal{G}_T)$ (as well as on $(\Omega, \mathcal{F}_T)$) through the formula

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}^*} = \frac{1}{B_T B(0,T)}\,, \quad \mathbb{Q}^*\text{-a.s.},$$

so that the process $W_t^T = W_t^* - \int_0^t b(u,T)\, du$ is a Brownian motion under $\mathbb{Q}_T$. Denote by $F(t,U,T) = B(t,U)(B(t,T))^{-1}$ the forward price of the U-maturity bond, so that

$$dF(t,U,T) = F(t,U,T)\big(b(t,U) - b(t,T)\big) dW_t^T.$$

Since the processes B_t and $B(t,T)$ are $\mathbb{F}$-adapted, it can be shown (see, e.g., Jamshidian (2002)) that Γ is also the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}_T$, and thus

$$\mathbb{Q}_T\{t < \tau \le T \, | \, \mathcal{G}_t\} = \mathbb{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}_T}(e^{\Gamma_t - \Gamma_T} \, | \, \mathcal{F}_t).$$

Let us define an auxiliary process $\Gamma(t,T) = \widetilde{D}^0(t,T)(B(t,T))^{-1}$ (for a fixed $T > 0$). The next result examines the basic properties of the process $\Gamma(t,T)$.

Lemma 1.2 *Assume that the $\mathbb{F}$-hazard process Γ is continuous. The process $\Gamma(t,T)$, $t \in [0,T]$, is a continuous $\mathbb{F}$-submartingale and*

$$d\Gamma(t,T) = \Gamma(t,T)\big(d\Gamma_t + \beta(t,T)\,dW_t^T\big) \tag{1.2}$$

for some $\mathbb{F}$-predictable process $\beta(t,T)$. The process $\Gamma(t,T)$ is of finite variation if and only if the hazard process Γ is deterministic. In the latter case, we have $\Gamma(t,T) = e^{\Gamma_t - \Gamma_T}$.

Proof. Recall that $\widehat{B}_t = B_t e^{\Gamma_t}$ and notice that

$$\Gamma(t,T) = \frac{\widetilde{D}^0(t,T)}{B(t,T)} = \frac{\widehat{B}_t\,\mathbb{E}_{\mathbb{Q}^*}(\widehat{B}_T^{-1}\,|\,\mathcal{F}_t)}{B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1}\,|\,\mathcal{F}_t)} = \mathbb{E}_{\mathbb{Q}_T}(e^{\Gamma_t - \Gamma_T}\,|\,\mathcal{F}_t) = e^{\Gamma_t} M_t,$$

where we set $M_t = \mathbb{E}_{\mathbb{Q}_T}(e^{-\Gamma_T}\,|\,\mathcal{F}_t)$. Recall that the filtration $\mathbb{F}$ is generated by a process W^*, which is a Wiener process with respect to $\mathbb{P}^*$ and $\mathbb{Q}^*$, and all martingales with respect to a Brownian filtration are continuous processes.

We conclude that $\Gamma(t,T)$ is the product of a strictly positive, increasing, right-continuous, $\mathbb{F}$-adapted process e^{Γ_t}, and a strictly positive, continuous, $\mathbb{F}$-martingale M. Furthermore, there exists an $\mathbb{F}$-predictable process $\widehat{\beta}(t,T)$ such that M satisfies

$$dM_t = M_t \widehat{\beta}(t,T)\,dW_t^T$$

with the initial condition $M_0 = \mathbb{E}_{\mathbb{Q}_T}(e^{-\Gamma_T})$. Formula (1.2) follows by an application of Itô's formula, by setting $\beta(t,T) = e^{-\Gamma_t}\widehat{\beta}(t,T)$. To complete the proof, it suffices to recall that a continuous martingale is never of finite variation, unless it is a constant process. $\square$

Suppose that $\Gamma_t = \int_0^t \gamma_u\,du$. Then (1.2) yields

$$d\Gamma(t,T) = \Gamma(t,T)\big(\gamma_t\,dt + \beta(t,T)\,dW_t^T\big).$$

Consequently, the pre-default price $\widetilde{D}^0(t,T) = \Gamma(t,T)B(t,T)$ is governed by

$$d\widetilde{D}^0(t,T) = \widetilde{D}^0(t,T)\Big(\big(r_t + \gamma_t + \beta(t,T)b(t,T)\big)\,dt + \widetilde{b}(t,T)\,dW_t^*\Big), \tag{1.3}$$

where the volatility process equals $\widetilde{b}(t,T) = \beta(t,T) + b(t,T)$.

1.3 Properties of Trading Strategies

In this section, we shall examine the most basic properties of the wealth process of a self-financing trading strategy. First, we concentrate on trading in default-free assets. In the next step, we also include defaultable assets in our portfolio.

1.3.1 Default-Free Primary Assets

Our goal in this section is to present some auxiliary results related to the concept of a self-financing trading strategy for a market model involving default-free and defaultable securities. For the sake of the reader's convenience, we shall first discuss briefly the classic concepts of self-financing cash and futures strategies in the context of default-free market model. It appears that in case of defaultable securities only minor adjustments of definitions and results are needed (see, Vaillant (2001) or Blanchet-Scalliet and Jeanblanc (2004)).

Cash Strategies

Let Y^1_t and Y^2_t stand for the cash prices at time $t \in [0,T]$ of two tradeable assets. We postulate that Y^1 and Y^2 are continuous semimartingales. We assume, in addition, that the process Y^1 is strictly positive, so that it can be used as a numeraire.

Remark. We chose the convention that price processes of default-free securities are continuous semimartingales. Results of this section can be extended to the case of general semimartingales (for instance, jump diffusions). Our choice was motivated by the desire of providing relatively simple closed-form expressions.

Let $\phi = (\phi^1, \phi^2)$ be a trading strategy for default-free market so that, in particular, processes ϕ^1 and ϕ^2 are predictable with respect to the reference filtration $\mathbb{F}$ (the same measurability assumption will be valid for components $\phi^1, \ldots, \phi^k$ of a k-dimensional trading strategy). The component ϕ^i_t represents the number of units of the i^{th} asset held in the portfolio at time t.

Let $V_t(\phi)$ denote the wealth of the cash strategy $\phi = (\phi^1, \phi^2)$ at time t, so that

$$V_t(\phi) = \phi^1_t Y^1_t + \phi^2_t Y^2_t, \quad \forall\, t \in [0,T].$$

We say that the cash strategy ϕ is *self-financing* if

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi^1_u \, dY^1_u + \int_0^t \phi^2_u \, dY^2_u, \quad \forall\, t \in [0,T],$$

that is,

$$dV_t(\phi) = \phi^1_t \, dY^1_t + \phi^2_t \, dY^2_t.$$

This yields

$$dV_t(\phi) = (V_t(\phi) - \phi^2_t Y^2_t)(Y^1_t)^{-1} \, dY^1_t + \phi^2_t \, dY^2_t.$$

Let us introduce the relative values:

$$V^1_t(\phi) = V_t(\phi)(Y^1_t)^{-1}, \quad Y^{2,1}_t = Y^2_t (Y^1_t)^{-1}.$$

A simple application of Itô's formula yields

$$V^1_t(\phi) = V^1_0(\phi) + \int_0^t \phi^2_u \, dY^{2,1}_u.$$

It is well known that a similar result holds for any finite number of cash assets. Let $Y_t^1, Y_t^2, \ldots, Y_t^k$ represent that cash values at time t of k assets. We postulate that Y^1, $Y^2, \ldots, Y^k$ are continuous semimartingales. Then the wealth $V_t(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ equals

$$V_t(\phi) = \sum_{i=1}^k \phi_t^i Y_t^i, \quad \forall\, t \in [0,T], \tag{1.4}$$

and ϕ is said to be a *self-financing cash strategy* if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^k \int_0^t \phi_u^i \, dY_u^i, \quad \forall\, t \in [0,T]. \tag{1.5}$$

Suppose that the process Y^1 is strictly positive. Then by combining the last two formulae, we obtain

$$dV_t(\phi) = \Big(V_t(\phi) - \sum_{i=2}^k \phi_t^i Y_t^i \Big) (Y_t^1)^{-1} \, dY_t^1 + \sum_{i=2}^k \phi_t^i \, dY_t^i.$$

The latter representation shows that the wealth process depends only on $k-1$ components of ϕ. Choosing Y^1 as a numeraire asset, and denoting $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$, $Y_t^{i,1} = Y_t^i (Y_t^1)^{-1}$, we get the following well-known result.

Lemma 1.3 *Let $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ be a self-financing cash strategy. Then we have*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^k \int_0^t \phi_u^i \, dY_u^{i,1}, \quad \forall\, t \in [0,T].$$

Cash-Futures Strategies

Let us first consider the special case of two assets. Assume that Y_t^1 and Y_t^2 represent the cash and futures prices at time $t \in [0,T]$ of some assets, respectively. As before, we postulate that Y^1 and Y^2 are continuous semimartingales. Moreover, Y^1 is assumed to be a strictly positive process. In view of specific features of a futures contract, it is natural to postulate that the wealth $V_t(\phi)$ satisfies

$$V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 0 = \phi_t^1 Y_t^1, \quad \forall\, t \in [0,T].$$

The cash-futures strategy $\phi = (\phi^1, \phi^2)$ is self-financing if

$$dV_t(\phi) = \phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2, \tag{1.6}$$

which yields, provided that Y^1 is strictly positive,

$$dV_t(\phi) = V_t(\phi)(Y_t^1)^{-1} \, dY_t^1 + \phi_t^2 \, dY_t^2.$$

Remark. Let us recall that the futures price Y_t^2 (that is, the quotation of a futures contract at time t) has different features than the cash price of an asset.

Specifically, we make the standard assumption that it is possible to enter a futures contract at no initial cost. The gains or losses from futures contracts are associated with *marking to market* (see, for instance, Duffie (2003) or Musiela and Rutkowski (1997)). Note that the 0 in the formula defining $V_t(\phi)$ is aimed to represent the value of a futures contract at time t, as opposed to the futures price Y_t^2 at this date.

Lemma 1.4 *Let $\phi = (\phi^1, \phi^2)$ be a self-financing cash-futures strategy. Suppose that the processes Y^1 and Y^2 are strictly positive. Then the relative wealth process $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ satisfies*

$$V_t^1(\phi) = V_0^1(\phi) + \int_0^t \widehat{\phi}_u^{2,1}\, d\widehat{Y}_u^{2,1}, \quad \forall\, t \in [0,T],$$

where $\widehat{\phi}_t^{2,1} = \phi_t^2 (Y_t^1)^{-1} e^{\alpha_t^{2,1}}$, $\widehat{Y}_t^{2,1} = Y_t^2 e^{-\alpha_t^{2,1}}$ and

$$\alpha_t^{2,1} = \langle \ln Y^2, \ln Y^1 \rangle_t = \int_0^t (Y_u^2)^{-1} (Y_u^1)^{-1}\, d\langle Y^2, Y^1 \rangle_u.$$

Proof. For brevity, we write $V_t = V_t(\phi)$ and $V_t^1 = V_t^1(\phi)$. The Itô formula, combined with (1.6), yields

$$\begin{aligned}
dV_t^1 &= (Y_t^1)^{-1} dV_t + V_t\, d(Y_t^1)^{-1} + d\langle (Y^1)^{-1}, V \rangle_t \\
&= \phi_t^1 (Y_t^1)^{-1}\, dY_t^1 + \phi_t^2 (Y_t^1)^{-1}\, dY_t^2 + \phi_t^1 Y_t^1\, d(Y_t^1)^{-1} \\
&\quad - \phi_t^1 (Y_t^1)^{-2}\, d\langle Y^1, Y^1 \rangle_t - \phi_t^2 (Y_t^1)^{-2}\, d\langle Y^1, Y^2 \rangle_t \\
&= \phi_t^2 (Y_t^1)^{-1} dY_t^2 - \phi_t^2 (Y_t^1)^{-2}\, d\langle Y^1, Y^2 \rangle_t \\
&= \phi_t^2 e^{\alpha_t^{2,1}} (Y_t^1)^{-1} \big(e^{-\alpha_t^{2,1}} dY_t^2 - Y_t^2 e^{-\alpha_t^{2,1}}\, d\alpha_t^{2,1} \big) = \widehat{\phi}_t^{2,1}\, d\widehat{Y}_t^{2,1}
\end{aligned}$$

and the result follows. $\square$

Let $Y^1, \ldots, Y^l$ be the cash prices of l assets, and let $Y^{l+1}, \ldots, Y^k$ represent the futures prices of $k - l$ assets. Then the wealth process of a trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ is given by the formula

$$V_t(\phi) = \sum_{i=1}^{l} \phi_t^i Y_t^i, \quad \forall\, t \in [0,T], \tag{1.7}$$

and ϕ is a *self-financing cash-futures strategy* whenever

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{k} \int_0^t \phi_u^i\, dY_u^i, \quad \forall\, t \in [0,T].$$

The proof of the next result relies on the similar calculations as the proofs of Lemmas 1.3 and 1.4.

Lemma 1.5 *Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing cash-futures strategy. Suppose that the processes Y^1 and $Y^{l+1}, \dots, Y^k$ are strictly positive. Then the relative wealth process $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ satisfies, for every $t \in [0, T]$,*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k} \int_0^t \widehat{\phi}_u^{i,1} \, d\widehat{Y}_u^{i,1},$$

where we denote $Y_t^{i,1} = Y_t^i (Y_t^1)^{-1}$, $\widehat{\phi}_t^{i,1} = \phi_t^i (Y_t^1)^{-1} e^{\alpha_t^{i1}}$, $\widehat{Y}_t^{i,1} = Y_t^i e^{-\alpha_t^{i1}}$, and

$$\alpha_t^{i1} = \langle \ln Y^i, \ln Y^1 \rangle_t = \int_0^t (Y_u^i)^{-1} (Y_u^1)^{-1} \, d\langle Y^i, Y^1 \rangle_u.$$

Constrained Cash Strategies

We continue the analysis of cash strategies for some $k \geq 3$. Price processes $Y^1, Y^2, \dots, Y^k$ are assumed to be continuous semimartingales. We postulate, in addition, that Y^1 and $Y^{l+1}, \dots, Y^k$ are strictly positive processes, where $1 < l + 1 \leq k$. Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing trading strategy, so that the wealth process $V(\phi)$ satisfies (1.4)-(1.5). We shall consider three particular cases of increasing generality.

Strategies with zero net investment in $Y^{l+1}, \dots, Y^k$. Assume first that at any time t there is zero net investment in assets $Y^{l+1}, \dots, Y^k$. Specifically, we postulate that the strategy is subject to the following constraint:

$$\sum_{i=l+1}^{k} \phi_t^i Y_t^i = 0, \quad \forall\, t \in [0, T], \tag{1.8}$$

so that the wealth process $V_t(\phi)$ is given by (1.7). Equivalently, we have $\phi_t^k = -\sum_{i=l+1}^{k-1} \phi_t^i Y_t^i (Y_t^k)^{-1}$. Combining the last equality with (1.5), we obtain

$$\begin{aligned} dV_t(\phi) &= \Big(V_t(\phi) - \sum_{i=2}^{l} \phi_t^i Y_t^i \Big) (Y_t^1)^{-1} \, dY_t^1 \\ &\quad + \sum_{i=2}^{l} \phi_t^i \, dY_t^i + \sum_{i=l+1}^{k-1} \phi_t^i \big(dY_t^i - Y_t^i (Y_t^k)^{-1} \, dY_t^k \big). \end{aligned}$$

It is thus clear that the wealth process $V(\phi)$ depends only on $k-2$ components $\phi^2, \dots, \phi^{k-1}$ of the k-dimensional trading strategy ϕ. The following result, which can be seen as an extension of Lemma 1.4, provides a more convenient representation for the (relative) wealth process.

Lemma 1.6 *Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing cash strategy such that (1.8) holds. Assume that the processes $Y^1, Y^{l+1}, \dots, Y^k$ are strictly positive. Then the relative wealth process $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ satisfies*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \widehat{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1}, \quad \forall\, t \in [0, T],$$

where we denote

$$\widehat{\phi}_t^{i,k,1} = \phi_t^i (Y_t^{1,k})^{-1} e^{\alpha_t^{i,k,1}}, \quad \widehat{Y}_t^{i,k,1} = Y_t^{i,k} e^{-\alpha_t^{i,k,1}}, \tag{1.9}$$

with $Y_t^{i,k} = Y_t^i (Y_t^k)^{-1}$ *and*

$$\alpha_t^{i,k,1} = \langle \ln Y^{i,k}, \ln Y^{1,k} \rangle_t = \int_0^t (Y_u^{i,k})^{-1} (Y_u^{1,k})^{-1} \, d\langle Y^{i,k}, Y^{1,k} \rangle_u. \tag{1.10}$$

Proof. Let us consider the relative values of all processes, with the price Y^k chosen as a numeraire, and let us consider the process

$$V_t^k(\phi) := V_t(\phi)(Y_t^k)^{-1} = \sum_{i=1}^k \phi_t^i Y_t^{i,k}.$$

In view of the constraint (1.8) we have that $V_t^k(\phi) = \sum_{i=1}^l \phi_t^i Y_t^{i,k}$. In addition, similarly as in Lemma 1.3, we obtain

$$dV_t^k(\phi) = \sum_{i=1}^{k-1} \phi_t^i \, dY_t^{i,k}.$$

Since

$$Y_t^{i,k} (Y_t^{1,k})^{-1} = Y_t^{i,1}, \quad V_t^1(\phi) = V_t^k(\phi)(Y_t^{1,k})^{-1},$$

using argument analogous as in proof of Lemma 1.4, we obtain

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^l \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \widehat{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1}, \quad \forall\, t \in [0,T],$$

where the processes $\widehat{\phi}_t^{i,k,1}$, $\widehat{Y}_t^{i,k,1}$ and $\alpha_t^{i,k,1}$ are given by (1.9)-(1.10). □

Strategies with a pre-specified net investment Z in $Y^{l+1}, \dots, Y^k$. We shall now postulate that the strategy ϕ is such that

$$\sum_{i=l+1}^k \phi_t^i Y_t^i = Z_t, \quad \forall\, t \in [0,T], \tag{1.11}$$

for a pre-specified, $\mathbb{F}$-progressively measurable, process Z. The following result is a rather straightforward extension of Lemma 1.6.

Lemma 1.7 *Let* $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ *be a self-financing cash strategy such that* (1.11) *holds. Assume that the processes* $Y^1, Y^{l+1}, \dots, Y^k$ *are strictly positive. Then the relative wealth process* $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ *satisfies*

$$\begin{aligned} V_t^1(\phi) &= V_0^1(\phi) + \sum_{i=2}^l \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \widehat{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} \\ &\quad + \int_0^t Z_u (Y_u^k)^{-1} \, d(Y_u^{1,k})^{-1}, \end{aligned}$$

where $\widehat{\phi}_t^{i,k,1}$, $\widehat{Y}_t^{i,k,1}$ *and* $\alpha_t^{i,k,1}$ *are given by* (1.9)-(1.10).

Proof. Let us sketch the proof of the lemma for $k = 3$. Then $l = 2$ and $\phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = Z_t$ for every $t \in [0, T]$. Consequently, for the process $V^3(\phi) = V(\phi)(Y^3)^{-1}$ we get

$$V_t^3(\phi) = \sum_{i=1}^{3} \phi_t^i Y_t^i (Y_t^3)^{-1} = \phi_t^1 Y_t^{1,3} + Z_t (Y_t^3)^{-1}, \quad \forall\, t \in [0, T].$$

Furthermore, the self-financing condition yields

$$dV_t^3(\phi) = \phi_t^1 \, dY_t^{1,3} + \phi_t^2 \, dY_t^{2,3}.$$

Proceeding in an analogous way as in the proof of Lemma 1.4, we obtain for $V_t^1(\phi) = V_t^3(\phi)(Y_t^{1,3})^{-1}$

$$\begin{aligned} dV_t^1(\phi) &= \phi_t^2 e^{\alpha_t^{2,3,1}} (Y_t^{1,3})^{-1} \big(e^{-\alpha_t^{2,3,1}} dY_t^{2,3} - Y_t^{2,3} e^{-\alpha_t^{2,3,1}} d\alpha_t^{2,3,1} \big) \\ &\quad + Z_t (Y_t^3)^{-1} d(Y_t^{1,3})^{-1} \\ &= \widehat{\phi}_u^{2,3,1} \, d\widehat{Y}_u^{2,3,1} + Z_t (Y_t^3)^{-1} d(Y_t^{1,3})^{-1}, \end{aligned}$$

where $\widehat{\phi}_t^{2,3,1} = \phi_t^2 (Y_t^{1,3})^{-1} e^{\alpha_t^{2,3,1}}$, $\widehat{Y}_t^{2,3,1} = Y_t^{2,3} e^{-\alpha_t^{2,3,1}}$ and

$$\alpha_t^{2,3,1} = \langle \ln Y^{2,3}, \ln Y^{1,3} \rangle_t = \int_0^t (Y_u^{2,3})^{-1} (Y_u^{1,3})^{-1} \, d\langle Y^{2,3}, Y^{1,3} \rangle_u.$$

The proof for the general case is based on similar calculations. □

Strategies with consumption A and a pre-specified net investment Z in $Y^{l+1}, \dots, Y^k$. Let A be an $\mathbb{F}$-adapted process of finite variation, with $A_0 = 0$. We consider a self-financing cash strategy ϕ with *consumption process* A, so that the wealth process $V(\phi)$ satisfies:

$$V_t(\phi) = \sum_{i=1}^{k} \phi_t^i Y_t^i = \sum_{i=1}^{l} \phi_t^i Y_t^i + Z_t, \quad \forall\, t \in [0, T],$$

and

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{k} \int_0^t \phi_u^i \, dY_u^i + A_t, \quad \forall\, t \in [0, T].$$

Then it suffices to modify the formula established in Lemma 1.7 by adding a term associated with the consumption process A. Specifically, for the relative wealth process $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ we obtain the following integral representation, which is valid for every $t \in [0, T]$

$$\begin{aligned} V_t^1(\phi) &= V_0^1(\phi) + \sum_{i=2}^{l} \int_0^t \phi_u^i \, dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \widehat{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} \\ &\quad + \int_0^t Z_u (Y_u^k)^{-1} \, d(Y_u^{1,k})^{-1} + \int_0^t (Y_u^1)^{-1} \, dA_u. \end{aligned}$$

Remark. We use here a generic term 'consumption' to reflect the impact of A on the wealth. The financial interpretation of A depends on particular circumstances. For instance, an increasing process A represents the inflows of cash, rather than the outflows of cash (the latter case is commonly referred to as consumption in the financial literature).

1.3.2 Defaultable and Default-Free Primary Assets

Let $Y^1, \dots, Y^m$ be prices of m defaultable assets, and let $Y^{m+1}, \dots, Y^k$ represent prices of $k-m$ default-free assets. Processes $Y^{m+1}, \dots, Y^k$ are assumed to be continuous semimartingales. We make here an essential assumption that τ is the default time for each defaultable asset Y^i, $i = 1, \dots, m$. Of course, in the case of defaultable assets with different default times (e.g., when dealing with the first-to-default claim), some definitions should be modified in a natural way. A special case of first-to-default claims is examined in Section 1.4.4.

Self-Financing Trading Strategies

The following definition is a rather obvious extension of conditions (1.4)-(1.5). We postulate here that the processes $\phi^1, \dots, \phi^k$ are $\mathbb{G}$-predictable processes, in general.

Definition 1.2 The wealth $V_t(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ equals $V_t(\phi) = \sum_{i=1}^k \phi_t^i Y_t^i$ for every $t \in [0,T]$. A strategy ϕ is said to be *self-financing* if for every $t \in [0,T]$

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^m \int_0^t \phi_{u-}^i \, dY_u^i + \sum_{i=m+1}^k \int_0^t \phi_u^i \, dY_u^i.$$

Although Definition 1.2 is formulated in a general setup, it can be simplified for our further purposes. Indeed, since we shall deal only with defaultable claims with default time τ, we shall only examine a particular trading strategy ϕ prior to and at default time τ or, more precisely, on the stochastic interval $[\![0, \tau \wedge T]\!]$, where $[\![0, \tau \wedge T]\!] = \{(t, \omega) \in \mathbb{R}_+ \times \Omega \,:\, 0 \le t \le \tau(\omega) \wedge T\}$.

In fact, we shall examine separately the following issues: (i) the behavior of the wealth process $V(\phi)$ on the random interval $[\![0, \tau \wedge T[\![= \{(t, \omega) \in \mathbb{R}_+ \times \Omega \,:\, 0 \le t < \tau(\omega) \wedge T\}$ and (ii) the size of its jump at the random time moment $\tau \wedge T$ or, equivalently, the value of $V_{\tau \wedge T}$. Such a study is, of course, sufficient in our setup, since we only consider the case where a recovery payment (if any) is made at the default time (and not after this date). Consequently, since we never deal with a trading strategy after the random time $\tau \wedge T$, we may and do assume from now on that all components $\phi^1, \phi^2, \dots, \phi^k$ of a portfolio ϕ are $\mathbb{F}$-predictable, rather than $\mathbb{G}$-predictable processes.

It is worthwhile to mention, that in the next two chapters we will examine the importance of the measurability property of an admissible trading strategy within the framework of optimization problems in incomplete market.

Remark. It can be formally shown that for any $\mathbb{R}^k$-valued $\mathbb{G}$-predictable process ϕ there exists a unique $\mathbb{F}$-predictable process ψ such that the equality $\mathbb{1}_{\{\tau \geq t\}}\phi_t = \mathbb{1}_{\{\tau \geq t\}}\psi_t$ holds for every $t \in [0, T]$. In addition, we find it convenient to postulate, by convention, that the price processes $Y^{m+1}, \ldots, Y^k$ are also stopped at the random time $\tau \wedge T$.

We have the following definition of a trading strategy.

Definition 1.3 By a *trading strategy* $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ we mean a family $\phi^1, \phi^2, \ldots, \phi^k$ of $\mathbb{F}$-predictable stochastic processes.

Let us stress that if a trading strategy considered in this section is self-financing on $[\![0, \tau \wedge T[\![$ then it is also self-financing on $[\![0, \tau \wedge T]\!]$. At the intuitive level, the portfolio is not rebalanced at time $\tau \wedge T$, but it is rather sold out in order to cover liabilities. Let $\widetilde{Y}^i_t$ stands for the pre-default value of the i^{th} defaultable asset at time t. We postulate throughout that processes $\widetilde{Y}^i$, $i = 1, \ldots, m$ are continuous $\mathbb{F}$-semimartingales.

Definition 1.4 The *pre-default wealth process* $\widetilde{V}(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ equals, for every $t \in [0, T]$,

$$\widetilde{V}_t(\phi) = \sum_{i=1}^{m} \phi^i_t \widetilde{Y}^i_t + \sum_{i=m+1}^{k} \phi^i_t Y^i_t.$$

A strategy ϕ is said to be *self-financing prior to default* if for every $t \in [0, T]$

$$\widetilde{V}_t(\phi) = \widetilde{V}_0(\phi) + \sum_{i=1}^{m} \int_0^t \phi^i_u \, d\widetilde{Y}^i_u + \sum_{i=m+1}^{k} \int_0^t \phi^i_u \, dY^i_u.$$

Note that $\widetilde{V}_0(\phi) = V_0(\phi)$, since $\mathbb{P}^*\{\tau > 0\} = 1$. Let us stress that if a trading strategy ϕ is self-financing prior to default then ϕ is also self-financing on $[0, T]$. Indeed, we always postulate that trading ceases at time of default, and the terminal wealth at time $\tau \wedge T$ equals

$$V_{\tau \wedge T}(\phi) = \sum_{i=1}^{k} \phi^i_{\tau \wedge T} Y^i_{\tau \wedge T}.$$

Of course, on the event $\{\tau > T\}$ we also have

$$V_{\tau \wedge T}(\phi) = V_T(\phi) = \widetilde{V}_T(\phi) = \sum_{i=1}^{m} \phi^i_T \widetilde{Y}^i_T + \sum_{i=m+1}^{k} \phi^i_T Y^i_T.$$

Hence, we shall not distinguish in what follows between the concept of a self-financing trading strategy and a trading strategy self-financing prior to default.

Zero Recovery for Defaultable Assets

The following assumption corresponds to the simplest situation of zero recovery for all defaultable primary assets that are used for replication. Manifestly, this assumption is not practical, and thus it will be later relaxed.

Assumption (A). The defaultable primary assets $Y^1, \dots, Y^m$ are all subject to the zero recovery scheme, and they have a common default time τ.

By virtue of Assumption (A), the prices $Y^1, \dots, Y^m$ vanish at default time τ, and thus also after this date. Consequently, for every $i = 1, \dots, m$ we have $Y_t^i = \mathbb{1}_{\{\tau > t\}} \widetilde{Y}_t^i$ for every $t \in [0, T]$ for some $\mathbb{F}$-predictable processes $\widetilde{Y}^1, \dots, \widetilde{Y}^m$. In other words, for any $i = 1, \dots, m$ the price Y^i jumps from $\widetilde{Y}^i_{\tau-}$ to $\widetilde{Y}^i_\tau = 0$ at the time of default. We make a technical assumption that the pre-default values $\widetilde{Y}^1, \dots, \widetilde{Y}^m$ are continuous $\mathbb{F}$-semimartingales.

In order to be able to use the price Y^1 as a numeraire prior to default, we assume that the pre-default price $\widetilde{Y}^1$ is a strictly positive continuous $\mathbb{F}$-semimartingale. Notice that $\widetilde{Y}^1_0 = Y^1_0$.

Assume first zero recovery for the defaultable contingent claim we wish to replicate. Thus, at time τ the wealth process of any strategy that is capable to replicate the claim $\mathbb{1}_{\{\tau > T\}} X$ should necessarily jump to zero, provided that $\tau \le T$. We can achieve this by considering only self-financing strategies $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ such that at any time the net investment in default-free assets $Y^{m+1}, \dots, Y^k$ equals zero, so that we have

$$\sum_{i=m+1}^{k} \phi_t^i Y_t^i = 0, \quad \forall\, t \in [0, T]. \tag{1.12}$$

In the general case, that is, when Z is a pre-specified non-zero recovery process for a defaultable claim under consideration, it suffices to consider self-financing strategies $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ such that

$$\sum_{i=m+1}^{k} \phi_t^i Y_t^i = Z_t, \quad \forall\, t \in [0, T]. \tag{1.13}$$

Notice that prior to default time (that is, on the event $\{\tau > t\}$) we have $V_t(\phi) = \sum_{i=1}^m \phi_t^i \widetilde{Y}_t^i + Z_t$, and the self-financing property of ϕ prior to default time τ takes the following form

$$dV_t(\phi) = \sum_{i=1}^{m} \phi_t^i \, d\widetilde{Y}_t^i + \sum_{i=m+1}^{k} \phi_t^i \, dY_t^i. \tag{1.14}$$

At default time τ, we have $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \le T\}$.

The next goal is to examine the existence of ϕ with the properties described above. To this end, we denote $\widetilde{Y}_t^{i,1} = \widetilde{Y}_t^i (\widetilde{Y}_t^1)^{-1}$ for $i = 2, \dots, m$ and $\widetilde{Y}_t^{1,k} = \widetilde{Y}_t^1 (Y_t^k)^{-1}$. As before, we write $Y_t^{i,k} = Y_t^i (Y_t^k)^{-1}$. Using Lemma 1.7, we obtain the following auxiliary result that will be later used to establish the existence of a replicating strategy for a defaultable claim.

Proposition 1.4 (i) *Let $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ be a self-financing strategy such that* (1.13) *holds. Assume that the processes $\widetilde{Y}^1, Y^{m+1}, \ldots, Y^k$ are strictly positive. Then the pre-default wealth process $\widetilde{V}(\phi)$ satisfies for every $t \in [0,T]$*

$$\begin{aligned}\widetilde{V}_t(\phi) &= \widetilde{Y}^1_t \Big(\widetilde{V}^1_0(\phi) + \sum_{i=2}^{m} \int_0^t \phi^i_u \, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\phi}^{i,k,1}_u \, d\widehat{Y}^{i,k,1}_u \\ &\quad + \int_0^t Z_u (Y^k_u)^{-1} \, d(\widetilde{Y}^{1,k}_u)^{-1} \Big),\end{aligned}$$

where we denote

$$\widetilde{\phi}^{i,k,1}_t = \phi^i_t (\widetilde{Y}^{1,k}_t)^{-1} e^{\widetilde{\alpha}^{i,k,1}_t}, \quad \widehat{Y}^{i,k,1}_t = Y^{i,k}_t e^{-\widetilde{\alpha}^{i,k,1}_t},$$

and

$$\widetilde{\alpha}^{i,k,1}_t = \langle \ln Y^{i,k}, \ln \widetilde{Y}^{1,k} \rangle_t = \int_0^t (Y^{i,k}_u)^{-1} (\widetilde{Y}^{1,k}_u)^{-1} \, d\langle Y^{i,k}, \widetilde{Y}^{1,k} \rangle_u.$$

In addition, at default time the wealth of ϕ equals $V_\tau(\phi) = Z_\tau$ on the event $\{\tau \leq T\}$.
(ii) *Suppose that the $\mathbb{F}$-predictable processes ψ^i, $i = 2, \ldots, m$ and $\widetilde{\psi}^{i,k,1}$, $i = m+1, \ldots, k-1$ are given. For an arbitrary constant $c \in \mathbb{R}$, we define the process $\widetilde{V}$ by setting, for $t \in [0,T]$,*

$$\widetilde{V}_t = c + \sum_{i=2}^{m} \int_0^t \psi^i_u \, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\psi}^{i,k,1}_u \, d\widehat{Y}^{i,k,1}_u + \int_0^t Z_u (Y^k_u)^{-1} \, d(\widetilde{Y}^{1,k}_u)^{-1}.$$

Then there exists a self-financing trading strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ such that:
(a) $\phi^i_t = \psi^i_t$ *for* $i = 2, \ldots, m$ *and* $\phi^i_t = \widetilde{\psi}^{i,k,1}_t \widetilde{Y}^{1,k}_t e^{-\widetilde{\alpha}^{i,k,1}_t}$ *for* $i = m+1, \ldots, k-1$,
(b) ϕ *satisfies* (1.13), *so that* $\sum_{i=m+1}^{k} \phi^i_t Y^i_t = Z_t$ *for every* $t \in [0,T]$,
(c) *the pre-default wealth* $\widetilde{V}(\phi)$ *of* ϕ *equals* $\widetilde{V}$,
(d) *at default time the wealth of* ϕ *equals* $V_\tau(\phi) = Z_\tau$ *on the event* $\{\tau \leq T\}$.

Proof. Part (i) is an almost immediate consequence of Lemma 1.7. Therefore, we shall focus on the second part. The idea of the proof of part (ii) is also rather clear. First, let ϕ^i, $i = 2, \ldots, m$ and ϕ^i, $i = m+1, \ldots, k-1$ be defined from processes ψ^i and $\widetilde{\psi}^{i,k,1}_t$ as in (a). Given the processes ϕ^i for $i = m+1, \ldots, k-1$, we observe that the component ϕ^k is uniquely specified by condition (1.13). Thus, it remains to check that there exists a (unique) component ϕ^1 such that the resulting k-dimensional trading strategy is self-financing prior to default, in the sense of Definition 1.4. Let us set

$$\phi^1_t = \Big(\widetilde{V}_t - \sum_{i=2}^{m} \phi^i_t Y^i_t - Z_t \Big) (\widetilde{Y}^1_t)^{-1} = \Big(\widetilde{V}_t - \sum_{i=2}^{k} \phi^i_t Y^i_t \Big) (\widetilde{Y}^1_t)^{-1}.$$

It is clear that $\widetilde{V}_t(\phi) = \widetilde{V}_t$ for every $t \in [0,T]$. To show that the strategy $(\phi^1, \phi^2, \dots, \phi^k)$ described above is self-financing prior to default, it suffices to show that for the discounted pre-default wealth

$$\widetilde{V}_t^1(\phi) = \sum_{i=1}^{m} \phi_t^i \widetilde{Y}_t^{i,1} + \sum_{i=m+1}^{k} \phi_t^i Y_t^{i,1}$$

we have for every $t \in [0,T]$

$$\widetilde{V}_t^1(\phi) = \widetilde{V}_0^1(\phi) + \sum_{i=2}^{m} \int_0^t \phi_u^i d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k} \int_0^t \phi_u^i dY_u^{i,1}.$$

Towards this end, it is enough observe that $\widetilde{V}_t^1(\phi) = (\widetilde{Y}_t^1)^{-1}\widetilde{V}_t = \widetilde{V}_t^1$, and then to verify that

$$\widetilde{V}_t^1 = \widetilde{V}_0^1 + \sum_{i=2}^{m} \int_0^t \phi_u^i d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k} \int_0^t \phi_u^i dY_u^{i,1}$$

for every $t \in [0,T]$. To establish that last equality, it suffices to use the definition of the process $\widetilde{V}^1$ and to observe that

$$\sum_{i=m+1}^{k-1} \widetilde{\psi}_t^{i,k,1}\, d\widehat{Y}_t^{i,k,1} = \sum_{i=m+1}^{k} \phi_t^i\, dY_t^{i,1},$$

which follows by direct calculations, using the definitions of ϕ^i, $i = m+1, \dots, k$. It is easy to see that the strategy ϕ satisfies conditions (a)-(d). $\square$

Remarks. Let us observe that the equality established in Proposition 1.4 is in fact valid on the random interval $[\![0,\tau[\![$ on the event $\{\tau \le T\}$ and on the interval $[0,T]$ on the event $\{\tau > T\}$. It is also important to notice that the assumption of zero recovery for $Y^1, \dots, Y^m$ is not essential for the validity of the statements in the last result, except for the last part, that is, the equality $V_\tau(\phi) = Z_\tau$. Indeed, the proof of Proposition 1.4 relies on conditions (1.13) and (1.14). Therefore, if defaultable primary assets $Y^1, \dots, Y^m$ are subject to non-zero recovery, it will be possible to modify Proposition 1.4 accordingly (see Section 1.3.2 below).

When dealing with defaultable claims with no recovery, that is, claims for which the recovery process Z vanishes, it will be convenient to use directly the following corollary to Proposition 1.4.

Corollary 1.1 *Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing strategy such that condition* (1.12) *holds.*
(i) *Assume that the processes $\widetilde{Y}^1, Y^{m+1}, \dots, Y^k$ are strictly positive. Then the wealth process $V(\phi)$ satisfies for every $t \in [0,T]$*

$$V_t(\phi) = Y_t^1 \Big(V_0^1(\phi) + \sum_{i=2}^{m} \int_0^t \phi_u^i\, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\phi}_u^{i,k,1}\, d\widehat{Y}_u^{i,k,1} \Big).$$

(ii) *Assume that all primary assets are defaultable, that is, $m = k$, and the pre-default value $\widetilde{Y}^1$ is a strictly positive process. Then the wealth process $V(\phi)$ satisfies for every $t \in [0,T]$*

$$V_t(\phi) = Y_t^1 \Big(V_0^1(\phi) + \sum_{i=2}^m \int_0^t \phi_u^i \, d\widetilde{Y}_u^{i,1} \Big).$$

Of course, the counterparts of part (ii) in Proposition 1.4 are also valid and they will be used in what follows, although they are not explicitly formulated here.

Remark. Consider the special case of two primary assets, defaultable and default-free, with prices $Y_t^1 = \mathbb{1}_{\{\tau > t\}} \widetilde{Y}_t^1$ and Y_t^2, respectively, where $\widetilde{Y}^1$ and Y^2 are strictly positive, continuous, $\mathbb{F}$-semimartingales. Suppose we wish to replicate a defaultable claim with zero recovery. We have

$$V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = \phi_t^1 \mathbb{1}_{\{\tau > t\}} \widetilde{Y}_t^1 + \phi_t^2 Y_t^2$$

and

$$dV_t(\phi) = (V_{t-}(\phi) - \phi_t^2 Y_t^2)(\widetilde{Y}_t^1)^{-1} dY_t^1 + \phi_t^2 \, dY_t^2.$$

It is rather clear that the equality $V_t(\phi) = 0$ on $\{\tau \le t\}$ implies that $\phi_t^2 = 0$ for every $t \in [0,T]$. Therefore,

$$dV_t(\phi) = V_{t-}(\phi)(\widetilde{Y}_t^1)^{-1} dY_t^1$$

and the existence of replicating strategy for a defaultable claim with zero-recovery is unlikely within the present setup (except for some trivial cases).

Non-Zero Recovery for Defaultable Assets

In this section, the assumption of zero recovery for defaultable primary assets $Y^1, \dots, Y^m$ is relaxed. To be more specific, Assumption (A) is replaced by the following weaker restriction.

Assumption (B). We assume that the defaultable assets $Y^1, \dots, Y^m$ are subject to an arbitrary recovery scheme, and they have a common default time τ.

Under Assumption (B), condition (1.13) no longer implies that $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \le T\}$. We can achieve this requirement by substituting (1.13) with the following constraint

$$\sum_{i=1}^m \phi_t^i \bar{Y}_t^i + \sum_{i=m+1}^k \phi_t^i Y_t^i = Z_t, \quad \forall \, t \in [0,T], \tag{1.15}$$

where $\bar{Y}^i$ represents the recovery payoff of the defaultable asset Y^i, so that $Y_\tau^i = \bar{Y}_\tau^i$ for $i = 1, 2, \dots, m$. In this general setup, condition (1.15) does not seem to be sufficiently restrictive for more explicit calculations. It is plausible, however, that it can be used to derive a replicating strategy in several non-trivial and practically interesting cases.

It is not difficult to see that Proposition 1.4 can be extended to the case of non-zero recovery for defaultable assets, provided, of course, that we are in a position to find a priori the wealth invested in non-defaultable assets, that is, if the process $\beta_t := \sum_{i=m+1}^{k} \phi_t^i Y_t^i$ is known beforehand. By arguing as in Proposition 1.4, we then obtain for every $t \in [0,T]$

$$\begin{aligned}\widetilde{V}_t(\phi) &= \widetilde{Y}_t^1 \Big(\widetilde{V}_0^1(\phi) + \sum_{i=2}^{m} \int_0^t \phi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} \\ &\quad + \int_0^t \beta_u (Y_u^k)^{-1} \, d(\widetilde{Y}_u^{1,k})^{-1} \Big).\end{aligned}$$

In view of (1.15), we also have that

$$\bar{\alpha}_t := \sum_{i=1}^{m} \phi_t^i \bar{Y}_t^i = Z_t - \beta_t, \quad \forall\, t \in [0,T], \tag{1.16}$$

thereby imposing an additional constraint on the wealth invested in defaultable assets. Condition (1.16) is not directly accounted for in the last formula for $\widetilde{V}(\phi)$, however, and thus the problem at hand is not completely solved. For further considerations related to non-zero recovery of defaultable primary assets, see Section 1.4.1 and 1.4.2.

Fractional recovery of market value. As an example of a non-zero recovery scheme, we consider the so-called fractional recovery of (pre-default) market value (FRMV) scheme with constant recovery rates $\delta_i \neq 1$ (typically, $0 \leq \delta_i < 1$). Then we have $\bar{Y}_t^i = \delta_i \widetilde{Y}_t^i$ for every $i = 1, 2, \ldots, m$, and thus (1.15) becomes

$$\sum_{i=1}^{m} \phi_t^i \delta_i \widetilde{Y}_t^i + \sum_{i=m+1}^{k} \phi_t^i Y_t^i = Z_t, \quad \forall\, t \in [0,T]. \tag{1.17}$$

Let us mention that in the case of a defaultable zero-coupon bond, the FRMV scheme results in the following expression for the pre-default value of a defaultable bond with unit face value (see, for instance, Section 2.2.4 in Bielecki et al. (2004a))

$$\widetilde{D}_M^\delta(t,T) = \mathbb{E}_{\mathbb{Q}^*}\Big(e^{-\int_t^T (r_u + (1-\delta)\gamma_u)du} \,\Big|\, \mathcal{F}_t \Big),$$

where the recovery rate δ may depend on the bond's maturity T, in general. In particular, if the default intensity γ is deterministic then we have

$$\widetilde{D}_M^\delta(t,T) = e^{-\int_t^T (1-\delta)\gamma(u)\, du}\, B(t,T).$$

Manifestly, we always have $D_M^\delta(\tau,T) = \delta D_M^\delta(\tau-,T)$ on the set $\{\tau \leq T\}$ under the FRMV scheme.

1.4 Replication of Defaultable Claims

We are in a position to examine the issue of an exact replication of a generic defaultable claim. By a *replicating strategy* we mean here a self-financing trading strategy ϕ such that the wealth process $V(\phi)$ matches exactly the pre-default value of the claim at any time prior to default (and prior to the maturity date), as well as coincides with the claim's payoff at default time or at maturity date, whichever comes first. Using our notation introduced in Section 1.2, this can be formalized as follows.

Definition 1.5 A self-financing trading strategy ϕ is a *replicating strategy* for a defaultable claim $(X, 0, Z, \tau)$ if and only if the following hold:
(i) $V_t(\phi) = \widetilde{U}_t(X) + \widetilde{U}_t(Z)$ on the random interval $[\![0, \tau \wedge T[\![$,
(ii) $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \leq T\}$,
(iii) $V_T(\phi) = X$ on the set $\{\tau > T\}$.
We say that a defaultable claim is *attainable* if it admits at least one replicating strategy.

The last definition is suitable only in the case of a defaultable claim with no promised dividends. Some comments regarding replication of promised dividends are given in Section 1.4.3.

1.4.1 Replication of a Promised Payoff

We shall first examine the possibility of an exact replication of a defaultable contingent claim of the form $(X, 0, 0, \tau)$, that is, a defaultable claim with zero recovery and with no promised dividends. Our approach will be based on Proposition 1.4. Thus, we assume that processes $Y^1, \ldots, Y^m$ represent prices of defaultable primary assets and $Y^{m+1}, \ldots, Y^k$ are prices of default-free primary assets. Processes $\widetilde{Y}^1, \ldots, \widetilde{Y}^m, Y^{m+1}, \ldots, Y^k$ are assumed to be continuous $\mathbb{F}$-semimartingales, and processes $\widetilde{Y}^1, Y^{m+1}, \ldots, Y^k$ are strictly positive.

Zero Recovery for Defaultable Primary Assets

Unless explicitly stated otherwise, we postulate that Assumption (A) is valid. Recall that $\widetilde{U}_t(X)$ stands for the pre-default value at time $t \in [0, T]$ of a defaultable claim $(X, 0, 0, \tau)$. In the statement of following result we preserve the notation of Proposition 1.4.

Proposition 1.5 *Suppose that there exist a constant $\widetilde{V}_0^1$, and $\mathbb{F}$-predictable processes ψ^i, $i = 2, \ldots, m$ and $\widetilde{\psi}^{i,k,1}$, $i = m+1, \ldots, k-1$ such that*

$$\widetilde{Y}_T^1 \Big(\widetilde{V}_0^1 + \sum_{i=2}^m \int_0^T \psi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^T \widetilde{\psi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} \Big) = X. \tag{1.18}$$

Let $\widetilde{V}_t = \widetilde{Y}^1_t \widetilde{V}^1_t$, *where the process* $\widetilde{V}^1_t$ *is defined as, for every* $t \in [0,T]$,

$$\widetilde{V}^1_t = \widetilde{V}^1_0 + \sum_{i=2}^{m} \int_0^t \psi^i_u \, d\widetilde{Y}^{i,1}_u + \sum_{i=m+1}^{k-1} \int_0^t \widetilde{\psi}^{i,k,1}_u \, d\widehat{Y}^{i,k,1}_u .$$

Then the trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ *defined by*

$$\begin{aligned}
\phi^1_t &= \Big(\widetilde{V}_t - \sum_{i=2}^{m} \psi^i_t Y^i_t\Big)(\widetilde{Y}^1_t)^{-1}, \\
\phi^i_t &= \psi^i_t, \quad i = 2, \dots, m, \\
\phi^i_t &= \widetilde{\psi}^{i,k,1}_t \widetilde{Y}^{1,k}_t e^{-\widetilde{\alpha}^{i,k,1}_t}, \quad i = m+1, \dots, k-1, \\
\phi^k_t &= -\sum_{i=m+1}^{k-1} \psi^i_t Y^i_t (Y^k_t)^{-1},
\end{aligned}$$

is self-financing and it replicates the claim $(X,0,0,\tau)$. *In particular, we have* $\widetilde{V}_t(\phi) = \widetilde{V}_t = \widetilde{U}_t(X)$, *so that* $\widetilde{V}$ *represents the pre-default value of* $(X,0,0,\tau)$.

Proof. The statement is an almost immediate consequence of part (ii) of Proposition 1.4 (see also Corollary 1.1). The strategy $(\phi^1, \phi^2, \dots, \phi^k)$ introduced in the statement of the proposition is self-financing, and at the default time τ the wealth $V(\phi)$ jumps to zero. Finally, $V_T(\phi) = \widetilde{V}_T(\phi) = X$ on the event $\{\tau > T\}$. We conclude that ϕ is self-financing and it replicates $(X,0,0,\tau)$. □

The following corollary to Proposition 1.5 provides the risk-neutral characterization of the process $\widetilde{U}_t(X)$, and thereby it furnishes a convenient method for the valuation of a promised payoff.

Corollary 1.2 *Assume that a defaultable claim* $(X,0,0,\tau)$ *is attainable. Suppose that there exists a probability measure* $\widetilde{\mathbb{Q}}$ *such that the processes* $\widetilde{Y}^{i,1}$, $i = 2, \dots, m-1$ *and processes* $\widehat{Y}^{i,k,1}$, $i = m+1, \dots, k-1$ *are* $\mathbb{F}$*-martingales under* $\widetilde{\mathbb{Q}}$. *If all stochastic integrals in* (1.18) *are* $\widetilde{\mathbb{Q}}$*-martingales, rather than* $\widetilde{\mathbb{Q}}$*-local martingales, then the pre-default value of* $(X,0,0,\tau)$ *equals, for every* $t \in [0,T]$,

$$\widetilde{U}_t(X) = \widetilde{Y}^1_t \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(X (\widetilde{Y}^1_T)^{-1} \,\big|\, \mathcal{F}_t\big).$$

Defaultable asset and two default-free assets. In the case when $m = 1$ and $k = 2$, Proposition 1.5 reduces to the following result. Recall that we denote

$$\widetilde{\alpha}^{2,3,1}_t = \langle \ln Y^{2,3}, \ln \widetilde{Y}^{1,3} \rangle_t = \int_0^t (Y^{2,3}_u)^{-1} (\widetilde{Y}^{1,3}_u)^{-1} \, d\langle Y^{2,3}, \widetilde{Y}^{1,3} \rangle_u ,$$

where in turn $\widetilde{Y}^{1,3}_t = \widetilde{Y}^1_t (Y^3_t)^{-1}$ and $Y^{2,3}_t = Y^2_t (Y^3_t)^{-1}$. Moreover, $\widehat{Y}^{2,3,1}_t = Y^{2,3}_t e^{-\widetilde{\alpha}^{2,3,1}_t}$. We postulate that the processes $\widetilde{Y}^1, Y^2$ and Y^3 are strictly positive.

Corollary 1.3 *Suppose that there exists a constant* $\widetilde{V}^1_0$ *and an* $\mathbb{F}$*-predictable process* $\widetilde{\psi}^{2,3,1}$ *such that*

$$\widetilde{Y}^1_T\Big(\widetilde{V}^1_0 + \int_0^T \widetilde{\psi}^{2,3,1}_u \, d\widehat{Y}^{2,3,1}_u\Big) = X. \tag{1.19}$$

Let us set $\widetilde{V}_t = \widetilde{Y}^1_t \widetilde{V}^1_t$*, where for every* $t \in [0,T]$ *the process* $\widetilde{V}^1_t$ *is given by*

$$\widetilde{V}^1_t = \widetilde{V}^1_0 + \int_0^t \widetilde{\psi}^{2,3,1}_u \, d\widehat{Y}^{2,3,1}_u. \tag{1.20}$$

Then the trading strategy $\phi = (\phi^1, \phi^2, \phi^3)$*, given by the expressions*

$$\phi^1_t = \widetilde{V}_t(\widetilde{Y}^1_t)^{-1}, \quad \phi^2_t = \widetilde{\psi}^{2,3,1}_t \widetilde{Y}^{1,3}_t e^{-\widetilde{\alpha}^{2,3,1}_t}, \quad \phi^3_t = -\phi^2_t Y^2_t (Y^3_t)^{-1},$$

is self-financing prior to default and it replicates a claim $(X, 0, 0, \tau)$*.*

Assume that a claim $(X, 0, 0, \tau)$ is attainable, and let $\widetilde{\mathbb{Q}}$ be a probability measure such that $\widehat{Y}^{2,3,1}$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. Then the pre-default value of $(X, 0, 0, \tau)$ equals, for every $t \in [0,T]$,

$$U_t(X) = \widetilde{Y}^1_t \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(X(\widetilde{Y}^1_T)^{-1} \,\big|\, \mathcal{F}_t\big), \tag{1.21}$$

provided that the integral in (1.20) is also a $\widetilde{\mathbb{Q}}$-martingale.

Example 1.1 Assume that

$$d\widetilde{Y}^1_t = Y^1_t(\mu_t \, dt + \sigma^1_t \, dW_t)$$

and

$$dY^i_t = Y^i_t(r_t \, dt + \sigma^i_t \, dW^*_t)$$

for $i = 2, 3$, where W^* is a one-dimensional standard Brownian motion with respect to the filtration $\mathbb{F} = \mathbb{F}^{W^*}$ under the martingale measure $\mathbb{Q}^*$. Then for the processes $\widetilde{Y}^{1,3}_t = \widetilde{Y}^1_t (Y^3_t)^{-1}$ and $Y^{2,3}_t = Y^2_t (Y^3_t)^{-1}$ we get

$$\begin{aligned} d\widetilde{Y}^{1,3}_t &= \widetilde{Y}^{1,3}_t \Big(\big(\mu_t - r_t + \sigma^3_t(\sigma^3_t - \sigma^1_t)\big)\, dt + (\sigma^1_t - \sigma^3_t)\, dW^*_t\Big), \\ dY^{2,3}_t &= Y^{2,3}_t \Big(\sigma^3_t(\sigma^3_t - \sigma^2_t)\, dt + (\sigma^2_t - \sigma^3_t)\, dW^*_t\Big), \end{aligned}$$

and thus

$$\widetilde{\alpha}^{2,3,1}_t = \int_0^t (\sigma^3_u - \sigma^1_u)(\sigma^3_u - \sigma^2_u)\, du.$$

Hence, the process $\widehat{Y}^{2,3,1}_t = Y^{2,3}_t e^{-\widetilde{\alpha}^{2,3,1}_t}$ satisfies

$$d\widehat{Y}^{2,3,1}_t = \widehat{Y}^{2,3,1}_t \Big(\sigma^1_t(\sigma^3_t - \sigma^2_t)\, dt + (\sigma^2_t - \sigma^3_t) dW^*_t\Big).$$

If $\sigma^2 \neq \sigma^3$ then, under mild technical assumptions, there exists a probability measure $\widetilde{\mathbb{Q}}$ such that $\widehat{Y}^{2,3,1}$ is a martingale. To conclude, it suffices to use the fact that an $\mathcal{F}_T$-measurable random variable $X(\widetilde{Y}^1_T)^{-1}$ can be represented (by virtue of the predictable representation theorem) as follows

$$X(\widetilde{Y}^1_T)^{-1} = \widetilde{U}_0(X) + \int_0^T \widetilde{\phi}^{2,3,1}_u \, d\widehat{Y}^{2,3,1}_u$$

for some $\mathbb{F}$-predictable process $\widetilde{\phi}^{2,3,1}$. It is natural to conjecture that within the present setup all defaultable claims with zero recovery and no promised dividends will be attainable, provided that the underlying default-free market is assumed to be complete, and provided we can use in our hedging portfolio a defaultable asset that is sensitive to the same default risk as the defaultable claim that we want to hedge.

Two defaultable assets. Let us examine the case when $m = k = 2$. We thus consider two defaultable primary assets Y^1 and Y^2 with zero recovery at default.

Corollary 1.4 *Suppose that there exists a constant $\widetilde{V}^1_0$ and an $\mathbb{F}$-predictable process ψ^2 such that*

$$\widetilde{Y}^1_T\Big(\widetilde{V}^1_0 + \int_0^T \psi^2_u \, d\widetilde{Y}^{2,1}_u\Big) = X, \tag{1.22}$$

where $\widetilde{Y}^{2,1}_t = \widetilde{Y}^2_t(\widetilde{Y}^1_t)^{-1}$. Let us set $\widetilde{V}_t = \widetilde{Y}^1_t\widetilde{V}^1_t$, where for every $t \in [0,T]$ the process $\widetilde{V}^1_t$ is given by

$$\widetilde{V}^1_t = \widetilde{V}^1_0 + \int_0^t \psi^2_u \, d\widetilde{Y}^{2,1}_u. \tag{1.23}$$

Then the trading strategy $\phi = (\phi^1, \phi^2)$ where, for every $t \in [0,T]$,

$$\phi^1_t = (\widetilde{V}^1_t - \psi^2_t\widetilde{Y}^2_t)(\widetilde{Y}^1_t)^{-1}, \quad \phi^2_t = \psi^2_t,$$

is self-financing and it replicates a defaultable claim $(X, 0, 0, \tau)$.

Suppose that $(X, 0, 0, \tau)$ is an attainable claim. Let $\widetilde{\mathbb{Q}}$ be a probability measure such that $\widetilde{Y}^{2,1}$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. If the stochastic integral in (1.23) is a $\widetilde{\mathbb{Q}}$-martingale, then the pre-default value of $(X, 0, 0, \tau)$ satisfies, for every $t \in [0,T]$,

$$\widetilde{U}_t(X) = \widetilde{Y}^1_0 \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(X(\widetilde{Y}^1_T)^{-1} \,\big|\, \mathcal{F}_t\big). \tag{1.24}$$

Remark. Under the assumptions of Corollary 1.4, a defaultable claim $(X, 0, 0, \tau)$ is attainable since the associated promised payoff X can be achieved by trading in the pre-default values $\widetilde{Y}^1$ and $\widetilde{Y}^2$. If we introduce, in addition, some default-free assets, a replicating strategy for an arbitrary defaultable claim $(X, 0, 0, \tau)$ will typically have a zero net investment in default-free assets. Therefore, default-free assets are not relevant if we restrict our attention to defaultable claims of the form $(X, 0, 0, \tau)$.

Non-Zero Recovery for Defaultable Primary Assets

We relax Assumption (A), and we postulate instead that Assumption (B) is valid. Specifically, let us consider m defaultable primary assets with a common default time τ that are subject to a fractional recovery of market value (see Section 1.3.2) with $\delta_i = \delta \neq 1$ for $i = 1, 2, \ldots, m$. Let us denote

$$\widetilde{\alpha}_t = \sum_{i=1}^{m} \phi_t^i \widetilde{Y}_t^i, \quad \beta_t = \sum_{i=m+1}^{k} \phi_t^i Y_t^i.$$

so that $\widetilde{\alpha}_t + \beta_t$ represents the pre-default wealth of ϕ. As usual, $\widetilde{U}_t(X)$ stands for the pre-default value at time t of the promised payoff X. It is rather clear that the processes $\widetilde{\alpha}_t$ and β_t should be chosen in such a way that $\widetilde{\alpha}_t + \beta_t = \widetilde{U}_t(X)$ and $\bar{\alpha}_t + \beta_t = \delta\widetilde{\alpha}_t + \beta_t = 0$ for every $t \in [0, T]$ (for the latter equality, see (1.16) and (1.17)). By solving these equations, we obtain, for every $t \in [0, T]$,

$$\widetilde{\alpha}_t = (1 - \delta)^{-1} \widetilde{U}_t(X), \quad \beta_t = (\delta - 1)^{-1} \delta \widetilde{U}_t(X).$$

We end up with the following equation

$$\widetilde{Y}_T^1 \Big(\widetilde{U}_0(X) + \sum_{i=2}^{m} \int_0^T \phi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^T \widetilde{\phi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} + \int_0^T \beta_u (Y_u^k)^{-1} \, d(\widetilde{Y}_u^{1,k})^{-1} \Big) = X.$$

Using the latter equation, one may try to establish a suitable extension of Proposition 1.5. Notice that the process β depends explicitly on the pre-default value $\widetilde{U}(X)$. In addition, we need to take care of the constraint $\widetilde{\alpha}_t = (1 - \delta)^{-1} \widetilde{U}_t(X)$ for every $t \in [0, T]$. Thus, the problem of replication of a promised payoff under non-zero recovery for defaultable primary assets seems to be rather difficult to solve, in general.

1.4.2 Replication of a Recovery Payoff

Let us now focus on the recovery payoff Z at time of default. As before, we write $\widetilde{U}_t(Z)$ to denote the pre-default value at time $t \in [0, T]$ of the claim $(0, 0, Z, \tau)$. Recall that $\widetilde{U}_T(Z) = 0$ (and $U_T(Z) = 0$ on the event $\{\tau > T\}$.

Zero Recovery for Defaultable Primary Assets

In order to examine the replicating strategy, we shall once again make use of Proposition 1.4. As already explained, in this case we need to assume that condition (1.11) is imposed on a strategy ϕ we are looking for, that is, we necessarily have $\sum_{i=m+1}^k \phi_t^i Y_t^i = Z_t$ for every $t \in [0, T]$.

Proposition 1.6 *Suppose that there exist a constant $\widetilde{V}_0^1$, and $\mathbb{F}$-predictable processes ψ^i, $i = 2, \ldots, m$ and $\widetilde{\psi}^{i,k,1}$, $i = m+1, \ldots, k-1$ such that*

$$\widetilde{Y}_T^1\Big(\widetilde{V}_0^1 + \sum_{i=2}^{m}\int_0^T \psi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1}\int_0^T \widetilde{\psi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} + \int_0^T Z_u (Y_u^k)^{-1} \, d(\widetilde{Y}_u^{1,k})^{-1}\Big) = 0. \tag{1.25}$$

Let $\widetilde{V}_t = \widetilde{Y}_t^1 \widetilde{V}_t^1$, where the process $\widetilde{V}_t^1$ is defined as

$$\widetilde{V}_t^1 = \widetilde{V}_0^1 + \sum_{i=2}^{m}\int_0^t \psi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1}\int_0^t \widetilde{\psi}_u^{i,k,1} \, d\widehat{Y}_u^{i,k,1} + \int_0^t Z_u (Y_u^k)^{-1} \, d(\widetilde{Y}_u^{1,k})^{-1}.$$

Then the replicating strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ for $(0, 0, Z, \tau)$ is given by

$$\begin{aligned}
\phi_t^1 &= \Big(\widetilde{V}_t - Z_t - \sum_{i=2}^{m}\phi_t^i Y_t^i\Big)(\widetilde{Y}_t^1)^{-1}, \\
\phi_t^i &= \psi_t^i, \quad \forall\, i = 2, \ldots, m, \\
\phi_t^i &= \widetilde{\psi}_t^{i,k,1} \widetilde{Y}_t^{1,k} e^{-\widetilde{\alpha}_t^{i,k,1}}, \quad \forall\, i = m+1, \ldots, k-1, \\
\phi_t^k &= \Big(Z_t - \sum_{i=m+1}^{k-1}\phi_t^i Y_t^i\Big)(Y_t^k)^{-1}.
\end{aligned}$$

Proof. The proof is based on an application of part (ii) of Proposition 1.4. First, notice that by virtue of the specification of the strategy ϕ we have $\widetilde{V}_t(\phi) = \widetilde{V}_t$ for every $t \in [0, T]$. Moreover, $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \le T\}$. Finally, $V_T(\phi) = \widetilde{V}_T(\phi) = 0$ on the event $\{\tau > T\}$. □

Defaultable asset and two default-free assets. For the ease of reference, we consider here a special case of Proposition 1.6. We take $m = 1$ and $k = 3$, and we postulate that the processes $\widetilde{Y}^1, Y^2$ and Y^3 are strictly positive. Recall that the recovery process Z, and thus also its pre-default value process $\widetilde{U}(Z)$, are prespecified.

Corollary 1.5 *Suppose that there exists a constant $\widetilde{V}_0^1$ and an $\mathbb{F}$-predictable process $\widetilde{\psi}^{2,3,1}$ such that*

$$\widetilde{Y}_T^1\Big(\widetilde{V}_0^1 + \int_0^T \widetilde{\psi}_u^{2,3,1} \, d\widehat{Y}_u^{2,3,1} + \int_0^T Z_u (Y_u^3)^{-1} \, d(\widetilde{Y}_u^{1,3})^{-1}\Big) = 0. \tag{1.26}$$

Let $\widetilde{V}_t = \widetilde{Y}_t^1 \widetilde{V}_t^1$, where the process $\widetilde{V}_t^1$ is defined as

$$\widetilde{V}_t = \widetilde{V}_0^1 + \int_0^t \widetilde{\psi}_u^{2,3,1} \, d\widehat{Y}_u^{2,3,1} + \int_0^t Z_u (Y_u^3)^{-1} \, d(\widetilde{Y}_u^{1,3})^{-1}.$$

Then the replicating strategy for the claim $(0,0,Z,\tau)$ *equals*

$$\phi_t^1 = (\widetilde{V}_t - Z_t)(\widetilde{Y}_t^1)^{-1}, \ \phi_t^2 = \widetilde{\psi}_t^{2,3,1}\widetilde{Y}_t^{1,3}e^{-\widetilde{\alpha}_t^{2,3,1}}, \ \phi_t^3 = (Z_t - \phi_t^2 Y_t^2)(Y_t^3)^{-1}.$$

The existence of $\widetilde{\psi}^{2,3,1}$, as well as the possibility of deriving a closed-form expression for ϕ are not obvious. One needs to impose more specific assumptions on the price processes of primary assets and the recovery process in order to obtain results that would be more practical.

If there exists a probability $\mathbb{Q}^*$ such that $\widehat{Y}^{2,3,1}$ is an $\mathbb{F}$-martingale, then the (ex-dividend) value of Z^0 equals

$$U_t(Z) = Y_t^1 \,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_t^T Z_u (Y_u^3)^{-1}\, d(\widetilde{Y}_u^{1,3})^{-1} \,\Big|\, \mathcal{F}_t\Big).$$

Two defaultable assets. Of course, if both defaultable primary assets are subject to the zero recovery scheme, and no other asset is available for trade, no replicating strategy exists in the case of a non-zero recovery process Z. Thus, we need to postulate a more general recovery scheme for defaultable assets if we wish to have a positive result.

Non-Zero Recovery for Defaultable Primary Assets

Suppose now that Assumption (B) is valid and $Y^1,\dots,Y^m$ are defaultable primary assets with a fractional recovery of market value. We assume that $\delta_i = \delta \neq 1$ for $i = 1,2,\dots,m$, and we proceed along the similar lines as in Section 1.4.1. Recall that we denote

$$\widetilde{\alpha}_t = \sum_{i=1}^m \phi_t^i \widetilde{Y}_t^i, \quad \beta_t = \sum_{i=m+1}^k \phi_t^i Y_t^i.$$

We now postulate that $\widetilde{\alpha}_t + \beta_t = \widetilde{U}_t(Z)$ and $\bar{\alpha}_t + \beta_t = \delta\widetilde{\alpha}_t + \beta_t = Z_t$ for every $t \in [0,T]$, where $\widetilde{U}_t(Z)$ is the pre-default value of $(0,0,Z,\tau)$. Consequently, for every $t \in [0,T]$ we have

$$\widetilde{\alpha}_t = (\delta-1)^{-1}(Z_t - \widetilde{U}_t(Z)), \quad \beta_t = (\delta-1)^{-1}(\delta\widetilde{U}_t(Z) - Z_t).$$

To find a replicating strategy for a defaultable claim $(0,0,Z,\tau)$, we need, in particular, to find $\mathbb{F}$-predictable processes ψ^i and $\widetilde{\psi}^{i,k,1}$ such that the equality

$$\begin{aligned} \widetilde{U}_t(Z) \quad &= \quad \widetilde{Y}_t^1\Big(U_0(Z) + \sum_{i=2}^m \int_0^t \psi_u^i \, d\widetilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1}\int_0^t \widetilde{\psi}_u^{i,k,1}\, d\widehat{Y}_u^{i,k,1} \\ &\quad + \int_0^t \beta_u (Y_u^k)^{-1}\, d(\widetilde{Y}_u^{1,k})^{-1}\Big) \end{aligned}$$

is satisfied for every $t \in [0,T]$. Similarly as in Section 1.4.2, we conclude that the considered problem is non-trivial, in general.

1.4.3 Replication of Promised Dividends

We return to the case of zero recovery for defaultable primary assets, and we consider a defaultable claim $(0, C, 0, \tau)$. In principle, replication of the stream of promised dividends can reduced to previously considered cases (that's why it was possible to postulate in Definition 1.5 that $C = 0$). Specifically, it suffices to introduce the recovery process Z^C generated by C by setting, for every $t \in [0, T]$,

$$Z_t^C = \int_{(0,t)} B^{-1}(u,t)\, dC_u,$$

and to combine it with the terminal payoff $\mathbb{1}_{\{\tau > T\}} X^C$, where the promised payoff X^C associated with C equals

$$X^C = \int_{(0,T]} B^{-1}(u,T)\, dC_u.$$

It should be stressed, however, that the pre-default price of an "equivalent" defaultable claim $(X^C, 0, Z^C, \tau)$ introduced above does not coincide with the pre-default price of the original claim $(0, C, 0, \tau)$, that is, processes $\widetilde{U}(C)$ and $\widetilde{U}(Z^C) + \widetilde{U}(X^C)$ are not identical. But, clearly, the equality $U_0(C) = U_0(Z^C) + U_0(X^C)$ is satisfied, and thus at time 0 the replicating strategies for both claims coincide.

Remark. It is apparent that the concept of the (ex-dividend) pre-default price $\widetilde{U}(C)$ does not fit well into study of replication of promised dividends if one only considers non-dividend paying primary assets. It would be much more convenient to use in the case of dividend-paying (default-free or defaultable) primary assets. For instance, it is sometimes legitimate to postulate the existence of a default-free version of the defaultable claim $(0, C, 0, \tau)$, that is, a default-free asset with the dividend stream C.

If we insist on working directly with the process $\widetilde{U}(C)$, then we derive the following set of necessary conditions for a self-financing trading strategy ϕ with the consumption process $A = -C$

$$\sum_{i=m+1}^{k} \phi_t^i Y_t^i = 0, \quad V_t(\phi) = \sum_{i=1}^{m} \phi_t^i \widetilde{Y}_t^i = \widetilde{U}_t(C), \tag{1.27}$$

and

$$dV_t(\phi) = \sum_{i=1}^{m} \phi_t^i\, d\widetilde{Y}_t^i + \sum_{i=m+1}^{k} \phi_t^i\, dY_t^i - dC_t = d\widetilde{U}_t(C). \tag{1.28}$$

The existence of a strategy $\phi = (\phi^1, \phi^2, \ldots, \phi^k)$ with consumption process $A = -C$, which satisfies (1.27)-(1.28) is not evident, however.

Example 1.2 Let us take, for instance, $m = 1$ and $k = 3$. Then conditions (1.27)-(1.28) become:

$$\phi_t^1 \widetilde{Y}_t^1 = \widetilde{U}_t(C), \quad \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = 0,$$

and

$$\phi_t^1 \, d\widetilde{Y}_t^1 + \phi_t^2 \, dY_t^2 + \phi_t^3 dY_t^3 = d\widetilde{U}_t(C) + dC_t.$$

Assume that under $\mathbb{Q}^*$ we have

$$\begin{aligned} d\widetilde{Y}^1 &= \mu_t \, dt + \sigma_t^1 \, dW_t^*, \\ dY_t^i &= r_t \, dt + \sigma_t^i \, dW_t^*, \quad i = 2, 3, \\ d\widetilde{U}_t(C) &= a_t \, dt + b_t \, dW_t^*. \end{aligned}$$

If, in addition, $dC_t = c_t dt$ then we obtain the following system of equations for $\phi = (\phi^1, \phi^2, \phi^3)$

$$\begin{aligned} \phi_t^1 \widetilde{Y}_t^1 &= \widetilde{U}_t(C), \\ \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 &= 0, \\ \phi_t^1 \mu_t^1 + \phi_t^2 \mu_t^2 + \phi_t^3 \mu_t^3 &= a_t + c_t, \\ \phi_t^1 \sigma_t^1 + \phi_t^2 \sigma_t^2 + \phi_t^3 \sigma_t^3 &= b_t. \end{aligned}$$

1.4.4 Replication of a First-to-Default Claim

Until now, we have always postulated that a random time τ represents a common default time for all defaultable primary assets, as well as for a defaultable contingent claim under consideration. This simplifying assumptions manifestly fails to hold in the case of a credit derivative that explicitly depends on default times of several (possibly independent) reference entities. Consequently, the issue of replication of a so-called *first-to-default claim* is more challenging, and the approach presented in the preceding sections needs to be extended.

Let the random times $\tau_1, \ldots, \tau_m$ represent the default times of m reference entities that underlie a given first-to-default claim. We assume that $\mathbb{Q}^*\{\tau_i = \tau_j\} = 0$ for every $i \neq j$, and we denote by $\tau_{(1)}$ the random moment of the first default, that is, we set $\tau_{(1)} = \min\{\tau_1, \tau_2, \ldots, \tau_n\} = \tau_1 \wedge \tau_2 \wedge \ldots \wedge \tau_n$. A *first-to-default claim* $(X, C, Z^1, \ldots, Z^m, \tau_1, \ldots, \tau_m)$ with maturity date T can be described as follows. If $\tau_{(1)} = \tau_i \leq T$ for some $i = 1, \ldots, m$, then it pays at time $\tau_{(1)}$ the amount $Z^i_{\tau_{(1)}}$, where Z^i is an $\mathbb{F}$-predictable recovery process. Otherwise, that is, if $\tau_{(1)} > T$, the claim pays at time T an $\mathcal{F}_T$-measurable promised amount X. Finally, a claim pays promised dividends stream C prior to the default time $\tau_{(1)}$, more precisely, on the random interval $\mathbb{1}_{\{\tau_{(1)} \leq T\}} [\![0, \tau_{(1)}[\![\cup \mathbb{1}_{\{\tau_{(1)} > T\}} [0, T]$. It is clear the dividend process of a generic first-to-default claim equals, for every $t \in [0, T]$,

$$D_t = X \mathbb{1}_{\{\tau_{(1)} > T\}} \mathbb{1}_{[T, \infty)}(t) + \int_{(0,t]} (1 - H_u^{(1)}) \, dC_u + \int_{(0,t]} Z_u^i \mathbb{1}_{\{\tau_{(1)} = \tau_i\}} \, dH_u^{(1)},$$

where $H_t^{(1)} = 1 - \prod_{i=1}^m (1 - H_t^i)$ or, equivalently, $H_t^{(1)} = \mathbb{1}_{\{\tau_{(1)} \leq t\}}$. Let $\mathbb{H}^i$ be the filtration generated by the process $H_t^i = \mathbb{1}_{\{\tau_i \leq t\}}$ for $i = 1, 2, \ldots, m$, and let the filtration $\mathbb{G}$ be given as $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2 \vee \ldots \vee \mathbb{H}^m$. Then, by definition, the (ex-dividend) price of $(X, C, Z^1, \ldots, Z^m, \tau_1, \ldots, \tau_m)$ equals, for every $t \in [0, T]$,

$$U_t = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{(t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big).$$

By a pre-default value of a claim we mean an $\mathbb{F}$-adapted process $\widetilde{U}$ such that $U_t = \widetilde{U}_t \mathbb{1}_{\{\tau_{(1)} > t\}}$ for every $t \in [0, T]$. The following definition is a direct extension of Definition 1.5 (thus, we maintain the assumption that $C = 0$). By a self-financing strategy we mean here a strategy which is self-financing prior to the first default (cf. Definition 1.4), and thus it is self-financing on $[0, T]$ as well.

Definition 1.6 A self-financing strategy ϕ is a *replicating strategy* for a first-to-default contingent claim $(X, 0, Z^1, \ldots, Z^m, \tau_1, \ldots, \tau_m)$ if and only if the following hold:
(i) $V_t(\phi) = \widetilde{U}_t$ on the random interval $[\![0, \tau_{(1)} \wedge T[\![$,
(ii) $V_\tau(\phi) = Z^i_\tau$ on the event $\{\tau_{(1)} = \tau_i \le T\}$,
(iii) $V_T(\phi) = X$ on the event $\{\tau_{(1)} > T\}$.

In order to provide a replicating strategy for a first-to-default claim we postulate the existence of m defaultable primary assets $Y^1, \ldots, Y^m$ with the corresponding default times $\widetilde{\tau}_1, \ldots, \widetilde{\tau}_m$. It is natural to postulate that the default times $\widetilde{\tau}_1, \ldots, \widetilde{\tau}_m$ are also the default times of m reference entities that underlie a first-to-default claim under consideration, so that, $\widetilde{\tau}_i = \tau_i$ for $i = 1, 2, \ldots, m$. It should be stressed that, typically, the pre-default value $\widetilde{Y}^j$ will follow a discontinuous process (for instance, it may have jumps at default times of other entities). Finally, let us recall that $\bar{Y}^i_t$ represents the recovery payoff of the i^{th} defaultable asset if its default occurs at time t.

Case of zero promised dividends. We shall assume from now on that $C = 0$. For arbitrary $i \ne j$, let $\widehat{Y}^{ij}_t$ represent the pre-default value of the i^{th} asset conditioned on the event $\{\tau_{(1)} = \tau_j = t\}$. More explicitly, $\widehat{Y}^{ij}_t$ is equal to $\widetilde{Y}^i_t$ on the random interval $[\![\tau_{(1)} \mathbb{1}_D, \tau_{(2)} \mathbb{1}_D[\![$, where $D = \{\tau_{(1)} = \tau_j\}$ and $\tau_{(2)}$ is the time of the second default ($\widehat{Y}^{ij}_t$ is not defined outside the random interval introduced above). At the intuitive level, the process $\widehat{Y}^{ij}_t$ gives the value at time t of the i^{th} defaultable asset, provided that the first default has occurred at time t, and the j^{th} entity is the first defaulting entity. Hence, $\widehat{Y}^{ij}_t$ is not a new process, but rather an additional notation introduced in order to simplify the formulae that follow.

Remark. It is important to stress that the notion of a 'defaultable asset' should not be understood literally. For instance, if the case of the so-called *flight to quality* the price of a default-free bond is discontinuous, and it jumps at the moment τ associated with some 'default event' (see, e.g., Collin-Dufresne et al. (2003)). Thus, from the perspective of hedging a default-free bond may be formally classified as a 'defaultable asset'.

In order to find a replicating strategy ϕ for a first-to-default claim within the present setup, we need to impose the following m conditions on its components

$\phi^1, \ldots, \phi^k$: for every $j = 1, \ldots, m$ and every $t \in [0, T]$

$$\sum_{i=1,\, i\neq j}^{m} \phi_t^i \widehat{Y}_t^{ij} + \phi_t^j \bar{Y}_t^j + \sum_{i=m+1}^{k} \phi_t^i Y_t^i = Z_t^j, \tag{1.29}$$

where $Z^1, \ldots, Z^m$ is a given family of recovery processes. Recall that Z^j specifies the payoff received by the owner of a claim if the first default occurs prior to or at T, and the first defaulting entity is the j^{th} entity.

For the sake of concreteness, assume that

$$Z_t^j = g_j(t, \widetilde{Y}_t^1, \ldots, \widetilde{Y}_t^m, \bar{Y}_t^1, \ldots, \bar{Y}_t^m, Y_t^{m+1}, \ldots, Y_t^k)$$

for some function $g : \mathbb{R}^{k+m+1} \to \mathbb{R}$. Under some additional assumptions, the system of equations (1.29) can be solved explicitly for $\phi^1, \ldots, \phi^m$. In the second step, we need to choose processes $\phi^{m+1}, \ldots, \phi^k$ in such a way that a strategy ϕ is self-financing prior to the first default, and thus also on the random interval $[\![0, \tau_{(1)} \wedge T]\!]$. Finally, the wealth of a strategy ϕ should match the promised payoff X at time T on the event $\{\tau_{(1)} > T\}$. Equivalently, the wealth of ϕ should coincide with the value of a considered claim prior to and at default, or up to time T if there is no default in $[0, T]$. It is apparent that the problem of existence of a replicating strategy is non-trivial, but it can be solved in some circumstances.

A detailed analysis of an explicit replication result for a particular example of a first-to-default claim is given in Section 1.5.2.

1.5 Vulnerable Claims and Credit Derivatives

In this section, we present a few examples of models and simple defaultable claims for which there exists explicit replicating strategy. We maintain our assumption that the default time τ admits a continuous hazard process Γ with respect to $\mathbb{F}$ under $\mathbb{Q}^*$, where $\mathbb{F} = \mathbb{F}^{W^*}$ is generated by a Brownian motion W^*. Recall that Γ is also assumed to be an increasing process.

1.5.1 Vulnerable Claims

Let us fix $T > 0$. We postulate that the T-maturity default-free bond and defaultable zero-coupon bond with zero recovery are also traded assets. As before, we assume that the risk-neutral dynamics of the discount default-free bond are

$$dB(t, T) = B(t, T)\big(r_t\, dt + b(t, T)\, dW_t^*\big)$$

for some $\mathbb{F}$-predictable volatility process $b(t, T)$.

Vulnerable Call Options

For a fixed $U > T$, we assume that the U-maturity default-free bond is also traded, and we consider a vulnerable European call option with the terminal payoff

$$\widehat{C}_T = 1\!\!1_{\{\tau>T\}}(B(T,U)-K)^+ = 1\!\!1_{\{\tau>T\}}X.$$

We thus deal with a defaultable claim $(X,0,0,\tau)$ with the promised payoff $X=(B(T,U)-K)^+$. The same method can be applied to an arbitrary $\mathcal{F}_T$-measurable promised payoff $X=g(B(T,U))$, where a function $g:\mathbb{R}\to\mathbb{R}$ satisfies usual technical assumptions.

We consider here the situation when one defaultable asset and two default-free assets are traded; we thus place ourselves within the framework of Corollary 1.3. Specifically, we take $Y_t^1=D^0(t,T)$, $Y_t^2=B(t,U)$ and $Y_t^3=B(t,T)$. Consider a strategy $\phi=(\phi^1,\phi^2,\phi^3)$ such that $V_t(\phi)=\phi_t^1\widetilde{D}^0(t,T)$ and $\phi_t^2B(t,U)+\phi_t^3B(t,T)=0$ for every $t\in[0,T]$. Observe that in view of the definition of $\Gamma(t,T)$ (see Section 1.2.3) we have

$$\widetilde{Y}_t^{1,3}=\widetilde{D}^0(t,T)(B(t,T))^{-1}=\Gamma(t,T).$$

Moreover, $Y_t^{2,3}=F(t,U,T)$ and $\widehat{Y}_t^{2,3,1}=F(t,U,T)e^{-\widetilde{\alpha}_t^{2,3,1}}$, where we denote $F(t,U,T)=B(t,U)(B(t,T))^{-1}$ and, by virtue of formula (1.2),

$$\widetilde{\alpha}_t^{2,3,1}=\langle \ln F(\cdot,U,T),\ln\Gamma(\cdot,T)\rangle_t=\int_0^t \big(b(u,U)-b(u,T)\big)\beta(u,T)\,du.$$

Therefore, the dynamics of $\widehat{Y}^{2,3,1}$ under $\mathbb{Q}_T$ are

$$\begin{aligned} d\widehat{Y}_t^{2,3,1} &= \widehat{Y}_t^{2,3,1}\Big(\big(b(t,T)-b(t,U)\big)\beta(t,T)\,dt+\big(b(t,U)-b(t,T)\big)\,dW_t^T\Big)\\ &= \widehat{Y}_t^{2,3,1}\big(b(t,U)-b(t,T)\big)\big(dW_t^T-\beta(t,T)\,dt\big). \end{aligned}$$

Let $\widetilde{\mathbb{Q}}$ be a probability measure such that $\widehat{Y}^{2,3,1}$ is a martingale under $\widetilde{\mathbb{Q}}$. By virtue of Girsanov's theorem, it is clear that the process $\widetilde{W}$, given by the formula

$$\widetilde{W}_t=W_t^T-\int_0^t\beta(u,T)\,du,\quad \forall\, t\in[0,T],$$

is a Brownian motion under $\widetilde{\mathbb{Q}}$. Thus, the process $F(t,U,T)$ satisfies under $\widetilde{\mathbb{Q}}$

$$dF(t,U,T)=F(t,U,T)\big(b(t,U)-b(t,T)\big)\big(d\widetilde{W}_t+\beta(t,T)\,dt\big). \tag{1.30}$$

Since $\widetilde{D}^0(T,T)=1$, equation (1.19) becomes

$$\widetilde{C}_0+\int_0^T\widetilde{\phi}_u^{2,3,1}\,d\widehat{Y}_u^{2,3,1}=X=(F(T,U,T)-K)^+. \tag{1.31}$$

By a simple extension of (1.21), for any $t\in[0,T]$ the pre-default value of the option equals

$$\widetilde{C}_t=\widetilde{D}^0(t,T)\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big((F(T,U,T)-K)^+\,\big|\,\mathcal{F}_t\big), \tag{1.32}$$

provided that the integral in (1.31) is a $\widetilde{\mathbb{Q}}$-martingale, rather than a $\widetilde{\mathbb{Q}}$-local martingale. Let us denote

$$f(t) = \beta(t,T)(b(t,U) - b(t,T)), \quad \forall\, t \in [0,T], \tag{1.33}$$

and let us assume that f is a deterministic function. Then we have the following result, which extends the valuation formula for a call option written on a default-free zero-coupon bond within the framework of the Gaussian HJM model.

Proposition 1.7 *The pre-default price $\widetilde{C}_t$ of a vulnerable call option written on a default-free zero-coupon bond equals*

$$\widetilde{C}_t = \widetilde{D}^0(t,T)\Big(F(t,U,T)e^{\int_t^T f(u)\,du} N\big(h_+(t,U,T)\big) - KN\big(h_-(t,U,T)\big)\Big),$$

where

$$h_\pm(t,U,T) = \frac{\ln F(t,U,T) - \ln K + \int_t^T f(u)\,du \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T |b(u,U) - b(u,T)|^2\,du.$$

The replicating strategy $\phi = (\phi^1, \phi^2, \phi^3)$ for the option satisfies

$$\begin{aligned} \phi_t^1 &= \widetilde{C}_t(\widetilde{D}^0(t,T))^{-1}, \\ \phi_t^2 &= e^{\widetilde{\alpha}_T^{2,3,1} - \widetilde{\alpha}_t^{2,3,1}}\Gamma(t,T)N\big(h_+(t,U,T)\big), \\ \phi_t^3 &= -\phi_t^2 F(t,U,T). \end{aligned}$$

Proof. Considering the Itô differential $d(\widetilde{C}_t/\widetilde{D}^0(t,T))$, and identifying terms in expression (1.31), we obtain that the process $\widetilde{\phi}^{2,3,1}$ in the integral representation (1.31) is given by the formula

$$\widetilde{\phi}_t^{2,3,1} = e^{\int_0^T f(u)\,du} N\big(h_+(t,U,T)\big) = e^{\widetilde{\alpha}_T^{2,3,1}} N\big(h_+(t,U,T)\big).$$

Consequently the valuation formula presented in the proposition is a rather straightforward consequence of (1.30) and (1.32). □

Remark. Although we consider here the bond $B(t,U)$ as the underlying asset, it is apparent that the method (and thus also the result) can be applied to a much wider class of underlying assets. For instance, a zero-coupon bond can be substituted with a non-dividend paying stock with the price S (this case was examined in Jeanblanc and Rutkowski (2003)). A suitable modification of formulae established in Proposition 1.7 can also be used to the valuation and hedging of vulnerable caplets, swaptions, and other vulnerable derivatives in lognormal market models of (non-defaultable) LIBORs and swap rates.

Case of a deterministic hazard process. Assume now that the $\mathbb{F}$-hazard process Γ of τ is deterministic. Then $\beta(t,T) = 0$ for every $t \in [0,T]$, and thus $\widetilde{\alpha}_t^{2,3,1} = 0$ and $\widehat{Y}_t^{2,3,1} = F(t,U,T)$ for every $t \in [0,T]$. We thus obtain the following result.

Corollary 1.6 *Let the $\mathbb{F}$-hazard process Γ and the volatility $b(t,U)-b(t,T)$ of the forward price $F(t,U,T)$ be deterministic. Then the pre-default price $\widetilde{C}_t$ of a vulnerable option satisfies $\widetilde{C}_t = \Gamma(t,T)C_t$, where C_t is the price of an equivalent non-vulnerable option*

$$C_t = B(t,U)N\big(h_+(t,U,T)\big) - KB(t,T)N\big(h_-(t,U,T)\big),$$

where

$$h_\pm(t,U,T) = \frac{\ln F(t,U,T) - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T |b(u,U) - b(u,T)|^2\,du.$$

The replicating strategy $\phi = (\phi^1, \phi^2, \phi^3)$ is given by

$$\begin{aligned} \phi_t^1 &= \widetilde{C}_t(\Gamma(t,T)B(t,T))^{-1}, \\ \phi_t^2 &= \Gamma(t,T)N\big(h_+(t,U,T)\big), \\ \phi_t^3 &= -\phi_t^2 F(t,U,T). \end{aligned}$$

Vulnerable Bonds

Let us consider the payoff of the form $\mathbb{1}_{\{\tau>T\}}$ which occurs at some date $U > T$. This payoff is, of course, equivalent to the payoff $B(T,U)\mathbb{1}_{\{\tau>T\}}$ at time T. We interpret this claim as a *vulnerable bond*; Vaillant (2001) proposes to term such a *delayed defaultable bond*. Although vulnerable bonds are not traded, under suitable assumptions one can show that they can be replicated by other liquid assets. Indeed, to replicate this claim within the framework of this section, it suffices to assume that default-free bonds with maturities T and U, as well as the defaultable bond with maturity T are among primary traded assets.

Specifically, we postulate that $\phi_t^2 B(t,U) + \phi_t^3 B(t,T) = 0$ for every $t \in [0,T]$ and thus the total wealth is invested in defaultable bonds of maturity T, so that $\phi_t^1 \widetilde{D}^0(t,T) = \widetilde{U}_t(X)$ for every $t \in [0,T]$, where $X = B(T,U) = F(T,U,T)$. Let $\widetilde{D}^0(t,T,U)$ stand for the pre-default value of a vulnerable bond at time $t < T$. Then formulae (1.31) and (1.32) become

$$\widetilde{D}^0(0,T,U) + \int_0^T \widetilde{\phi}_u^{2,3,1}\,d\widehat{Y}_u^{2,3,1} = F(T,U,T)$$

and

$$\widetilde{D}^0(t,T,U) = \widetilde{D}^0(t,T)\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(F(T,U,T)\,|\,\mathcal{F}_t\big),$$

respectively. Using dynamics (1.30), we obtain

$$\begin{aligned} \widetilde{D}^0(t,T,U) &= \widetilde{D}^0(t,T)F(t,T,U)\,e^{\int_t^T f(u)\,du} \\ &= \widetilde{D}^0(t,T)F(t,T,U)\,e^{\widetilde{\alpha}_T^{2,3,1} - \widetilde{\alpha}_t^{2,3,1}} \end{aligned} \tag{1.34}$$

provided that $\widetilde{\alpha}^{2,3,1}$ is deterministic.

1.5.2 Credit Derivatives

The most widely traded credit derivatives are credit default swaps and swaptions, total rate of return swaps and credit linked notes. Furthermore, a large class of basket credit derivatives have a special feature of being linked to the default risk of several reference entities. We shall consider here only two examples: a credit default swap and a first-to-default contract. Before proceeding to the analysis of more complex contract, we shall first examine a standard (non-vulnerable) option written on a defaultable asset.

Options on a Defaultable Asset

We shall now consider a non-vulnerable call option written on a defaultable bond with maturity date U and zero recovery. Let T be the expiration date and let $K > 0$ stand for the strike. Formally, we deal with the terminal payoff $\bar{C}_T$ given by

$$\bar{C}_T = (D^0(T,U) - K)^+.$$

To replicate this option, we postulate that defaultable bonds of maturities U and T are primary assets. Notice also that

$$\bar{C}_T = \big(\mathbb{1}_{\{\tau>T\}}\widetilde{D}^0(T,U) - K\big)^+ = \mathbb{1}_{\{\tau>T\}}\big(\widetilde{D}^0(T,U) - K\big)^+ = \mathbb{1}_{\{\tau>T\}}X,$$

where $X = (\widetilde{D}^0(T,U)-K)^+$, so that once again we deal with a defaultable claim of the form $(X, 0, 0, \tau)$. It should be stressed, however, that since the underlying asset is now defaultable, the valuation result will differ from Proposition 1.7.

We shall use two defaultable primary assets for replication. Specifically, we shall now apply Corollary 1.4, by choosing $Y_t^1 = D^0(t,T)$ and $Y_t^2 = D^0(t,U)$ as primary assets. As before, we denote by $\widetilde{C}_t$ the pre-default value of the option under consideration. By virtue of Corollary 1.4, it suffices to show that there exists a process ϕ^2 such that

$$\widetilde{C}_0 + \int_0^T \phi_u^2\, d\widetilde{Y}_u^{2,1} = X = (\widetilde{D}^0(T,U) - K)^+ = (\widetilde{Y}_T^{2,1} - K)^+, \tag{1.35}$$

where $\widetilde{Y}_t^{2,1} = \widetilde{D}^0(t,U)(\widetilde{D}^0(t,T))^{-1}$. Then the trading strategy $\phi = (\phi^1, \phi^2)$ where

$$\phi_t^1 = (\widetilde{C}_t - \phi_t^2\widetilde{D}^0(t,U))(\widetilde{D}^0(t,T))^{-1}$$

is self-financing and it replicates the option. To derive the valuation formula, it suffices to find the probability measure $\widetilde{\mathbb{Q}}$ such that the process $\widetilde{Y}^{2,1}$ is a $\widetilde{\mathbb{Q}}$-martingale, and to use the generic representation

$$\widetilde{C}_t = \widetilde{D}^0(t,T)\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big((\widetilde{Y}_T^{2,1} - K)^+ \,\big|\, \mathcal{F}_t\big).$$

Recall that the price process $D^0(t,U)$ admits the representation $D^0(t,U) = \mathbb{1}_{\{\tau>t\}}\widetilde{D}^0(t,U)$ where $\widetilde{D}^0(t,U) = \Gamma(t,U)B(t,T)$. Assume that τ has a stochastic intensity γ. Then we have (see (1.3))

$$d\widetilde{D}^0(t,U) = \widetilde{D}^0(t,U)\Big(\big(r_t + \gamma_t + \beta(t,U)b(t,U)\big)\,dt + \big(\beta(t,U) + b(t,U)\big)\,dW_t^*\Big),$$

and the dynamics of $\widetilde{Y}^{2,1}_t = \widetilde{D}^0(t,U)(\widetilde{D}^0(t,T))^{-1}$ under $\mathbb{Q}^*$ are

$$\begin{aligned} d\widetilde{Y}^{2,1}_t &= \widetilde{Y}^{2,1}_t \Big(\big(r_t + \gamma_t + \beta(t,U)b(t,U)\big)\,dt \\ &\quad + \big(\beta(t,U) + b(t,U) - b(t,T)\big)\big(dW^*_t - b(t,T)dt\big)\Big). \end{aligned}$$

As we said above, it suffices to find the probability measure $\widetilde{\mathbb{Q}}$ such that the process $\widetilde{Y}^{2,1}$ is a $\widetilde{\mathbb{Q}}$-martingale. By applying standard Girsanov's transformation, we can construct a measure $\widetilde{\mathbb{Q}}$ so that we have

$$d\widetilde{Y}^{2,1}_t = \widetilde{Y}^{2,1}_t \big(\beta(t,U) + b(t,U) - b(t,T)\big)\,d\widetilde{W}_t,$$

where $\widetilde{W}$ is a Brownian motion under $\widetilde{\mathbb{Q}}$.

Proposition 1.8 *Assume that $\beta(t,U) + b(t,U) - b(t,T)$, $t \in [0,T]$, is a deterministic function. Then the pre-default price $\widetilde{C}_t$ of a call option written on a U-maturity defaultable bond equals*

$$\widetilde{C}_t = \widetilde{D}^0(t,U)N\big(k_+(t,U,T)\big) - K\widetilde{D}^0(t,T)N\big(k_-(t,U,T)\big),$$

where

$$k_\pm(t,U,T) = \frac{\ln \widetilde{D}^0(t,U) - \ln \widetilde{D}^0(t,T) - \ln K \pm \frac{1}{2}\widetilde{v}^2(t,T)}{\widetilde{v}(t,T)}$$

and

$$\widetilde{v}^2(t,T) = \int_t^T |\beta(u,U) + b(u,U) - b(u,T)|^2\,du.$$

The replicating strategy $\phi = (\phi^1, \phi^2)$ for the option is given by

$$\phi^1_t = (\widetilde{C}_t - \phi^2_t \widetilde{D}^0(t,U))(\widetilde{D}^0(t,T))^{-1}, \quad \phi^2_t = N\big(k_+(t,U,T)\big).$$

Case of a deterministic hazard process. Assume that the $\mathbb{F}$-hazard process Γ and the volatility $b(t,U) - b(t,T)$, $t \in [0,T]$, of the forward price $F(t,U,T)$ are deterministic.

Corollary 1.7 *The pre-default price $\widetilde{C}_t$ of a call option written on a U-maturity defaultable bond equals*

$$\begin{aligned} \widetilde{C}_t &= e^{-\int_t^U \gamma(u)\,du} B(t,U) N\big(k_+(t,U,T)\big) \\ &\quad - K e^{-\int_t^T \gamma(u)\,du} B(t,T) N\big(k_-(t,U,T)\big), \end{aligned}$$

where

$$k_\pm(t,U,T) = \frac{\ln B(t,U) - \ln B(t,T) - \ln K - \int_T^U \gamma(u)\,du \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T |b(u,U) - b(u,T)|^2\, du.$$

The replicating strategy $\phi = (\phi^1, \phi^2)$ *for the option is given by*

$$\phi_t^1 = (\widetilde{C}_t - \phi_t^2 \widetilde{D}^0(t,U))(\widetilde{D}^0(t,T))^{-1}, \quad \phi_t^2 = N\big(k_+(t,U,T)\big).$$

Notice that this is exactly the same result as in the case of a call option written on a zero-coupon bond in a default-free term structure model with the interest rate r_t substituted with the default-risk adjusted rate $r_t + \gamma(t)$.

Credit Default Swaps

A generic *credit default swap* (CDS, for short) is a derivative contract which allows to directly transfer the credit risk of the reference entity from one party (the risk seller) to another party (the risk buyer). The contingent payment is triggered by the pre-specified default event, provided that it happens before the maturity date T. The standard version of a credit default swap stipulates that the contract is settled at default time τ of the reference entity, and the recovery payoff equals $Z_\tau = 1 - \delta B(\tau, T)$ where δ represents the recovery rate at default of a reference entity. It is usually assumed that $0 \le \delta < 1$ is non-random, and known in advance. This convention corresponds to the fractional recovery of Treasury value scheme for a defaultable bond issued by the reference entity. Otherwise, that is, in case of no default prior to or at T, the contract expires at time T worthless. The following alternative market conventions are encountered in practice:

- The buyer of the insurance pays a lump sum at inception, and the contract is termed a *default option*,
- The buyer of the insurance pays annuities κ at the predetermined dates $0 < T_1 < \ldots < T_{n-1} < T_n = T$ prior to τ, so that the contract represents a plain-vanilla *default swap*.

In the former case, the (pre-default) value $\widetilde{U}_0(Z)$ at time 0 of the default option equals

$$\widetilde{U}_0(Z) = \mathbb{E}_{\mathbb{Q}^*}\Big(B_\tau^{-1}\left(1 - \delta B(\tau,T)\right)\mathbb{1}_{\{\tau \le T\}}\Big). \tag{1.36}$$

In the latter case, the level of the annuity κ should be chosen in such a way that the value of the contract at time 0 equals zero. The annuity κ can thus be specified by solving the following equation

$$\widetilde{U}_0(Z) = \kappa\, \mathbb{E}_{\mathbb{Q}^*}\Big(\sum_{i=1}^n B_{T_i}^{-1}\, \mathbb{1}_{\{\tau > T_i\}}\Big),$$

where the value $\widetilde{U}_0(Z)$ is given by (1.36).

Digital credit default swap. The fixed leg of a CDS can be represented as the sequence of payoffs $c_i = \kappa 1\!\!1_{\{\tau > T_i\}}$ at the dates T_i for $i = 1, \ldots, n$. The fixed leg of a CDS can thus be seen as a portfolio of defaultable zero-coupon bonds with zero recovery, and thus the valuation of the fixed leg is rather straightforward. To simplify the valuation of the floating leg, we shall consider a digital CDS. Specifically, we postulate that the constant payoff δ is received at time T_{i+1} if default occurs between T_i and T_{i+1}. Therefore, the floating leg is represented by the following sequence of payoffs:

$$d_i = \delta 1\!\!1_{\{T_i < \tau \le T_{i+1}\}} = \delta 1\!\!1_{\{\tau \le T_{i+1}\}} - \delta 1\!\!1_{\{\tau \le T_i\}}$$

at the dates T_{i+1} for $i = 1, \ldots, n-1$. Clearly

$$d_i = \delta(1 - 1\!\!1_{\{\tau > T_{i+1}\}}) - \delta(1 - 1\!\!1_{\{\tau > T_i\}}).$$

We conclude that in order to analyze the floating leg of a digital CDS, it suffices to focus on the valuation and replication of the payoff $1\!\!1_{\{\tau > T_i\}}$ that occurs at time T_{i+1}, that is, a vulnerable bond. The latter problem was already examined in Section 1.5.1, however (see, in particular, the valuation formula (1.34)).

First-to-Default Claims

We shall now focus on the issue of modeling dependent ("correlated") defaults, which arises in the context of *basket credit derivatives.* In order to model dependent default times, we shall employ Kusuoka's (1999) setting with $n = 2$ default times (for related results, see Jarrow and Yu (2001), Gregory and Laurent (2002, 2003), Bielecki and Rutkowski (2003), or Collin-Dufresne et al. (2003)). Our main goal is to show that the jump risk of a first-to-default claim can be perfectly hedged using the underlying defaultable zero-coupon bonds. Recovery schemes and the associated values of (deterministic) recovery rates should be specified a priori.

Construction of dependent defaults. Following Kusuoka (1999), we postulate that under the original probability $\mathbb{Q}$ the random times τ_i, $i = 1, 2$, given on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, are assumed to be mutually independent random variables with exponential laws with parameters λ_1 and λ_2, resp. Let $\mathbb{F}$ be some reference filtration (generated by a Wiener process W, say) such that τ_1 and τ_2 are independent of $\mathbb{F}$ under $\mathbb{Q}$. We write $\mathbb{H}^i$ to denote the filtration generated by the process $H_t^i = 1\!\!1_{\{\tau_i \le t\}}$ for $i = 1, 2$, and we set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$. Notice that the process $M_t^i = H_t^i - \int_0^{t \wedge \tau_i} \lambda_i \, du = H_t^i - \lambda(\tau_i \wedge t)$ is a $\mathbb{G}$-martingale for $i = 1, 2$.

For a fixed $T > 0$, we define a probability measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{G}_T)$ by setting

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \eta_T, \quad \mathbb{Q}\text{-a.s.},$$

where the Radon-Nikodym density process η_t, $t \in [0, T]$, satisfies

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{(0,t]} \eta_{u-} \kappa_u^i \, dM_u^i$$

with auxiliary processes κ^1, κ^2 given by

$$\kappa_t^1 = 1\!\!1_{\{\tau_2<t\}} \left(\frac{\alpha_1}{\lambda_1} - 1 \right), \quad \kappa_t^2 = 1\!\!1_{\{\tau_1<t\}} \left(\frac{\alpha_2}{\lambda_2} - 1 \right).$$

Let $B(t,T)$ be the price of zero-coupon bond, and let $\mathbb{Q}_T$ be the forward martingale measure for the date T. It appears that the 'martingale intensities' under $\mathbb{Q}^*$ and under $\mathbb{Q}_T$ are

$$\lambda_t^1 = \lambda_1 1\!\!1_{\{\tau_2>t\}} + \alpha_1 1\!\!1_{\{\tau_2\le t\}}, \quad \lambda_t^2 = \lambda_2 1\!\!1_{\{\tau_1>t\}} + \alpha_2 1\!\!1_{\{\tau_1\le t\}}.$$

Specifically, the process $\bar{M}_t^i = H_t^i - \int_0^{t\wedge\tau_i} \lambda_u^i \, du$ is a $\mathbb{G}$-martingale under $\mathbb{Q}^*$ and under $\mathbb{Q}_T$ for $i = 1,2$. Moreover, it is easily seen that the random times τ_1 and τ_2 are independent of the filtration $\mathbb{F}$ under $\mathbb{Q}^*$ and $\mathbb{Q}_T$. The following result shows that intensities λ^1 and λ^2 can be interpreted as *local intensities* of default with respect to the information available at time t. Therefore, the model can be reformulated as a two-dimensional Markov chain.

Proposition 1.9 *For $i = 1,2$ and every $t \in [0,T]$ we have*

$$\lambda_i = \lim_{h\downarrow 0} h^{-1} \, \mathbb{Q}_T\{t < \tau_i \le t+h \,|\, \mathcal{F}_t, \tau_1 > t, \, \tau_2 > t\}.$$

Moreover

$$\alpha_1 = \lim_{h\downarrow 0} h^{-1} \, \mathbb{Q}_T\{t < \tau_1 \le t+h \,|\, \mathcal{F}_t, \tau_1 > t, \, \tau_2 \le t\}$$

and

$$\alpha_2 = \lim_{h\downarrow 0} h^{-1} \, \mathbb{Q}_T\{t < \tau_2 \le t+h \,|\, \mathcal{F}_t, \tau_2 > t, \, \tau_1 \le t\}.$$

Assume that defaultable zero-coupon bonds are subject to zero recovery rule. Then the price of the bond issued by the i^{th} entity is given by

$$D_i^0(t,T) = B(t,T) \, \mathbb{Q}_T\{\tau_i > T \,|\, \mathcal{G}_t\} = 1\!\!1_{\{\tau_i>t\}} \widetilde{D}_i^0(t,T),$$

where, as usual, $\widetilde{D}_i^0(t,T)$ stands for the pre-default value of the bond. Let us denote $\lambda = \lambda_1 + \lambda_2$ and let us assume that $\lambda - \alpha_1 \neq 0$. Then straightforward calculations lead to an explicit formula for $\widetilde{D}_i^0(t,T)$ (for details, see Bielecki and Rutkowski (2003)). Of course, an analogous expression holds for the pre-default price $\widetilde{D}_2^0(t,T)$ provided that $\lambda - \alpha_2 \neq 0$.

Proposition 1.10 *Assume that $\lambda - \alpha_1 \neq 0$. Then for every $t \in [0,T]$ the pre-default price $\widetilde{D}_1^0(t,T)$ equals*

$$\widetilde{D}_1^0(t,T) = 1\!\!1_{\{\tau_2>t\}} \, D_1^*(t,T) + 1\!\!1_{\{\tau_2\le t\}} \, \widehat{D}_1(t,T),$$

where

$$D_1^*(t,T) = \frac{B(t,T)}{\lambda - \alpha_1} \left(\lambda_2 e^{-\alpha_1(T-t)} + (\lambda_1 - \alpha_1) e^{-\lambda(T-t)} \right)$$

represents the value of the bond prior to the first default, that is, on the random interval $[\![0, \tau_{(1)} \wedge T[\![$*, and* $\widehat{D}_1(t,T) = B(t,T)e^{-\alpha_1(T-t)}$ *is the value of the bond after the default of the second entity, but prior to default of the issuer, that is, on* $[\![\tau_2 \wedge T, \tau_1 \wedge T[\![$.

Let $\tau_{(1)} = \tau_1 \wedge \tau_2$ be the date of the first default. Consider a first-to-default claim with the terminal payoff $X1\!\!1_{\{\tau_{(1)}>T\}}$, where X is an $\mathcal{F}_T$-adapted random variable, and $\mathbb{F}$-predictable recovery processes Z^1 and Z^2. As primary traded assets, we take defaultable zero-coupon bonds $D_1^0(t,T)$ and $D_2^0(t,T)$ with respective default times τ_1 and τ_2, as well as the default-free zero-coupon bond $B(t,T)$.

In Section 1.4.4, we have examined the basic features of a replicating strategy for a first-to-default claim. Under the present assumptions, (1.29) yields

$$\phi_t^1 B(t,T)e^{-\alpha_1(T-t)} + \phi_t^3 B(t,T) = Z_t^2$$

and

$$\phi_t^2 B(t,T)e^{-\alpha_2(T-t)} + \phi_t^3 B(t,T) = Z_t^1.$$

A strategy ϕ should be self-financing prior to the first default (and thus also on the random interval $[\![0, \tau_{(1)} \wedge T]\!]$). In other words, we are looking for ϕ such that the pre-default wealth process $\widetilde{V}(\phi)$, given by the formula

$$\widetilde{V}_t(\phi) = \phi_t^1 D_1^*(t,T) + \phi_t^2 D_2^*(t,T) + \phi_t^3 B(t,T), \quad \forall\, t \in [0,T],$$

satisfies

$$d\widetilde{V}_t(\phi) = \phi_t^1\, dD_1^*(t,T) + \phi_t^2\, dD_2^*(t,T) + \phi_t^3\, dB(t,T). \tag{1.37}$$

Finally, at time T the wealth of ϕ should coincide with the promised payoff X on the event $\{\tau_{(1)} > T\}$. This means that the pre-default wealth needs to satisfy $\widetilde{V}_T(\phi) = X$, so that (1.37) becomes

$$\widetilde{V}_0(\phi) + \int_0^T \phi_t^1\, dD_1^*(t,T) + \int_0^T \phi_t^2\, dD_2^*(t,T) + \int_0^T \phi_t^3\, dB(t,T) = X.$$

Equivalently, the pre-default wealth should coincide with the pre-default value of a first-to-default claim on the random interval $[\![0, \tau_{(1)} \wedge T[\![$ and the jump of the wealth at default time $\tau_{(1)}$ should adequately reproduce the behavior at $\tau_{(1)}$ of a first-to-default claim.

First-to-default credit swap. For the sake of concreteness, let us consider a *first-to-default credit swap*. Specifically, we shall examine replication of a first-to-default claim with $X = 0$ and $Z_t^i = \delta B(t,T)$ for $i = 1, 2$, where $0 \leq \delta \leq 1$. Let U_t be the value of this claim at time $t \in [0,T]$. It can be shown that

$$\mathbb{Q}_T\{\tau_{(1)} > T \,|\, \mathcal{G}_t\} = 1\!\!1_{\{\tau_{(1)}>t\}}\, e^{-\lambda(T-t)}.$$

Consequently, for every $t \in [0,T]$ we have

$$U_t = 1\!\!1_{\{\tau_{(1)}>t\}}\, \delta\big(1 - e^{-\lambda(T-t)}\big)B(t,T) + 1\!\!1_{\{\tau_{(1)}\leq t\}}\, \delta B(t,T),$$

and thus the pre-default value equals

$$\widetilde{U}_t = \delta(1 - e^{-\lambda(T-t)})B(t,T).$$

To find the replicating strategy ϕ, we first observe that ϕ needs to satisfy, for every $t \in [0,T]$,

$$\phi_t^1 e^{-\alpha_1(T-t)} + \phi_t^3 = \delta, \quad \phi_t^2 e^{-\alpha_2(T-t)} + \phi_t^3 = \delta, \tag{1.38}$$

Moreover, the pre-default wealth process $\widetilde{V}(\phi)$, given by

$$\widetilde{V}_t(\phi) = \phi_t^1 D_1^*(t,T) + \phi_t^2 D_2^*(t,T) + \phi_t^3 B(t,T), \tag{1.39}$$

should satisfy $\widetilde{V}_t(\phi) = \widetilde{U}_t$ and

$$d\widetilde{V}_t(\phi) = \phi_t^1\, dD_1^*(t,T) + \phi_t^2\, dD_2^*(t,T) + \phi_t^3\, dB(t,T). \tag{1.40}$$

It is convenient to work with relative prices, by taking $B(t,T)$ as a numeraire, so that (1.39)-(1.40) become

$$\widetilde{V}_t^B(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 = \delta\big(1 - e^{-\lambda(T-t)}\big) \tag{1.41}$$

and

$$\widetilde{V}_t^B(\phi) = \widetilde{V}_0^B(\phi) + \int_0^t \phi_u^1\, dY_u^1 + \int_0^t \phi_u^2\, dY_u^2, \tag{1.42}$$

where $\widetilde{V}_t^B(\phi) = \widetilde{V}_t(\phi)B^{-1}(t,T)$ and

$$Y_t^1 = \frac{D_1^*(t,T)}{B(t,T)} = \frac{1}{\lambda - \alpha_1}\Big(\lambda_2 e^{-\alpha_1(T-t)} + (\lambda_1 - \alpha_1)e^{-\lambda(T-t)}\Big)$$

and

$$Y_t^2 = \frac{D_2^*(t,T)}{B(t,T)} = \frac{1}{\lambda - \alpha_2}\Big(\lambda_1 e^{-\alpha_2(T-t)} + (\lambda_2 - \alpha_2)e^{-\lambda(T-t)}\Big).$$

Working with relative values is here equivalent to setting $B(t,T) = 1$ for every $t \in [0,T]$ in equations (1.39)-(1.40), as well as in the pricing formulae of Proposition 1.10.

¿From (1.38) it follows that ϕ^3 equals

$$\phi_t^3 = \delta - \phi_t^1 e^{-\alpha_1(T-t)} = \delta - \phi_t^2 e^{-\alpha_2(T-t)}, \tag{1.43}$$

where ϕ^1 and ϕ^2 are related to each other through the formula

$$\phi_t^2 = \phi_t^1 e^{(\alpha_2 - \alpha_1)(T-t)}, \quad \forall\, t \in [0,T]. \tag{1.44}$$

By substituting the last equality in (1.41), we obtain the following expression for ϕ^1

$$\phi_t^1 = -\delta e^{-\lambda(T-t)}\Big(Y_t^1 + Y_t^2 e^{(\alpha_2-\alpha_1)(T-t)} - e^{-\alpha_1(T-t)}\Big)^{-1}.$$

More explicitly,

$$\phi_t^1 = -\delta\xi_1\xi_2 e^{-\xi_1(T-t)}(g(t))^{-1}, \tag{1.45}$$

where we denote $\xi_i = \lambda - \alpha_i$ for $i = 1, 2$ and where $g(t)$ equals

$$g(t) = \lambda_2\xi_2 + (\lambda_1 - \alpha_1)\xi_2 e^{-\xi_1(T-t)} + \lambda_1\xi_1 + (\lambda_2 - \alpha_2)\xi_1 e^{-\xi_2(T-t)} - \xi_1\xi_2.$$

To determine ϕ^2 we may either use (1.44) with (1.45), or to observe that by the symmetry of the problem

$$\phi_t^2 = -\delta e^{-\lambda(T-t)}\Big(Y_t^2 + Y_t^1 e^{(\alpha_1-\alpha_2)(T-t)} - e^{-\alpha_2(T-t)}\Big)^{-1}.$$

Of course, both methods yield, as expected, the same expression for ϕ^2, namely,

$$\phi_t^2 = -\delta\xi_1\xi_2 e^{-\xi_2(T-t)}(g(t))^{-1}.$$

Moreover, straightforward calculations show that for ϕ^1, ϕ^2 as above, we have

$$\phi_t^1\, dY_t^1 + \phi_t^2\, dY_t^2 = d\widetilde{V}_t^B(\phi) = -\delta\lambda e^{-\lambda(T-t)}.$$

Finally, the component ϕ^3 can be found from (1.43), and thus the calculation of a replicating strategy for the considered example of first-to-default credit swap is completed.

1.6 PDE Approach

Let us assume that two (defaultable, in general) assets are tradeable, with respective price processes

$$\begin{aligned} dY_t^1 &= Y_{t-}^1\big(\nu_1\, dt + \sigma_1\, dW_t + \varrho_1\, dM_t\big), \quad Y_0^1 > 0, & (1.46)\\ dY_t^2 &= Y_{t-}^2\big(\nu_2\, dt + \sigma_2\, dW_t + \varrho_2\, dM_t\big), \quad Y_0^2 > 0, & (1.47)\end{aligned}$$

under the real-world probability $\mathbb{Q}$, where W is a one-dimensional standard Brownian motion and the $\mathbb{G}$-martingale M is given by

$$M_t = H_t - \int_0^t \mathbb{1}_{\{\tau>u\}}\varsigma_u\, du, \quad \forall\, t \in [0,T],$$

and the $\mathbb{F}$-adapted intensity ς of the default time τ is strictly positive. We postulate that the interest rate is equal to a constant r, so that the money market account equals $Y_t^3 = B_t = e^{rt}$. We assume that $\sigma_1 \neq 0$, $\sigma_2 \neq 0$ and the constants ϱ_1 and ϱ_2 are greater or equal to -1 so that the price process Y^i is non-negative for $i = 1, 2$.

Remark. It may happen that either ϱ_1 or ϱ_2 equals 0, and thus the corresponding asset is default-free. The case when $\varrho_1 = \varrho_2 = 0$ will be excluded, however (see condition (1.48) below).

We shall now examine the no-arbitrage property of this market. Specifically, we shall impose additional conditions on the model's coefficients that will ensure the existence of an equivalent martingale measure. From Kusuoka's (1999) representation theorem, any equivalent martingale measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{G}_T)$ is of the form $d\mathbb{Q}^*|_{\mathcal{G}_t} = \widetilde{\eta}_t \, d\mathbb{Q}|_{\mathcal{G}_t}$ for $t \in [0,T]$, where

$$d\widetilde{\eta}_t = \widetilde{\eta}_{t-}(\psi_t \, dW_t + \kappa_t \, dM_t), \quad \widetilde{\eta}_0 = 1,$$

for some $\mathbb{G}$-predictable processes ψ and κ. By applying Itô's formula, we obtain for $i = 1, 2$,

$$Y_t^i \widetilde{\eta}_t e^{-rt} = Y_0^i + \int_0^t Y_u^i \widetilde{\eta}_u e^{-ru} \big(\nu_i - r + \psi_u \sigma_1 + \kappa_u \varrho_i \xi_u\big) du + \text{martingale},$$

where we denote $\xi_t = \varsigma_t \mathbb{1}_{\{\tau > t\}}$. Hence, the process $Y_t^i \widetilde{\eta}_t e^{-rt}$ is a (local) $\mathbb{G}$-martingale under $\mathbb{Q}$ for $i = 1, 2$ if and only if

$$\nu_i - r + \psi_t \sigma_i + \kappa_t \varrho_i \xi_t = 0$$

for $i = 1, 2$ and almost every $t \in [0,T]$. Hence, a density process $\widetilde{\eta}$ determines an equivalent martingale measure $\mathbb{Q}^*$ for the processes $Y_t^i e^{-rt}$, $i = 1, 2$ if and only if the processes ψ and κ are such that for every $t \in [0,T]$

$$\begin{aligned} \nu_1 - r + \psi_t \sigma_1 + \kappa_t \varrho_1 \xi_t &= 0, \\ \nu_2 - r + \psi_t \sigma_2 + \kappa_t \varrho_2 \xi_t &= 0. \end{aligned}$$

Assume that $\varrho_1 \sigma_2 - \varrho_2 \sigma_1 \neq 0$. Then the unique solution is the pair of processes (ψ_t, κ_t), $t \in [0,T]$, such that

$$\psi_t = \frac{(\nu_2 - r)\varrho_1 - (\nu_1 - r)\varrho_2}{\varrho_1 \sigma_2 - \varrho_2 \sigma_1}$$

and

$$\kappa_t \xi_t = \frac{(\nu_2 - r)\sigma_1 - (\nu_1 - r)\sigma_2}{\varrho_1 \sigma_2 - \varrho_2 \sigma_1}.$$

Since $\widetilde{\eta}$ is a strictly positive process, we restrict our attention to parameters such that the process κ is greater than -1. Obviously, the value of the process κ after the default time τ is irrelevant. However, the pre-default value of κ is uniquely given as

$$\kappa_t = \frac{(\nu_2 - r)\sigma_1 - (\nu_1 - r)\sigma_2}{\varsigma_t(\varrho_1 \sigma_2 - \varrho_2 \sigma_1)},$$

and thus we postulate that the last formula holds for every $t \in [0,T]$. We thus have the following auxiliary result. Let us set $\gamma_t = \varsigma_t(1 + \kappa_t)$.

Lemma 1.8 *Assume that* $\varrho_1 \sigma_2 - \varrho_2 \sigma_1 \neq 0$ *and*

$$\frac{(\nu_2 - r)\sigma_1 - (\nu_1 - r)\sigma_2}{\varsigma_t(\varrho_1 \sigma_2 - \varrho_2 \sigma_1)} > -1, \quad \forall\, t \in [0,T]. \tag{1.48}$$

Then the market model defined by (1.46)-(1.47) *and the money market account* $Y_t^3 = e^{rt}$ *is complete and arbitrage-free. Moreover, under the unique equivalent martingale measure* $\mathbb{Q}^*$ *we have*

$$\begin{aligned} dY_t^1 &= Y_{t-}^1\big(r\,dt + \sigma_1\,dW_t^* + \varrho_1\,dM_t^*\big), \\ dY_t^2 &= Y_{t-}^2\big(r\,dt + \sigma_2\,dW_t^* + \varrho_2\,dM_t^*\big), \\ dY_t^3 &= rY_t^3\,dt, \end{aligned} \tag{1.49}$$

where W^* *is a Brownian motion under* $\mathbb{Q}^*$*, and where the process* M^**, given by*

$$M_t^* = M_t - \int_0^t \xi_u \kappa_u\,du = H_t - \int_0^t \mathbb{1}_{\{\tau > u\}}\gamma_u\,du,$$

follows a martingale under $\mathbb{Q}^*$*.*

¿From now on, we shall conduct the analysis of the model given by (1.49) under the martingale measure $\mathbb{Q}^*$.

1.6.1 Markovian Case

To proceed further it would be convenient to assume that ς, and thus also κ, are deterministic functions of the time parameter. In this case, the default intensity γ under $\mathbb{Q}^*$ would be a deterministic function as well. More generally, it suffices to postulate that the $\mathbb{F}$-intensity of default under $\mathbb{Q}^*$ is of the form $\gamma_t = \gamma(t, Y_t^1, Y_t^2)$ for some sufficiently smooth function γ. For instance, $\gamma(t,x,y)$ may be assumed to be piecewise continuous with respect to t and Lipschitz continuous with respect to x and y. Under this assumption, the process (Y^1, Y^2, H), where the two-dimensional process (Y^1, Y^2) is the unique solution to the SDE (1.49), is Markovian under $\mathbb{Q}^*$ (since Y^3 is deterministic, it is not essential here).

For the sake of concreteness, we shall frequently focus on a defaultable claim represented by the following payoff at the maturity date T

$$Y = \mathbb{1}_{\{\tau > T\}} g(Y_T^1, Y_T^2) + \mathbb{1}_{\{\tau \le T\}} h(Y_T^1, Y_T^2) \tag{1.50}$$

for some functions $g, h : \mathbb{R}_+^2 \to \mathbb{R}$ satisfying suitable integrability conditions. Hence, the price of Y is given by the risk-neutral valuation formula

$$\pi_t(Y) = B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} Y \,|\, \mathcal{G}_t), \quad \forall\, t \in [0,T]. \tag{1.51}$$

Notice that $\pi_t(Y)$ represents the standard (cum-dividend) price of a European contingent claim Y, which settles at time T. Our goal is to find a quasi-explicit representation for a self-financing trading strategy ψ such that $\pi_t(Y) = V_t(\psi)$ for every $t \in [0,T]$, where $V_t(\psi) = \sum_{i=1}^3 \psi_t^i Y_t^i$ (see Section 1.6.3).

We shall first prove an auxiliary result, which shows that the arbitrage price of the claim Y splits in a natural way into the pre-default price and the post-default price.

Lemma 1.9 *The price $\pi_t(Y)$ of the claim Y given by (1.50) satisfies*

$$\pi_t(Y) = (1 - H_t)\widetilde{v}(t, Y_t^1, Y_t^2) + H_t \bar{v}(t, Y_t^1, Y_t^2), \quad \forall\, t \in [0, T], \tag{1.52}$$

for some functions $\widetilde{v}$, $\bar{v}$: $[0, T] \times \mathbb{R}_+^2 \to \mathbb{R}$ such that $\widetilde{v}(T, x, y) = g(x, y)$ and $\bar{v}(T, x, y) = h(x, y)$.

Proof. We have

$$\begin{aligned}
\pi_t(Y) &= B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1} Y \,\big|\, \mathcal{G}_t\big) \\
&= B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1} \mathbb{1}_{\{\tau > T\}} g(Y_T^1, Y_T^2) \,\big|\, \mathcal{G}_t\big) + B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1} \mathbb{1}_{\{\tau \le T\}} h(Y_T^1, Y_T^2) \,\big|\, \mathcal{G}_t\big) \\
&= \mathbb{1}_{\{\tau > t\}} B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{\tau > T\}} B_T^{-1} g(Y_T^1, Y_T^2) + \mathbb{1}_{\{t < \tau \le T\}} B_T^{-1} h(Y_T^1, Y_T^2) \,\big|\, \mathcal{G}_t\big) \\
&\quad + \mathbb{1}_{\{\tau \le t\}} B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{\tau \le t\}} B_T^{-1} h(Y_T^1, Y_T^2) \,\big|\, \mathcal{G}_t\big).
\end{aligned}$$

This shows that

$$\pi_t(Y) = \mathbb{1}_{\{\tau > t\}} \widetilde{u}(t, Y_t^1, Y_t^2, 0) + \mathbb{1}_{\{\tau \le t\}} \bar{u}(t, Y_t^1, Y_t^2, 1),$$

where

$$\begin{aligned}
\widetilde{u}(t, Y_t^1, Y_t^2, H_t) &= B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{\tau > T\}} B_T^{-1} g(Y_T^1, Y_T^2) \,\big|\, \mathcal{G}_t\big) \\
&\quad + B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t < \tau \le T\}} B_T^{-1} h(Y_T^1, Y_T^2) \,\big|\, \mathcal{G}_t\big) \\
&= B_t\, \mathbb{E}_{\mathbb{Q}^*}\big((1 - H_T) B_T^{-1} g(Y_T^1, Y_T^2) \\
&\quad + (H_T - H_t) B_T^{-1} h(Y_T^1, Y_T^2) \,\big|\, Y_t^1, Y_t^2, H_t\big)
\end{aligned}$$

and

$$\begin{aligned}
\bar{u}(t, Y_t^1, Y_t^2, H_t) &= B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{\tau \le t\}} B_T^{-1} h(Y_T^1, Y_T^2) \,\big|\, \mathcal{G}_t\big) \\
&= B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(H_t B_T^{-1} h(Y_T^1, Y_T^2) \,\big|\, Y_t^1, Y_t^2, H_t\big).
\end{aligned}$$

Let us set

$$\begin{aligned}
\widetilde{v}(t, Y_t^1, Y_t^2) &= \widetilde{u}(t, Y_t^1, Y_t^2, 0) \\
&= B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1} Y \,\big|\, Y_t^1, Y_t^2, H_t = 0\big)
\end{aligned} \tag{1.53}$$

and

$$\begin{aligned}
\bar{v}(t, Y_t^1, Y_t^2) &= \bar{u}(t, Y_t^1, Y_t^2, 1) \\
&= B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1} h(Y_T^1, Y_T^2) \,\big|\, Y_t^1, Y_t^2, H_t = 1\big).
\end{aligned} \tag{1.54}$$

It is clear that $\widetilde{v}(T, Y_T^1, Y_T^2) = g(Y_T^1, Y_T^2)$ and $\bar{v}(T, Y_T^1, Y_T^2) = h(Y_T^1, Y_T^2)$. We conclude that the price of the claim Y is of the form $v(t, Y_t^1, Y_t^2)$, where

$$v(t, Y_t^1, Y_t^2) = \mathbb{1}_{\{\tau > t\}} \widetilde{v}(t, Y_t^1, Y_t^2) + \mathbb{1}_{\{\tau \le t\}} \bar{v}(t, Y_t^1, Y_t^2).$$

Notice that $\widetilde{v}(t, Y_t^1, Y_t^2)$ and $\bar{v}(t, Y_t^1, Y_t^2)$ represent the pre-default and post-default values of Y, respectively. $\square$

Post-default value. It should be stressed that the conditional expectation in (1.53) is to be evaluated using the dynamics of (Y^1, Y^2, Y^3) given by (1.49). To compute the conditional expectation in (1.54), however, it is manifestly sufficient to make use of the post-default dynamics of (Y^1, Y^2, Y^3), which is given by the following expressions, which are valid if $\varrho_1 > -1$ and $\varrho_2 > -1$,

$$\begin{aligned} dY_t^1 &= Y_{t-}^1 \big(r\,dt + \sigma_1\,dW_t^* \big), \\ dY_t^2 &= Y_{t-}^2 \big(r\,dt + \sigma_2\,dW_t^* \big), \\ dY_t^3 &= rY_t^3\,dt. \end{aligned} \tag{1.55}$$

Using standard arguments, we conclude that if the function $\bar v = \bar v(t,x,y)$ is sufficiently regular then it satisfies the following PDE:

$$\begin{aligned} -r\bar v + \partial_t \bar v + rx\partial_x \bar v + ry\partial_y \bar v + \tfrac{1}{2}\big(\sigma_1^2 x^2 \partial_{xx}^2 \bar v + \sigma_2^2 y^2 \partial_{yy}^2 \bar v \big) \\ + \sigma_1\sigma_2 xy \partial_{xy}^2 \bar v = 0 \end{aligned} \tag{1.56}$$

with the terminal condition $\bar v(T,x,y) = h(x,y)$. Hence, the equation (1.56) can be referred to as the *post-default pricing* PDE for our claim. Of course, since after the default time our model becomes a default-free model, the use of a such a PDE to arbitrage valuation of path-independent European claims is fairly standard.

If $\varrho_1 > -1$ and $\varrho_2 = -1$, then the process Y^2 jumps to zero at time of default, and thus the post-default pricing PDE becomes:

$$-r\bar v + \partial_t \bar v + rx\partial_x \bar v + \tfrac{1}{2}\sigma_1^2 x^2 \partial_{xx}^2 \bar v = 0 \tag{1.57}$$

with the terminal condition $\bar v(T,x) = \bar h(x)$ for some function $\bar h : \mathbb{R}_+ \to \mathbb{R}$ (formally, $\bar h(x) = h(x,0)$).

Recovery process. Following Jamshidian (2002) (see Theorem 2.1), one may check that for any $t \in [0,T]$ we have

$$B_t\,\mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1} \mathbb{1}_D h(Y_T^1, Y_T^2) \,\big|\, \mathcal{G}_t \big) = B_t\,\mathbb{E}_{\mathbb{Q}^*}\big(B_\tau^{-1} \mathbb{1}_D \bar v(\tau, Y_\tau^1, Y_\tau^2) \,\big|\, \mathcal{G}_t \big),$$

where $D = \{t < \tau \le T\}$. Hence, if we wish to compute the pre-default value of Y, it is tempting to consider the process $\bar v(t, Y_t^1, Y_t^2)$ as the recovery process Z. According to our convention, the recovery process Z should necessarily be an $\mathbb{F}$-predictable process, and the process $\bar v(t, Y_t^1, Y_t^2)$ is not $\mathbb{F}$-predictable, in general. Therefore, we formally define the recovery process Z associated with the claim Y by setting

$$Z_t = z(t, \widetilde Y_t^1(1+\varrho_1), \widetilde Y_t^2(1+\varrho_2)) = \bar v(t, \widetilde Y_t^1(1+\varrho_1), \widetilde Y_t^2(1+\varrho_2))), \tag{1.58}$$

where $\widetilde Y^i$ is the pre-default value of the i^{th} asset (so that $\widetilde Y^i$ is manifestly an $\mathbb{F}$-adapted, continuous process). It is clear that

$$Z_\tau = \bar v(\tau, \widetilde Y_\tau^1(1+\varrho_1), \widetilde Y_\tau^2(1+\varrho_2)) = \bar v(\tau, Y_\tau^1, Y_\tau^2), \quad \mathbb{Q}^*\text{-a.s.}$$

Notice that the pre-default value of the claim Y given by (1.50) coincides with the pre-default value of $(X, Z, 0, \tau)$, where the promised payoff $X = g(Y_T^1, Y_T^2)$ and the $\mathbb{F}$-predictable recovery process Z is given by (1.58).

1.6.2 Pricing PDE for the Pre-Default Value

Recall that the price process of the claim Y given by (1.50) admits the following representation, for every $t \in [0,T]$,

$$\pi_t(Y) = (1-H_t)\widetilde{v}(t, Y_t^1, Y_t^2) + H_t \bar{v}(t, Y_t^1, Y_t^2) \tag{1.59}$$

for some functions $\widetilde{v}$, $\bar{v} : [0,T] \times \mathbb{R}_+^2 \to \mathbb{R}$ such that $\widetilde{v}(T,x,y) = g(x,y)$ and $\bar{v}(T,x,y) = h(x,y)$. We assume that processes $\widetilde{v}(t, Y_t^1, Y_t^2)$ and $\bar{v}(t, Y_t^1, Y_t^2)$ are semimartingales. We shall need the following simple version of the Itô integration by parts formula for (discontinuous) semimartingales.

Lemma 1.10 *Assume that Z is a semimartingale and A is a bounded process of finite variation. Then*

$$\begin{aligned} Z_t A_t &= Z_0 A_0 + \int_0^t Z_{u-}\, dA_u + \int_0^t A_u \, dZ_u \\ &= Z_0 A_0 + \int_0^t Z_u \, dA_u + \int_0^t A_{u-}\, dZ_u. \end{aligned}$$

Proof. Both formulae are almost immediate consequences of the general Itô formula for semimartingales (see, for instance, Protter (2003)), and the fact that under the present assumptions we have $[Z,A]_t = \sum_{0<s\le t} \Delta Z_s \Delta A_s$. □

Our next goal is to derive the partial differential equation satisfied by the pre-default pricing function $\widetilde{v}$. The post-default pricing function $\bar{v}$ (or, equivalently, the recovery function z) is taken here as an input. Hence, the only unknown function at this stage is the pre-default pricing function $\widetilde{v}$.

In view of the financial interpretation of the function $\widetilde{v}$, the PDE derived in Proposition 1.11 will be referred to as the *pre-default pricing PDE* for a defaultable claim Y. For a related result, see Proposition 3.4 in Lukas (2001).

Proposition 1.11 *Suppose that the function $\widetilde{v} = \widetilde{v}(t,x,y)$ belong to the class $C^{1,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$. Assume, in addition, that $\widetilde{v}$ satisfies the PDE*

$$\begin{aligned} -r\widetilde{v} + \partial_t \widetilde{v} + rx\partial_x \widetilde{v} + ry\partial_y \widetilde{v} + \tfrac{1}{2}\big(\sigma_1^2 x^2 \partial_{xx}^2 \widetilde{v} + \sigma_2^2 y^2 \partial_{yy}^2 \widetilde{v}\big) + \sigma_1\sigma_2 xy \partial_{xy}^2 \widetilde{v} \\ + \gamma(t,x,y)\big(\bar{v}(t, x(1+\varrho_1), y(1+\varrho_2)) - \widetilde{v} - \varrho_1 x \partial_x \widetilde{v} - \varrho_2 y \partial_y \widetilde{v}\big) = 0 \end{aligned}$$

with the terminal condition $\widetilde{v}(T,x,y) = g(x,y)$. Let the process $\pi(Y)$ be given by (1.59). Then the process $V_t^* = B_t^{-1}\pi_t(Y)$ stopped at τ is a $\mathbb{G}$-martingale under $\mathbb{Q}^*$.

Proof. By applying the Itô integration by parts formula to both terms in the right-hand side of (1.59), we obtain

$$\begin{aligned} d\pi_t(Y) &= (1-H_t)\, d\widetilde{v}(t, Y_t^1, Y_t^2) - \widetilde{v}(t, Y_{t-}^1, Y_{t-}^2)\, dH_t \\ &\quad + H_{t-}\, d\bar{v}(t, Y_t^1, Y_t^2) + \bar{v}(t, Y_t^1, Y_t^2)\, dH_t \\ &= \mathbb{1}_{\{\tau>t\}}\, d\widetilde{v}(t, Y_t^1, Y_t^2) + \big(\bar{v}(t, Y_t^1, Y_t^2) - \widetilde{v}(t, Y_{t-}^1, Y_{t-}^2)\big)\, dH_t \\ &\quad + \mathbb{1}_{\{\tau<t\}}\, d\bar{v}(t, Y_t^1, Y_t^2). \end{aligned}$$

Hence, the process $V^*_t = e^{-rt}\pi_t(Y)$ satisfies for every $t \in [0,T]$

$$\begin{aligned}
V^*_t &= V^*_0 - \int_0^{\tau\wedge t} re^{-ru}\widetilde{v}(u, Y^1_u, Y^2_u)\,du + \int_{(0,\tau\wedge t)} e^{-ru}\,d\widetilde{v}(u, Y^1_u, Y^2_u) \\
&\quad + \int_{(0,\tau\wedge t]} e^{-ru}\left(\bar{v}(u, Y^1_u, Y^2_u) - \widetilde{v}(u, Y^1_{u-}, Y^2_{u-})\right) dH_u \\
&\quad + \int_{(\tau\wedge t,t]} e^{-ru}\,d\bar{v}(u, Y^1_u, Y^2_u).
\end{aligned}$$

It is clear that if $\pi(Y)$ is given by (1.51) then the process V^* is a $\mathbb{G}$-martingale under $\mathbb{Q}^*$ (see also Corollary 1.8 below). To derive the pre-default pricing PDE, it suffices to make use of the martingale property of the stopped process

$$V^*_{\tau\wedge t} = \mathbb{1}_{\{\tau>t\}}e^{-rt}\widetilde{v}(t, Y^1_t, Y^2_t) + \mathbb{1}_{\{\tau\le t\}}e^{-r\tau}\bar{v}(\tau, Y^1_\tau, Y^2_\tau).$$

By applying Itô's formula to $\widetilde{v}(t, Y^1_t, Y^2_t)$ on $\{\tau > t\}$, we obtain

$$\begin{aligned}
V^*_{\tau\wedge t} &= V^*_0 \\
&+ \int_0^{\tau\wedge t} e^{-ru}\Big(-r\widetilde{v}_u + \partial_t\widetilde{v}_u + rY^1_u\partial_x\widetilde{v}_u + rY^2_u\partial_y\widetilde{v}_u\Big)du \\
&+ \int_0^{\tau\wedge t} \frac{1}{2}\,e^{-ru}\Big(\sigma_1^2(Y^1_u)^2\partial_{xx}\widetilde{v}_u + \sigma_2^2(Y^2_u)^2\partial_{yy}\widetilde{v}_u + 2\sigma_1\sigma_2Y^1_uY^2_u\partial_{xy}\widetilde{v}_u\Big)du \\
&- \int_0^{\tau\wedge t} e^{-ru}\Big(\varrho_1Y^1_u\partial_x\widetilde{v}_u + \varrho_2Y^2_u\partial_y\widetilde{v}_u\Big)\,\gamma_u\,du \\
&+ \int_{(0,\tau\wedge t]} e^{-ru}\Big(\bar{v}(u, Y^1_{u-}(1+\varrho_1), Y^2_{u-}(1+\varrho_2)) - \widetilde{v}(u, Y^1_{u-}, Y^2_{u-})\Big)\,dH_u \\
&+ \int_0^{\tau\wedge t} e^{-ru}\Big(\sigma_1Y^1_u\partial_x\widetilde{v}_u + \sigma_2Y^2_u\partial_y\widetilde{v}_u\Big)\,dW^*_u,
\end{aligned}$$

where $\widetilde{v}_u = \widetilde{v}(u, Y^1_u, Y^2_u)$, $\partial_x\widetilde{v}_u = \partial_x\widetilde{v}(u, Y^1_u, Y^2_u)$, $\gamma_u = \gamma(u, Y^1_u, Y^2_u)$, etc. The last formula can be rewritten as follows:

$$\begin{aligned}
V^*_{\tau\wedge t} &= V^*_0 \\
&+ \int_0^{\tau\wedge t} e^{-ru}\Big(-r\widetilde{v}_u + \partial_t\widetilde{v}_u + rY^1_u\partial_x\widetilde{v}_u + rY^2_u\partial_y\widetilde{v}_u\Big)du \\
&+ \int_0^{\tau\wedge t} \frac{1}{2}\,e^{-ru}\Big(\sigma_1^2(Y^1_u)^2\partial_{xx}\widetilde{v}_u + \sigma_2^2(Y^2_u)^2\partial_{yy}\widetilde{v}_u + 2\sigma_1\sigma_2Y^1_uY^2_u\partial_{xy}\widetilde{v}_u\Big)du \\
&- \int_0^{\tau\wedge t} e^{-ru}\Big(\varrho_1Y^1_u\partial_x\widetilde{v}_u + \varrho_2Y^2_u\partial_y\widetilde{v}_u\Big)\,\gamma_u\,du \\
&+ \int_0^{\tau\wedge t} e^{-ru}\Big(\bar{v}(u, Y^1_{u-}(1+\varrho_1), Y^2_{u-}(1+\varrho_2)) - \widetilde{v}(u, Y^1_{u-}, Y^2_{u-})\Big)\gamma_u\,du \\
&+ \int_{(0,\tau\wedge t]} e^{-ru}\Big(\bar{v}(u, Y^1_{u-}(1+\varrho_1), Y^2_{u-}(1+\varrho_2)) - \widetilde{v}(u, Y^1_{u-}, Y^2_{u-})\Big)\,dM^*_u \\
&+ \int_0^{\tau\wedge t} e^{-ru}\Big(\sigma_1Y^1_u\partial_x\widetilde{v}_u + \sigma_2Y^2_u\partial_y\widetilde{v}_u\Big)\,dW^*_u.
\end{aligned}$$

Recall that the processes W^* and M^* are $\mathbb{G}$-martingales under $\mathbb{Q}^*$. Thus, the stopped process $V^*_{t\wedge\tau}$ is a $\mathbb{G}$-martingale if and only if for every $t \in [0,T]$

$$\begin{aligned}
&\int_0^{\tau\wedge t} e^{-ru}\Big(-r\widetilde{v}_u + \partial_t\widetilde{v}_u + rY^1_u\partial_x\widetilde{v}_u + rY^2_u\partial_y\widetilde{v}_u\Big)\,du \\
&\quad + \int_0^{\tau\wedge t} \frac{1}{2}\, e^{-ru}\Big(\sigma_1^2(Y^1_u)^2\partial_{xx}\widetilde{v}_u + \sigma_2^2(Y^2_u)^2\partial_{yy}\widetilde{v}_u + 2\sigma_1\sigma_2 Y^1_u Y^2_u\partial_{xy}\widetilde{v}_u\Big)\,du \\
&\quad - \int_0^{\tau\wedge t} e^{-ru}\Big(\varrho_1 Y^1_u\partial_x\widetilde{v}_u + \varrho_2 Y^2_u\partial_y\widetilde{v}_u\Big)\,\gamma_u\,du \\
&\quad + \int_0^{\tau\wedge t} e^{-ru}\Big(\bar{v}(u, Y^1_u(1+\varrho_1), Y^2_u(1+\varrho_2)) - \widetilde{v}(u, Y^1_u, Y^2_u)\Big)\gamma_u\,du = 0.
\end{aligned}$$

The last equality is manifestly satisfied if the function $\widetilde{v}$ solves the PDE given in the statement of the proposition. Conversely, if the function $\widetilde{v}$ in representation (1.59) is sufficiently regular, then it necessarily satisfies the last equation. □

Corollary 1.8 *Assume that the pricing functions $\bar{v}$ and $\widetilde{v}$ belong to the class $C^{1,2,2}([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ and satisfy the post-default and pre-default pricing PDEs, respectively. Then the discounted price process V^*_t, $t \in [0,T]$, is a $\mathbb{G}$-martingale under $\mathbb{Q}^*$ and the dynamics of V^* under $\mathbb{Q}^*$ are*

$$\begin{aligned}
dV^*_t \;=\;& \mathbb{1}_{\{\tau>t\}} e^{-rt}(\sigma_1 Y^1_t\partial_x\widetilde{v}_t + \sigma_2 Y^2_t\partial_y\widetilde{v}_t)\,dW^*_t \\
&+ \mathbb{1}_{\{\tau<t\}} e^{-rt}(\sigma_1 Y^1_t\partial_x\bar{v}_t + \sigma_2 Y^2_t\partial_y\bar{v}_t)\,dW^*_t \\
&+ e^{-rt}\big(\bar{v}(t, Y^1_{t-}(1+\varrho_1), Y^2_{t-}(1+\varrho_2)) - \widetilde{v}(t, Y^1_{t-}, Y^2_{t-})\big)dM^*_t.
\end{aligned}$$

Generic defaultable claim. Technique described above can be applied to the case of a general defaultable claim. Consider a generic defaultable claim $(X, Z, 0, \tau)$ with the promised payoff $X = g(Y^1_T, Y^2_T)$ and the recovery process $Z_t = z(t, Y^1_t, Y^2_t)$, where z is a continuous function. Then the discounted price process stopped at τ equals

$$V^*_{\tau\wedge t} = \mathbb{1}_{\{\tau>t\}} e^{-rt}\widetilde{v}(t, Y^1_t, Y^2_t) + \mathbb{1}_{\{\tau\le t\}} e^{-r\tau} z(\tau, Y^1_\tau, Y^2_\tau).$$

The latter formula can also be rewritten as follows (note that the pre-default prices $\widetilde{Y}^1$ and $\widetilde{Y}^2$ are continuous)

$$V^*_{\tau\wedge t} = \mathbb{1}_{\{\tau>t\}} e^{-rt}\widetilde{v}(t, \widetilde{Y}^1_t, \widetilde{Y}^2_t) + \mathbb{1}_{\{\tau\le t\}} e^{-r\tau} z(\tau, \widetilde{Y}^1_\tau(1+\varrho_1), \widetilde{Y}^2_\tau(1+\varrho_2)).$$

In this case, the pre-default pricing PDE reads

$$\begin{aligned}
-r\widetilde{v} + \partial_t\widetilde{v} + rx\partial_x\widetilde{v} + ry\partial_y\widetilde{v} + \tfrac{1}{2}\big(\sigma_1^2x^2\partial^2_{xx}\widetilde{v} + \sigma_2^2y^2\partial^2_{yy}\widetilde{v}\big) + \sigma_1\sigma_2 xy\partial^2_{xy}\widetilde{v} \\
+ \gamma(t,x,y)\big(z(t, x(1+\varrho_1), y(1+\varrho_2)) - \widetilde{v} - \varrho_1 x\partial_x\widetilde{v} - \varrho_2 y\partial_y\widetilde{v}\big) = 0
\end{aligned}$$

with the terminal condition $\widetilde{v}(T,x,y) = g(x,y)$. According to our interpretation of the pre-default value $\widetilde{U} = \widetilde{U}(X) + \widetilde{U}(Z)$ of the claim $(X, Z, 0, \tau)$, the solution to the last equation is expected to satisfy $\widetilde{v}(t, \widetilde{Y}^1_t, \widetilde{Y}^2_t) = \widetilde{U}_t$ for every $t \in [0,T]$.

1.6.3 Replicating Strategy

Consider a claim Y of the form (1.50), and assume that any $t \in [0,T]$ we have

$$\pi_t(Y) = \widetilde{v}(t, \widetilde{Y}^1_t, \widetilde{Y}^2_t)\mathbb{1}_{\{\tau > t\}} + \bar{v}(t, Y^1_t, Y^2_t)\mathbb{1}_{\{\tau \le t\}},$$

where the functions $\bar{v}$ and $\widetilde{v}$ satisfy the post-default and pre-default pricing PDEs, respectively. It view of Corollary 1.8, we have (recall that the process M^* is stopped at τ and processes $\widetilde{Y}^1$ and $\widetilde{Y}^2$ are continuous)

$$\begin{aligned} dV^*_t &= \mathbb{1}_{\{\tau \ge t\}} e^{-rt} \widetilde{V}_t \, dW^*_t + \mathbb{1}_{\{\tau < t\}} e^{-rt} \bar{V}_t \, dW^*_t \\ &\quad + e^{-rt}\big[\bar{v}(t, \widetilde{Y}^1_t(1+\varrho_1), \widetilde{Y}^2_t(1+\varrho_2)) - \widetilde{v}(t, \widetilde{Y}^1_t, \widetilde{Y}^2_t)\big] \, dM^*_t, \end{aligned}$$

where the $\mathbb{F}$-adapted process $\widetilde{V}$ is given by

$$\widetilde{V}_t = \sigma_1 \widetilde{Y}^1_t \partial_x \widetilde{v}(t, \widetilde{Y}^1_t, \widetilde{Y}^2_t) + \sigma_2 \widetilde{Y}^2_t \partial_y \widetilde{v}(t, \widetilde{Y}^1_t, \widetilde{Y}^2_t) \tag{1.60}$$

and $\bar{V}$ is the $\mathbb{G}$-adapted process:

$$\bar{V}_t = \sigma_1 Y^1_t \partial_x \bar{v}(t, Y^1_t, Y^2_t) + \sigma_2 Y^2_t \partial_y \bar{v}(t, Y^1_t, Y^2_t). \tag{1.61}$$

As before, we denote the discounted prices by

$$Y^{1,3}_t = Y^1_t / Y^3_t = Y^1_t e^{-rt}, \quad Y^{2,3}_t = Y^2_t / Y^3_t = Y^2_t e^{-rt}.$$

Some algebra leads to

$$\begin{aligned} dW^*_t &= \frac{1}{\varrho_2\sigma_1 - \varrho_1\sigma_2} \left(\frac{\varrho_2}{Y^{1,3}_{t-}} \, dY^{1,3}_t - \frac{\varrho_1}{Y^{2,3}_{t-}} \, dY^{2,3}_t \right), \\ dM^*_t &= \frac{1}{\varrho_1\sigma_2 - \varrho_2\sigma_1} \left(\frac{\sigma_2}{Y^{1,3}_{t-}} \, dY^{1,3}_t - \frac{\sigma_1}{Y^{2,3}_{t-}} \, dY^{2,3}_t \right). \end{aligned}$$

It should be stressed that the above representation for W^* and M^* is always valid, under the present assumptions, on the stochastic interval $[\![0, \tau \wedge T]\!]$. It also holds after default, provided that neither Y^1 nor Y^2 jumps to zero at time τ. Hence, the case when Y^1 (or Y^2) becomes worthless at time τ (and thus also after τ) should be considered separately. It is worthwhile to emphasize that the strategy ϕ derived below is always the replicating strategy for the claim Y up to default time τ. Recall that we work under the standing assumption that $c = \varrho_2\sigma_1 - \varrho_1\sigma_2 \neq 0$. Hence, under the assumption that $\varrho_1 > -1$ and $\varrho_2 > -1$, we obtain

$$\begin{aligned} V^*_t &= V^*_0 + \frac{1}{c} \int_{(0,t]} \Big[\varrho_2 \big(\mathbb{1}_{\{\tau \ge u\}} \widetilde{V}_u + \mathbb{1}_{\{\tau < u\}} \bar{V}_u \big) \\ &\quad - \sigma_2 \Big(\bar{v}(u, \widetilde{Y}^1_u(1+\varrho_1), \widetilde{Y}^2_u(1+\varrho_2)) - \widetilde{v}(u, \widetilde{Y}^1_u, \widetilde{Y}^2_u) \Big) \Big] \frac{dY^{1,3}_u}{Y^1_{u-}} \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{c}\int_{(0,t]}\Big[\varrho_1\big(1\!\!1_{\{\tau\geq u\}}\widetilde{V}_u+1\!\!1_{\{\tau<u\}}\bar{V}_u\big) \\
& \quad -\sigma_1\Big(\bar{v}(u,\widetilde{Y}^1_u(1+\varrho_1),\widetilde{Y}^2_u(1+\varrho_2))-\widetilde{v}(u,\widetilde{Y}^1_u,\widetilde{Y}^2_u)\Big)\Big]\,\frac{dY^{2,3}_u}{Y^2_{u-}} \\
= \;& V^*_0+\int_{(0,t]}\psi^1_u\,dY^{1,3}_u+\int_{(0,t]}\psi^2_u\,dY^{2,3}_u,
\end{aligned}
$$

where the processes ψ^1 and ψ^2 are $\mathbb{G}$-predictable. If we do not postulate that $\varrho_1>-1$ and $\varrho_2>-1$, then we obtain

$$
\begin{aligned}
V^*_{t\wedge\tau} &= V^*_0+\int_0^{t\wedge\tau}\widetilde{V}_u\,dW^*_u \\
&\quad +\int_{(0,t\wedge\tau]}e^{-ru}\big[\bar{v}(u,\widetilde{Y}^1_u(1+\varrho_1),\widetilde{Y}^2_u(1+\varrho_2))-\widetilde{v}(u,\widetilde{Y}^1_u,\widetilde{Y}^2_u)\big]\,dM^*_u
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
V^*_{t\wedge\tau} &= V^*_0+\frac{1}{c}\int_{(0,t\wedge\tau]}\varrho_2\widetilde{V}_u\,\frac{dY^{1,3}_u}{\widetilde{Y}^1_u}-\frac{1}{c}\int_{(0,t\wedge\tau]}\varrho_1\widetilde{V}_u\,\frac{dY^{2,3}_u}{\widetilde{Y}^2_u} \\
&\quad -\frac{\sigma_2}{c}\int_{(0,t\wedge\tau]}\Big(\bar{v}(u,\widetilde{Y}^1_u(1+\varrho_1),\widetilde{Y}^2_u(1+\varrho_2))-\widetilde{v}(u,\widetilde{Y}^1_u,\widetilde{Y}^2_u)\Big)\,\frac{dY^{1,3}_u}{\widetilde{Y}^1_u} \\
&\quad +\frac{\sigma_1}{c}\int_{(0,t\wedge\tau]}\Big(\bar{v}(u,\widetilde{Y}^1_u(1+\varrho_1),\widetilde{Y}^2_u(1+\varrho_2))-\widetilde{v}(u,\widetilde{Y}^1_u,\widetilde{Y}^2_u)\Big)\,\frac{dY^{2,3}_u}{\widetilde{Y}^2_u} \\
&= V^*_0+\int_{(0,t\wedge\tau]}\phi^1_u\,dY^{1,3}_u+\int_{(0,t\wedge\tau]}\phi^2_u\,dY^{2,3}_u,
\end{aligned}
$$

where the processes ϕ^1 and ϕ^2 are $\mathbb{F}$-predictable.

We are in a position to state the following result, which establishes the formula for the replicating strategy for Y.

Proposition 1.12 *Assume that $\varrho_1>-1$ and $\varrho_2>-1$. Then the replicating strategy for the defaultable claim Y defined by* (1.50) *is given as $\psi=(\psi^1,\psi^2,\pi(Y)-\psi^1Y^1-\psi^2Y^2)$, where the $\mathbb{G}$-predictable processes ψ^1 and ψ^2 are given by the expressions*

$$
\begin{aligned}
\psi^1_t \;=\;& (cY^1_{t-})^{-1}\Big(\varrho_2\big(1\!\!1_{\{\tau\geq t\}}\widetilde{V}_t+1\!\!1_{\{\tau<t\}}\bar{V}_t\big) \\
& -\sigma_2\big(\bar{v}(t,\widetilde{Y}^1_t(1+\varrho_1),\widetilde{Y}^2_t(1+\varrho_2))-\widetilde{v}(t,\widetilde{Y}^1_t,\widetilde{Y}^2_t)\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi^2_t \;=\;& -(cY^2_{t-})^{-1}\Big(\varrho_1\big(1\!\!1_{\{\tau\geq t\}}\widetilde{V}_t+1\!\!1_{\{\tau<t\}}\bar{V}_t\big) \\
& -\sigma_1\big(\bar{v}(t,\widetilde{Y}^1_t(1+\varrho_1),\widetilde{Y}^2_t(1+\varrho_2))-\widetilde{v}(t,\widetilde{Y}^1_t,\widetilde{Y}^2_t)\big)\Big)
\end{aligned}
$$

with the processes $\widetilde{V}$ and $\bar{V}$ given by (1.60) *and* (1.61), *respectively. The wealth process of ψ satisfies $V_t(\psi)=\pi_t(Y)$ for every $t\in[0,T]$.*

It is worthwhile to stress that the replicating strategy ψ is understood in the standard sense, that is, it duplicates the payoff Y at the maturity date T. If we wish instead to use the convention adopted in Section 1.4, then we should focus on the defaultable claim $(X, Z, 0, \tau)$ associated with Y through equality (1.58) (in this case, $z(t, x, y) = \bar{v}(t, x, y)$), and thus it is a replicating strategy for the associated defaultable claim (X, Z, τ). The latter convention is particularly convenient if the assumption that both ϱ_1 and ϱ_2 are strictly greater than -1 is relaxed. Let us focus on the replication of the claim $(X, Z, 0, \tau)$ with the pre-default value

$$\widetilde{U}_t = \widetilde{U}_t(X) + \widetilde{U}_t(Z) = \widetilde{v}(t, \widetilde{Y}^1_t, \widetilde{Y}^2_t).$$

Proposition 1.13 *Assume that either $\varrho_1 > -1$ or $\varrho_2 > -1$, and let the process $\widetilde{V}$ be given by* (1.60). *Then the replicating strategy for the defaultable claim $(X, Z, 0, \tau)$ is $\phi = (\phi^1, \phi^2, \widetilde{U} - \phi^1 \widetilde{Y}^1 - \phi^2 \widetilde{Y}^2)$, where the $\mathbb{F}$-predictable processes ϕ^1 and ϕ^2 are given by the formulae*

$$\begin{aligned}
\phi^1_t &= (c\widetilde{Y}^1_t)^{-1}\Big(\varrho_2 \widetilde{V}_t - \sigma_2\big(z(t, \widetilde{Y}^1_t(1+\varrho_1), \widetilde{Y}^2_t(1+\varrho_2)) - \widetilde{U}_t\big)\Big)\\
\phi^2_t &= -(c\widetilde{Y}^2_t)^{-1}\Big(\varrho_1 \widetilde{V}_t - \sigma_1\big(z(t, \widetilde{Y}^1_t(1+\varrho_1), \widetilde{Y}^2_t(1+\varrho_2)) - \widetilde{U}_t\big)\Big).
\end{aligned}$$

Survival claim. Assume that the first tradeable asset is a default-free asset (that is, $\varrho_1 = 0$), and the second asset is a defaultable asset with zero recovery (hence, $\varrho_2 = -1$). Then we have

$$\begin{aligned}
dY^1_t &= Y^1_t\big(r\,dt + \sigma_1\,dW^*_t\big), \quad Y^1_0 > 0,\\
dY^2_t &= Y^2_{t-}\big(r\,dt + \sigma_2\,dW^*_t - dM^*_t\big), \quad Y^2_0 > 0,\\
dY^3_t &= rY^3_t\,dt, \quad Y^3_0 = 1.
\end{aligned}$$

Notice that $c = \varrho_2\sigma_1 - \varrho_1\sigma_2 = -\sigma_1 \neq 0$. Consider a survival claim Y of the form $Y = \mathbb{1}_{\{\tau > T\}} g(Y^1_T)$, that is, a vulnerable claim with zero recovery written on the default-free asset Y^1. It is obvious that we may formally identify Y with the defaultable claim $(X, 0, 0, \tau)$ with the promised payoff $X = g(Y^1_T)$ and $Z = 0$. For the replicating strategy ϕ we obtain that

$$\phi^2_t Y^2_t = \widetilde{v}(t, \widetilde{Y}^1_t, \widetilde{Y}^2_t) - \bar{v}(t, \widetilde{Y}^1_t(1+\varrho_1), \widetilde{Y}^2_t(1+\varrho_2)y) = \widetilde{U}_t,$$

since $\bar{v}(t, x, y) = 0$. We conclude that the net investment in default-free assets equals 0 at any time $t \in [0, T]$. One can check, by inspection, that the strategy ϕ replicates the claim Y also after default (formally, we set $\psi^i_t = 0$ for $i = 1, 2, 3$ on the event $\{\tau > t\}$).

Suppose that the risk-neutral intensity of default is of the form $\gamma_t = \gamma(t, Y^1_t)$. In this case, it is rather obvious that the pre-default pricing function $\widetilde{v}$ does not depend on the variable y. In particular, the volatility coefficient σ_2 of the second asset plays no role in the risk-neutral valuation of Y; only the properties of the default time τ really matter. This feature of the function $\widetilde{v}$ can be formally deduced from the representation (1.53) and the observation that if $\gamma_t = \gamma(t, Y^1_t)$

then the two-dimensional process (Y^1, H) is Markovian with respect to the filtration $\mathbb{G}$. We conclude that the function $\widetilde{v} = \widetilde{v}(t, x)$ satisfies the following simple version of the pre-default pricing PDE

$$-r\widetilde{v} + \partial_t \widetilde{v} + rx\partial_x \widetilde{v} + \tfrac{1}{2}\sigma_1^2 x^2 \partial_{xx}^2 \widetilde{v} - \gamma(t, x)\widetilde{v} = 0$$

with the terminal condition $\widetilde{v}(T, x) = g(x)$.

1.6.4 Generalizations

For the sake of simplicity, we have postulated that the prices Y^1, Y^2 and Y^3 are given by the SDE (1.49) with constant coefficients. In order to cover a large class of defaultable assets, we should relax these restrictive assumptions by postulating, for instance, that the processes Y^1 and Y^2 are governed under $\mathbb{Q}^*$ by

$$dY_t^i = Y_{t-}^i \big(r_t \, dt + \sigma_t^i \, dW_t^* + \varrho_t^i \, dM_t^* \big), \quad Y_0^i > 0,$$

where

$$\sigma_t^i = \widetilde{\sigma}_i(t, T) \mathbb{1}_{\{\tau < t\}} + \bar{\sigma}_2(t, T) \mathbb{1}_{\{\tau \geq t\}}$$

for some pre-default and post-default volatilities $\widetilde{\sigma}_i(t, T)$ and $\bar{\sigma}_i(t, T)$, and where $\varrho_t^i = \varrho_i(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3)$ for some functions $\varrho_i : [0, T] \times \mathbb{R}_+^3 \to [-1, \infty)$. The proposed dynamics for Y^1 and Y^2 has the following practical consequences. First, the choice of $\widetilde{\sigma}_i$ and $\bar{\sigma}_i$ allows us to model the real-life fact that the character of a defaultable security may change essentially after default. Second, through a judicious specification of the function ϱ_i, we are able to examine various alternative recovery schemes at time of default. As the process Y^3, we may take the price of a zero-coupon default-free bond. Hence, $Y^3 = B(t, T)$ satisfies under $\mathbb{Q}^*$

$$dY_t^3 = Y_t^3 \big(r_t \, dt + b(t, T) \, dW_t^* \big).$$

Example 1.3 Suppose that the process Y^1 represents the price of a generic defaultable zero-coupon bond with maturity date T. Then the bond is subject to the fractional recovery of market value scheme with recovery rate $\delta_1 \in [0, 1]$ if the process ϱ^1 is constant, specifically,

$$\varrho_t^1 = \varrho_1(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3) = \delta_1 - 1.$$

To model a defaultable bond with the fractional recovery of par value at default, we set

$$\varrho_t^1 = \varrho_1(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3) = \delta_1 (Y_{t-}^1)^{-1} - 1.$$

Finally, the fractional recovery of Treasury value scheme corresponds to the following choice of the process ϱ_t^1 (recall that $Y_t^3 = B(t, T)$, and thus it is a continuous process)

$$\varrho_t^1 = \varrho_1(t, Y_{t-}^1, Y_{t-}^2, Y_{t-}^3) = \delta_1 Y_t^3 (Y_{t-}^1)^{-1} - 1.$$

In all cases, the post-default volatility $\bar{\sigma}_1(t, T)$ should coincide with the volatility of the default-free zero-coupon bond of maturity T. This corresponds to the natural interpretation that after default the recovery payoff is invested in default-free bonds.

Chapter 2

Mean-Variance Approach

In this chapter, we formulate a new paradigm for pricing and hedging financial risks in incomplete markets, rooted in the classical Markowitz mean-variance portfolio selection principle. We consider an underlying market of liquid financial instruments that are available to an investor (also called an agent) for investment. We assume that the underlying market is arbitrage-free and complete. We also consider an investor who is interested in dynamic selection of her portfolio, so that the expected value of her wealth at the end of the pre-selected planning horizon is no less then some floor value, and so that the associated risk, as measured by the variance of the wealth at the end of the planning horizon, is minimized.

When a new investment opportunity becomes available for the agent, in a form of some contingent claim, she needs to decide how much she is willing to pay for acquiring the opportunity. More specifically, she has to decide what portion of her current endowment she is willing to invest in a new opportunity. It is assumed that the new claim, if acquired, is held until the horizon date, and the remaining part of the endowment is dynamically invested in primary (liquid) assets. If the cash-flows generated by the new opportunity can be perfectly replicated by the existing liquid market instruments already available for trading, then the price of the opportunity will be uniquely determined by the wealth of the replicating strategy. However, if perfect replication is not possible, then the determination of a purchase (or bid) price that the investor is willing to pay for the opportunity, will become subject to the investor's overall attitude towards trading. In case of our investor, the bid price and the corresponding hedging strategy will be determined in accordance with the mean-variance paradigm. Analogous remarks apply to an investor who engages in creation of an investment opportunity and needs to decide about its selling (or ask) price.

As explained above, it suffices to focus on a situation when the newly available investment opportunity can not be perfectly replicated by the instruments existing in the underlying market. Thus, the emerging investment opportunity is not *attainable*, and consequently the market model (that is the underlying market and new investment opportunities) is *incomplete*.

It is well known (see, e.g., El Karoui and Quenez (1995) or Kramkov (1996)) that when a market is incomplete, then for any non-attainable contingent claim X there exists a non-empty interval of arbitrage prices, referred to as the *no-arbitrage interval,* determined by the maximum bid price $\pi^u(X)$ (the upper price) and the minimum ask price $\pi^l(X)$ (the lower price) The maximum bid price represents the cost of the most expensive dynamic portfolio that can be used to perfectly hedge the long position in the contingent claim. The minimum ask price represents the initial cost of the cheapest dynamic portfolio that can be used to perfectly hedge the short position in the contingent claim.

Put another way, the maximum bid price is the maximum amount that the agent purchasing the contingent claim can afford to pay for the claim, and still be sure to find an admissible portfolio that would fully manage her debt and repay it with cash flows generated by the strategy and the contingent claim, and end up with a non-negative wealth at the maturity date of the claim. Likewise, the minimum ask price is the minimum amount that the agent selling the claim can afford to accept to charge for the claim, and still be sure to find an admissible portfolio that would generate enough cash flow to make good on her commitment to buyer of the claim, and end up with a non-negative wealth at the maturity date of the claim.

As is well known, the arbitrage opportunities are precluded if and only if the actual price of the contingent claim belongs to the no-arbitrage interval. But this means, of course, that perfect hedging will not be accomplished by neither the short party, nor by the long party. Thus, any price that precludes arbitrage, enforces possibility of a financial loss for either party at the maturity date. This observation gave rise to quite abundant literature regarding the judicious choice of a specific price within the no-arbitrage interval by means of minimizing some functional that assesses the risk associated with potential losses.

We shall not be discussing this extensive literature here. Let us only observe that much work within this line of research has been done with regard to the so-called *mean-variance hedging*; we refer to the recent paper by Schweizer (2001) for an exhaustive survey of relevant results. It is worth stressing that the interpretation of the term "mean-variance hedging", as defined in these works, is entirely different from what is meant here by mean-variance hedging.

The optimization techniques used in this chapter are based on mean-variance portfolio selection in continuous time. Probably the first work in this area was the paper by Zhou and Li (2000) who used the embedding technique and linear-quadratic (LQ) optimal control theory to solve the continuous-time, mean-variance problem with assets having deterministic diffusion coefficients. They essentially ended up with a problem that was inherently an *indefinite stochastic LQ control problem*, the theory of which has been developed only very recently (see, e.g., Yong and Zhou (1999), Chapter 6). In subsequent works, the techniques of stochastic LQ optimal control were heavily exploited in order to solve more sophisticated variants of the mean-variance portfolio selection in continuous time. For instance, Li et al. (2001) introduced a constraint on short-selling, Lim and Zhou (2002) allowed for stocks which are modeled by processes having random drift and diffusion coefficients, Zhou and Yin (2004) featured assets in a regime switching market, and Bielecki et al. (2004b) solved the problem with positivity constraint imposed on the wealth process. An excellent survey of most of these results is presented in Zhou (2003), who also provided a number of examples that illustrate the similarities as well as differences between the continuous-time and single-period settings.

2.1 Mean-Variance Pricing and Hedging

We consider an economy in continuous time, $t \in [0, T^*]$, and the underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a one-dimensional standard Brownian motion W (with respect to its natural filtration). The probability $\mathbb{P}$ plays the role of the statistical probability. We denote by $\mathbb{F}$ the $\mathbb{P}$-augmentation of the filtration generated by W. Consider an agent who initially has two liquid assets available to invest in:

- a risky asset whose price dynamics are
$$dZ_t^1 = Z_t^1\big(\nu\, dt + \sigma\, dW_t\big), \quad Z_0^1 > 0,$$
for some constants ν and $\sigma > 0$,
- a money market account whose price dynamics under $\mathbb{P}$ are
$$dZ_t^2 = rZ_t^2\, dt, \quad Z_0^2 = 1,$$
where r is a constant interest rate.

Suppose for the moment that $\mathcal{G} = \mathcal{F}_{T^*}$. It is well known that in this case the *underlying market*, consisting of the two above assets, is complete. Thus the fair value of any claim contingent X which settles at time $T \leq T^*$, and thus is formally defined as an $\mathcal{F}_T$-measurable random variable, is the (unique) arbitrage price of X, denoted as $\pi_0(X)$ in what follows.

Now let $\mathbb{H}$ be another filtration in $(\Omega, \mathcal{G}, \mathbb{P})$, which satisfies the usual conditions. We consider the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ and we postulate that $\mathcal{G} = \mathcal{G}_{T^*}$. We shall refer to $\mathbb{G}$ as to the *full filtration*; the Brownian filtration $\mathbb{F}$ will be called the *reference filtration*. We make an important assumption that W is a standard Brownian motion with respect to the full filtration $\mathbb{G}$ under the probability $\mathbb{P}$.

Let ϕ_t^i represent the number of shares of asset i held in the agent's portfolio at time t. We consider trading strategies $\phi = (\phi^1, \phi^2)$, where ϕ^1 and ϕ^2 are $\mathbb{G}$-predictable processes. A strategy ϕ is *self-financing* if

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^1\, dZ_u^1 + \int_0^t \phi_u^2\, dZ_u^2, \quad \forall\, t \in [0, T^*],$$

where $V_t(\phi) = \phi_t^1 Z_t^1 + \phi_t^2 Z_t^2$ is the *wealth* of ϕ at time t. Thus, we postulate the absence of outside endowments and/or consumption.

Definition 2.1 We say that a self-financing strategy ϕ is *admissible* on the interval $[0,T]$ if and only if for any $t \in [0,T]$ the wealth $V_t(\phi)$ is a $\mathbb{P}$-square-integrable random variable.

The condition

$$\mathbb{E}_{\mathbb{P}}\Big(\int_0^T (\phi_u^i Z_u^i)^2 du\Big) < \infty, \quad i = 1, 2,$$

is manifestly sufficient for the admissibility of ϕ on $[0,T]$. Let us fix T and let us denote by $\Phi(\mathbb{G})$ the linear space of all admissible trading strategies on the finite interval $[0,T]$.

Suppose that the agent has at time $t = 0$ a positive amount $v > 0$ available for investment (we shall refer to v as the *initial endowment*). It is easily seen that for any $\phi \in \Phi(\mathbb{G})$ the wealth process satisfies the following SDE

$$dV_t^v(\phi) = rV_t^v(\phi)\,dt + \phi_t^1\big(dZ_t^1 - rZ_t^1 dt\big), \quad V_0^v(\phi) = v.$$

This shows that the wealth at time t depends exclusively on the initial endowment v and the component ϕ^1 of a self-financing strategy ϕ.

Now, imagine that a new investment opportunity becomes available for the agent. Namely, the agent may purchase at time $t = 0$ a contingent claim X, whose corresponding cash-flow of X units of cash occurs at time T. We assume that X is not an $\mathcal{F}_T$-measurable random variable. Notice that this requirement alone may not suffice for the non-attainability of X. Indeed, in the present setup, we have the following definition of attainability.

Definition 2.2 A contingent claim X is *attainable* if there exists a strategy $\phi \in \Phi(\mathbb{G})$ such that $X = V_T(\phi)$ or, equivalently,

$$X = V_0(\phi) + \int_0^T \phi_u^1\, dZ_u^1 + \int_0^T \phi_u^2\, dZ_u^2.$$

If a claim X can be replicated by means of a trading strategy $\phi \in \Phi(\mathbb{F})$, we shall say that X is $\mathbb{F}$-*attainable*. According to the definition of admissibility, the square-integrability of X under $\mathbb{P}$ is a necessary condition for attainability. Notice, however, that it may happen that X is not an $\mathcal{F}_T$-measurable random variable, but it represents an attainable contingent claim according to the definition above.

Suppose now that a considered claim X is not attainable. The main question that we want to study is: how much would the agent be willing to pay at time $t = 0$ for X, and how the agent should hedge her investment? A symmetric study can be conducted for an agent creating such an investment opportunity by selling the claim. In what follows, we shall first present our results in a general framework of a generic $\mathcal{G}_T$-measurable claim; then we shall examine a particular case of defaultable claims.

2.1.1 Mean-Variance Portfolio Selection

We postulate that the agent's objective for investment is based on the classical *mean-variance portfolio selection*. Let $\mathbb{V}_{\mathbb{P}}(Z)$ be the variance under $\mathbb{P}$ of a random variable Z. For any fixed date T, any initial endowment $v > 0$, and any given $d \in \mathbb{R}$, the agent is interested in solving the following problem:

Problem MV(d, v)**:** Minimize $\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))$ over all strategies $\phi \in \Phi(\mathbb{G})$, subject to $\mathbb{E}_{\mathbb{P}} V_T^v(\phi) \geq d$.

We shall show that, given the parameters d and v satisfy certain additional conditions, the above problem admits a solution, so that there exists an optimal trading strategy, say $\phi^*(d, v)$. Let $V^*(d, v) = V(\phi^*(d, v))$ stand for the optimal wealth process, and let us denote by $\mathbf{v}^*(d, v)$ the value of the variance $\mathbb{V}_{\mathbb{P}}(V_T^*(d, v))$.

For simplicity of presentation, we did not postulate above that agent's wealth should be non-negative at any time. Problem $\mathrm{MV}(d, v)$ with this additional restriction has been recently studied in Bielecki et al. (2004b).

Remark. It is apparent that the problem $\mathrm{MV}(d, v)$ is non-trivial only if $d > ve^{rT}$. Otherwise, investing in the money market alone generates the wealth process $V_t^v(\phi) = ve^{rt}$, that obviously satisfies the terminal condition $\mathbb{E}_{\mathbb{P}} V_T^v(\phi) = ve^{rT} \geq d$, and for which the variance of the terminal wealth $V_T^v(\phi)$ is zero. Thus, when considering the problem $\mathrm{MV}(d, v)$ we shall always assume that $d > ve^{rT}$. Put another way, we shall only consider trading strategies ϕ for which the expected return satisfies $\mathbb{E}_{\mathbb{P}}(V_T^v(\phi)/v) \geq e^{rT}$, that is, it is strictly higher than the return on the money market account.

Assume that a claim X is available for purchase at time $t = 0$. We postulate that the random variable X is $\mathcal{G}_T$-measurable and square-integrable under $\mathbb{P}$. The agent shall decide whether to purchase X, and what is the maximal price she could offer for X. According to the mean-variance paradigm, her decision will be based on the following reasoning. First, for any $p \in [0, v]$ the agent needs to solve the related mean-variance problem.

Problem MV(d, v, p, X)**:** Minimize $\mathbb{V}_{\mathbb{P}}(V_T^{v-p}(\phi) + X)$ over all trading strategies $\phi \in \Phi(\mathbb{G})$, subject to $\mathbb{E}_{\mathbb{P}}(V_T^{v-p}(\phi) + X) \geq d$.

We shall show that if d, v, p and X satisfy certain sufficient conditions, then there exists an optimal strategy, say $\phi^*(d, v, p, X)$, for this problem. We denote by $V_T^*(d, v, p, X)$ the value of $V_T^{v-p}(\phi^*(d, v, p, X))$ and we set $\mathbf{v}^*(d, v, p, X) = \mathbb{V}_{\mathbb{P}}\big(V_T^*(d, v, p, X) + X\big)$.

It is reasonable to expect that the agent will be willing to pay for the claim X the price that is no more than (by convention, $\sup \emptyset = -\infty$)

$$\begin{aligned} p^{d,v}(X) \quad := \quad & \sup \{\, p \in [0, v] \ : \mathrm{MV}(d, v, p, X) \text{ admits a solution} \\ & \text{and } \mathbf{v}^*(d, v, p, X) \leq \mathbf{v}^*(d, v) \}. \end{aligned}$$

This leads to the following definition of mean-variance price and hedging strategy.

Definition 2.3 The number $p^{d,v}(X)$ is called the *buying agent's mean-variance price* of X. The optimal trading strategy $\phi^*(d,v,p^{d,v}(X),X)$ is called the agent's *mean-variance hedging strategy* for X.

Of course, in order to make the last definition operational, we need to be able to solve explicitly problems $\mathrm{MV}(d,v)$ and $\mathrm{MV}(d,v,p,X)$, at least in some special cases of a common interest. These issues will be examined in some detail in the remaining part of this note, first for the special case of $\mathbb{F}$-adapted trading strategies (see Section 2.2), and subsequently, in the general case of $\mathbb{G}$-adapted strategies (see Section 2.3).

Remark. Let us denote $\mu_X = \mathbb{E}_{\mathbb{P}}X$. Inequality $\mathbb{E}_{\mathbb{P}}(V_T^{v-p}(\phi) + X) \geq d$ is equivalent to $\mathbb{E}_{\mathbb{P}}V_T^{v-p}(\phi) \geq d - \mu_X$. Observe that, unlike as in the case of the problem $\mathrm{MV}(d,v)$, the problem $\mathrm{MV}(d,v,p,X)$ may be non-trivial even if $d - \mu_X \leq e^{rT}(v-p)$. Although investing in a money market alone will produce in this case a wealth process for which the condition $\mathbb{E}_{\mathbb{P}}V_T^{v-p}(\phi) \geq d - \mu_X$ is manifestly satisfied, the corresponding variance $\mathbb{V}_{\mathbb{P}}\big(V_T^{v-p}(\phi) + X\big) = \mathbb{V}_{\mathbb{P}}(X)$ is not necessarily minimal.

Financial Interpretation

Let us denote by $\mathcal{N}(X)$ the no-arbitrage interval for the claim X, that is, $\mathcal{N}(X) = [\pi^l(X), \pi^u(X)]$. It may well happen that the mean-variance price $p^{d,v}(X)$ is outside this interval. Since this possibility may appear as an unwanted feature of the approach to pricing and hedging presented in this note, we shall comment briefly on this issue. When we consider the valuation of a claim X from the perspective of the entire market, then we naturally apply the no-arbitrage paradigm.

According to the no-arbitrage paradigm, the financial market as a whole will accept only those prices of a financial asset, which fall into the no-arbitrage interval. Prices from outside this interval can't be sustained in a longer term due to market forces, which will tend to eliminate any arbitrage opportunity.

Now, let us consider the same issue from the perspective of an individual. Suppose that an individual investor is interested in putting some of her initial endowment $v > 0$ into an investment opportunity provided by some claim X. Thus, the investor needs to decide whether to acquire the investment opportunity, and if so then how much to pay for it, based on her overall attitude towards risk and reward.

The number $p^{d,v}(X)$ is the price that investor is willing to pay for the investment opportunity X, given her initial capital v, given her attitude towards risk and reward, and given the primary market. The investor "submits" her price to the market. Now, suppose that the market recognized no-arbitrage interval for X is $\mathcal{N}(X)$. If it happens that $p \in \mathcal{N}(X)$ then the investor's bid price for X can be accepted by the market. In the opposite case, the investor's bid price may not be accepted by the market, and the investor may not enter into the investment opportunity.

2.2 Strategies Adapted to the Reference Filtration

In this section, we shall solve the problem $\mathrm{MV}(d,v)$ under the restriction that trading strategies are based on the reference filtration $\mathbb{F}$. In other words, we postulate that ϕ belongs to the class $\Phi(\mathbb{F})$ of all admissible and $\mathbb{F}$-predictable strategies ϕ. In this case, we shall say that a strategy ϕ is $\mathbb{F}$-*admissible*. The assumption that ϕ is $\mathbb{F}$-admissible implies, of course, that the terminal wealth $V_T^v(\phi)$ is an $\mathcal{F}_T$-measurable random variable.

2.2.1 Solution to MV(d,v) in the Class $\Phi(\mathbb{F})$

A general version of the problem $\mathrm{MV}(d,v)$ has been studied in Bielecki et al. (2004b). Because our problem is a very special version of the general one, we give below a complete solution tailored to present set-up.

Reduction to Zero Interest Rate Case

Recall our standing assumption that $d > ve^{rT}$. Problem $\mathrm{MV}(d,v)$ is clearly equivalent to: minimize the variance $\mathbb{V}_{\mathbb{P}}(e^{-rT}V_T(\phi))$ under the constraint

$$\mathbb{E}_{\mathbb{P}}(e^{-rT}V_T(\phi)) \geq e^{-rT}d.$$

For the sake of notational simplicity, we shall write V_t instead of $V_t^v(\phi)$. We set $\widetilde{V}_t = V_t(Z_t^2)^{-1} = e^{-rt}V_t$, so that

$$d\widetilde{V}_t = \phi_t^1\, d\widetilde{Z}_t^1 = \phi_t^1 \widetilde{Z}_t^1 \big(\widehat{\nu}\, dt + \sigma\, dW_t\big), \tag{2.1}$$

where we denote $\widehat{\nu} = \nu - r$. So we can and do restrict our attention to the case $r = 0$. Thus, in what follows, we shall have $Z_t^2 = 1$ for every $t \in \mathbb{R}_+$. In the rest of this note, unless explicitly stated otherwise, we assume that $d > v$.

Decomposition of Problem MV(d,v)

Let $\mathbb{Q}$ be a (unique) equivalent martingale measure on $(\Omega, \mathcal{F}_{T^*})$ for the underlying market. It is easily seen that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \eta_t, \quad \forall\, t \in [0, T^*],$$

where we denote by η the Radon-Nikodym density process. Specifically, we have

$$d\eta_t = -\theta \eta_t\, dW_t, \quad \eta_0 = 1, \tag{2.2}$$

or, equivalently,

$$\eta_t = \exp\big(-\theta W_t - \tfrac{1}{2}\theta^2 t\big),$$

where $\theta = \nu/\sigma$ (recall that we have formally reduced the problem to the case $r = 0$). The process η is a $\mathbb{F}$-martingale under $\mathbb{P}$. Moreover,

$$\mathbb{E}_{\mathbb{P}}(\eta_T^2 \,|\, \mathcal{F}_t) = \eta_t^2 \exp(\theta^2(T-t)), \tag{2.3}$$

and thus $\mathbb{E}_{\mathbb{P}}(\eta_t^2) = \exp(\theta^2 t)$ for $t \in [0, T^*]$. It is easily seen that the price Z^1 is an $\mathbb{F}$-martingale under $\mathbb{Q}$, since

$$dZ_t^1 = \sigma Z_t^1 \, d(W_t + \theta t) = \sigma Z_t^1 \, d\widetilde{W}_t \tag{2.4}$$

for the $\mathbb{Q}$-Brownian motion $\widetilde{W}_t = W_t + \theta t$. The measure $\mathbb{Q}$ is thus the equivalent martingale measure for our primary market.

From (2.1), we have that

$$V_t = v + \int_0^t \phi_u^1 \, dZ_u^1 = v + \int_0^t \phi_u^1 \sigma Z_u^1 \, d\widetilde{W}_u. \tag{2.5}$$

Recall that if ϕ is an $\mathbb{F}$-admissible strategy, that is, $\phi \in \Phi(\mathbb{F})$, then V_T is an $\mathcal{F}_T$-measurable random variable, which is $\mathbb{P}$-square-integrable.

Let X be a $\mathbb{P}$-square-integrable and $\mathcal{F}_T$-measurable random variable. It is easily seen that X is integrable with respect to $\mathbb{Q}$ (since η_T is square-integrable with respect to $\mathbb{P}$). The existence of a self-financing trading strategy that replicates X can be justified by the predictable representation theorem combined with the Bayes formula. We thus have the following result.

Lemma 2.1 *Let X be a $\mathbb{P}$-square-integrable and $\mathcal{F}_T$-measurable random variable. Then X is an $\mathbb{F}$-attainable contingent claim, i.e., there exists a strategy ϕ^X in $\Phi(\mathbb{F})$ such that $V_T(\phi) = X$.*

We shall argue that problem MV(d, v) can be split into two problems (see also Pliska (2001) and Bielecki et al. (2004b) in this regard). We first focus on the optimal terminal wealth $V_T^*(d, v)$. Let $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ denote the collection of $\mathbb{P}$-square-integrable random variables that are $\mathcal{F}_T$-measurable. Thus the first problem we need to solve is:

Problem MV1: Minimize $\mathbb{V}_{\mathbb{P}}(\xi)$ over all $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, subject to $\mathbb{E}_{\mathbb{P}}\xi \geq d$ and $\mathbb{E}_{\mathbb{Q}}\xi = v$.

Lemma 2.2 *Suppose that $\phi^* = \phi^*(d, v)$ solves the problem MV(d, v), and let $V^*(d, v) = V(\phi^*)$. Then the random variable $\xi^* = V_T^*(d, v)$ solves the problem MV1.*

Proof. We argue by contradiction. Suppose that there exists a random variable $\widehat{\xi} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ such that $\mathbb{E}_{\mathbb{P}}\widehat{\xi} \geq d$, $\mathbb{E}_{\mathbb{Q}}\widehat{\xi} = v$ and $\mathbb{V}_{\mathbb{P}}(\widehat{\xi}) < \mathbb{V}_{\mathbb{P}}(\xi^*)$. Since $\widehat{\xi}$ is $\mathbb{P}$-square-integrable and $\mathcal{F}_T$-measurable, it represents an attainable contingent claim, so that there exists an $\mathbb{F}$-admissible strategy $\widehat{\phi}$ such that $\widehat{\xi} = V_T(\widehat{\phi})$. Of course, this contradicts the assumption that ϕ^* solves MV(d, v). $\square$

Denoting by ξ^* the optimal solution to problem MV1, the second problem is:

Problem MV2: Find an $\mathbb{F}$-admissible strategy ϕ^* such that $V_T(\phi^*) = \xi^*$.

Since the next result is analogous to Theorem 2.1 in Bielecki et al. (2004b), its proof is omitted. It demonstrates that solving problem MV(d, v) is indeed equivalent to successful solving problems MV1 and MV2. In the formulation of the result below we make use of a backward stochastic differential equation (BSDE). The reader can refer to El Karoui and Mazliak (1997), El Karoui and Quenez (1997), El Karoui et al. (1997), Ma and Yong (1999) or to the survey by Buckdahn (2000) for an introduction to the theory of backward stochastic differential equations and its applications in finance.

Proposition 2.1 *Suppose that the problem MV1 has a solution ξ^*. The following BSDE*

$$dv_t = -\theta z_t\, dt + z_t\, dW_t,\ v_T = \xi^*,\ t \in [0, T], \tag{2.6}$$

has a unique, $\mathbb{P}$-square-integrable solution, denoted as (v^*, z^*), which is adapted to $\mathbb{F}$. Moreover, if we define a process ϕ^{1*} by

$$\phi_t^{1*} = z_t^*(\sigma Z_t^1)^{-1}, \quad \forall\, t \in [0, T],$$

then the $\mathbb{F}$-admissible trading strategy $\phi^* = (\phi^{1*}, \phi^{2*})$ with the wealth process $V_t(\phi^*) = v_t^*$ solves the problem MV(d, v).

For the last statement, recall that if the first component of a self-financing strategy ϕ and its wealth process $V(\phi)$ is known, then the component ϕ^2 is uniquely determined through the equality $V_t(\phi) = \phi_t^1 Z_t^1 + \phi_t^2 Z_t^2$.

Remark. In what follows, we shall derive closed-form expressions for ϕ^* and $V(\phi^*)$. It will be easily seen that the process $V(\phi^*)$ is not only $\mathbb{P}$-square-integrable, but also $\mathbb{Q}$-square-integrable. It should be stressed that Proposition 2.1 will not be used in the derivation of a solution to problem MV(d, v). In fact, we shall find a solution to MV(d, v) through explicit calculations.

Solution of Problem MV1

In order to make the problem MV1 non-trivial, we need to make an additional assumption that $\theta \neq 0$. Indeed, if $\theta = 0$ then we have $\mathbb{P} = \mathbb{Q}$, and thus the problem MV1 becomes:

Problem MV1: Minimize $\mathbb{V}_{\mathbb{P}}(\xi)$ over all $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, subject to $\mathbb{E}_{\mathbb{P}}\xi \geq d$ and $\mathbb{E}_{\mathbb{Q}}\xi = v$.

It is easily seen that this problem admits a solution for $d = v$ only, and the optimal solution is trivial, in the sense that the optimal variance is null. Consequently, for $\theta = 0$, the solution to MV(d, v) exists if and only if $d = v$, and it is trivial: $\phi^* = (0, 1)$. Let us reiterate that we postulate that $d > v$ in order to avoid trivial solutions to MV(d, v).

¿From now on, we assume that $\theta \neq 0$. We begin with the following auxiliary problem:

Problem MV1A: Minimize $\mathbb{V}_{\mathbb{P}}(\xi)$ over all $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, subject to $\mathbb{E}_{\mathbb{P}}\xi = d$ and $\mathbb{E}_{\mathbb{P}}\xi = v$.

The previous problem is manifestly equivalent to:

Problem MV1B: Minimize $\mathbb{E}_{\mathbb{P}}\xi^2$ over all $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, subject to $\mathbb{E}_{\mathbb{P}}\xi = d$ and $\mathbb{E}_{\mathbb{Q}}\xi = v$.

Since $\mathbb{E}_{\mathbb{Q}}\xi = \mathbb{E}_{\mathbb{P}}(\eta_T \xi)$, the corresponding Lagrangian is

$$\mathbb{E}_{\mathbb{P}}(\xi^2 - \lambda_1 \xi - \lambda_2 \eta_T \xi) - d^2 + \lambda_1 d + \lambda_2 v.$$

The optimal random variable is given by $2\xi^* = \lambda_1 + \lambda_2 \eta_T$, where the Lagrange multipliers satisfy

$$2d = \lambda_1 + \lambda_2, \quad 2v = \lambda_1 + \lambda_2 \exp(\theta^2 T).$$

Hence, we have

$$\xi^* = \big(de^{\theta^2 T} - v + (v - d)\eta_T\big)\big(e^{\theta^2 T} - 1\big)^{-1}, \tag{2.7}$$

and the corresponding minimal variance is

$$\mathbb{V}_{\mathbb{P}}(\xi^*) = \mathbb{E}_{\mathbb{P}}(\xi^*)^2 - d^2 = (d - v)^2\big(e^{\theta^2 T} - 1\big)^{-1}. \tag{2.8}$$

Since we assumed that $d > v$, the minimal variance is an increasing function of the parameter d for any fixed value of the initial endowment v, we conclude that we have solved not only the problem MV1A, but the problem MV1 as well. We thus have the following result.

Proposition 2.2 *The solution ξ^* to problem MV1 is given by* (2.7) *and the minimal variance $\mathbb{V}_{\mathbb{P}}(\xi^*)$ is given by* (2.8).

For an alternative approach to Problem MV1, in a fairly general setup, see Jankunas (2001).

Solution of Problem MV2

We maintain the assumption that $\theta \neq 0$. Thus, the optimal wealth for the terminal time T is given by (2.7), that is, $V_T(\phi^*) = \xi^*$. Our goal is to determine an $\mathbb{F}$-admissible strategy ϕ^* for which the last equality is indeed satisfied. In view if (2.5), it suffices to find ϕ^{1*} such that the process V_t^* given by

$$V_t^* = v + \int_0^t \phi_u^{1*}\, dZ_u^1 \tag{2.9}$$

satisfies $V_T = \xi^*$, and the strategy $\phi^* = (\phi^{1*}, \phi^{2*})$, where ϕ^{2*} is derived from $V_t = \phi_t^{1*} Z_t^1 + \phi_t^{2*} Z_t^2$, is $\mathbb{F}$-admissible.

To this end, let us introduce an $\mathbb{F}$-martingale V under $\mathbb{Q}$ by setting $V_t = \mathbb{E}_{\mathbb{Q}}(\xi^* \,|\, \mathcal{F}_t)$ (the integrability of ξ^* under $\mathbb{Q}$ is rather obvious).

It is easy to see that $V_T^* = \xi^*$ and $V_0^* = v$. It thus remains to find the process ϕ^{1*}. Using (2.3), we obtain

$$V_t^* = \big(de^{\theta^2 T} - v + (v-d)\eta_t e^{\theta^2(T-t)}\big)\big(e^{\theta^2 T} - 1\big)^{-1}.$$

Consequently, in view of (2.2) and (2.4), we have

$$\begin{aligned} dV_t^* &= \frac{v-d}{e^{\theta^2 T}-1}\big(e^{\theta^2(T-t)}d\eta_t - \eta_t e^{\theta^2(T-t)}\theta^2\,dt\big) \\ &= e^{\theta^2(T-t)}\,\frac{\theta\eta_t(v-d)}{e^{\theta^2 T}-1}(dW_t - \theta dt) \\ &= e^{\theta^2(T-t)}\,\frac{d-v}{e^{\theta^2 T}-1}\frac{\nu\eta_t}{\sigma^2}\frac{dZ_t^1}{Z_t^1}\,. \end{aligned}$$

This shows that we may choose

$$\phi_t^{1*} = e^{\theta^2(T-t)}\,\frac{d-v}{e^{\theta^2 T}-1}\frac{\nu}{\sigma^2}\frac{\eta_t}{Z_t^1}\,. \tag{2.10}$$

It is clear that ϕ^* is $\mathbb{F}$-admissible, since it is $\mathbb{F}$-adapted, self-financing, and $V_t(\phi^*)$ is $\mathbb{P}$-square-integrable for every $t \in [0, T]$.

Solution of Problem MV(d, v)

By virtue of Lemma 2.2, we conclude that ϕ^* solves MV(d, v). In view of (2.8), the variance under $\mathbb{P}$ of the terminal wealth of the optimal strategy is

$$\mathbf{v}^*(d,v) = \mathbb{E}_{\mathbb{P}}(V_T^*)^2 - d^2 = \frac{(d-v)^2}{e^{\theta^2 T}-1}\,.$$

Let us stress that since we did not impose any no-bankruptcy condition, that is we do no require that the agent's wealth is non-negative, we see that d can be any number greater then v.

We are in a position to state the following result, which summarizes the analysis above. For a fixed $T > 0$, we denote $\rho(\theta) = e^{\theta^2 T}(e^{\theta^2 T} - 1)^{-1}$ and $\eta_t(\theta) = \eta_t e^{-\theta^2 t}$, so that $\eta_0(\theta) = 1$.

Proposition 2.3 *Assume that $\theta \neq 0$ and let $d > v$. Then a solution $\phi^*(d, v) = (\phi^{*1}(d, v), \phi^{*2}(d, v))$ to MV(d, v) is given by*

$$\phi_t^{1*}(d,v) = (d-v)\rho(\theta)\frac{\nu\eta_t(\theta)}{\sigma^2 Z_t^1} \tag{2.11}$$

and $V_t^*(d, v) = V_t(\phi^*(d, v)) = \phi_t^{*1}(d, v)Z_t^1 + \phi_t^{*2}(d, v)$, where the optimal wealth process equals

$$V_t^*(d,v) = v + (d-v)\rho(\theta)\big(1 - \eta_t(\theta)\big). \tag{2.12}$$

The minimal variance $\mathbf{v}^*(d, v)$ is given by

$$\mathbf{v}^*(d,v) = \mathbb{E}_{\mathbb{P}}(V_T^*(d,v))^2 - d^2 = \frac{(d-v)^2}{e^{\theta^2 T}-1}. \tag{2.13}$$

Notice that the optimal trading strategy $\phi^*(d, v)$, the minimal variance $\mathbf{v}^*(d, v)$ and the optimal gains process $G^*_t(d, v) = V^*_t(d, v) - v$ depend exclusively on the difference $d - v > 0$, rather than on parameters d and v themselves.

Efficient Portfolio

As it was observed above the function $f(d) := \mathbf{v}^*(d, v)$ is (strictly) increasing for $d \geq v$. Consider the following problem (as usual, for $d \geq v$):

Problem ME(d, v)**:** Maximize $\mathbb{E}_{\mathbb{P}} V^v_T(\phi)$ over all strategies $\phi \in \Phi(\mathbb{G})$, subject to $\mathbb{V}_{\mathbb{P}}(V^v_T(\phi)) = \mathbf{v}^*(d, v)$.

Denote the maximal expectation in the above problem by $\mu^*(d, v)$. In view of the strict monotonicity of the function $f(d)$ for $d \geq v$, it is clear that $\mu^*(d, v) = d$. Consequently, the minimum variance portfolio ϕ^* is in fact an *efficient portfolio.*

2.2.2 Solution to MV(d, v, p, X) in the Class $\Phi(\mathbb{F})$

Consider first the special case of an attainable claim, which is $\mathcal{F}_T$-measurable. Subsequently, we shall show that in general it suffices to decompose a general claim X into an attainable component $\widetilde{X} = \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, and a component $X - \widetilde{X}$ which is orthogonal in $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ to the subspace $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ of admissible terminal wealths.

Case of an Attainable Claim

We shall verify that the mean-variance price coincides with the (unique) arbitrage price for any contingent claim that is attainable. Of course, this feature is a standard requirement for any reasonable valuation mechanism for contingent claims. Since in this section we consider only $\mathbb{F}$-adapted strategies, we postulate here that a claim X is $\mathcal{F}_T$-measurable; the general case of a $\mathcal{G}_T$-measurable claim is considered in Section 2.3.1. Let $\phi^X \in \Phi(\mathbb{F})$ be a replicating strategy for X, so that X is $\mathbb{F}$-attainable, and let $\pi_0(X) = \mathbb{E}_{\mathbb{Q}} X$ be the arbitrage price of X. Since $\Phi(\mathbb{F})$ is a linear space, it is easily seen that $\Phi(\mathbb{F}) = \Phi(\mathbb{F}) + \phi^X = \Phi(\mathbb{F}) - \phi^X$. The following lemma is thus easy to prove.

Lemma 2.3 *Let X be an $\mathbb{F}$-attainable contingent claim. Then the problem MV(d, v, p, X) is equivalent to the problem MV$(d, \widehat{v})$ with $\widehat{v} = v - p + \pi_0(X)$.*

Equivalence of problems MV(d, v, p, X) and MV$(d, \widehat{v})$ is understood in the following way: first, the minimal variance for both problems is identical. Second, if a strategy ψ^* is a solution to MV$(d, v - p + \pi_0(X))$, then a strategy $\phi^* = \psi^* - \phi^X$ is a solution to the original problem MV(d, v, p, X).

Corollary 2.1 *Suppose that an $\mathcal{F}_T$-measurable random variable X represents an $\mathbb{F}$-attainable claim.* (i) *If the arbitrage price $\pi_0(X)$ satisfies $\pi_0(X) \in [0, v]$ then $p^{d,v}(X) = \pi_0(X)$.*
(ii) *If the arbitrage price $\pi_0(X)$ is strictly greater than v then $p^{d,v}(X) = v$.*

Proof. By definition, the mean-variance price of X is the maximal value of $p \in [0, v]$ for which $\mathbf{v}^*(d, v, p, X) = \mathbf{v}^*(d, \widehat{v}) \le \mathbf{v}^*(d, v)$. Recall that we assume that $d > v$ so that, in view of (2.17),

$$\mathbf{v}^*(d, v) = \frac{(d - v)^2}{e^{\theta^2 T} - 1}.$$

By applying this result to $\mathrm{MV}(d, \widehat{v})$ we obtain

$$\mathbf{v}^*(d, v, p, X) = \frac{(d - v + p - \pi_0(X))^2}{e^{\theta^2 T} - 1}$$

provided that $d > v - p + \pi_0(X)$. Assume that $p > \pi_0(X)$. Then $d > v - p + \pi_0(X)$ and thus $\mathbf{v}^*(d, v, p, X) > \mathbf{v}^*(d, v)$ since manifestly $(d - v)^2 > (d - v + p - \pi_0(X))^2$ in this case. This shows that $p^{d,v}(X) \le \pi_0(X)$. Of course, for $p = \pi_0(X)$ we have the equality of minimal variances. We conclude that $p^{d,v}(X) = \pi_0(X)$ provided that $\pi_0(X) \in [0, v]$. This completes the proof of part (i).

To prove part (ii), let us assume that $\pi_0(X) > v$. In this case, it suffices to take $p = v$ and to check that $\mathbf{v}^*(d, v, v, X) = \mathbf{v}^*(d, \pi_0(X)) \le \mathbf{v}^*(d, v)$. This is again rather obvious since for $v < \pi_0(X) < d$ we have $(d - \pi_0(X))^2 < (d - v)^2$, and for $\pi_0(X) \ge d$ we have $\mathbf{v}^*(d, \pi_0(X)) = 0$. $\square$

Case of a Generic Claim

Consider an arbitrary $\mathcal{G}_T$-measurable claim X, which is $\mathbb{P}$-square-integrable. Recall that our goal is to solve the following problem for $0 \le p \le v$.

Problem MV(d, v, p, X)**:** Minimize $\mathbb{V}_{\mathbb{P}}(V_T^{v-p}(\phi) + X)$ over all trading strategies $\phi \in \Phi(\mathbb{F})$, subject to $\mathbb{E}_{\mathbb{P}}(V_T^{v-p}(\phi) + X) \ge d$.

Let us denote by $\widetilde{X}$ the conditional expectation $\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_T)$. Then, of course, $\mathbb{E}_{\mathbb{P}}\widetilde{X} = \mathbb{E}_{\mathbb{P}}X$. Moreover, $\widetilde{X}$ is an attainable claim and its arbitrage price at time 0 equals

$$\pi_0(\widetilde{X}) = \mathbb{E}_{\mathbb{Q}}\widetilde{X} = \mathbb{E}_{\mathbb{P}}(\eta_T \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_T)) = \mathbb{E}_{\mathbb{P}}(\eta_T X) = \mathbb{E}_{\mathbb{Q}}X,$$

where $\mathbb{Q}$ is the martingale measure introduced in Section 2.2.1. Let $\phi^{\widetilde{X}}$ stand for the replicating strategy for $\widetilde{X}$ in the class $\Phi(\mathbb{F})$. Arguing as in the previous case, we conclude that the problem $\mathrm{MV}(d, v, p, X)$ is equivalent to the following problem. We set here $\widetilde{p} = p - \pi_0(\widetilde{X})$.

Problem MV$(d, v, \widetilde{p}, X - \widetilde{X})$**:** Minimize $\mathbb{V}_{\mathbb{P}}(V_T^{v-\widetilde{p}}(\phi) + X - \widetilde{X})$ over all trading strategies $\phi \in \Phi(\mathbb{F})$, subject to $\mathbb{E}_{\mathbb{P}}(V_T^{v-\widetilde{p}}(\phi) + X - \widetilde{X}) \ge d$.

Recall that $\mathbb{E}_{\mathbb{P}}\widetilde{X} = \mathbb{E}_{\mathbb{P}}X$ and denote $\gamma_X = \mathbb{V}_{\mathbb{P}}(X - \widetilde{X})$. Observe that for any $\phi \in \Phi(\mathbb{F})$ we have

$$\mathbb{V}_{\mathbb{P}}(V_T^{v-\widetilde{p}}(\phi) + X - \widetilde{X}) = \mathbb{V}_{\mathbb{P}}(V_T^{v-\widetilde{p}}(\phi)) + \mathbb{V}_{\mathbb{P}}(X - \widetilde{X}) = \mathbb{V}_{\mathbb{P}}(V_T^{v-\widetilde{p}}(\phi)) + \gamma_X.$$

The problem $\mathrm{MV}(d, v, \widetilde{p}, X - \widetilde{X})$ can thus be represented as follows. We denote $\widetilde{v} = v - \widetilde{p}$.

Problem MV$(d, \widetilde{v}; \gamma_X)$**:** Minimize $\mathbb{V}_{\mathbb{P}}(V_T^{\widetilde{v}}(\phi)) + \gamma_X$ over all trading strategies $\phi \in \Phi(\mathbb{F})$, subject to $\mathbb{E}_{\mathbb{P}}(V_T^{\widetilde{v}}(\phi)) \geq d$.

Observe that the problem $\mathrm{MV}(d, \widetilde{v}; \gamma_X)$ is formally equivalent to the original problem $\mathrm{MV}(d, v, p, X)$ in the following sense: first, the minimal variances for both problems are identical, more precisely, we have

$$\mathbf{v}^*(d, v, p, X) = \mathbf{v}^*(d, \widetilde{v}) + \gamma_X,$$

where $\mathbf{v}^*(d, \widetilde{v})$ is the minimal variance for $\mathrm{MV}(d, \widetilde{v})$. Second, if a strategy ψ^* is a solution to problem $\mathrm{MV}(d, \widetilde{v})$, then $\phi^* = \psi^* - \phi^{\widetilde{X}}$ is a solution to $\mathrm{MV}(d, v, p, X)$.

Remark. It is interesting to notice that a solution $\mathrm{MV}(d, \widetilde{v}; \gamma_X)$ does not depend explicitly on the expected value of X under $\mathbb{P}$. Hence, the minimal variance for the problem $\mathrm{MV}(d, v, p, X)$ is independent of μ_X as well, but, of course, it depends on the price $\pi_0(\widetilde{X}) = \mathbb{E}_{\mathbb{Q}} X$, which may in fact coincide with μ_X under some circumstances.

In view of the arguments above, it suffices to consider the problem $\mathrm{MV}(d, \widetilde{v})$, where $\widetilde{v} = v - p + \mathbb{E}_{\mathbb{Q}} X$. Since the problem of this form has been already solved in Section 2.2.1, we are in a position to state the following result, which is an immediate consequence of Proposition 2.3. Recall that $\rho(\theta) = e^{\theta^2 T}(e^{\theta^2 T} - 1)^{-1}$ and $\eta_t(\theta) = \eta_t e^{-\theta^2 t}$, so that $\eta_0(\theta) = 1$. Finally, $\widetilde{v} = v - p + \mathbb{E}_{\mathbb{Q}} X = v - p + \mathbb{E}_{\mathbb{Q}} \widetilde{X}$.

Proposition 2.4 *Assume that* $\theta \neq 0$. (i) *Suppose that* $d > \widetilde{v}$. *Then a solution* $\phi^*(d, v, p, X)$ *to* $MV(d, v, p, X)$ *is given as* $\phi^*(d, v, p, X) = \psi^*(d, \widetilde{v}) - \phi^{\widetilde{X}}$, *where* $\psi^*(d, \widetilde{v}) = (\psi^{1*}(d, \widetilde{v}), \psi^{2*}(d, \widetilde{v}))$ *is such that* $\psi^{1*}(d, \widetilde{v})$ *equals*

$$\psi_t^{1*}(d, \widetilde{v}) = (d - \widetilde{v})\rho(\theta) \frac{\nu \eta_t(\theta)}{\sigma^2 Z_t^1} \tag{2.14}$$

and $\psi^{2*}(d, \widetilde{v})$ *satisfies* $\psi_t^{*1}(d, \widetilde{v}) Z_t^1 + \psi_t^{*2}(d, \widetilde{v}) = V_t^*(d, \widetilde{v})$ *for* $t \in [0, T]$, *where in turn*

$$V_t^*(d, \widetilde{v}) = \widetilde{v} + (d - \widetilde{v})\rho(\theta)\big(1 - \eta_t(\theta)\big). \tag{2.15}$$

Thus the optimal wealth for the problem $MV(d, v, p, X)$ *equals*

$$V_t^*(d, v, p, X) = v - p + (d - \widetilde{v})\rho(\theta)\big(1 - \eta_t(\theta)\big) + \mathbb{E}_{\mathbb{Q}} \widetilde{X} - \mathbb{E}_{\mathbb{Q}}(\widetilde{X} \,|\, \mathcal{F}_t) \tag{2.16}$$

and the minimal variance $\mathbf{v}^*(d, v, p, X)$ *is given by*

$$\mathbf{v}^*(d, v, p, X) = \frac{(d - \widetilde{v})^2}{e^{\theta^2 T} - 1} + \gamma_X. \tag{2.17}$$

(ii) *If* $d \leq \widetilde{v}$ *then the optimal wealth process equals*

$$V_t^*(d, v, p, X) = v - p + \mathbb{E}_{\mathbb{Q}} \widetilde{X} - \mathbb{E}_{\mathbb{Q}}(\widetilde{X} \,|\, \mathcal{F}_t)$$

and the minimal variance equals γ_X.

Remark. Let us comment briefly on the assumption $\theta \neq 0$. Recall that if it fails to hold, the problem $\mathrm{MV}(d, \widetilde{v})$ has no solution, unless $d = \widetilde{v}$. Hence, for $\theta = 0$ we need to postulate that $d = v - p + \mathbb{E}_{\mathbb{P}} X$ (recall that $\theta = 0$ if and only if $\mathbb{Q} = \mathbb{P}$). The optimal strategy $\phi^* = (0, 1)$ and thus the solution to $\mathrm{MV}(d, v, p, X)$ is exactly the same as in part (ii) of Proposition 2.4.

Mean-Variance Pricing and Hedging of a Generic Claim

Our next goal is to provide explicit representations for the mean-variance price of X. We maintain the assumption that the problem $\mathrm{MV}(d, v, p, X)$ is examined in the class $\Phi(\mathbb{F})$. Thus, the mean-variance price considered in this section, denoted as $p_{\mathbb{F}}^{d,v}(X)$ in what follows, is relative to the reference filtration $\mathbb{F}$.

Assume that $d > \widetilde{v} = v - p + \mathbb{E}_{\mathbb{Q}} X$ (recall that $\mathbb{E}_{\mathbb{Q}} X = \mathbb{E}_{\mathbb{Q}} \widetilde{X} = \pi_0(\widetilde{X})$). Then, by virtue of Proposition 2.4, we see that the minimal variance for the problem $\mathrm{MV}(d, v, p, X)$ equals

$$\mathbf{v}^*(d, v, p, X) = \frac{(d - v + p - \mathbb{E}_{\mathbb{Q}} X)^2}{e^{\theta^2 T} - 1} + \gamma_X,$$

where

$$\gamma_X = \mathbb{V}_{\mathbb{P}}(X - \widetilde{X}).$$

Of course, if $d \leq \widetilde{v} = v - p + \mathbb{E}_{\mathbb{Q}} X$ then we have $\mathbf{v}^*(d, v, p, X) = \gamma_X$. Recall that we postulate that $d > v$, and thus the minimal variance for the problem $\mathrm{MV}(d, v)$ equals

$$\mathbf{v}^*(d, v) = \frac{(d - v)^2}{e^{\theta^2 T} - 1}.$$

Let us denote

$$\kappa = d - v - \mathbb{E}_{\mathbb{Q}} X, \quad \rho = (d - v)^2 - \gamma_X (e^{\theta^2 T} - 1).$$

Proposition 2.5 (i) *Suppose that $\pi_0(\widetilde{X}) \geq d$ so that $\kappa \leq -v$. If $\gamma_X \leq \mathbf{v}^*(d, v)$ then the mean variance price equals $p_{\mathbb{F}}^{d,v}(X) = v$. Otherwise, $p_{\mathbb{F}}^{d,v}(X) = -\infty$.*
(ii) *Suppose that $d - v \leq \pi_0(\widetilde{X}) < d$ so that $-v < \kappa \leq 0$. If, in addition, $\rho \geq 0$ then we have*

$$p_{\mathbb{F}}^{d,v}(X) = \min\{-\kappa + \sqrt{\rho}, v\} \vee 0. \tag{2.18}$$

Otherwise, i.e., when $\rho < 0$, we have $p_{\mathbb{F}}^{d,v}(X) = -\kappa$ if $\gamma_X \leq \mathbf{v}^*(d, v)$, and $p^{d,v}(X) = -\infty$ if $\gamma_X > \mathbf{v}^*(d, v)$.
(iii) *Suppose that $\pi_0(\widetilde{X}) < d - v$ so that $\kappa > 0$. If $\rho \geq 0$ then $p_{\mathbb{F}}^{d,v}(X)$ is given by* (2.18). *Otherwise, we have $p_{\mathbb{F}}^{d,v}(X) = -\infty$.*

Proof. In case (i), we have $d - v - \mathbb{E}_{\mathbb{Q}} X \leq -p$ for every $p \in [0, v]$. Thus $d \leq v - p + \mathbb{E}_{\mathbb{Q}} X$, so that $\mathbf{v}^*(d, v, p, X) = \gamma_X$. Therefore, if $\gamma_X \leq \mathbf{v}^*(d, v)$ it is clear that $p_{\mathbb{F}}^{d,v}(X) = v$. Otherwise, for every $p \in [0, v]$ we have $\mathbf{v}^*(d, v, p, X) = \gamma_X > \mathbf{v}^*(d, v)$ and thus $p_{\mathbb{F}}^{d,v}(X) = -\infty$.

In case (ii), it suffices to notice that $d \le v - p + \mathbb{E}_{\mathbb{Q}} X$ for any $p \in [0, -\kappa]$, and $d > v - p + \mathbb{E}_{\mathbb{Q}} X$ for any $p \in (-\kappa, v]$. Thus the maximal $p \in [0, v]$ for which $\mathbf{v}^*(d, v, p, X) \le \mathbf{v}^*(d, v)$ can be found from the equation

$$(\kappa + p)^2 + \gamma_X \big(e^{\theta^2 T} - 1\big) = (d - v)^2,$$

which admits the solution $p = -\kappa + \sqrt{\rho}$ provided that $\rho \ge 0$. If $\rho < 0$, then we need to examine the case $p \in [0, -\kappa]$, and we see that $p_{\mathbb{F}}^{d,v}(X)$ equals either $-\kappa$ or $-\infty$, depending on whether $\gamma_X \le \mathbf{v}^*(d, v)$ or $\gamma_X > \mathbf{v}^*(d, v)$.

In case (iii), we have $d - v - \mathbb{E}_{\mathbb{Q}} X > 0$, which yields $d > v - p + \mathbb{E}_{\mathbb{Q}} X$ for any $p \in [0, v]$. Inequality $\mathbf{v}^*(d, v, p, X) \le \mathbf{v}^*(d, v)$ becomes

$$(d - v + p - \mathbb{E}_{\mathbb{Q}} X)^2 + \gamma_X \big(e^{\theta^2 T} - 1\big) \le (d - v)^2$$

If $\rho \ge 0$ then $p_{\mathbb{F}}^{d,v}(X)$ is given by (2.18). Otherwise, we have $p_{\mathbb{F}}^{d,v}(X) = -\infty$. $\square$

The mean variance hedging strategy for a claim X is now obtained as $\phi^{\mathrm{MV}} = \phi^*(d, v, p_{\mathbb{F}}^{d,v}(X), X)$ for all cases above when $p_{\mathbb{F}}^{d,v}(X) \ne -\infty$.

2.2.3 Defaultable Claims

In order to provide a better intuition, we shall now examine in some detail two special cases. First, we shall assume that X is independent of the σ-field $\mathcal{F}_T$. Since X is $\mathcal{G}_T$-measurable, but obviously it is not $\mathcal{G}_T$-measurable, we shall refer to X as a *defaultable claim* (a more general interpretation of X is possible, however).

Although this case may look rather trivial at the first glance, we shall see that some interesting conclusions can be obtained. Second, we shall analyze the case of a defaultable zero-coupon bond with fractional recovery of Treasury value. Of course, both examples are merely simple illustrations of Proposition 2.4, and thus they should not be considered as real-life applications.

Claim Independent of the Reference Filtration

Consider a $\mathcal{G}_T$-measurable contingent claim X, such that X is independent of the σ-field $\mathcal{F}_T$. Then for any strategy $\phi \in \Phi(\mathbb{F})$, the terminal wealth $V_T(\phi)$ and the payoff X are independent random variables, so that

$$\mathbb{V}_{\mathbb{P}}\big(V_T(\phi) + X\big) = \mathbb{V}_{\mathbb{P}}(V_T(\phi)) + \mathbb{V}_{\mathbb{P}}(X).$$

It is clear that if the variance $\mathbb{V}_{\mathbb{P}}(X)$ satisfies $\mathbb{V}_{\mathbb{P}}(X) > \mathbf{v}^*(d, v)$, then $p_{\mathbb{F}}^{d,v}(X) = -\infty$ for every $v > 0$. Moreover, if $\mathbb{V}_{\mathbb{P}}(X) \le \mathbf{v}^*(d, v)$ and $\mathbb{E}_{\mathbb{P}} X \ge d$, then $p^{d,v}(X) = v$ for every $v > 0$.

It thus remains to examine the case when $\mathbb{V}_{\mathbb{P}}(X) \le \mathbf{v}^*(d, v)$ and $\mathbb{E}_{\mathbb{P}} X < d$. Notice that $\widetilde{X} = \mathbb{E}_{\mathbb{P}} X$ and thus $\pi_0(\widetilde{X}) = \mathbb{E}_{\mathbb{P}} X$. In particular, since $\widetilde{X}$ is constant, its replicating strategy is trivial, i.e. $\phi^{\widetilde{X}} = 0$.

In view of Proposition 2.4, if $d > v - p + \mathbb{E}_{\mathbb{P}}X$ then the minimal variance for the problem $\mathrm{MV}(d, v, p, X)$ equals

$$\mathbf{v}^*(d, v, p, X) = \frac{(d - v + p - \mu_X)^2}{e^{\theta^2 T} - 1} + \sigma_X^2,$$

where $\mu_X = \mathbb{E}_{\mathbb{P}}X$ and $\sigma_X^2 = \mathbb{V}_{\mathbb{P}}(X) = \gamma_X$. Let us denote

$$\widetilde{p}^{d,v}(X) = -d + v + \mu_X + \sqrt{(d - v)^2 - \sigma_X^2(e^{\theta^2 T} - 1)}.$$

Proposition 2.6 *The mean variance price of the claim X equals*

$$p_{\mathbb{F}}^{d,v}(X) = \min\{\widetilde{p}^{d,v}(X), v\} \vee 0$$

if $(d - v)^2 - \sigma_X^2(e^{\theta^2 T} - 1) \geq 0$, and $-\infty$ otherwise. The mean-variance hedging strategy $\phi^{MV} = \psi^*$, where ψ^* is such that

$$\psi_t^{1*} = e^{\theta^2(T-t)} \frac{d - v + p^{d,v}(X) - \mu_X}{e^{\theta^2 T} - 1} \frac{\nu}{\sigma^2} \frac{\eta_t}{Z_t^1}, \quad \forall\, t \in [0, T].$$

The mean-variance price depends, of course, on the initial value v of the investor's capital. This dependence has very intuitive and natural properties, though. Let us denote

$$k = d - \sqrt{(d - \mu_X)^2 + \sigma_X^2(e^{\theta^2 T} - 1)}, \quad l = d - \sigma_X \sqrt{e^{\theta^2 T} - 1}.$$

We fix all parameters, except for v. Notice that the function $p(v) = p_{\mathbb{F}}^{d,v}(X)$ is non-negative and finite for $v \in [0, l \vee 0]$. Moreover, the function $p(v)$ is increasing for $v \in [0, k \vee 0)$, and it is decreasing on the interval $[k \vee 0, l \vee 0]$. Specifically,

$$p(v) = \begin{cases} v, & \text{if } 0 \leq v < k \vee 0, \\ \mu_X - d + v + \sqrt{(d - v)^2 - \sigma_X^2(e^{\theta^2 T} - 1)}, & \text{if } k \vee 0 \leq v \leq l \vee 0\,. \end{cases}$$

This conclusion is quite intuitive: once the initial level of investor's capital is big enough (that is, $v \geq l$) the investor is less and less interested in purchasing the claim X. This is because when the initial endowment is sufficiently close to the expected terminal wealth level, the investor has enough leverage to meet this terminal objective at minimum risk; therefore, the investor is increasingly reluctant to purchase the claim X as this would introduce unwanted additional risk (unless of course $\sigma_X = 0$). For example, if $v = d$ then the investor is not at all interested in purchasing the claim ($p_{\mathbb{F}}^{v,v}(X) = -\infty$ if $\sigma_X > 0$ and $\theta \neq 0$). For further properties of the mean-variance price of a claim X independent of $\mathcal{F}_T$, we refer to Bielecki and Jeanblanc (2003).

Defaultable Bond

Let τ be a random time on the underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We define the indicator process H associated with τ by setting $H_t = \mathbb{1}_{\{\tau \leq t\}}$ for $t \in \mathbb{R}_+$, and we denote by $\mathbb{H}$ the natural filtration of H ($\mathbb{P}$-completed). We take $\mathbb{H}$ to serve as the auxiliary filtration, so that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. We assume that the default time τ is defined as follows:

$$\tau = \inf \{ t \in \mathbb{R}_+ \,:\, \Gamma_t > \zeta \}, \tag{2.19}$$

where Γ is an increasing, $\mathbb{F}$-adapted process, with $\Gamma_0 = 0$, and ζ is an exponentially distributed random variable with parameter 1, independent of $\mathbb{F}$. It is well known that any Brownian motion W with respect to $\mathbb{F}$ is also a Brownian motion with respect to $\mathbb{G}$ within the present setup (the latter property is closely related to the so-called hypothesis (H) frequently used in the modeling of default event, see Jeanblanc and Rutkowski (2000) or Bielecki et al. (2004)).

Now, suppose that a new investment opportunity becomes available for the agent. Namely, the agent may purchase a defaultable bond that matures at time $T \in (0, T^*]$. We postulate that the terminal payoff at time T of the bond is $X = L\mathbb{1}_{\{\tau > T\}} + \delta L \mathbb{1}_{\{\tau \le T\}}$, where $L > 0$ is the bond's notional amount and $\delta \in [0, 1)$ is the (constant) recovery rate. In other words, we deal with a defaultable zero-coupon bond that is subject to the fractional recovery of Treasury value.

Notice that the payoff X can be represented as follows $X = \delta L + Y$, where $Y = L(1-\delta)\mathbb{1}_{\{\tau > T\}}$. According to our general definition, we associate to X an $\mathcal{F}_T$-measurable random variable $\widetilde{X}$ by setting

$$\widetilde{X} = \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_T) = \delta L + \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{F}_T).$$

In view of (2.19), we have

$$\mathbb{E}_{\mathbb{P}}(Y \,|\, F_T) = \mathbb{P}\{\tau > T \,|\, \mathcal{F}_T\} = e^{-\Gamma_T},$$

and thus the arbitrage price at time 0 of the attainable claim $\widetilde{X}$ equals (recall that we have reduced our problem to the case $r = 0$)

$$\pi_0(\widetilde{X}) = \mathbb{E}_{\mathbb{Q}}\widetilde{X} = \delta L + \mathbb{E}_{\mathbb{P}}\big(\eta_T e^{-\Gamma_T}\big).$$

Since clearly

$$X - \widetilde{X} = L(1-\delta)\big(\mathbb{1}_{\{\tau > T\}} - \mathbb{P}\{\tau > T \,|\, \mathcal{F}_T\}\big),$$

we obtain

$$\gamma_X = \mathbb{V}_{\mathbb{P}}(X - \widetilde{X}) = L^2(1-\delta)^2\, \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_{\{\tau > T\}} - e^{-\Gamma_T}\big)^2.$$

In order to find the mean-variance price $p^{d,v}_{\mathbb{F}}(X)$ at time 0 of a defaultable bond with respect to the reference filtration $\mathbb{F}$, it suffices to make use of Proposition 2.4 (or Proposition 2.5). If we wish to describe the mean-variance hedging strategy with respect to $\mathbb{F}$, we need also to know an explicit representation for the replicating strategy $\phi^{\widetilde{X}}$ for the claim $\widetilde{X}$. To this end, it suffices to find the integral representation of the random variable $\mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{F}_T)$ with respect to the price process Z^1 or, equivalently, to find a process $\phi^{\widetilde{X}}$ for which

$$\widetilde{X} = \pi_0(\widetilde{X}) + \int_0^T \phi_t^{\widetilde{X}} \, dZ_t^1.$$

Example 2.1 In practical applications of the reduced-form approach, it is fairly common to postulate that the $\mathbb{F}$-hazard process Γ is given as $\Gamma_t = \int_0^t \gamma_t \, dt$, where γ is a non-negative process, progressively measurable with respect to $\mathbb{F}$, referred to as the $\mathbb{F}$-intensity of default. Suppose, for the sake of simplicity, that the intensity of default γ is deterministic, and let us set

$$p_\gamma = \mathbb{P}\{\tau > T\} = \widetilde{\mathbb{Q}}\{\tau > T\} = \exp\Big(- \int_0^T \gamma(t) \, dt \Big).$$

Then we get

$$\pi_0(\widetilde{X}) = \mathbb{E}_{\mathbb{Q}} \widetilde{X} = \delta L + p_\gamma$$

and

$$\gamma_X = L^2 (1 - \delta)^2 p_\gamma \big(1 - p_\gamma\big).$$

Of course, in the case of a deterministic default intensity γ, in order to replicate the claim $\widetilde{X}$, it suffices to invest the amount $\pi_0(\widetilde{X})$ in the savings account. For a more detailed analysis of the mean-variance price of a defaultable bond, the reader may consult Bielecki and Jeanblanc (2003).

2.3 Strategies Adapted to the Full Filtration

In this section, the mean-variance hedging and pricing is examined in the case of trading strategies adapted to the full filtration. Recall that W is assumed to be a one-dimensional Brownian motion with respect to $\mathbb{F}$ under $\mathbb{P}$. We postulated, in addition, that W is also a Brownian motion with respect to the filtration $\mathbb{G}$ under the probability $\mathbb{P}$. We define a new probability $\widetilde{\mathbb{Q}}$ on $(\Omega, \mathcal{G}_{T^*})$ by setting

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\Big|_{\mathcal{G}_t} = \eta_t, \quad \forall\, t \in [0, T^*],$$

where the process η is given by (2.2). Clearly, $\widetilde{\mathbb{Q}}$ is an equivalent martingale probability for our primary market and the process η is a $\mathbb{G}$-martingale under $\mathbb{P}$. Moreover, we have (cf. (2.3))

$$\mathbb{E}_{\mathbb{P}}(\eta_T^2 \,|\, \mathcal{G}_t) = \eta_t^2 e^{\theta^2 (T-t)},$$

and thus $\mathbb{E}_{\mathbb{P}}(\eta_t^2) = \exp(\theta^2 t)$ for every $t \in [0, T^*]$. It is easy to check that the process $\widetilde{W_t} = W_t - \theta t$ is a martingale, and thus a Brownian motion, with respect to $\mathbb{G}$ under $\widetilde{\mathbb{Q}}$.

¿From the $\mathbb{P}$-square-integrability of η_T, it follows that for any strategy $\phi \in \Phi(\mathbb{G})$ the terminal wealth $V_T(\phi)$ is $\widetilde{\mathbb{Q}}$-integrable. In fact, we have the following useful result. Recall that a $\mathbb{G}$-predictable process ϕ^1 uniquely determines a self-financing strategy $\phi = (\phi^1, \phi^2)$, and thus we may formally identify ϕ^1 with the associated strategy ϕ (and vice versa). The following lemma will prove useful.

Lemma 2.4 *Let $\mathcal{A}(\widetilde{\mathbb{Q}})$ be the linear space of all $\mathbb{G}$-predictable processes ψ such that the process $\int_0^t \psi_u \, dZ_u^1$ is a $\widetilde{\mathbb{Q}}$-martingale and the integral $\int_0^T \psi_u \, dZ_u^1$ is in $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$. Then $\mathcal{A}(\widetilde{\mathbb{Q}}) = \Phi(\mathbb{G})$.*

Proof. It is clear that $\mathcal{A}(\widetilde{\mathbb{Q}}) \subseteq \Phi(\mathbb{G})$. For the proof of the inclusion $\Phi(\mathbb{G}) \subseteq \mathcal{A}(\widetilde{\mathbb{Q}})$, see Lemma 9 in Rheinländer and Schweizer (1997). $\square$

It is worthwhile to note that the class $\mathcal{A}(\widetilde{\mathbb{Q}})$ corresponds to the set Θ_{GLP} ($\widetilde{\Theta}$, respectively) considered in Schweizer (2001) (in Rheinländer and Schweizer (1997), respectively). The class $\Phi(\mathbb{G})$ corresponds with the class Θ_{S} (Θ, respectively) considered in Schweizer (2001) (in Rheinländer and Schweizer (1997), respectively).

Let us denote by $\mathbb{G}^1$ the filtration generated by all wealth processes:

$$V_t^v(\phi) = v + \int_0^t \phi_u^1 \, dZ_u^1,$$

where $v \in \mathbb{R}$ and $\phi = (\phi^1, \phi^2)$ belongs to $\Phi(\mathbb{G})$. Equivalently, $\mathbb{G}^1$ is generated by the processes

$$x + \int_0^t \psi_u \, dZ_u^1$$

with $x \in \mathbb{R}$ and $\psi \in \mathcal{A}(\widetilde{\mathbb{Q}})$. Also, we denote by $\mathcal{P}^0$ the following set of random variables:

$$\mathcal{P}^0 = \big\{ \xi \in L^2(\Omega, \mathcal{G}_T^1, \mathbb{P}) \,\big|\, \xi = \int_0^T \psi_u \, dZ_u^1, \ \psi \in \mathcal{A}(\widetilde{\mathbb{Q}}) \big\}.$$

We write $\Pi_{\mathbb{P}}^0$ to denote the orthogonal projection (in the norm of the space $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$) from $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ on the space $\mathcal{P}^0$. A similar notation will be also used for orthogonal projections on $\mathcal{P}^0$ under $\widetilde{\mathbb{Q}}$. Let us mention that, in general, we shall have $\Pi_{\mathbb{P}}^0(Y) \neq \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}_T^1)$ for $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ and $\Pi_{\widetilde{\mathbb{Q}}}^0(Y) \neq \mathbb{E}_{\widetilde{\mathbb{Q}}}(Y \,|\, \mathcal{G}_T^1)$ for $Y \in L^2(\Omega, \mathcal{G}_T, \widetilde{\mathbb{Q}})$ (see Section 2.3.3 for more details).

2.3.1 Solution to MV(d, v) in the Class $\Phi(\mathbb{G})$

Recall that our basic mean-variance problem has the following form:

Problem MV(d, v): Minimize $\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))$ over all strategies $\phi \in \Phi(\mathbb{G})$, subject to $\mathbb{E}_{\mathbb{P}} V_T^v(\phi) \geq d$.

As in Section 2.2.1, we postulate that $d > v$, since otherwise the problem is trivial. We shall argue that it suffices to solve a simpler problem:

Problem MVA(d, v)**:** Minimize $\mathbb{E}_{\mathbb{P}}(V_T^v(\phi))^2$ over all strategies $\phi \in \Phi(\mathbb{G})$, subject to $\mathbb{E}_{\mathbb{P}} V_T^v(\phi) = d$.

In view of the definition of class $\mathcal{A}(\widetilde{\mathbb{Q}})$, Lemma 2.4, and the fact that $\mathbb{E}_{\mathbb{Q}}\xi = 0$ for any $\xi \in \mathcal{P}^0$, we see that it suffices to solve the problem

Problem MVB(d, v)**:** Minimize $\mathbb{E}_{\mathbb{P}}(v + \xi)^2$ over all random variables $\xi \in \mathcal{P}^0$, subject to $\mathbb{E}_{\mathbb{P}}\xi = d - v$.

Solution to the last problem is exactly the same as in the case of strategies from $\Phi(\mathbb{F})$. Indeed, by solving the last problem in the class $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ (rather than in $\mathcal{P}^0$), and with additional constraint $\mathbb{E}_{\mathbb{Q}}\xi = 0$, we see that the optimal solution, given by (2.7), is in fact $\mathcal{F}_T$-measurable, and thus it belongs to the class $\mathcal{P}^0$ as well. In view of (2.8), the same random variable is a solution to $\mathrm{MV}(d, v)$, that is, it represents the optimal terminal wealth. We conclude that a solution to $\mathrm{MV}(d, v)$ in the class $\Phi(\mathbb{G})$ is given by the formulae (2.11)-(2.13) of Proposition 2.3, i.e., it coincides with a solution in the class $\Phi(\mathbb{F})$.

Assume that X is an attainable contingent claim, in the sense that there exists a trading strategy $\phi \in \Phi(\mathbb{G})$ which replicates X. Then, arguing along the same lines as in Section 2.2.2, we get the following result.

Corollary 2.2 *Let a $\mathcal{G}_T$-measurable random variable X represent an attainable contingent claim. Then*
(i) *If the arbitrage price $\pi_0(X)$ satisfies $\pi_0(X) \in [0, v]$ then $p^{d,v}(X) = \pi_0(X)$.*
(ii) *If the arbitrage price $\pi_0(X)$ is strictly greater than v then $p^{d,v}(X) = v$.*

2.3.2 Solution to MV(d, v, p, X) in the Class $\Phi(\mathbb{G})$

We shall study the problem $\mathrm{MV}(d, v, p, X)$ for an arbitrary $\mathcal{G}_T$-measurable claim X, which is $\mathbb{P}$-square-integrable. Recall that we deal with the following problem:

Problem MV(d, v, p, X)**:** Minimize $\mathbb{V}_{\mathbb{P}}(V_T^{v-p}(\phi) + X)$ over all trading strategies $\phi \in \Phi(\mathbb{G})$, subject to $\mathbb{E}_{\mathbb{P}}(V_T^{v-p}(\phi) + X) \geq d$.

Basic idea of solving the problem $\mathrm{MV}(d, v, p, X)$ with respect to $\mathbb{G}$-predictable strategies is similar to that used in the case of $\mathbb{F}$-predictable strategies. The main difference is that the auxiliary random variable $\widetilde{X}$ will now be defined as the orthogonal projection $\Pi_{\mathbb{P}}(X)$ of X on $\mathcal{P}^0$, rather than the conditional expectation $\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}_T)$.

Let us denote $\widehat{d} = d - v + p$. The problem $\mathrm{MV}(d, v, p, X)$ can be reformulated as follows:

Problem MV$(\widehat{d}, 0, 0, X)$**:** Minimize $\mathbb{V}_{\mathbb{P}}(V_T^0(\phi) + X)$ over all trading strategies $\phi \in \Phi(\mathbb{G})$, subject to $\mathbb{E}_{\mathbb{P}}(V_T^0(\phi) + X) \geq \widehat{d}$.

That is, if $V_T^{0,*}$ is the optimal wealth in problem $\mathrm{MV}(\widehat{d}, 0, 0, X)$ then $V_T^{v-p,*} = V_T^{0,*} + v - p$ is the optimal wealth in problem $\mathrm{MV}(d, v, p, X)$, and the optimal strategies as well as the optimal variances are the same in both problems.

Let $\widetilde{X}^0 = \Pi^0_{\mathbb{P}}(X)$ stand for the orthogonal projection of X on $\mathcal{P}^0$, so that $\psi^{\widetilde{X}^0}$ is a process from $\mathcal{A}(\widetilde{\mathbb{Q}}) = \Phi(\mathbb{G})$, for which

$$\widetilde{X}^0 = \int_0^T \psi_t^{1,\widetilde{X}^0} \, dZ_t^1 \tag{2.20}$$

and $X - \widetilde{X}^0 = X - \Pi^0_{\mathbb{P}}(X)$ is orthogonal to $\mathcal{P}^0$. The price of $\widetilde{X}^0$ equals

$$\pi_t(\widetilde{X}^0) = \int_0^t \psi_u^{1,\widetilde{X}^0} \, dZ_u^1 = \mathbb{E}_{\widetilde{\mathbb{Q}}}(\widetilde{X}^0 \,|\, \mathcal{G}_t), \quad \forall\, t \in [0,T]. \tag{2.21}$$

Let $\psi^{\widetilde{X}^0} \in \Phi(\mathbb{G})$ be a replicating strategy for the claim $\widetilde{X}^0$. Explicitly, $\psi^{\widetilde{X}^0} = (\psi^{1,\widetilde{X}^0}, \psi^{2,\widetilde{X}^0})$, where $\psi^{2,\widetilde{X}^0}$ satisfies $\psi_t^{1,\widetilde{X}^0} Z_t^1 + \psi_t^{2,\widetilde{X}^0} = \pi_t(\widetilde{X}^0)$. Notice that $\pi_0(\widetilde{X}^0) = \mathbb{E}_{\widetilde{\mathbb{Q}}} \widetilde{X}^0 = 0$ and, of course, $\pi_T(\widetilde{X}^0) = \widetilde{X}^0$. It thus suffices to consider the following problem:

Problem MV$(\widehat{d}, 0, 0, X - \widetilde{X}^0)$**:** Minimize $\mathbb{V}_{\mathbb{P}}(V_T^0(\phi) + X - \widetilde{X}^0)$ over all trading strategies $\phi \in \Phi(\mathbb{G})$, subject to $\mathbb{E}_{\mathbb{P}}(V_T^0(\phi) + X - \widetilde{X}^0) \geq \widehat{d}$.

Since $X - \widetilde{X}^0$ is orthogonal to $\mathcal{P}^0$, for any strategy $\phi \in \Phi(\mathbb{G})$ we have

$$\mathbb{V}_{\mathbb{P}}(V_T^0(\phi) + X - \widetilde{X}^0) = \mathbb{V}_{\mathbb{P}}(V_T^0(\phi)) + \mathbb{V}_{\mathbb{P}}(X - \widetilde{X}^0) = \mathbb{V}_{\mathbb{P}}(V_T^0(\phi)) + \gamma_X^0,$$

where $\gamma_X^0 = \mathbb{V}_{\mathbb{P}}(X - \widetilde{X}^0)$. Let us denote $\widetilde{d} = d - v + p - \mathbb{E}_{\mathbb{P}} X + \mathbb{E}_{\mathbb{P}} \widetilde{X}^0$. Then the problem MV$(\widehat{d}, 0, 0, X - \widetilde{X}^0)$ can thus be simplified as follows:

Problem MV$(\widetilde{d}, 0; \gamma_X^0)$**:** Minimize $\mathbb{V}_{\mathbb{P}}(V_T^0(\phi)) + \gamma_X^0$ over all trading strategies $\phi \in \Phi(\mathbb{G})$, subject to $\mathbb{E}_{\mathbb{P}}(V_T^0(\phi)) \geq \widetilde{d} = d - v + p - \mathbb{E}_{\mathbb{P}} X + \mathbb{E}_{\mathbb{P}} \widetilde{X}^0$.

Let us write $\widetilde{v} = v - p - \mathbb{E}_{\mathbb{P}} X + \mathbb{E}_{\mathbb{P}} \widetilde{X}^0$, so that $\widetilde{d} = d - \widetilde{v}$. Then the minimal variance for the problem MV(d, v, p, X) equals

$$\mathbf{v}^*(d, v, p, X) = \mathbf{v}^*(\widetilde{d}, 0) + \gamma_X^0 = \mathbf{v}^*(d, \widetilde{v}) + \gamma_X^0.$$

Moreover, if ψ^* is an optimal strategy to MV$(\widetilde{d}, 0)$, then $\phi^{1*} = \psi^{1*} - \psi^{\widetilde{X}^0}$ is a solution to MV(d, v, p, X). The proof of the next proposition is based on the considerations above, combined with Proposition 2.3. We use the standard notation $\rho(\theta) = e^{\theta^2 T}(e^{\theta^2 T} - 1)^{-1}$ and $\eta_t(\theta) = \eta_t e^{-\theta^2 t}$, so that $\eta_0(\theta) = 1$. Recall that $\mathbb{E}_{\widetilde{\mathbb{Q}}} \widetilde{X}^0 = 0$.

Proposition 2.7 *Assume that $\theta \neq 0$ and let $\psi^{\widetilde{X}^0} \in \Phi(\mathbb{G})$ be a replicating strategy for $\widetilde{X}^0 = \Pi^0_{\mathbb{P}}(X)$.*
(i) *Suppose that $d > \widetilde{v}$. Then an optimal strategy $\phi^*(d, v, p, X)$ for the problem MV(d, v, p, X) is given as $\phi^{1*}(d, v, p, X) = \psi^{1*}(\widetilde{d}, 0) - \psi^{1,\widetilde{X}^0}$ with $\psi^*(\widetilde{d}, 0) = (\psi^{1*}(\widetilde{d}, 0), \psi^{2*}(\widetilde{d}, 0))$ such that $\psi^{1*}(\widetilde{d}, 0)$ equals*

$$\psi_t^{1*}(\widetilde{d}, 0) = (d - \widetilde{v})\rho(\theta) \frac{\nu \eta_t(\theta)}{\sigma^2 Z_t^1} \tag{2.22}$$

and $\psi^{2*}(\widetilde{d},0)$ *satisfies* $\psi_t^{*1}(\widetilde{d},0)Z_t^1 + \psi^{*2}(\widetilde{d},0) = V_t^*(\widetilde{d},0)$*, where in turn*

$$V_t^*(\widetilde{d},0) = (d-\widetilde{v})\rho(\theta)\big(1-\eta_t(\theta)\big). \tag{2.23}$$

Thus the optimal wealth for the problem $MV(d,v,p,X)$ *equals*

$$V_t^*(d,v,p,X) = v - p + (d-\widetilde{v})\rho(\theta)\big(1-\eta_t(\theta)\big) - \mathbb{E}_{\widetilde{\mathbb{Q}}}(\widetilde{X}^0 \,|\, \mathcal{G}_t). \tag{2.24}$$

The minimal variance $\mathbf{v}^*(d,v,p,X)$ *is given by*

$$\mathbf{v}^*(d,v,p,X) = \frac{(d-\widetilde{v})^2}{e^{\theta^2 T}-1} + \gamma_X^0. \tag{2.25}$$

(ii) *If* $d \le \widetilde{v}$ *then the optimal wealth process equals*

$$V_t^*(d,v,p,X) = v - p - \mathbb{E}_{\widetilde{\mathbb{Q}}}(\widetilde{X}^0 \,|\, \mathcal{G}_t)$$

and the minimal variance equals γ_X^0.

Remark. It is natural to expect that the optimal variance given in (2.25) is not greater than the optimal variance given in (2.17). In fact, this is the case (see Proposition 5.4 in Bielecki and Jeanblanc (2003)).

Of course, the practical relevance of the last result hinges on the availability of explicit representation for the orthogonal projection $\widetilde{X}^0 = \Pi_{\mathbb{P}}^0(X)$ of X on the space $\mathcal{P}^0$. This important issue will be examined in the next section in a general setup. We shall continue the study of this question in the framework of defaultable claims in Section 2.3.5.

2.3.3 Projection of a Generic Claim

Let us first recall two well-known result concerning the decomposition of a $\mathcal{G}_T$-measurable random variable, which represents a generic contingent claim in our financial model.

Galtchouk-Kunita-Watanabe decomposition under $\widetilde{\mathbb{Q}}$. Suppose first that we work under $\mathbb{Q}$, so that the process Z^1 is a continuous martingale. Recall that by assumption W is a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{P}$; hence, the process $\widetilde{W}$ is a Brownian motion with respect to $\mathbb{G}$ under $\widetilde{\mathbb{Q}}$.

It is well known that any random variable $Y \in L^2(\Omega,\mathcal{G}_T,\widetilde{\mathbb{Q}})$ can be represented by means of the *Galtchouk-Kunita-Watanabe decomposition* with respect to the martingale Z^1 under $\widetilde{\mathbb{Q}}$. To be more specific, for any random variable $Y \in L^2(\Omega,\mathcal{G}_T,\widetilde{\mathbb{Q}})$ there exists a $\mathbb{G}$-martingale $N^{Y,\widetilde{\mathbb{Q}}}$, which is strongly orthogonal in the martingale sense to Z^1 under $\widetilde{\mathbb{Q}}$, and a $\mathbb{G}$-adapted process $\psi^{Y,\widetilde{\mathbb{Q}}}$, such that Y can be represented as follows:

$$Y = \mathbb{E}_{\widetilde{\mathbb{Q}}} Y + \int_0^T \psi_t^{Y,\widetilde{\mathbb{Q}}}\, dZ_t^1 + N_T^{Y,\widetilde{\mathbb{Q}}}. \tag{2.26}$$

Furthermore, the process $\psi^{Y,\widetilde{\mathbb{Q}}}$ can be represented as follows:

$$\psi_t^{Y,\widetilde{\mathbb{Q}}} = \frac{d\langle \mathcal{Y}, Z^1 \rangle_t}{d\langle Z^1 \rangle_t}\,, \tag{2.27}$$

where the $\mathbb{G}$-martingale $\mathcal{Y}$ is defined as $\mathcal{Y}_t = \mathbb{E}_{\widetilde{\mathbb{Q}}}(Y \,|\, \mathcal{G}_t)$.

Föllmer-Schweizer decomposition under $\mathbb{P}$. Let us now consider the same issue, but under the original probability $\mathbb{P}$. The process Z^1 is a (continuous) semimartingale with respect to $\mathbb{G}$ under $\mathbb{P}$, and thus it admits a unique continuous martingale part under $\mathbb{P}$.

Any random variable $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ can be represented by means of the *Föllmer-Schweizer decomposition*. Specifically, there exists a $\mathbb{G}$-adapted process $\psi^{Y,\mathbb{P}}$, a $(\mathbb{G}, \mathbb{P})$-martingale $N^{Y,\mathbb{P}}$, strongly orthogonal in the martingale sense to the continuous martingale part of Z^1, and a constant $y^{Y,\mathbb{P}}$, so that

$$Y = y^{Y,\mathbb{P}} + \int_0^T \psi_t^{Y,\mathbb{P}}\, dZ_t^1 + N_T^{Y,\mathbb{P}}. \tag{2.28}$$

We shall see that it will be not necessary to compute the process $\psi^{Y,\mathbb{P}}$ for the purpose of finding a hedging strategy for the problem considered in this section.

Projection on $\mathcal{P}^0$. As already mentioned, $\Pi^0_{\widetilde{\mathbb{Q}}}(Y) \neq \mathbb{E}_{\widetilde{\mathbb{Q}}}(Y \,|\, \mathcal{G}^1_T)$ for $Y \in L^2(\Omega, \mathcal{G}_T, \widetilde{\mathbb{Q}})$, as well as $\Pi^0_{\mathbb{P}}(Y) \neq \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}^1_T)$ for $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{P})$, in general. For instance, for any random variable Y as in (2.26) we get $\Pi^0_{\widetilde{\mathbb{Q}}}(Y) = \int_0^T \psi_t^{Y,\widetilde{\mathbb{Q}}}\, dZ_t^1$, whereas

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}(Y \,|\, \mathcal{G}^1_T) = Y = \Pi^0_{\widetilde{\mathbb{Q}}}(Y) - \mathbb{E}_{\widetilde{\mathbb{Q}}} Y.$$

The projection $\Pi^0_{\widetilde{\mathbb{Q}}}(Y)$ differs here from the conditional expectation just by the expected value $\mathbb{E}_{\widetilde{\mathbb{Q}}} Y$. Consequently, we have $\Pi^0_{\widetilde{\mathbb{Q}}}(Y) = \mathbb{E}_{\widetilde{\mathbb{Q}}}(Y \,|\, \mathcal{G}^1_T)$ for any $Y \in L^2(\Omega, \mathcal{G}_T, \widetilde{\mathbb{Q}})$ with $\mathbb{E}_{\widetilde{\mathbb{Q}}} Y = 0$. More importantly, observe that for Y as in (2.28) we shall have, in general,

$$\Pi^0_{\mathbb{P}}(Y) \neq \int_0^T \psi_t^{Y,\mathbb{P}}\, dZ_t^1,$$

so that, in particular, $\Pi^0_{\mathbb{P}}(Y) \neq \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}^1_T)$ even if $\mathbb{E}_{\mathbb{P}} Y = 0$.

Our next goal is to compute the projection $\Pi^0_{\mathbb{P}}(Y)$ for any random variable $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$. We know that any such Y can be represented as in (2.26). Due to linearity of the projection, it is enough to compute the projection of each component in the right-hand side of (2.26). Let us set $\widetilde{\eta}_t = \mathbb{E}_{\widetilde{\mathbb{Q}}}(\eta_T \,|\, \mathcal{G}_t)$ for every $t \in [0, T]$, so that, in particular, $\widetilde{\eta}_T = \eta_T$. Since $\widetilde{\eta}$ is a square-integrable $\mathbb{G}$-martingale under $\widetilde{\mathbb{Q}}$, there exists a process $\widetilde{\psi}$ in $\mathcal{A}(\widetilde{\mathbb{Q}})$ such that

$$\widetilde{\eta}_t = \mathbb{E}_{\widetilde{\mathbb{Q}}} \widetilde{\eta}_T + \int_0^t \widetilde{\psi}_u\, dZ_u^1 = \mathbb{E}_{\widetilde{\mathbb{Q}}} \widetilde{\eta}_T + Z_t^\eta, \quad \forall\, t \in [0, T], \tag{2.29}$$

where we denote

$$Z_t^\eta = \int_0^t \widetilde{\psi}_u \, dZ_u^1.$$

Lemma 2.5 *We have*

$$\widetilde{\psi}_t = -\frac{\theta \widetilde{\eta}_t}{\sigma Z_t^1} = -\frac{\theta e^{\theta^2 T}}{\sigma Z_t^1} \exp\big(- \theta \widetilde{W}_t - \tfrac{1}{2}\, \theta^2 (t - 2T) \big) \tag{2.30}$$

and the process $\widetilde{W}_t = W_t + \theta t$ *is a Brownian motion under* $\widetilde{\mathbb{Q}}$.

Proof. Direct calculations show that for every $t \in [0, T]$

$$\widetilde{\eta}_t = \exp\Big(- \frac{\theta}{\sigma} \int_0^t \frac{dZ_u^1}{Z_u^1} - \frac{1}{2}\, \theta^2 (t - 2T) \Big) = e^{\theta^2 T} \exp\big(- \theta \widetilde{W}_t - \tfrac{1}{2}\, \theta^2 t \big). \tag{2.31}$$

Hence, $\widetilde{\eta}$ solves the SDE

$$d\widetilde{\eta}_t = -\theta \widetilde{\eta}_t \, d\widetilde{W}_t = -\frac{\theta}{\sigma}\, \frac{\widetilde{\eta}_t}{Z_t^1}\, dZ_t^1$$

with the initial condition $\widetilde{\eta}_0 = \mathbb{E}_{\widetilde{\mathbb{Q}}} \widetilde{\eta}_T = \mathbb{E}_{\widetilde{\mathbb{Q}}} \eta_T = e^{\theta^2 T}$. □

In the next result, we provide a general representation for the projection $\Pi_{\mathbb{P}}^0(Y)$ for a $\mathcal{G}_T$-measurable random variable Y, which is $\mathbb{P}$-square-integrable.

Proposition 2.8 *Let* $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{P})$. *Then we have*

$$\Pi_{\mathbb{P}}^0(Y) = \int_0^T \widetilde{\psi}_t^{Y,\mathbb{P}} \, dZ_t^1,$$

where

$$\widetilde{\psi}_t^{Y,\mathbb{P}} = \psi_t^{Y,\widetilde{\mathbb{Q}}} - \widetilde{\psi}_t \Big(\widetilde{\eta}_0^{-1} \, \mathbb{E}_{\widetilde{\mathbb{Q}}} Y + \int_0^t \widetilde{\eta}_u^{-1} \, dN_u^{Y,\widetilde{\mathbb{Q}}} \Big) \tag{2.32}$$

and where processes $\psi^{Y,\widetilde{\mathbb{Q}}}$ *and* $N^{Y,\widetilde{\mathbb{Q}}}$ *are given by the Galtchouk-Kunita-Watanabe decomposition* (2.26) *of* Y *under* $\widetilde{\mathbb{Q}}$.

Proof. First, we compute projection of the constant $c = \mathbb{E}_{\widetilde{\mathbb{Q}}} Y$. To this end, recall that $\widetilde{\eta}_T = \eta_T$ and by virtue of (2.29) we have $\widetilde{\eta}_T = \widetilde{\eta}_0 + Z_T^\eta$. Hence, for any $\psi \in \mathcal{A}(\widetilde{\mathbb{Q}})$ we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \Big(\big(1 + \widetilde{\eta}_0^{-1} Z_T^\eta \big) \int_0^T \psi_t \, dZ_t^1 \Big) &= \widetilde{\eta}_0^{-1} \mathbb{E}_{\mathbb{P}} \Big(\eta_T \int_0^T \psi_t \, dZ_t^1 \Big) \\ &= \widetilde{\eta}_0^{-1} \mathbb{E}_{\widetilde{\mathbb{Q}}} \Big(\int_0^T \psi_t \, dZ_t^1 \Big) = 0, \end{aligned}$$

and thus $\Pi_{\mathbb{P}}^0(1) = -\widetilde{\eta}_0^{-1} Z_T^\eta$. We conclude that for any $c \in \mathbb{R}$

$$\Pi_{\mathbb{P}}^0(c) = c \, \Pi_{\mathbb{P}}^0(1) = -c \widetilde{\eta}_0^{-1} Z_T^\eta = -c \widetilde{\eta}_0^{-1} \int_0^T \widetilde{\psi}_t \, dZ_t^1. \tag{2.33}$$

Next, it is obvious that the projection of the second term, that is, the projection of $\int_0^T \psi_t^{Y,\widetilde{\mathbb{Q}}}\,dZ_t^1$, on $\mathcal{P}^0$ is equal to itself, so that

$$\Pi^0_{\mathbb{P}}\Big(\int_0^T \psi_t^{Y,\widetilde{\mathbb{Q}}}\,dZ_t^1\Big) = \int_0^T \psi_t^{Y,\widetilde{\mathbb{Q}}}\,dZ_t^1. \tag{2.34}$$

Finally, we shall compute the projection $\Pi^0_{\mathbb{P}}(N_T^{Y,\widetilde{\mathbb{Q}}})$. Recall that the process $N^{Y,\widetilde{\mathbb{Q}}}$ is a $\widetilde{\mathbb{Q}}$-martingale strongly orthogonal to Z^1 under $\widetilde{\mathbb{Q}}$. Hence, for any $N^{Y,\widetilde{\mathbb{Q}}}$-integrable process ν and any process $\psi \in \mathcal{A}(\widetilde{\mathbb{Q}})$ we have

$$\mathbb{E}_{\mathbb{P}}\Big(\eta_T \int_0^T \nu_t\,dN_t^{Y,\widetilde{\mathbb{Q}}} \int_0^T \psi_t\,dZ_t^1\Big) = 0.$$

Thus, it remains to find processes $\widehat{\nu}$ and $\widehat{\psi} \in \mathcal{A}(\widetilde{\mathbb{Q}})$ for which

$$\eta_T \int_0^T \widehat{\nu}_t\,dN_t^{Y,\widetilde{\mathbb{Q}}} = N_T^{Y,\widetilde{\mathbb{Q}}} - \int_0^T \widehat{\psi}_t\,dZ_t^1, \tag{2.35}$$

in which case we shall have that $\Pi^0_{\mathbb{P}}(N_T^{Y,\widetilde{\mathbb{Q}}}) = \int_0^T \widehat{\psi}_t\,dZ_t^1$.

Let us set $U_t = \widetilde{\eta}_t \int_0^t \nu_u\,dN_u^{Y,\widetilde{\mathbb{Q}}}$ for every $t \in [0,T]$. Recall that (see (2.29)) there exists a process $\widetilde{\psi}$ in $\Phi(\mathbb{G}) = \mathcal{A}(\widetilde{\mathbb{Q}})$ such that $d\widetilde{\eta}_t = \widetilde{\psi}_t\,dZ_t^1$. Using the product rule, and taking into account the orthogonality of $\widetilde{\eta}$ and $N^{Y,\widetilde{\mathbb{Q}}}$ under $\widetilde{\mathbb{Q}}$, we find that U is a local martingale under $\widetilde{\mathbb{Q}}$, and it satisfies

$$U_t = \int_0^t \widetilde{\eta}_{u-}\nu_u\,dN_u^{Y,\widetilde{\mathbb{Q}}} + \int_0^t \Big(\int_0^u \nu_s\,dN_s^{Y,\widetilde{\mathbb{Q}}}\Big)\widetilde{\psi}_u\,dZ_u^1. \tag{2.36}$$

Consequently, upon letting

$$\widehat{\nu}_t = (\widetilde{\eta}_{t-})^{-1}, \quad \forall\, t \in [0,T], \tag{2.37}$$

we obtain from (2.36)

$$U_t = N_t^{Y,\widetilde{\mathbb{Q}}} + \int_0^t \widetilde{\psi}_u\Big(\int_0^u \widehat{\nu}_s\,dN_s^{Y,\widetilde{\mathbb{Q}}}\Big)dZ_u^1. \tag{2.38}$$

Note that the left-hand side of (2.35) is equal to U_T. Thus, comparing (2.35) and (2.38), we see that we may take

$$\widehat{\psi}_t = -\widetilde{\psi}_t \int_0^t \widehat{\nu}_u\,dN_u^{Y,\widetilde{\mathbb{Q}}} = -\widetilde{\psi}_t \int_0^t (\widetilde{\eta}_{u-})^{-1}\,dN_u^{Y,\widetilde{\mathbb{Q}}}. \tag{2.39}$$

It is clear that with $\widehat{\nu}$ defined in (2.37) the integral $\int_0^t \widehat{\nu}_u\,dN_u^{Y,\widetilde{\mathbb{Q}}}$ is a $\widetilde{\mathbb{Q}}$-martingale. Thus, the process U is a martingale, rather than a local martingale, under $\widetilde{\mathbb{Q}}$. Together with (2.38) this implies that the process

$$\int_0^t \widetilde{\psi}_u\Big(\int_0^u \widehat{\nu}_s\,dN_s^{Y,\widetilde{\mathbb{Q}}}\Big)dZ_u^1$$

is a $\widetilde{\mathbb{Q}}$-martingale. Consequently, the process $\widehat{\psi}$ defined in (2.39) belongs to the class $\mathcal{A}(\widetilde{\mathbb{Q}})$. To complete the proof, it suffices to combine (2.33), (2.34) and (2.39). □

It should be acknowledged that the last result is not new. In fact, it is merely a special case of Theorem 6 in Rheinländer and Schweizer (1997). We believe, however, that our derivation of the result sheds a new light on the structure of the orthogonal projection computed above.

Remark. Although the above proposition provides us with the structure of the projection $\Pi^0_{\mathbb{P}}(Y)$, it is not easy in general to obtain closed-form expressions for the components on the right-hand side of (2.32) in terms of the initial data for the problem. Thus, one may need to resort to numerical approximations, which in principle can be obtained by solving the following problem

$$\min_{\xi \in \mathcal{P}^0} \mathbb{E}_{\mathbb{P}}(Y - \xi)^2. \tag{2.40}$$

An approximate solution to the last problem yields a process, say $\psi^{Y,\mathbb{P}}$, so that $\Pi^0_{\mathbb{P}}(Y) \approx \int_0^T \psi^{Y,\mathbb{P}}_t \, dZ^1_t$.

2.3.4 Mean-Variance Pricing and Hedging of a Generic Claim

Let us define

$$\widetilde{\kappa} = \widetilde{d} - v = d - v - \mathbb{E}_{\mathbb{P}} X + \mathbb{E}_{\mathbb{P}} \widetilde{X}^0.$$

For simplicity, we shall only consider the case when $\widetilde{\kappa} > 0$. This is equivalent to assuming that $\widetilde{d} > v - p$ for all $p \in [0, v]$. Thus, the results of Proposition 2.7 (i) apply. Consequently, denoting

$$\widetilde{\rho} = (d - v)^2 - \gamma^0_X (e^{\theta^2 T} - 1),$$

we obtain the following result.

Proposition 2.9 *Suppose that* $\gamma^0_X \leq (d - v)^2 (e^{\theta^2 T} - 1)^{-1}$. *Then the buyer's mean variance price is*

$$p^{d,v}(X) = \min\{-\widetilde{\kappa}_1 + \sqrt{\widetilde{\rho}}, v\} \vee 0. \tag{2.41}$$

Otherwise, $p^{d,v}(X) = -\infty$.

In case when $\gamma^0_X \leq (d-v)^2 (e^{\theta^2 T} - 1)^{-1}$, the mean-variance hedging strategy for a generic claim X is given by $\phi^*(d, v, p^{d,v}(X), X)$, where the process ϕ^* is defined in Proposition 2.7. The projection part of the strategy $\phi^*(d, v, p^{d,v}(X), X)$, that is, the process $\psi^{1,\widetilde{X}^0}$, can be computed according to (2.32).

2.3.5 Projections of Defaultable Claims

In this section, we adopt the framework of Section 2.2.3. In particular, the default time τ is a random time on $(\Omega, \mathcal{G}, \mathbb{P})$ given by formula (2.19), and the process H is given as $H_t = \mathbb{1}_{\{\tau \le t\}}$ for every $t \in [0,T]$. The natural filtration $\mathbb{H}$ of H is an auxiliary filtration, so that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Recall that we have assumed that τ admits the $\mathbb{F}$-hazard process Γ under $\mathbb{P}$ and thus also, in view of the construction (2.19), under $\widetilde{\mathbb{Q}}$. Suppose, in addition, that the hazard process Γ is an increasing continuous process. Then the process $M_t = H_t - \Gamma_{t\wedge\tau}$ is known to be a $\mathbb{G}$-martingale under $\widetilde{\mathbb{Q}}$. Any $\mathcal{G}_T$-measurable random variable X is referred to as a *defaultable claim*.

Recall that the process $\widetilde{W}_t = W_t + \theta t$ is a Brownian motion with respect to $\mathbb{F}$ under $\widetilde{\mathbb{Q}}$, and thus the process Z^1 is a square-integrable $\mathbb{G}$-martingale under $\widetilde{\mathbb{Q}}$, since

$$dZ^1_t = Z^1_t \sigma \, d\widetilde{W}_t, \quad Z^1_0 > 0.$$

The following proposition is an important technical result.

Proposition 2.10 *The filtration $\mathbb{G}^1$ is equal to the filtration $\mathbb{G}$, that is, $\mathcal{G}^1_t = \mathcal{G}_t$ for every $t \in \mathbb{R}_+$.*

Proof. It is clear that $\mathbb{G}^1 \subseteq \mathbb{G}$. For a fixed $T > 0$, let $y_1, y_2 \in \mathbb{R}$ and let the processes ψ^1, ψ^2 belong to $\mathcal{A}(\widetilde{\mathbb{Q}})$. Thus the processes

$$Y^1_t = y_1 + \int_0^t \psi^1_u \, dZ^1_u, \quad Y^2_t = y_2 + \int_0^t \psi^2_u \, dZ^1_u$$

be $\mathbb{G}^1$-adapted processes. Then the process

$$Y^1_t Y^2_t = y_1 y_2 + \int_0^t Y^1_u \psi^2_u \, dZ^1_u + \int_0^t Y^2_u \psi^1_u \, dZ^1_u + \int_0^t \psi^1_u \psi^2_u \, d\langle Z^1 \rangle_u$$

is also $\mathbb{G}^1$-adapted. It is easy to check that the processes

$$\int_0^t Y^1_u \psi^2_u \, dZ^1_u, \quad \int_0^t Y^2_u \psi^1_u \, dZ^1_u$$

are $\mathbb{G}^1$-adapted. We thus conclude that for any processes ϕ and ψ from $\mathcal{A}(\widetilde{\mathbb{Q}})$, the process

$$\int_0^t \psi^1_u \psi^2_u \, d\langle Z^1 \rangle_u = \int_0^t \psi^1_u \psi^2_u (Z^1_u)^2 \sigma^2 du$$

is $\mathbb{G}^1$-adapted as well. In particular, it follows that for any bounded $\mathbb{G}$-adapted process ζ the integral $\int_0^t \zeta_u \, du$ defines a $\mathbb{G}^1$-adapted process. Let us take $\zeta_u = H_u$. Then we obtain that the process $\tau \wedge t$ is $\mathbb{G}^1$-adapted. Hence, it is easily seen that $\mathcal{G}_t \subseteq \mathcal{G}^1_t$ for $t \in [0,T]$. Since T was an arbitrary positive number, we have shown that $\mathbb{G} = \mathbb{G}^1$. $\square$

Projection of a Survival Claim

We shall now compute the process $\psi^{Y,\mathbb{P}}$, which occurs in the projection $\Pi^0_{\mathbb{P}}(Y)$ for a random variable $Y = Z1\!\!1_{\{\tau>T\}}$, where $Z \in L^2(\Omega, \mathcal{F}_T, \widetilde{\mathbb{Q}})$. It is known that any random variable Y from $L^2(\Omega, \mathcal{G}_T, \widetilde{\mathbb{Q}}) = L^2(\Omega, \mathcal{G}^1_T, \widetilde{\mathbb{Q}})$, which vanishes on the set $\{\tau > T\}$, can indeed be represented in this way. Any random variable Y of the form $Z1\!\!1_{\{\tau>T\}}$ is referred to as a *survival claim* with maturity date T, and a random variable Z is said to be the *promised payoff* associated with Y.

It is known (see, e.g., Bielecki and Rutkowski (2004)) that

$$\begin{aligned}\mathbb{E}_{\widetilde{\mathbb{Q}}}(Y \,|\, \mathcal{G}_t) &= \mathbb{E}_{\widetilde{\mathbb{Q}}}(Z1\!\!1_{\{\tau>T\}} \,|\, \mathcal{G}_t) = \mathbb{E}_{\widetilde{\mathbb{Q}}}(Z1\!\!1_{\{\tau>T\}} \,|\, \mathcal{G}^1_t) \\ &= 1\!\!1_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\widetilde{\mathbb{Q}}}(Ze^{-\Gamma_T} \,|\, \mathcal{F}_t) = L_t m^Z_t,\end{aligned}$$

where $L_t := 1\!\!1_{\{\tau>t\}} e^{\Gamma_t}$ is a $\mathbb{G}$-martingale and $m^Z_t = \mathbb{E}_{\widetilde{\mathbb{Q}}}(Ze^{-\Gamma_T} \,|\, \mathcal{F}_t)$ is an $\mathbb{F}$-martingale. From the predictable representation theorem for a Brownian motion (or since the default-free market is complete), it follows that there exists an $\mathbb{F}$-adapted process μ^Z such that

$$m^Z_t = m^Z_0 + \int_0^t \mu^Z_u \, dZ^1_u. \tag{2.42}$$

In Proposition 2.8, we have already described the structure of the process $\psi^{Y,\mathbb{P}}$ that specifies the projection of Y on $\mathcal{P}^0$. In the next two results, we shall give more explicit formulae for $\psi^{Y,\widetilde{\mathbb{Q}}}$ and $N^{Y,\widetilde{\mathbb{Q}}}$ within the present setup.

Lemma 2.6 *Consider a survival claim $Y = Z1\!\!1_{\{\tau>T\}}$ with the promised payoff $Z \in L^2(\Omega, \mathcal{F}_T, \widetilde{\mathbb{Q}})$. It holds that $\psi^{Y,\widetilde{\mathbb{Q}}}_t = L_{t-}\mu^Z_t$ for every $t \in [0, T]$, where by convention $L_{0-} = 0$.*

Proof. It is easy to check that $dL_t = -L_{t-} dM_t$. Since Γ is increasing, the process L is of finite variation, and thus

$$d(L_t m^Z_t) = L_{t-}\, dm^Z_t + m^Z_t \, dL_t = L_{t-}\mu^Z_t \, dZ^1_t + m^Z_t \, dL_t,$$

and thus we obtain

$$d\langle \mathcal{Y}, Z^1 \rangle_t = L_{t-}\mu^Z_t \, d\langle Z^1 \rangle_t$$

and $\psi^{Y,\widetilde{\mathbb{Q}}}_t = L_{t-}\mu^Z_t$, which proves the result. □

For the proof of the next auxiliary result, the reader is referred, for instance, to Jeanblanc and Rutkowski (2000) or Bielecki and Rutkowski (2004).

Lemma 2.7 *Consider a survival claim $Y = Z1\!\!1_{\{\tau>T\}}$ with the promised payoff $Z \in L^2(\Omega, \mathcal{F}_T, \widetilde{\mathbb{Q}})$. The process $N^{Y,\widetilde{\mathbb{Q}}}$ in the Galtchouk-Kunita-Watanabe decomposition of Y with respect to Z^1 under $\widetilde{\mathbb{Q}}$ is given by the expression*

$$N^{Y,\widetilde{\mathbb{Q}}}_t = \int_{[0,t)} n^Z_u \, dM_u,$$

where the process $M_t = H_t - \Gamma_{t\wedge\tau}$ is a $\mathbb{G}$-martingale, strongly orthogonal in the martingale sense to $\widetilde{W}$ under $\widetilde{\mathbb{Q}}$, and where

$$n_t^Z = -\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(Ze^{\Gamma_t - \Gamma_T} \,\big|\, \mathcal{F}_t\big). \tag{2.43}$$

By combining Proposition 2.8 with the last two result, we obtain the following corollary, which furnishes an almost explicit representation for the process $\widetilde{\psi}^{Y,\mathbb{P}}$ associated with the projection on $\mathcal{P}^0$ of a survival claim.

Corollary 2.3 *Let $Y = Z1\!\!1_{\{\tau>T\}}$ be a survival claim, where Z belongs to $L^2(\Omega, \mathcal{F}_T, \widetilde{\mathbb{Q}})$. Then $\Pi^0_{\mathbb{P}}(Y)$ is given by the following expression*

$$\Pi^0_{\mathbb{P}}(Y) = \int_0^T \widetilde{\psi}_t^{Y,\mathbb{P}}\, dZ_t^1,$$

where for every $t \in [0,T]$

$$\widetilde{\psi}_t^{Y,\mathbb{P}} = L_{t-}\mu_t^Z - \widetilde{\psi}_t\Big(\widetilde{\eta}_0^{-1}\,\mathbb{E}_{\widetilde{\mathbb{Q}}}Y + \int_0^t \widetilde{\eta}_u^{-1}\, n_u^Z\, dM_u\Big) \tag{2.44}$$

where in turn $L_t = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}$ and the processes $\widetilde{\psi}, \widetilde{\eta}, \mu^Z$ and n^Z are given by (2.30), (2.31), (2.42) *and* (2.43)*, respectively.*

Projection of a Defaultable Bond

According to the adopted convention regarding the recovery scheme, the terminal payoff at time T of a defaultable bond equals $X = L1\!\!1_{\{\tau>T\}} + \delta L 1\!\!1_{\{\tau\le T\}}$ for some $L > 0$ and $\delta \in [0,1)$. Notice that the payoff X can be represented as follows $X = \delta L + (1-\delta)LY$, where $Y = 1\!\!1_{\{\tau>T\}}$ is a simple survival claim, with the promised payoff $Z = 1$. Using the linearity of the projection $\Pi^0_{\mathbb{P}}$, we notice that $\Pi^0_{\mathbb{P}}(X)$ can be evaluated as follows

$$\Pi^0_{\mathbb{P}}(X) = \delta L\, \Pi^0_{\mathbb{P}}(1) + (1-\delta)L\, \Pi^0_{\mathbb{P}}(Y).$$

By virtue of Corollary 2.3, we conclude that

$$\Pi^0_{\mathbb{P}}(X) = -\delta L e^{\theta^2 T}\, \Pi^0_{\widetilde{\mathbb{Q}}}(\eta_T) + (1-\delta)L \int_0^T \psi_t\, dZ_t^1,$$

where (cf. (2.44))

$$\psi_t = 1\!\!1_{\{\tau>t\}}e^{\Gamma_t}\mu_t - \widetilde{\psi}_t\Big(e^{-\theta^2 T}\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\big(e^{-\Gamma_T}\big) + \int_0^t \widetilde{\eta}_u^{-1}\, n_u\, dM_u\Big), \tag{2.45}$$

where in turn the process $\widetilde{\psi}$ is given by (2.30), the process n equals $n_t = -\mathbb{E}_{\widetilde{\mathbb{Q}}}(e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t)$, and the process μ is such that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}(e^{-\Gamma_T} \,|\, \mathcal{F}_t) = \mathbb{E}_{\widetilde{\mathbb{Q}}}Y + \int_0^t \mu_u\, dZ_u^1, \quad \forall\, t \in [0,T].$$

Example 2.2 Consider the special case when Γ is deterministic. It is easily seen that we now have $\mu = 0$ and $n_t = -e^{\Gamma_t - \Gamma_T}$. Consequently, (2.45) becomes

$$\psi_t = -\widetilde{\psi}_t e^{-\Gamma_T}\Big(e^{-\theta^2 T} - \int_0^t \widetilde{\eta}_u^{-1}\, e^{\Gamma_t}\, dM_u\Big),$$

and thus

$$\begin{aligned}\Pi^0_{\mathbb{P}}(X) &= -\delta L e^{\theta^2 T}\, \Pi^0_{\widetilde{\mathbb{Q}}}(\eta_T)\\ &\quad - (1-\delta)L \int_0^T \widetilde{\psi}_t e^{-\Gamma_T}\Big(e^{-\theta^2 T} - \int_0^t \widetilde{\eta}_u^{-1}\, e^{\Gamma_t}\, dM_u\Big) dZ^1_t,\end{aligned}$$

where the processes $\widetilde{\psi}$ and $\widetilde{\eta}$ are given by (2.30) and (2.31), respectively.

2.4 Risk-Return Portfolio Selection

In the preceding sections, we have examined the Markowitz-type mean-variance hedging problem from the particular perspective of valuation of non-attainable contingent claims. In view of the dependence of the mean-variance price obtained through this procedure on agent's preferences, (formally reflected, among others, by the values of parameters d and v), this specific application of Markowitz-type methodology suffers from deficiencies, which may undermine its practical implementations.

In this section, we shall take a totally different perspective, and we shall assume that a given claim X can be purchased by an agent (an asset management fund, say) for some pre-specified price. For instance, the price of X can be given by an investment bank that is able to hedge this claim using some arbitrage-free model, or it can be simply given by the OTC market.

Let us emphasize that an agent is now assumed to be a pricetaker, so that the issue of preference-based valuation of a non-attainable claim will not be considered in this section.

We postulate that an agent would like to invest in X, but will not be able (or willing) to hedge this claim using the underlying primary assets (if any such assets are available). As a consequence, an agent will only have in its portfolio standard instruments that are widely available for trading. The two important issues we would like to address in this section are:

- What proportion of the initial endowment v should an agent invest in the claim X if the goal is to lower the standard deviation (or, equivalently, the variance) of return, and to keep the expected rate of return at the desired level.

- How much should an agent invest in X in order to enhance the expected rate of return, and to preserve at the same time the pre-specified level of risk, as measured by the standard deviation of the rate of return.

We shall argue that mathematical tools and results presented in the previous sections are sufficient to solve both these problems. It seems to us that this alternative application of the mean-variance methodology can be of practical importance as well.

For the sake of simplicity, we shall solve the optimization problems formulated above in the class $\Phi(\mathbb{F})$ of $\mathbb{F}$-admissible trading strategies. A similar study can be conducted for the case of $\mathbb{G}$-admissible strategies. For any $v > 0$ and any trading strategy $\phi \in \Phi(\mathbb{F})$, let $r(\phi)$ be the simple rate of return, defined as

$$r(\phi) = \frac{V_T^v(\phi) - v}{v}.$$

The minimization of the standard deviation of the rate of return, which equals

$$\sigma(r(\phi)) = \sqrt{\mathbb{V}_{\mathbb{P}}\Big(\frac{V_T^v(\phi) - v}{v}\Big)} = v^{-1}\sqrt{\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))},$$

is, of course, equivalent to the minimization of the variance $\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))$. Within the present context, it is natural to introduce the constraint

$$\mathbb{E}_{\mathbb{P}}(v^{-1}V_T^v(\phi)) \ge d = 1 + d_r,$$

where $d_r > 0$ represents the desired minimal level of the expected rate of return.

2.4.1 Auxiliary Problems

The following auxiliary problem $\mathrm{MV}(dv, v)$ is merely a version of the previously considered problem $\mathrm{MV}(d, v)$:

Problem MV(dv, v): For a fixed $v > 0$ and $d > 1$, minimize the variance $\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))$ over all strategies $\phi \in \Phi(\mathbb{F})$, subject to $\mathbb{E}_{\mathbb{P}}V_T^v(\phi) \ge dv$.

We assume from now on that $\theta \neq 0$, and we denote by Θ the constant

$$\Theta = (e^{\theta^2 T} - 1)^{-1} > 0.$$

Recall that for the problem $\mathrm{MV}(dv, v)$, the risk-return trade-off can be summarized by the minimal variance curve $\mathbf{v}^*(dv, v)$. By virtue of Proposition 2.3, we have

$$\mathbf{v}^*(dv, v) = \Theta v^2 (d-1)^2 = \Theta v^2 d_r^2. \tag{2.46}$$

Equivalently, the minimal standard deviation of the rate of return satisfies

$$\sigma_r^* = \sigma(r(\phi^*(dv, v))) = \sqrt{\mathbf{v}^*(dv, v)} = \sqrt{\Theta} d_r,$$

so, as expected, it is independent of the value of the initial endowment v.

Suppose now that a claim X is available for some price $p_X \neq 0$, referred to as the *market price*. It is convenient to introduce the normalized claim $\bar{X} = X p_X^{-1}$. Under this convention, by the postulated linearity property of the market price, the price $p_{\bar{X}}$ of one unit of $\bar{X}$ is manifestly equal to 1.

The next auxiliary problem we wish to solve reads: find $p \in \mathbb{R}$ such that the solution to the problem $\mathrm{MV}(dv, v, p, p\bar{X})$ has the minimal variance. This means, of course, that we are looking for $p \in \mathbb{R}$ for which $\mathbf{v}^*(dv, v, p, p\bar{X})$ is minimal. Notice that the constraint on the expected rate of return becomes

$$\mathbb{E}_{\mathbb{P}}(v^{-1}V_T^{v-p}(\phi) + p\bar{X}) \geq d = 1 + d_r,$$

where $d_r > 0$. It is clear that the curve $\mathbf{v}^*(dv, v, p, p\bar{X})$ can be derived from the general expression for $\mathbf{v}^*(d, v, p, X)$, which was established in Proposition 2.4. Let us denote

$$\gamma_{\bar{X}} = \mathbb{V}_{\mathbb{P}}\big(\bar{X} - \mathbb{E}_{\mathbb{P}}(\bar{X} \,|\, \mathcal{F}_T)\big)$$

and

$$\nu_{\bar{X}} = \mathbb{E}_{\mathbb{Q}}\bar{X} - 1.$$

Let us notice that the condition $d - v + p - \mathbb{E}_{\mathbb{Q}}X > 0$, which was imposed in part (i) of Proposition 2.4, now corresponds to the following inequality: $vd_r > p\nu_{\bar{X}}$. We shall assume from now on that $\bar{X} \neq 1$ (this assumption means simply that the claim $\bar{X}$ does not represent the savings account). Recall that $v > 0$ and $d_r = d - 1 > 1$.

Proposition 2.11 (i) *Assume that $\gamma_{\bar{X}} > 0$ and $\nu_{\bar{X}} \neq 0$. Then the problem $MV(dv, v, p, p\bar{X})$ has a solution with the minimal variance with respect to p. The minimal variance equals*

$$\mathbf{v}^*(dv, v, p^*, p^*\bar{X}) = \Theta v^2 d_r^2 \left(1 - \frac{\nu_{\bar{X}}^2}{\Theta^{-1}\gamma_{\bar{X}} + \nu_{\bar{X}}^2}\right) \tag{2.47}$$

and the optimal value of p equals

$$p^* = \frac{vd_r\nu_{\bar{X}}}{\Theta^{-1}\gamma_{\bar{X}} + \nu_{\bar{X}}^2}\,. \tag{2.48}$$

(ii) *Let $\gamma_{\bar{X}} > 0$ and $\nu_{\bar{X}} = 0$. Then we have $p^* = 0$ and the minimal variance equals*

$$\mathbf{v}^*(dv, v, p^*, p^*\bar{X}) = \Theta v^2 d_r^2.$$

(iii) *Let $\gamma_{\bar{X}} = 0$ and $\nu_{\bar{X}} \neq 0$. If the inequality $\nu_{\bar{X}} > 0$ ($\nu_{\bar{X}} < 0$, respectively) holds then for any $p \geq vd_r\nu_{\bar{X}}^{-1}$ ($p \leq vd_r\nu_{\bar{X}}^{-1}$, respectively) the minimal variance $\mathbf{v}^*(dv, v, p, p\bar{X})$ is minimal with respect to p and it equals 0.*
(iv) *Let $\gamma_{\bar{X}} = \nu_{\bar{X}} = 0$. Then $\bar{X}$ is an attainable claim and $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$. In this case, for any $p \in \mathbb{R}$ the minimal variance equals*

$$\mathbf{v}^*(dv, v, p, p\bar{X}) = \Theta v^2 d_r^2.$$

Proof. Let us first prove parts (i)-(ii). It suffices to observe that, by virtue of Proposition 2.4, the minimal variance for the problem $\mathrm{MV}(dv, v, p, p\bar{X})$ is given by the expression:

$$\mathbf{v}^*(dv, v, p, p\bar{X}) = \Theta(d_r v - p\nu_{\bar{X}})^2 + p^2\gamma_{\bar{X}} \tag{2.49}$$

provided that $vd_r > p\nu_{\bar{X}}$. A simple argument shows that the minimal value for the right-hand side in (2.49) is obtained by setting $p = p^*$, where p^* is given by (2.48), and the minimal variance is given by (2.47). Moreover, it is easily seen that for p^* given by (2.48) the inequality $vd_r > p^*\nu_{\bar{X}}$ is indeed satisfied, provided that $\gamma_{\bar{X}} > 0$. Notice also that if $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$, we obviously have $vd_r > p\nu_{\bar{X}} = 0$ for any $p \in \mathbb{R}_+$, and thus we obtain the following optimal values:

$$p^* = 0, \quad \mathbf{v}^*(dv, v, p^*, p^*\bar{X}) = \Theta v^2 d_r^2.$$

Assume now that $vd_r \leq p\nu_{\bar{X}}$, so that the case $\nu_{\bar{X}} = 0$ (i.e., the case $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$) is excluded. Then, by virtue of part (ii) in Proposition 2.4, the minimal variance equals $p^2\gamma_{\bar{X}}$ (notice that the assumption that $\gamma_{\bar{X}}$ is strictly positive is not needed here). Assume first that $\mathbb{E}_{\mathbb{Q}}\bar{X} < 1$, so that $\nu_{\bar{X}} < 0$. Then the condition $vd_r \leq p\nu_{\bar{X}}$ becomes $p \leq vd_r\nu_{\bar{X}}^{-1}$, and thus p is necessarily negative. The minimal variance corresponds to $p^* = vd_r\nu_{\bar{X}}^{-1}$, and it equals

$$\mathbf{v}^*(dv, v, p^*, p^*\bar{X}) = (p^*)^2\gamma_{\bar{X}} = v^2 d_r^2 \nu_{\bar{X}}^{-2}\gamma_{\bar{X}}. \tag{2.50}$$

In, on the contrary, $\mathbb{E}_{\mathbb{Q}}\bar{X} > 1$, then $\nu_{\bar{X}} > 0$ and we obtain $p \geq vd_r\nu_{\bar{X}}^{-1}$, so that p is strictly positive. Again, the minimal variance corresponds to $p^* = vd_r\nu_{\bar{X}}^{-1}$, and it is given by (2.50). It is easy to check that the following inequality holds:

$$\Theta v^2 d_r^2 \left(1 - \frac{\nu_{\bar{X}}^2}{\Theta^{-1}\gamma_{\bar{X}} + \nu_{\bar{X}}^2}\right) < v^2 d_r^2 \nu_{\bar{X}}^{-2}\gamma_{\bar{X}}.$$

By combining the considerations above, we conclude that statements (i)-(ii) are valid. The proof of part (iii) is also based on the analysis above. We thus proceed to the proof of the last statement.

Notice that $\Theta^{-1}\gamma_{\bar{X}} + \nu_{\bar{X}}^2 = 0$ if and only if $\gamma_{\bar{X}} = 0$ and $\nu_{\bar{X}} = 0$. This means that $\bar{X}$ is $\mathcal{F}_T$-adapted (and thus $\mathbb{F}$-attainable) and $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$ (so that the arbitrage price of $\bar{X}$ coincides with its market price $p_{\bar{X}}$). Condition $vd_r - p\nu_{\bar{X}} > 0$ is now satisfied, and thus the minimal variance is given by (2.49), which now becomes

$$\mathbf{v}^*(dv, v, p, p\bar{X}) = \Theta v^2 d_r^2, \quad \forall\, p \in \mathbb{R}_+.$$

Obviously, the result does not depend on p. This proves part (iv). □

In the last proposition, no a priori restriction on the value of the parameter p was imposed. Of course, one can also consider a related constrained problem by postulating, for instance, that the price p belongs to the interval $[0, v]$.

2.4.2 Minimization of Risk

We are in a position to examine the first question, which reads: how much to invest in the new opportunity in order to minimize the risk and to preserve at the same time the pre-specified level $d_r > 0$ of the expected rate of return.

Case of an attainable claim. Assume first that $\bar{X}$ is an $\mathbb{F}$-attainable contingent claim, so that $\mathbb{E}_{\mathbb{P}}(\bar{X} \,|\, \mathcal{F}_T) = \bar{X}$, and thus $\gamma_{\bar{X}} = 0$. If the claim $\bar{X}$ is correctly priced by the market, i.e., if $\mathbb{E}_{\mathbb{Q}}\bar{X} = p_{\bar{X}} = 1$ then, by virtue of part (iv) in Proposition 2.11, for any choice of p the minimal variance is the same as in the problem $\mathrm{MV}(dv, v)$. Hence, as expected, the possibility of investing in the claim $\bar{X}$ has no bearing on the efficiency of trading.

Let us now consider the case where $\mathbb{E}_{\mathbb{Q}}\bar{X} \neq 1$, that is, the market price $p_{\bar{X}}$ does not coincide with the arbitrage price $\pi_0(\bar{X})$. Suppose first that $\mathbb{E}_{\mathbb{Q}}\bar{X} > 1$, that is, $\bar{X}$ is underpriced by the market. Then, in view of part (iii) in Proposition 2.11, the variance of the rate of return can be reduced to 0 by choosing p which satisfies

$$p \geq vd_r(\mathbb{E}_{\mathbb{Q}}\bar{X} - 1)^{-1} > 0.$$

Similarly, if $\mathbb{E}_{\mathbb{Q}}\bar{X} < 1$ then for any p such that

$$p \leq vd_r(\mathbb{E}_{\mathbb{Q}}\bar{X} - 1)^{-1} < 0$$

the variance equals 0. Off course, this feature is due to the presence of arbitrage opportunities in the market. We conclude that, as expected, in the case of an attainable claim the solution to the problem considered is this section is rather trivial, and thus it has no practical appeal.

Case of a non-attainable claim. We now assume that $\gamma_{\bar{X}} > 0$. Suppose first that $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$. By virtue of part (ii) in Proposition 2.11, under this assumption it is optimal not to invest in $\bar{X}$. To better appreciate this result, notice that for the conditional expectation $\widetilde{X} = \mathbb{E}_{\mathbb{P}}(\bar{X} \mid \mathcal{F}_T)$ we have $\mathbb{E}_{\mathbb{Q}}\widetilde{X} = \mathbb{E}_{\mathbb{Q}}\bar{X} = 1$ and $\mathbb{E}_{\mathbb{P}}\widetilde{X} = \mathbb{E}_{\mathbb{P}}\bar{X}$ (cf. Section 2.2.2). Therefore, trading in $\bar{X}$ is essentially equivalent to trading in an attainable claim $\widetilde{X}$, but trading in $\bar{X}$ results in the residual variance $p^2\gamma_{\bar{X}}$. This observations explains why the solution $p^* = 0$ is optimal.

Suppose now that $\mathbb{E}_{\mathbb{Q}}\bar{X} \neq 1$. Then part (i) of Proposition 2.11 shows that the variance of the rate of return can always be reduced by trading in $\bar{X}$. Specifically, p^* is strictly positive provided that $\mathbb{E}_{\mathbb{Q}}\bar{X} > 1 = p_{\bar{X}}$, that is, the expected value of $\bar{X}$ under the martingale measure $\mathbb{Q}$ for the underlying market is greater than its market price.

Case of an independent claim. Assume that the claim $\bar{X}$ is independent of $\mathcal{F}_T$, so that $\gamma_{\bar{X}} > 0$ is the variance of $\bar{X}$. In this case $\mathbb{E}_{\mathbb{Q}}\bar{X} = \mathbb{E}_{\mathbb{P}}\bar{X}$ and thus (2.47) becomes

$$\mathbf{v}^* = \Theta v^2 d_r^2 \left(1 - \frac{(\mathbb{E}_{\mathbb{P}}\bar{X} - 1)^2}{\Theta^{-1}\mathbb{V}_{\mathbb{P}}(\bar{X}) + (\mathbb{E}_{\mathbb{P}}\bar{X} - 1)^2}\right).$$

From the last formula, it is clear that an agent should always to invest either a positive or negative amount of initial endowment v in an independent claim X, except for the case where $\mathbb{E}_{\mathbb{P}}\bar{X} = 1$. If $\mathbb{E}_{\mathbb{P}}\bar{X} \neq 1$ then the optimal value of p equals (cf. (2.48))

$$p^* = \frac{vd_r(\mathbb{E}_{\mathbb{P}}\bar{X} - 1)}{\Theta^{-1}\mathbb{V}_{\mathbb{P}}(\bar{X}) + (\mathbb{E}_{\mathbb{P}}\bar{X} - 1)^2}$$

so that it is positive if and only if $\mathbb{E}_{\mathbb{P}}\bar{X} > 1$.

Case of a claim with zero market price. The case when the market price of X is zero (that is, the equality $p_X = 0$ holds) is also of practical interest, since such a feature is typical for forward contracts. It should be stressed that this particular case is not covered by Proposition 2.11, however.

In fact, we deal here with the following variant of the mean-variance problem:

Find $\alpha \in \mathbb{R}$ such that the solution to the problem $\mathrm{MV}(dv, v, 0, \alpha X)$ has the minimal variance.

Under the assumption that $vd_r > \alpha \mathbb{E}_{\mathbb{Q}} X$, we have

$$\mathbf{v}^*(dv, v, 0, \alpha X) = \Theta(vd_r - \alpha \mathbb{E}_{\mathbb{Q}} X)^2 + \alpha^2 \gamma_X.$$

If, on the contrary, the inequality $vd_r \leq \alpha \mathbb{E}_{\mathbb{Q}} X$ is valid, then the minimal variance equals $\alpha^2 \gamma_X$. Of course, we necessarily have $\alpha \neq 0$ here (since $vd_r > 0$).

2.4.3 Maximization of Expected Return

Let us focus on part (i) in Proposition 2.11, that is, let us assume that $\gamma_{\bar{X}} > 0$ and $\nu_{\bar{X}} \neq 0$ (as was explained above, other cases examined in Proposition 2.11 are of minor practical interest). The question of maximization of the expected rate return for a pre-specified level of risk, can be easily solved by comparing (2.46) with (2.47). Indeed, for a given level d_r of the expected rate of return, and thus a given level $\mathbf{v}^*(dv, v)$ of the minimal variance, it suffices to find a number $\widehat{d_r}$ which solves the following equation

$$\Theta v^2 d_r^2 = \Theta v^2 \widehat{d}_r^2 \left(1 - \frac{\nu_{\bar{X}}^2}{\Theta^{-1}\gamma_{\bar{X}} + \nu_{\bar{X}}^2}\right).$$

It is obvious that the last equation has the unique solution

$$\widehat{d_r} = d_r \sqrt{1 + \frac{\nu_{\bar{X}}^2}{\Theta^{-1}\gamma_{\bar{X}}}} > d_r.$$

The corresponding value of p^* is given by (2.48) with d_r substituted with $\widehat{d_r}$. It is thus clear that, under the present assumptions, a new investment opportunity can be used to enhance the expected rate of return. If we insist, in addition, that $p > 0$, then the latter statement remains valid, provided that $\mathbb{E}_{\mathbb{Q}} \bar{X} > 1$.

Chapter 3

Indifference Pricing

In this chapter, we present a few alternative ways of pricing defaultable claims in the situation when perfect hedging is not possible. In the previous chapter, we have presented the mean-variance hedging framework. Now, we study the indifference price approach that was initiated by Hodges and Neuberger (1989). We shall refer to this approach as the "Hodges price" approach. This will lead us to solving portfolio optimization problems in incomplete market, and we shall use the dynamic programming (DP) approach.

We also present the Hamilton-Jacobi-Bellman (HJB) equations, when appropriate, even though this method typically requires strong assumptions to give closed-form solutions. In particular, when dealing with the general DP approach, we need not make any Markovian assumption about the underlying processes; such assumptions are fundamental for the HJB methodology to work.

In Section 3.1, we define the Hodges indifference price associated to strategies adapted with the reference filtration $\mathbb{F}$, and we solve the problem for exponential preferences and for some particular defaultable claims. We shall use results obtained here to provide basis for a comparison between the historical spread and the risk-neutral one.

In Section 3.2, using backward stochastic differential equations (BSDEs), we work with $\mathbb{G}$-adapted strategies, and we solve portfolio optimization problems for exponential utility functions. Our method relies on the ideas of Rouge and El Karoui (2000) and Musiela and Zariphopoulou (2004). The reader can refer to El Karoui and Mazliak (1997), El Karoui and Quenez (1997), El Karoui et al. (1997), or to the survey by Buckdahn (2000) for an introduction to the theory of backward stochastic differential equations and its applications in finance.

Section 3.3 is devoted to the study of a particular indifference price, based on the quadratic criterion; we call such a price the *quadratic hedging price* (see the introduction to Chapter 2). In particular, we compare the indifference prices obtained using strategies adapted to the reference filtration $\mathbb{F}$ to the indifference prices obtained using strategies based on the enlarged filtration $\mathbb{G}$. It is worthwhile to stress, though, that the quadratic utility alone is not quite adequate for the pricing purposes, although it represents a good criterion for hedging purposes. This is one of the reasons we presented the mean-variance approach to pricing and hedging of defaultable claims in Chapter 2.

In the last section, we present a very particular case of the duality approach for exponential utilities.

We emphasize that in this chapter, similarly as in the previous chapter, a very important aspect of our analysis is the distinction between the case when admissible portfolios are adapted to the filtration $\mathbb{F}$, and the case when admissible portfolios are adapted to the filtration $\mathbb{G}$.

3.1 Hedging in Incomplete Markets

We recall briefly the probabilistic setting of Chapter 2. The default-free asset is Z^1 with the dynamics

$$dZ_t^1 = Z_t^1(\nu dt + \sigma dW_t), \quad Z_0^1 > 0,$$

and the price process of the money market account has the dynamics

$$dZ_t^2 = rZ_t^2\, dt, \quad Z_0^2 = 1,$$

where r is the constant interest rate. The default-free market is complete and arbitrage free: one can hedge perfectly any square-integrable contingent claim $X \in \mathcal{F}_T$. The default time is some random time τ, and the default process is denoted as $H_t = \mathbb{1}_{\{\tau \le t\}}$. The reference filtration is the Brownian filtration $\mathcal{F}_t = \sigma(W_u,\, u \le t)$ and the enlarged filtration is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathcal{H}_t = \sigma(H_u,\, u \le t)$.

We assume that the hazard process $F_t = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\}$ is absolutely continuous with respect to Lebesgue measure, so that $F_t = \int_0^t f_u\, du$ (hence, it is an increasing process). Therefore, the process

$$M_t = H_t - \int_0^{t\wedge\tau} \gamma_u\, du = H_t - \int_0^{t\wedge\tau} \frac{f_u}{1-F_u}\, du$$

is a $\mathbb{G}$-martingale, where γ is the *default intensity.* Note that the stochastic intensity γ is the intensity of the default time τ with respect to the reference filtration $\mathbb{F}$ generated by the Brownian motion W.

For a fixed $T > 0$, we introduce a *risk-neutral probability* $\mathbb{Q}$ for the market model (Z^1, Z^2) by setting $d\mathbb{Q}|_{\mathcal{G}_t} = \eta_t\, d\mathbb{P}|_{\mathcal{G}_t}$ for $t \in [0, T]$, where the Radon-Nikodym density η is the $\mathbb{F}$-martingale defined as

$$d\eta_t = -\theta\eta_t\, dW_t, \quad \eta_0 = 1,$$

where $\theta = (\nu - r)/\sigma$. Under $\mathbb{Q}$, the discounted process $\widetilde{Z}_t^1 = e^{-rt}Z_t^1$ is a martingale. It should be emphasized that $\mathbb{Q}$ is not necessarily a martingale measure for defaultable assets. Let us recall, however, that if $\widetilde{\mathbb{Q}}$ is any equivalent martingale measure on $\mathbb{G}$ for the default-free and defaultable market, then the restriction of $\widetilde{\mathbb{Q}}$ to $\mathbb{F}$ is equal to the restriction of $\mathbb{Q}$ to $\mathbb{F}$. A *defaultable claim* is simply any random variable X, which is $\mathcal{G}_T$-measurable. Hence, default-free claims are formally considered as special cases of defaultable claims.

3.1.1 Hodges Indifference Price

We present a general framework of the Hodges and Neuberger (1989) approach with some strictly increasing, strictly concave and continuously differentiable mapping u, defined on $\mathbb{R}$. We solve explicitly the problem in the case of exponential utility for portfolios adapted to the reference filtration.

The Hodges approach to pricing of unhedgeable claims is a utility-based approach and can be summarized as follows: the issue at hand is to assess the value of some (defaultable) claim X as seen from the perspective of an economic agent who optimizes his behavior relative to some utility function, say u. In order to provide such an assessment one can argue that one should first consider the following possible modes of agent's behavior and the associated optimization problems:

Problem $(\mathcal{P})$**:** Optimization in the default-free market.

The agent invests his initial wealth $v > 0$ in the default-free financial market using a self-financing strategy. The associated optimization problem is,

$$(\mathcal{P}) \ : \ \mathcal{V}(v) := \sup_{\phi \in \Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^v(\phi)\big)\big\},$$

where the wealth process $V_t = V_t^v(\phi)$, $t \in \mathbb{R}_+$, is solution of

$$dV_t = rV_t\,dt + \phi_t(dZ_t^1 - rZ_t^1 dt), \quad V_0 = v. \tag{3.1}$$

Recall that $\Phi(\mathbb{F})$ is the class of all admissible, $\mathbb{F}$-adapted, self-financing trading strategies (for the definition of this class, see Chapter 2).

Problem $(\mathcal{P}_{\mathbb{F}}^X)$**:** Optimization in the default-free market using $\mathbb{F}$-adapted strategies and buying the defaultable claim.

The agent buys the contingent claim X at price p, and invests the remaining wealth $v - p$ in the financial market, using a trading strategy $\phi \in \Phi(\mathbb{F})$. The resulting *global terminal wealth* will be

$$V_T^{v-p,X}(\phi) = V_T^{v-p}(\phi) + X.$$

The associated optimization problem is

$$(\mathcal{P}_{\mathbb{F}}^X) \ : \ \mathcal{V}_X(v-p) := \sup_{\phi \in \Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p}(\phi) + X\big)\big\},$$

where the process $V^{v-p}(\phi)$ is a solution of (3.1) with the initial condition $V_0^{v-p}(\phi) = v - p$. We emphasize that the class $\Phi(\mathbb{F})$ of admissible strategies is the same as in the problem $(\mathcal{P})$, that is, we restrict here our attention to trading strategies that are adapted to the reference filtration $\mathbb{F}$.

Problem $(\mathcal{P}_{\mathbb{G}}^X)$**:** Optimization in the default-free market using $\mathbb{G}$-adapted strategies and buying the defaultable claim.

The agent buys the contingent claim X at price p, and invests the remaining wealth $v-p$ in the financial market, using a strategy adapted to the enlarged filtration $\mathbb{G}$. The associated optimization problem is

$$(\mathcal{P}_{\mathbb{G}}^X)\ :\ \mathcal{V}_X^{\mathbb{G}}(v-p) := \sup_{\phi\in\Phi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p}(\phi)+X\big)\big\},$$

where $\Phi(\mathbb{G})$ is the class of all $\mathbb{G}$-admissible trading strategies (for the definition of the class $\Phi(\mathbb{G})$, see Chapter 2). Next, the utility based assessment of the value (price) of the claim X, as seen from the agent's perspective, is given in terms of the following definition.

Definition 3.1 For a given initial endowment v, the $\mathbb{F}$-*Hodges buying price* of a defaultable claim X is the real number $p_{\mathbb{F}}^*(v)$ such that $\mathcal{V}(v)=\mathcal{V}_X\big(v-p_{\mathbb{F}}^*(v)\big)$. Similarly, the $\mathbb{G}$-*Hodges buying price* of X is the real number $p_{\mathbb{G}}^*(v)$ such that $\mathcal{V}(v)=\mathcal{V}_X^{\mathbb{G}}\big(v-p_{\mathbb{G}}^*(v)\big)$.

Remark. We can define the $\mathbb{F}$-*Hodges selling price* $p_*^{\mathbb{F}}(v)$ of X by considering $-p$, where p is the buying price of $-X$, as specified in Definition 3.1.

If the contingent claim X is $\mathcal{F}_T$-measurable, then the $\mathbb{F}$- and the $\mathbb{G}$-Hodges prices coincide with the hedging price of X, i.e., $p_{\mathbb{F}}^*(v)=p_{\mathbb{G}}^*(v)=\pi_0(X)=\mathbb{E}_{\mathbb{P}}(\zeta_T X)$, where we denote $\zeta_t=\eta_t R_t$ with $R_t=(Z_t^2)^{-1}=e^{-rt}$. Indeed, assume that there exists a self-financing portfolio $\widehat{\phi}$ such that $X=V_T^{\pi_0(X)}(\widehat{\phi})$, and let h be the $\mathbb{F}$-Hodges buying price. Suppose first that $h<\pi_0(X)$. Then for any ϕ we obtain

$$V_T^{v-h}(\phi)+X=V_T^{v-h}(\phi)+V_T^{\pi_0(X)}(\widehat{\phi})=V_T^{v-h+\pi_0(X)}(\psi),$$

where we denote $\psi=\widehat{\phi}+\phi\in\Phi(\mathbb{F})$. Hence

$$\begin{aligned}\mathcal{V}_X(v-h) &= \sup_{\phi\in\Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-h}(\phi)+X\big)\big\}\\ &= \sup_{\psi\in\Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-h+\pi_0(X)}(\psi)\big)\big\}\geq \mathcal{V}(v),\end{aligned}$$

where the last inequality (which is a strict inequality) follows from $v<v-h+\pi_0(X)$ and the arbitrage principle. Therefore, the supremum over $\phi\in\Phi(\mathbb{F})$ of $\mathbb{E}_{\mathbb{P}}(u(V_T^{v-h}(\phi)+X))$ is greater than $\mathcal{V}(v)$. We conclude that the $\mathbb{F}$-Hodges buying price can not be smaller than the hedging price. Arguing in a similar way, one can show that the $\mathbb{F}$-Hodges selling price of an $\mathcal{F}_T$-measurable claim can not be smaller than the hedging price. Finally, almost identical arguments show that the $\mathbb{G}$-Hodges buying and selling price of an $\mathcal{F}_T$-measurable claim are equal to the hedging price of X (see Section 3.2.2).

Remark. It can be shown (see Rouge and El Karoui (2000), or Collin-Dufresne and Hugonnier (2002)) that in the general case of non-hedgeable contingent claim, the Hodges price belongs to the open interval

$$\big(\inf_{\widetilde{\mathbb{Q}}}\mathbb{E}_{\widetilde{\mathbb{Q}}}(Xe^{-rT}),\ \sup_{\widetilde{\mathbb{Q}}}\mathbb{E}_{\widetilde{\mathbb{Q}}}(Xe^{-rT})\big),$$

where $\widetilde{\mathbb{Q}}$ runs over the set of all equivalent martingale measures, and thus it can not induce arbitrage opportunities.

3.1.2 Solution of Problem $(\mathcal{P})$

We briefly recall one of the solution methods for the problem $(\mathcal{P})$. To this end, we first observe that in view of (3.1) the process $e^{-rt}V_t^{v-p}(\phi)$, $t \in \mathbb{R}_+$, is a martingale under any equivalent martingale measure, hence $\zeta_t V_t^{v-p}(\phi)$, $t \in \mathbb{R}_+$, is a $\mathbb{P}$-martingale and, in particular, $\mathbb{E}_{\mathbb{P}}(V_T^v(\phi)\zeta_T) = v$. It follows that in order to obtain a terminal wealth equal to, say V, the initial endowment v has to be greater or equal to $\mathbb{E}_{\mathbb{P}}(V\zeta_T)$; this condition is commonly referred to as the *budget constraint*.

Now, let us denote by I the inverse of the monotonic mapping u' (the first derivative of u). It is well known (see, e.g., Karatzas and Shreve (1998)) that the optimal terminal wealth in the problem $(\mathcal{P})$ is given by the formula

$$V_T^{v,*} = I(\mu\zeta_T), \quad \mathbb{P}\text{-a.s.}, \tag{3.2}$$

where μ is a real number such that the budget constraint is binding, that is,

$$v = \mathbb{E}_{\mathbb{P}}\big(\zeta_T V_T^{v,*}\big). \tag{3.3}$$

Consequently, the optimal value of the objective criterion for the problem $(\mathcal{P})$ is $\mathcal{V}(v) = \mathbb{E}_{\mathbb{P}}(u(V_T^{v,*}))$.

The above results are obtained by means of convex duality theory. The disadvantage of this approach, however, is the fact that it is typically very difficult to identify an optimal trading strategy. Thus, in general, using the convex duality approach we can only partially solve the problem $(\mathcal{P})$. Specifically, we can compute the optimal value of the objective criterion, but we can't identify the optimal strategy. Later in this chapter, we shall use the BSDE approach in a more general setting. It will be seen that this approach will allow us to identify (at least in principle) an optimal trading strategy.

3.1.3 Solution of Problem $(\mathcal{P}_{\mathbb{F}}^X)$

In this subsection, we shall examine the problem $(\mathcal{P}_{\mathbb{F}}^X)$ for a defaultable claim of a particular form. First, we shall provide a solution $\mathcal{V}_X(v-p)$ to the related optimization problem. Next, we shall establish a quasi-explicit representation for the Hodges price of X in the case of exponential utility. Finally, we shall compare the spread obtained via the risk-neutral valuation with the spread determined by the Hodges price of a defaultable zero-coupon bond. The reader can refer to Bernis and Jeanblanc (2003) for other comments.

Particular Form of a Defaultable Claim

We restrict our attention to the case when X is of the form

$$X = X_1 \mathbb{1}_{\{\tau > T\}} + X_2 \mathbb{1}_{\{\tau \le T\}}, \tag{3.4}$$

where $X_i, i = 1, 2$ are $\mathbb{P}$-square-integrable and $\mathcal{F}_T$-measurable random variables. In this case, we have

$$V_T^{v-p,X}(\phi) = V_T^{v-p}(\phi) + X_1$$

if the default did not occur before maturity date T, that is, on the set $\{\tau > T\}$, and

$$V_T^{v-p,X}(\phi) = V_T^{v-p}(\phi) + X_2$$

otherwise. In other words,

$$V_T^{v-p,X}(\phi) = \mathbb{1}_{\{\tau>T\}}(V_T^{v-p}(\phi) + X_1) + \mathbb{1}_{\{\tau\leq T\}}(V_T^{v-p}(\phi) + X_2).$$

Observe that the pay-off X_2 is not paid at time of default τ, but at the terminal time T.

Since the trading strategies are $\mathbb{F}$-adapted, the terminal wealth $V_T^{v-p}(\phi)$ is an $\mathcal{F}_T$-measurable random variable. Consequently, it holds that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p,X}(\phi)\big)\big\} \\
&= \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p}(\phi) + X_1\big)\mathbb{1}_{\{\tau>T\}} + u\big(V_T^{v-p}(\phi) + X_2\big)\mathbb{1}_{\{\tau\leq T\}}\big\} \\
&= \mathbb{E}_{\mathbb{P}}\left\{\mathbb{E}_{\mathbb{P}}\left(u\left(V_T^{v-p}(\phi) + X_1\right)\mathbb{1}_{\{\tau>T\}} + u\left(V_T^{v-p}(\phi) + X_2\right)\mathbb{1}_{\{\tau\leq T\}}|\mathcal{F}_T\right)\right\} \\
&= \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^{v-p}(\phi) + X_1\big)(1 - F_T) + u\big(V_T^{v-p}(\phi) + X_2\big)F_T\big\},
\end{aligned}$$

where $F_T = \mathbb{P}\{\tau \leq T \,|\, \mathcal{F}_T\}$. Define, for every $\omega \in \Omega$ and $y \in \mathbb{R}$,

$$J_X(y, \omega) = u(y + X_1(\omega))(1 - F_T(\omega)) + u(y + X_2(\omega))F_T(\omega).$$

Notice that under the present assumptions, the problem $(\mathcal{P}_{\mathbb{F}}^X)$ is equivalent to the following problem:

$$(\mathcal{P}_{\mathbb{F}}^X) \ : \ \mathcal{V}_X(v - p) := \sup_{\phi\in\Phi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big\{J_X\big(V_T^{v-p}(\phi), \omega\big)\big\}.$$

The mapping $J_X(\cdot, \omega)$ is a strictly concave and increasing real-valued mapping. Consequently, for any $\omega \in \Omega$ we can define the mapping $I_X(z, \omega)$ by setting $I_X(z, \omega) = \big(J'_X(\cdot, \omega)\big)^{-1}(z)$ for $z \in \mathbb{R}$, where $(J'_X(\cdot, \omega))^{-1}$ denotes the inverse mapping of the derivative of J_X with respect to the first variable. To simplify the notation, we shall usually suppress the second variable, and we shall write $I_X(\cdot)$ in place of $I_X(\cdot, \omega)$.

The following lemma provides the form of the optimal solution.

Lemma 3.1 *The optimal terminal wealth for the problem* $(\mathcal{P}_{\mathbb{F}}^X)$ *is given by* $V_T^{v-p,*} = I_X(\lambda^*\zeta_T)$, $\mathbb{P}$*-a.s., for some* λ^* *such that*

$$v - p = \mathbb{E}_{\mathbb{P}}\big(\zeta_T V_T^{v-p,*}\big). \tag{3.5}$$

Thus the optimal global wealth equals $V_T^{v-p,X,*} = V_T^{v-p,*} + X = I_X(\lambda^*\zeta_T) + X$ *and the optimal value of the objective criterion for the problem* $(\mathcal{P}_{\mathbb{F}}^X)$ *is*

$$\mathcal{V}_X(v - p) = \mathbb{E}_{\mathbb{P}}(u(V_T^{v-p,X,*})) = \mathbb{E}_{\mathbb{P}}(u(I_X(\lambda^*\zeta_T) + X)). \tag{3.6}$$

Proof. As a consequence of predictable representation property (see, e.g., Karatzas and Shreve (1991)), one knows that in order to find the optimal wealth it is enough to maximize $u(\Delta)$ over the set of square-integrable and $\mathcal{F}_T$-measurable random variables Δ, subject to the budget constraint, given by

$$\mathbb{E}_{\mathbb{P}}(\zeta_T \Delta) \leq v - p.$$

The associated Lagrange multiplier, say λ^*, is non-negative. Moreover, by the strict monotonicity of u, we know that, at optimum, the constraint is binding, and thus $\lambda^* > 0$. We check that $I_X(\lambda^* \zeta_T)$ is the optimal wealth.

The mapping $J_X(\cdot)$ is strictly concave (for all ω). Hence, for every wealth process $V^{v-p}(\phi)$, starting from $v-p$, by tangent inequality, we have

$$\mathbb{E}_{\mathbb{P}}\big\{J_X(V_T^{v-p}(\phi)) - J_X(V_T^{v-p,*})\big\} \;\leq\; \mathbb{E}_{\mathbb{P}}\big\{(V_T^{v-p}(\phi) - V_T^{v-p,*})J'_X(V_T^{v-p,*})\big\}.$$

Replacing $V^{v-p,*}$ by its expression given in Lemma 3.1 yields for any $\phi \in \Phi(\mathbb{F})$

$$\mathbb{E}_{\mathbb{P}}\big\{J_X(V_T^{v-p}(\phi)) - J_X(V_T^{v-p,*})\big\} \leq \lambda^* \, \mathbb{E}_{\mathbb{P}}\big\{\zeta_T(V_T^{v-p}(\phi) - V_T^{v-p,*})\big\} \leq 0,$$

where the last inequality follows from (3.5) and the budget constraint. To end the proof, it remains to observe that the first order conditions are also sufficient in the case of a concave criterion. Moreover, by virtue of strict concavity of the function J_X, the optimum is unique. □

Exponential Utility: Explicit Computation of the Hodges Price

For the sake of simplicity, we assume here that $r = 0$. Let us state the following result, the proof of which stems from Lemma 3.1, by direct computations.

Proposition 3.1 *Let $u(x) = 1 - \exp(-\varrho x)$ for some $\varrho > 0$. Assume that for $i = 1, 2$ the random variable $\zeta_T e^{-\varrho X^i}$ is $\mathbb{P}$-integrable. Then we have*

$$p^*_{\mathbb{F}}(v) = -\frac{1}{\varrho}\, \mathbb{E}_{\mathbb{P}}\left(\zeta_T \ln\big((1-F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}\big)\right) = \mathbb{E}_{\mathbb{P}}(\zeta_T \Psi),$$

where the $\mathcal{F}_T$-measurable random variable Ψ equals

$$\Psi = -\frac{1}{\varrho} \ln\big((1-F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}\big). \tag{3.7}$$

Thus, the $\mathbb{F}$-Hodges buying price $p^*_{\mathbb{F}}(v)$ is the arbitrage price of the associated claim Ψ. In addition, the claim Ψ enjoys the following meaningful property

$$\mathbb{E}_{\mathbb{P}}\big\{u\big(X - \Psi\big) \,\big|\, \mathcal{F}_T\big\} = 0. \tag{3.8}$$

Proof. In view of the form of the solution to the problem $(\mathcal{P})$, we obtain (cf. (3.2))

$$V_T^{v,*} = -\frac{1}{\varrho} \ln\left(\frac{\mu^* \zeta_T}{\varrho}\right).$$

The budget constraint $\mathbb{E}_{\mathbb{P}}(\zeta_T V_T^{v,*}) = v$ implies that the Lagrange multiplier μ^* satisfies

$$\frac{1}{\varrho}\ln\left(\frac{\mu^*}{\varrho}\right) = -\frac{1}{\varrho}\mathbb{E}_{\mathbb{P}}\big(\zeta_T \ln \zeta_T\big) - v. \tag{3.9}$$

In the case of an exponential utility, we have (recall that the variable ω is suppressed)

$$J_X(y) = (1 - e^{-\varrho(y+X_1)})(1 - F_T) + (1 - e^{-\varrho(y+X_2)})F_T,$$

so that

$$J'_X(y) = \varrho e^{-\varrho y}(e^{-\varrho X_1}(1 - F_T) + e^{-\varrho X_2} F_T).$$

Thus, setting

$$A = e^{-\varrho X_1}(1 - F_T) + e^{-\varrho X_2} F_T = e^{-\varrho \Psi},$$

we obtain

$$I_X(z) = -\frac{1}{\varrho}\ln\left(\frac{z}{A\varrho}\right) = -\frac{1}{\varrho}\ln\left(\frac{z}{\varrho}\right) - \Psi.$$

It follows that the optimal terminal wealth for the initial endowment $v - p$ is

$$V_T^{v-p,*} = -\frac{1}{\varrho}\ln\left(\frac{\lambda^*\zeta_T}{A\varrho}\right) = -\frac{1}{\varrho}\ln\left(\frac{\lambda^*}{\varrho}\right) - \frac{1}{\varrho}\ln\zeta_T - \Psi,$$

where the Lagrange multiplier λ^* is chosen to satisfy the budget constraint $\mathbb{E}_{\mathbb{P}}(\zeta_T V_T^{v-p,*}) = v - p$, that is,

$$\frac{1}{\varrho}\ln\left(\frac{\lambda^*}{\varrho}\right) = -\frac{1}{\varrho}\,\mathbb{E}_{\mathbb{P}}\big(\zeta_T \ln\zeta_T\big) - \mathbb{E}_{\mathbb{P}}\big(\zeta_T \Psi\big) - v + p. \tag{3.10}$$

The $\mathbb{F}$-Hodges buying price is a real number $p^* = p^*_{\mathbb{F}}(v)$ such that

$$\mathbb{E}_{\mathbb{P}}\big(\exp(-\varrho V_T^{v,*})\big) = \mathbb{E}_{\mathbb{P}}\big(\exp(-\varrho(V_T^{v-p^*,*} + X))\big),$$

where μ^* and λ^* are given by (3.9) and (3.10), respectively. After substitution and simplifications, we arrive at the following equality

$$\mathbb{E}_{\mathbb{P}}\Big\{\exp\Big(-\varrho\big(\mathbb{E}_{\mathbb{P}}(\zeta_T\Psi) - p^* + X - \Psi\big)\Big)\Big\} = 1. \tag{3.11}$$

Using (3.4), it is easy to check that

$$\mathbb{E}_{\mathbb{P}}\big(e^{-\varrho(X-\Psi)}\,\big|\,\mathcal{F}_T\big) = 1 \tag{3.12}$$

so that equality (3.8) holds, and $\mathbb{E}_{\mathbb{P}}\big(e^{-\varrho(X-\Psi)}\big) = 1$. Combining (3.11) and (3.12), we conclude that $p^*_{\mathbb{F}}(v) = \mathbb{E}_{\mathbb{P}}(\zeta_T\Psi)$. □

We briefly provide the analog of (3.7) for the $\mathbb{F}$-Hodges selling price of X . We have $p_*^{\mathbb{F}}(v) = \mathbb{E}_{\mathbb{P}}(\zeta_T\widetilde{\Psi})$, where

$$\widetilde{\Psi} = \frac{1}{\varrho}\ln\big((1 - F_T)e^{\varrho X_1} + F_T e^{\varrho X_2}\big). \tag{3.13}$$

Remark. It is important to notice that the $\mathbb{F}$-Hodges prices $p^*_{\mathbb{F}}(v)$ and $p^{\mathbb{F}}_*(v)$ do not depend on the initial endowment v. This is an interesting property of the exponential utility function. In view of (3.8), the random variable Ψ will be called the *indifference conditional hedge.*

Comparison with the Davis price. Let us present the results derived from the marginal utility pricing approach. The *Davis price* (see Davis (1997)) is given by

$$d^*(v) = \frac{\mathbb{E}_{\mathbb{P}}\big\{u'\big(V_T^{v,*}\big)X\big\}}{\mathcal{V}'(v)} .$$

In our context, this yields

$$d^*(v) = \mathbb{E}_{\mathbb{P}}\big\{\zeta_T\big(X_1 F_T + X_2(1 - F_T)\big)\big\}.$$

In this case, the risk aversion ϱ has no influence on the pricing of the contingent claim. In particular, when F is deterministic, the Davis price reduces to the arbitrage price of each (default-free) financial asset X^i, $i = 1, 2$, weighted by the corresponding probabilities F_T and $1 - F_T$.

Risk-Neutral Spread Versus Hodges Spreads

Let us consider the case of a defaultable bond with zero recovery, so that $X_1 = 1$ and $X_2 = 0$. It follows from (3.13) that the $\mathbb{F}$-Hodges buying and selling prices of the bond are (it will be convenient here to indicate the dependence of the Hodges price on maturity T)

$$D^*_{\mathbb{F}}(0, T) = -\frac{1}{\varrho}\, \mathbb{E}_{\mathbb{P}}\big\{\zeta_T \ln(e^{-\varrho}(1 - F_T) + F_T)\big\}$$

and

$$D^{\mathbb{F}}_*(0, T) = \frac{1}{\varrho}\, \mathbb{E}_{\mathbb{P}}\big\{\zeta_T \ln(e^{\varrho}(1 - F_T) + F_T)\big\},$$

respectively. Let $\widetilde{\mathbb{Q}}$ be a risk-neutral probability for the filtration $\mathbb{G}$, that is, for the enlarged market. The "market" price at time $t = 0$ of defaultable bond, denoted as $D^0(0, T)$, is thus equal to the expectation under $\widetilde{\mathbb{Q}}$ of its discounted pay-off, that is,

$$D^0(0, T) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(\mathbb{1}_{\{\tau > T\}} R_T\big) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\big((1 - \widetilde{F}_T) R_T\big),$$

where $\widetilde{F}_t = \widetilde{\mathbb{Q}}\,\{\tau \le t \,|\, \mathcal{F}_t\}$ for every $t \in [0, T]$. Let us emphasize that the risk-neutral probability $\widetilde{\mathbb{Q}}$ is chosen by the market, via the price of the defaultable asset. Hence, it should not be confused with the probability measure $\mathbb{Q}$, which combines, in a sense, the risk-neutral probability for the default-free market (Z^1, Z^2) with the real-life intensity of default.

Let us recall that in our setting the price process of the T-maturity unit discount Treasury (default-free) bond is $B(t, T) = e^{-r(T-t)}$. The Hodges buying and selling spreads at time $t = 0$ are defined as

$$S^*(0, T) = -\frac{1}{T} \ln \frac{D^*_{\mathbb{F}}(0, T)}{B(0, T)}$$

and

$$S_*(0,T) = -\frac{1}{T} \ln \frac{D_*^{\mathbb{F}}(0,T)}{B(0,T)} \,,$$

respectively. Likewise, the *risk-neutral spread* at time $t = 0$ is given as

$$S^0(0,T) = -\frac{1}{T} \ln \frac{D^0(0,T)}{B(0,T)} \,.$$

Since $D_{\mathbb{F}}^*(0,0) = D_*^{\mathbb{F}}(0,0) = D^0(0,0) = 1$, the respective *backward short spreads* at time $t = 0$ are given by the following limits (provided the limits exist)

$$s^*(0) = \lim_{T \downarrow 0} S^*(0,T) = -\frac{d^+ \ln D_{\mathbb{F}}^*(0,T)}{dT}\Big|_{T=0} - r$$

and

$$s_*(0) = \lim_{T \downarrow 0} S_*(0,T) = -\frac{d^+ \ln D_*^{\mathbb{F}}(0,T)}{dT}\Big|_{T=0} - r,$$

respectively. We also set

$$s^0(0) = \lim_{T \downarrow 0} S^0(0,T) = -\frac{d^+ \ln D^0(0,T)}{dT}\Big|_{T=0} - r.$$

Assuming, as we do, that the processes $\widetilde{F}_T$ and F_T are absolutely continuous with respect to the Lebesgue measure, and using the observation that the restriction of $\widetilde{\mathbb{Q}}$ to $\mathcal{F}_T$ is equal to $\mathbb{Q}$, we find out that

$$\begin{aligned}
\frac{D_{\mathbb{F}}^*(0,T)}{B(0,T)} &= -\frac{1}{\varrho}\, \mathbb{E}_{\mathbb{Q}} \big\{ \ln \big(e^{-\varrho}(1 - F_T) + F_T \big) \big\} \\
&= -\frac{1}{\varrho}\, \mathbb{E}_{\mathbb{Q}} \Big\{ \ln \Big(e^{-\varrho} \Big(1 - \int_0^T f_t \, dt \Big) + \int_0^T f_t \, dt \Big) \Big\},
\end{aligned}$$

and

$$\begin{aligned}
\frac{D_*^{\mathbb{F}}(0,T)}{B(0,T)} &= \frac{1}{\varrho}\, \mathbb{E}_{\mathbb{Q}} \big\{ \ln \big(e^{\varrho}(1 - F_T) + F_T \big) \big\} \\
&= \frac{1}{\varrho}\, \mathbb{E}_{\mathbb{Q}} \Big\{ \ln \Big(e^{\varrho} \Big(1 - \int_0^T f_t \, dt \Big) + \int_0^T f_t \, dt \Big) \Big\}.
\end{aligned}$$

Furthermore,

$$\frac{D^0(0,T)}{B(0,T)} = \mathbb{E}_{\mathbb{Q}}(1 - \widetilde{F}_T) = \mathbb{E}_{\mathbb{Q}} \Big(1 - \int_0^T \widetilde{f}_t \, dt \Big).$$

Consequently,

$$s^*(0) = \frac{1}{\varrho} \big(e^{\varrho} - 1 \big) f_0, \quad s_*(0) = \frac{1}{\varrho} \big(1 - e^{-\varrho} \big) f_0,$$

and $s^0(0) = \widetilde{f}_0$. Now, if we postulate, for instance, that $s_*(0) = s^0(0)$ (it would be the case if the market price is the selling Hodges price), then we must have

$$\widetilde{\gamma}_0 = \widetilde{f}_0 = \frac{1}{\varrho}\big(1 - e^{-\varrho}\big) f_0 = \frac{1}{\varrho}\big(1 - e^{-\varrho}\big)\gamma_0$$

so that $\widetilde{\gamma}_0 < \gamma_0$. Observe, however, that the case when the market price were equal to the buying Hodges price, that is $s^*(0) = s^0(0)$ would necessitate that $\widetilde{\gamma}_0 > \gamma_0$. Similar calculations can be made for any $t \in [0, T)$.

3.2 Optimization Problems and BSDEs

The major distinction between this section and the previous one is that here we consider strategies ϕ that are predictable with respect to the full filtration $\mathbb{G}$. Unless explicitly stated otherwise, the underlying probability measure is the real-world probability $\mathbb{P}$. We consider the following dynamics for the risky asset Z^1

$$dZ^1_t = Z^1_{t-}(\nu dt + \sigma dW_t + \varphi dM_t), \tag{3.14}$$

where $M_t = H_t - \int_0^{t\wedge\tau} \gamma_s\, ds$, and where we impose the condition $\varphi > -1$, which ensures that the price Z^1_t remains strictly positive.

In order to simplify notation, we shall denote by ξ the process such that $dM_t = dH_t - \xi_t\, dt$ is a $\mathbb{G}$-martingale, i.e., $\xi_t = \gamma_t(1 - H_t)$. We assume that the hypothesis (H) holds, that is, any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale as well.

Throughout most of the section, we shall deal with the same market model as in the previous section, that is, we shall set $\varphi = 0$. Only in Section 3.4 we generalize the dynamics of the risky asset to the case when $\varphi \neq 0$, so that the dynamics of the risky asset Z^1 are sensitive to the default risk. In particular, the limit case $\varphi = -1$ corresponds to the case where the underlying risky asset has value 0 after the default.

We assume for simplicity that $r = 0$, and we change the notational convention for an admissible portfolio to the one that will be more suitable for problems considered here: instead of using the number of shares ϕ as before, we set $\pi = \phi Z^1$, so that π represents the value invested in the risky asset. The portfolio process π_t should not be confused with the arbitrage price process $\pi_t(X)$. In addition, we adopt here the following relaxed definition of admissibility of a self-financing trading strategy.

Definition 3.2 The class $\Pi(\mathbb{F})$ (the class $\Pi(\mathbb{G})$, respectively) of $\mathbb{F}$-*admissible* ($\mathbb{G}$-*admissible*, respectively) trading strategies is the set of all $\mathbb{F}$-predictable ($\mathbb{G}$-predictable, respectively) processes π such that $\int_0^T \pi_t^2\, dt < \infty$, $\mathbb{P}$-a.s.

The wealth process of a strategy π satisfies

$$dV_t(\pi) = \pi_t\big(\nu dt + \sigma dW_t + \varphi dM_t\big). \tag{3.15}$$

Note that with the present definition of admissible strategies the "martingale part" of the wealth process is a local martingale, in general.

Let X be a given contingent claim, represented by a $\mathcal{G}_T$-measurable random variable. We shall study the following problem:

$$\sup_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big\{u\big(V_T^v(\pi) + X\big)\big\}.$$

3.2.1 Exponential Utility

In this section, we shall examine the problem introduced above in the case of the exponential utility, and setting $\varphi = 0$ in dynamics (3.14). First, we examine the existence and the form of a solution to the optimization problem, under additional technical assumptions. Subsequently, we shall derive the expression for the Hodges buying price.

Optimization Problem

Let $X \in \mathcal{G}_T$ be a given non-negative contingent claim, and let v be the initial endowment of an agent. Our first goal is to solve an optimization problem for an agent who buys a claim X. To this end, it suffices to find a strategy $\pi \in \Pi(\mathbb{G})$ that maximizes $\mathbb{E}_{\mathbb{P}}(u(V_T^v(\pi) + X))$, where the wealth process $V_t = V_t^v(\pi)$ (for simplicity, we shall frequently skip v and π from the notation) satisfies

$$dV_t = \phi_t \, dZ_t^1 = \pi_t(\nu dt + \sigma dW_t), \ V_0 = v.$$

We consider the exponential utility function $u(x) = 1 - e^{-\varrho x}$, with $\varrho > 0$. Therefore, we deal with the following problem:

$$\sup_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big\{u(V_T^v(\pi) + X)\big\} = 1 - \inf_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X}\big).$$

Let us describe the idea of a solution. Suppose that we can find a process Z with $Z_T = e^{-\varrho X}$, which depends only on the claim X and parameters ϱ, σ, ν, and such that the process $e^{-\varrho V_t^v(\pi)} Z_t$ is a $\mathbb{G}$-submartingale under $\mathbb{P}$ for any admissible strategy π and is a martingale under $\mathbb{P}$ for some admissible strategy $\pi^* \in \Pi(\mathbb{G})$. Then, we would have

$$\mathbb{E}_{\mathbb{P}}(e^{-\varrho V_T^v(\pi)} Z_T) \geq e^{-\varrho V_0^v(\pi)} Z_0 = e^{-\varrho v} Z_0$$

for any $\pi \in \Pi(\mathbb{G})$, with equality for some strategy $\pi^* \in \Pi(\mathbb{G})$. Consequently, we would obtain

$$\inf_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X}\big) = \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi^*)} e^{-\varrho X}\big) = e^{-\varrho v} Z_0, \tag{3.16}$$

and thus we would be in a position to conclude that π^* is an optimal strategy. In fact, it will turn out that in order to implement the above idea we shall need to restrict further the class of $\mathbb{G}$-admissible trading strategies.

We shall search for an auxiliary process Z in the class of all processes satisfying the following backward stochastic differential equation (BSDE)

$$dZ_t = f_t \, dt + \widehat{z}_t \, dW_t + \widetilde{z}_t \, dM_t, \ t \in [0, T), \ Z_T = e^{-\varrho X}, \tag{3.17}$$

where the process f will be determined later (see equation (3.19) below). By applying Itô's formula, we obtain

$$d(e^{-\varrho V_t}) = e^{-\varrho V_t}\left(\left(\tfrac{1}{2}\varrho^2\pi_t^2\sigma^2 - \varrho\pi_t\nu\right)dt - \varrho\pi_t\sigma\,dW_t\right),$$

so that

$$\begin{aligned} d(e^{-\varrho V_t}Z_t) &= e^{-\varrho V_t}\big(f_t + Z_t(\tfrac{1}{2}\varrho^2\pi_t^2\sigma^2 - \varrho\pi_t\nu) - \varrho\pi_t\sigma\widehat{z}_t\big)dt \\ &\quad + e^{-\varrho V_t}\big((\widehat{z}_t - \varrho\pi_t\sigma Z_t)\,dW_t + \widetilde{z}_t\,dM_t\big). \end{aligned}$$

Let us choose π^* such that it minimizes, for every t, the following expression

$$Z_t\big(\tfrac{1}{2}\varrho^2\pi_t^2\sigma^2 - \varrho\pi_t\nu\big) - \varrho\pi_t\sigma\widehat{z}_t = -\varrho\pi_t(\nu Z_t + \sigma\widehat{z}_t) + \tfrac{1}{2}\varrho^2\pi_t^2\sigma^2 Z_t.$$

It is easily seen that

$$\pi_t^* = \frac{\nu Z_t + \sigma\widehat{z}_t}{\varrho\sigma^2 Z_t} = \frac{1}{\varrho\sigma}\left(\theta + \frac{\widehat{z}_t}{Z_t}\right). \tag{3.18}$$

Now, let us choose the process f, by postulating that

$$\begin{aligned} f_t = f(Z_t,\widehat{z}_t) &= Z_t\left(\varrho\pi_t^*\nu - \tfrac{1}{2}\varrho^2(\pi_t^*)^2\sigma^2\right) + \varrho\pi_t^*\sigma\widehat{z}_t \\ &= \varrho\pi_t^*(Z_t\nu + \sigma\widehat{z}_t) - \tfrac{1}{2}\varrho^2(\pi_t^*)^2\sigma^2 Z_t = \frac{(\nu Z_t + \sigma\widehat{z}_t)^2}{2\sigma^2 Z_t}. \end{aligned} \tag{3.19}$$

In other words, we shall focus on the following BSDE:

$$dZ_t = \frac{(\nu Z_t + \sigma\widehat{z}_t)^2}{2\sigma^2 Z_t}\,dt + \widehat{z}_t\,dW_t + \widetilde{z}_t\,dM_t,\ t\in[0,T[,\ Z_T = e^{-\varrho X}. \tag{3.20}$$

Recall that W is a Brownian motion under $\mathbb{P}$, and that the risk-neutral probability $\mathbb{Q}$ is given by $d\mathbb{Q}|_{\mathcal{F}_t} = \eta_t\,d\mathbb{P}|_{\mathcal{F}_t}$, where $d\eta_t = -\eta_t\theta\,dW_t$ with $\theta = \nu/\sigma$ and $\eta_0 = 1$. Thus the process $W_t^{\mathbb{Q}} = W_t + \theta t$, $t\in[0,T]$, is a Brownian motion under $\mathbb{Q}$. It will be convenient to write equation (3.20) as

$$dZ_t = \big(\tfrac{1}{2}\theta^2 Z_t + \theta\widehat{z}_t + \tfrac{1}{2}Z_t^{-1}\widehat{z}_t^2\big)dt + \widehat{z}_t\,dW_t + \widetilde{z}_t\,dM_t,\ t\in[0,T[,\ Z_T = e^{-\varrho X}.$$

Equivalently,

$$dZ_t = \big(\tfrac{1}{2}\theta^2 Z_t + \tfrac{1}{2}Z_t^{-1}\widehat{z}_t^2\big)dt + \widehat{z}_t\,dW_t^{\mathbb{Q}} + \widetilde{z}_t\,dM_t,\ t\in[0,T[,\ Z_T = e^{-\varrho X}. \tag{3.21}$$

Remark. To the best of out knowledge, no general theorem, which would establish the existence of a solution to equation (3.21), is available. The comparison theorem works for BSDEs driven by a jump process when the drift satisfies some Lipschitz condition (see Royer (2003)). Hence, the proofs of Lepeltier and San-Martin (1997) and Kobylanski (2000), which rely on comparison results, may not be directly carried to the case of quadratic BSDEs driven by a jump process. We shall solve the BSDE (3.21) under rather restrictive assumptions on X. Hence, the general case remains an open problem.

Lemma 3.2 *Assume that there exists $\mathbb{G}$-predictable processes $\widehat{k}$, $\widetilde{k} > -1$ and a constant c such that*

$$\exp(K_T)\mathcal{E}_T(\widetilde{M}) = e^{-\varrho X}, \tag{3.22}$$

where

$$K_t = c + \int_0^t \widehat{k}_u \, dW_u^{\mathbb{Q}}, \quad \widetilde{M}_t = \int_0^t \widetilde{k}_u \, dM_u,$$

and $\mathcal{E}(\widetilde{M})$ is the Doléans exponential of $\widetilde{M}$. Then $U_t = \exp(K_t)\mathcal{E}_t(\widetilde{M})$ solves the following BSDE

$$dU_t = \tfrac{1}{2} U_t^{-1} \widehat{u}_t^2 \, dt + \widehat{u}_t \, dW_t^{\mathbb{Q}} + \widetilde{u}_t \, dM_t, \ t \in [0, T[, \ U_T = e^{-\varrho X}. \tag{3.23}$$

Proof. Since $d\mathcal{E}_t(\widetilde{M}) = \mathcal{E}_{t-}(\widetilde{M}) \, d\widetilde{M}_t$, the process U defined above satisfies

$$dU_t = \tfrac{1}{2} U_t \widehat{k}_t^2 \, dt + U_t \widehat{k}_t \, dW_t^{\mathbb{Q}} + U_{t-} \widetilde{k}_t \, dM_t$$

and thus

$$dU_t = \tfrac{1}{2} U_t^{-1} \widehat{u}_t^2 \, dt + \widehat{u}_t \, dW_t^{\mathbb{Q}} + \widetilde{u}_t \, dM_t$$

where we denote $\widehat{u}_t = U_t \widehat{k}_t$ and $\widetilde{u}_t = U_{t-} \widetilde{k}_t$. Since obviously $U_T = e^{-\varrho X}$, this ends the proof. □

Corollary 3.1 *Let X be a $\mathcal{G}_T$-measurable claim such that* (3.22) *holds for some $\mathbb{G}$-predictable processes $\widehat{k}$, $\widetilde{k} > -1$ and some constant c. Then there exists a solution $(Z, \widehat{z}, \widetilde{z})$ of the BSDE* (3.21)*. Moreover, the process Z is strictly positive.*

Proof. Let us set $Y_t = e^{-(T-t)\theta^2/2}$ and let U be the process introduced in Lemma 3.2. Then the process $Z_t = U_t Y_t$ satisfies

$$\begin{aligned}
dZ_t &= Y_t \, dU_t + \tfrac{1}{2}\theta^2 Y_t U_t \, dt \\
&= \tfrac{1}{2}\theta^2 Y_t U_t \, dt + \tfrac{1}{2} Y_t U_t^{-1} \widehat{u}_t^2 \, dt + Y_t \widehat{u}_t \, dW_t^{\mathbb{Q}} + Y_t \widetilde{u}_t \, dM_t \\
&= \tfrac{1}{2}\theta^2 Z_t \, dt + \tfrac{1}{2} Z_t^{-1} Y_t^2 \widehat{u}_t^2 \, dt + Y_t \widehat{u}_t \, dW_t^{\mathbb{Q}} + Y_t \widetilde{u}_t \, dM_t \\
&= \tfrac{1}{2}\theta^2 Z_t \, dt + \tfrac{1}{2} Z_t^{-1} \widehat{z}_t^2 \, dt + \widehat{z}_t \, dW_t^{\mathbb{Q}} + \widetilde{z}_t \, dM_t
\end{aligned}$$

where we set $\widehat{z}_t = Y_t \widehat{u}_t$ and $\widetilde{z}_t = Y_t \widetilde{u}_t$. It is also clear that $Z_T = U_T = e^{-\varrho X}$ and Z is strictly positive. □

Recall that the process Z depends on the choice of a contingent claim X, as well as on the model's parameters ϱ, σ and ν. The next lemma shows that the processes Z and π^* introduced above have indeed the desired properties that were described at the beginning of this section. To achieve our goal, we need to restrict the class of admissible trading strategies, however. We say that an admissible strategy π is *regular with respect to* X if the martingale part of the process $e^{-\varrho V_t^v(\pi)} Z_t$ is a martingale under $\mathbb{P}$, rather than a local martingale. We denote by $\Pi_X(\mathbb{G})$ the class of all admissible trading strategies, which are regular with respect to X.

Lemma 3.3 *Let X be a $\mathcal{G}_T$-measurable claim such that* (3.22) *holds for some $\mathbb{G}$-predictable processes $\widehat{k}$, $\widetilde{k}$ and some constant c. Assume that the default intensity γ and the processes $\widetilde{k}$, $\widehat{k}$ are bounded. Suppose that the process $Z = Z(X, \varrho, \sigma, \nu)$ is a solution to the BSDE* (3.20) *given in Corollary 3.1. Then:*
(i) *The process $e^{-\varrho V_t^v(\pi)} Z_t$ is a submartingale for any strategy $\pi \in \Pi_X(\mathbb{G})$.*
(ii) *The process $e^{-\varrho V_t^v(\pi^*)} Z_t$ is a martingale for the process π^* given by expression* (3.18).
(iii) *The process π^* belongs to the class $\Pi_X(\mathbb{G})$ of admissible trading strategies regular with respect to X.*

Proof. In view of the definition of π^* and the choice of the process f (see formula (3.19)), the validity of part (i) is rather clear. To establish (ii), we shall first check that the process $e^{-\varrho V_t^*} Z_t$ is a martingale (and not only a local martingale) under $\mathbb{P}$, where $V_t^* = V_t^v(\pi^*)$. From the choice of π^*, we obtain

$$\begin{aligned} d(e^{-\varrho V_t^*} Z_t) &= e^{-\varrho V_t^*} \big((\widehat{z}_t - \varrho \pi_t^* \sigma Z_t)\, dW_t + \widetilde{z}_t\, dM_t \big) \\ &= -\theta e^{-\varrho V_t^*} Z_t\, dW_t + e^{-\varrho V_t^*} \widetilde{z}_t\, dM_t. \end{aligned}$$

This means that

$$e^{-\varrho V_t^*} Z_t = e^{-\varrho v} Z_0 \exp\big(-\theta W_t - \tfrac{1}{2}\theta^2 t \big) \exp\Big(-\int_0^t \frac{\widetilde{z}_s}{Z_s}\, \xi_s\, ds \Big) \Big(1 + \frac{\widetilde{z}_{\tau-}}{Z_{\tau-}}\, H_t \Big).$$

The quantity $e^{-\varrho v} Z_0 \exp\big(-\theta W_t - \frac{1}{2}\theta^2 t \big)$ is clearly a continuous martingale under $\mathbb{P}$. Recall that

$$\widetilde{z}_t = Y_t \widetilde{u}_t = \widetilde{k}_t Z_t.$$

and thus $\widetilde{z}_t / Z_t = \widetilde{k}_t$ is a bounded process. We conclude that the process

$$\exp\Big(-\int_0^t \frac{\widetilde{z}_s}{Z_s}\, \xi_s\, ds \Big) \Big(1 + \frac{\widetilde{z}_{\tau-}}{Z_{\tau-}}\, H_t \Big)$$

is a bounded, purely discontinuous martingale under $\mathbb{P}$. To complete the proof, it remains to check that the process π^* given by (3.18) is $\mathbb{G}$-admissible, in the sense of Definition 3.2. To this end, it suffices to check that

$$\int_0^T \widehat{z}_t^2 Z_t^{-2}\, dt < \infty, \quad \mathbb{P}\text{-a.s.}$$

This is clear since the process $\widehat{z}_t / Z_t = \widehat{k}_t$ is bounded. We conclude that the strategy π^* belongs to the class $\Pi_X(\mathbb{G})$. $\Box$

Recall now that in this section we examine the following problem:

$$\sup_{\pi \in \Pi_X(\mathbb{G})} \mathbb{E}_{\mathbb{P}} \big(u(V_T^v(\pi) + X) \big) = 1 - \inf_{\pi \in \Pi_X(\mathbb{G})} \mathbb{E}_{\mathbb{P}} \big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X} \big).$$

We are in a position to state the following result.

Proposition 3.2 *Let X be a $\mathcal{G}_T$-measurable claim such that (3.22) holds for some $\mathbb{G}$-predictable processes $\widehat{k}$, $\widetilde{k}$ and some constant c. Assume that the default intensity γ and the processes $\widetilde{k}$, $\widehat{k}$ are bounded. Then*

$$\inf_{\pi \in \Pi_X(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X}\big) = \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V_T^v(\pi^*)} e^{-\varrho X}\big) = e^{-\varrho v} Z_0^X,$$

where the optimal strategy $\pi^* \in \Pi_X(\mathbb{G})$ is given by the formula, for every $t \in [0,T]$,

$$\pi_t^* = \frac{1}{\varrho\sigma}\Big(\theta + \frac{\widehat{z}_t^X}{Z_t^X}\Big) = \frac{\theta + \widehat{k}_t}{\varrho\sigma},$$

where $Z_t^X = Z_t$ and $\widehat{z}_t^X = \widehat{z}_t$ are the two first components of a solution $(Z_t, \widehat{z}_t, \widetilde{z})$ of the BSDE

$$dZ_t = \frac{(\nu Z_t + \sigma \widehat{z}_t)^2}{2\sigma^2 Z_t}\, dt + \widehat{z}_t\, dW_t + \widetilde{z}_t\, dM_t, \; Z_T = e^{-\varrho X}. \tag{3.24}$$

More explicitly (see Corollary 3.1), we have $\widehat{z}_t = \widehat{k}_t Z_t$ and

$$Z_t = e^{-(T-t)\theta^2/2} \exp(K_t) \mathcal{E}_t(\widetilde{M}).$$

Proof. The proof is rather straightforward. We know that the process Z which solves (3.24) is such that: (i) the process $Z_t e^{-\varrho V_t^v(\pi^*)}$ is a martingale, and (ii) for any strategy $\pi \in \Pi_X(\mathbb{G})$ the process $Z_t e^{-\varrho V_t^v(\pi)}$ is equal to a martingale minus an increasing process (since the drift term is non-positive), and thus it is a submartingale. This shows that (3.16) holds with $\Pi(\mathbb{G})$ substituted with $\Pi_X(\mathbb{G})$. □

It should be acknowledged that the assumptions of Proposition 3.2 are restrictive, so that it covers only a very special case of a claim X. Let us now comment briefly on the case of a general claim; we do not pretend here to give strict results, our aim is merely to give some hints how one can deal with the general case.

Recall that our aim is to find a solution $(Z, \widehat{z}, \widetilde{z})$ of the following BSDE

$$dZ_t = \big(\tfrac{1}{2}\theta^2 Z_t + \tfrac{1}{2} Z_t^{-1} \widehat{z}_t^2\big)dt + \widehat{z}_t\, dW_t^{\mathbb{Q}} + \widetilde{z}_t\, dM_t, \; t \in [0,T[, \; Z_T = e^{-\varrho X},$$

or equivalently, of the equation

$$dU_t = \tfrac{1}{2} U_t^{-1} \widehat{u}_t^2\, dt + \widehat{u}_t\, dW_t^{\mathbb{Q}} + \widetilde{u}_t\, dM_t, \; t \in [0,T[, \; U_T = e^{-\varrho X},$$

Assume that the process U is strictly positive and set $X_t = \ln U_t$. Then, denoting $\widehat{x}_t = \widehat{u}_t U_t^{-1}, \widetilde{x}_t = \widetilde{u}_t U_{t-}^{-1}$ and applying Itô's formula, we obtain (recall that we denote $\xi_t = \gamma_t \mathbb{1}_{\{\tau > t\}}$)

$$\begin{aligned} dX_t &= \widehat{x}_t\, dW_t^{\mathbb{Q}} + \widetilde{x}_t\, dM_t + (\ln(1+\widetilde{x}_t) - \widetilde{x}_t)\, dH_t \\ &= \widehat{x}_t\, dW_t^{\mathbb{Q}} + \widetilde{x}_t\, dM_t + (\ln(1+\widetilde{x}_t) - \widetilde{x}_t)(dM_t + \xi_t\, dt) \\ &= \widehat{x}_t\, dW_t^{\mathbb{Q}} + \ln(1+\widetilde{x}_t)\, dM_t + (\ln(1+\widetilde{x}_t) - \widetilde{x}_t)\xi_t\, dt \\ &= \widehat{x}_t\, dW_t^{\mathbb{Q}} + x_t^*\, dM_t + (1 - e^{x_t^*} + x_t^*)\xi_t\, dt \\ &= \widehat{x}_t\, dW_t^{\mathbb{Q}} + x_t^*\, dH_t + (1 - e^{x_t^*})\xi_t\, dt, \end{aligned}$$

where $x_t^* = \ln(1+\widetilde{x}_t)$ and the terminal condition is $X_T = -\varrho X$. It thus suffices to solve the following BSDE

$$dX_t = \widehat{x}_t \, dW_t^{\mathbb{Q}} + x_t^* \, dH_t + (1 - e^{x_t^*})\xi_t \, dt, \ t \in [0,T[, \ X_T = -\varrho X. \tag{3.25}$$

Assume first that $X \in \mathcal{F}_T$. In that case, it is obvious that we may take $\widehat{x} = \widetilde{x}^* = 0$ and thus $X_t = -\mathbb{E}_{\mathbb{Q}}(\varrho X \,|\, \mathcal{G}_t) = -\mathbb{E}_{\mathbb{Q}}(\varrho X \,|\, \mathcal{F}_t)$ is a solution. In the general case, we note that the continuous $\mathbb{G}$-martingales are stochastic integrals with respect to the Brownian motion $W^{\mathbb{Q}}$. We may thus transform the problem: it suffices to find a process x^* such that the process R, defined through the formula

$$R_t = \mathbb{E}_{\mathbb{Q}}\Big(- \varrho X + \int_0^T (e^{x_s^*} - 1)\xi_s \, ds - x_\tau^* \mathbb{1}_{\{\tau \le T\}} \,\Big|\, \mathcal{G}_t \Big),$$

is a continuous $\mathbb{G}$-martingale, so that $dR_t = \widehat{x}_t \, dW_t^{\mathbb{Q}}$ for some $\mathbb{G}$-predictable process $\widehat{x}$. Suppose that we can find a process x^* for which the last property is valid. Then, by setting

$$\begin{aligned} X_t &= R_t - \int_0^t (e^{x_s^*} - 1)\xi_s \, ds - x_\tau^* \mathbb{1}_{\{\tau \le t\}} \\ &= \mathbb{E}_{\mathbb{Q}}\Big(- \varrho X + \int_t^T (e^{x_s^*} - 1)\xi_s \, ds - x_\tau^* \mathbb{1}_{\{t < \tau \le T\}} \,\Big|\, \mathcal{G}_t \Big) \end{aligned}$$

we obtain a solution $(X, \widehat{x}, x^*)$ to (3.25).

Case of a survival claim. From now on, we shall focus on a survival claim $X = Y \mathbb{1}_{\{\tau > T\}}$, where Y is an $\mathcal{F}_T$-measurable random variable. Let us fix $t \in [0,T]$. On the set $\{t \le \tau\}$ we obtain

$$\mathbb{E}_{\mathbb{Q}}(\varrho Y \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t) = e^{\Gamma_t} \, \mathbb{E}_{\mathbb{Q}}(e^{-\Gamma_T} \varrho Y \,|\, \mathcal{F}_t)$$

and on the set $\{\tau < t\}$, we have $\mathbb{E}_{\mathbb{Q}}(\varrho Y \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t) = 0$. The jump of the term A_t, defined as

$$A_t = \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T (e^{x_s^*} - 1)\xi_s \, ds - x_\tau^* \mathbb{1}_{\{\tau \le T\}} \,\Big|\, \mathcal{G}_t \Big),$$

can be computed as follows. On the set $\{t \le \tau\}$, we obtain

$$\begin{aligned} A_t &= \int_t^T \mathbb{E}_{\mathbb{Q}}\big((e^{x_s^*} - 1)\gamma_s \mathbb{1}_{\{\tau > s\}} \,|\, \mathcal{G}_t\big) \, ds - \mathbb{E}_{\mathbb{Q}}\big(x_\tau^* \mathbb{1}_{\{\tau \le T\}} \,|\, \mathcal{G}_t\big) \\ &= \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \, \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T \big(e^{x_s^*} - 1 - x_s^*\big) e^{-\Gamma_s} \gamma_s \, ds \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

On the set $\{\tau < t\}$ for A_t we have

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T (e^{x_s^*} - 1)\gamma_s \mathbb{1}_{\{\tau > s\}} \, ds - x_\tau^* \mathbb{1}_{\{\tau \le T\}} \,\Big|\, \mathcal{G}_t \Big) = -\mathbb{E}_{\mathbb{Q}}\big(x_\tau^* \mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{G}_t\big) = -x_\tau^*.$$

We conclude that our problem is to find a process x^* such that

$$-\mathbb{E}_{\mathbb{Q}}\big(e^{-\Gamma_T}\varrho Y|\mathcal{F}_t\big) = -e^{-\Gamma_t}x^*_t - \mathbb{E}_{\mathbb{Q}}\Big(\int_t^T (e^{x^*_s} - 1 - x^*_s)e^{-\Gamma_s}\gamma_s\, ds \,\Big|\, \mathcal{F}_t\Big).$$

In other words, we need to solve the following BSDE with $\mathbb{F}$-adapted processes x^* and κ

$$d\big(x^*_t e^{-\Gamma_t}\big) = \big(e^{x^*_t} - 1 - x^*_t\big)e^{-\Gamma_t}\gamma_t\, dt + \kappa_t\, dW^{\mathbb{Q}}_t,\ t \in [0, T[,\ x^*_T = \varrho Y.$$

¿From integration by parts, this BSDE can be written

$$dx^*_t = \big(e^{x^*_t} - 1\big)e^{-\Gamma_t}\gamma_t\, dt + \kappa_t\, dW^{\mathbb{Q}}_t,\ t \in [0, T[,\ x^*_T = \varrho X.$$

Unfortunately, the standard results for existence of solutions to BSDEs do not apply here because the drift term is not of a linear growth with respect to x^*.

3.2.2 Hodges Buying and Selling Prices

Particular case. Assume, as before, that $r = 0$ and let us check that the Hodges buying price is the hedging price in case of attainable claims. Assume that a claim X is $\mathcal{F}_T$-measurable. By virtue of the predictable representation theorem, there exists a pair $(x, \widehat{x})$, where x is a constant and $\widehat{x}_t$ is an $\mathbb{F}$-adapted process, such that $X = x + \int_0^T \widehat{x}_u\, dW^{\mathbb{Q}}_u$, where $W^{\mathbb{Q}}_t = W_t + \theta t$. Here $x = \mathbb{E}_{\mathbb{Q}}X$ is the arbitrage price $\pi_0(X)$ of X and the replicating portfolio is obtained through $\widehat{x}$. Hence, the time t value of X is $X_t = x + \int_0^t \widehat{x}_u\, dW^{\mathbb{Q}}_u$. Then $dX_t = \widehat{x}_t\, dW^{\mathbb{Q}}_t$ and the process

$$Z_t = e^{-\theta^2(T-t)/2}e^{-\varrho X_t}$$

satisfies

$$\begin{aligned} dZ_t &= Z_t\Big(\big(\tfrac{1}{2}\theta^2 + \tfrac{1}{2}\varrho^2\widehat{x}^2_t\big)\, dt + \varrho\widehat{x}_t\, dW^{\mathbb{Q}}_t\Big) \\ &= \frac{1}{2\sigma^2 Z_t}(\nu Z_t + \sigma\varrho Z_t\widehat{x}_t)^2\, dt + \varrho Z_t\widehat{x}_t\, dW_t. \end{aligned}$$

Hence $(Z_t, \varrho Z_t\widehat{x}_t, 0)$ is the solution of (3.24) with the terminal condition $e^{-\varrho X}$, and

$$Z_0 = e^{-\theta^2 T/2}e^{-\varrho x}$$

Note that, for $X = 0$, we get $Z_0 = e^{-\theta^2 T/2}$, therefore

$$\inf_{\pi\in\Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V^v_T(\pi)}\big) = e^{-\varrho v}e^{-\theta^2 T/2}.$$

The $\mathbb{G}$-Hodges buying price of X is the value of p such that

$$\inf_{\pi\in\Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho V^v_T(\pi)}\big) = \inf_{\pi\in\Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big(e^{-\varrho(V^{v-p}_T(\pi)+X)}\big),$$

that is,

$$e^{-\varrho v} e^{-\theta^2 T/2} = e^{-\varrho(v-p+\pi_0(X))} e^{-\theta^2 T/2}.$$

We conclude easily that $p_*^{\mathbb{G}}(X) = \pi_0(X) = \mathbb{E}_{\mathbb{Q}} X$. Similar arguments show that $p_{\mathbb{G}}^*(X) = \pi_0(X)$.

General case. Assume now that a claim X is $\mathcal{G}_T$-measurable and the assumptions of Proposition 3.2 are satisfied. Since the process Z introduced in Corollary 3.1 is strictly positive, we can use its logarithm. Let us assume that the processes $\widehat{k}$ and $\widetilde{k}$ are strictly positive, and let us denote $\widehat{\psi}_t = Z_t/\widehat{z}_t = \widehat{k}_t^{-1}$, $\widetilde{\psi}_t = Z_t/\widetilde{z}_t = \widetilde{k}_t^{-1}$ and

$$\kappa_t = \frac{\widetilde{\psi}_t}{\ln(1+\widetilde{\psi}_t)} \geq 0.$$

Then we get

$$d(\ln Z_t) = \tfrac{1}{2}\,\theta^2 dt + \widehat{\psi}_t\, dW_t^{\mathbb{Q}} + \ln(1+\widetilde{\psi}_t)\big(dM_t + \xi_t(1-\kappa_t)\,dt\big),$$

and thus

$$d(\ln Z_t) = \tfrac{1}{2}\,\theta^2 dt + \widehat{\psi}_t\, dW_t^{\mathbb{Q}} + \ln(1+\widetilde{\psi}_t)\, d\widehat{M}_t,$$

where

$$d\widehat{M}_t = dM_t + \xi_t(1-\kappa_t)\,dt = dH_t - \xi_t\kappa_t\,dt.$$

The process $\widehat{M}$ is a martingale under the probability measure $\widehat{\mathbb{Q}}$ defined as $d\widehat{\mathbb{Q}}|_{\mathcal{G}_t} = \widehat{\eta}_t\, d\mathbb{P}|_{\mathcal{G}_t}$, where $\widehat{\eta}$ satisfies

$$d\widehat{\eta}_t = -\widehat{\eta}_{t-}\big(\theta\, dW_t + \xi_t(1-\kappa_t)\,dM_t\big)$$

with $\widehat{\eta}_0 = 1$.

Proposition 3.3 *The $\mathbb{G}$-Hodges buying price of X with respect to the exponential utility is the real number p such that $e^{-\varrho(v-p)} Z_0^X = e^{-\varrho v} Z_0^0$, that is, $p_{\mathbb{G}}^*(X) = \varrho^{-1}\ln(Z_0^0/Z_0^X)$ or, equivalently, $p_{\mathbb{G}}^*(X) = \mathbb{E}_{\widehat{\mathbb{Q}}} X$.*

Our previous study establishes that the dynamic hedging price of a claim X is the process $X_t = \mathbb{E}_{\widehat{\mathbb{Q}}}(X \,|\, \mathcal{G}_t)$. This price is the expectation of the payoff, under some martingale measure, as is any price in the range of no-arbitrage prices.

3.3 Quadratic Hedging

We assume here that the wealth process follows

$$dV_t^v(\pi) = \pi_t\big(\nu\, dt + \sigma\, dW_t\big), \quad V_0^v(\pi) = v,$$

where we assume that $\pi \in \Pi(\mathbb{F})$ or $\pi \in \Pi(\mathbb{G})$, depending on the case studied below. The more general case

$$dV_t^v(\pi) = \pi_t\big(\nu\, dt + \sigma\, dW_t + \varphi\, dM_t\big), \quad V_0^v(\pi) = v$$

is too long to be presented here. In this section, we examine the issue of the quadratic pricing and hedging, specifically, for a given $\mathbb{P}$-square-integrable claim X we solve the following minimization problems:

• For a given initial endowment v, solve the minimization problem:

$$\min_{\pi} \mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi) - X)^2\big).$$

A solution to this problem provides the portfolio which, among the portfolios with a given initial wealth, has the closest terminal wealth to a given claim X, in sense of L^2-norm under $\mathbb{P}$.

• Solve the minimization problem:

$$\min_{\pi, v} \mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi) - X)^2\big).$$

The minimal value of v is called the *quadratic hedging price* and the optimal π the *quadratic hedging strategy*.

The mean-variance hedging problem was examined in a fairly general framework of incomplete markets by means of BSDEs in several papers; see, for example, Mania (2000), Mania and Tevzadze (2003), Bobrovnytska and Schweizer (2004), Hu and Zhou (2004) or Lim (2004). Since this list is by no means exhaustive, the interested reader is referred to the references quoted in the above-mentioned papers.

3.3.1 Quadratic Hedging with $\mathbb{F}$-Adapted Strategies

We shall first solve, for a given initial endowment v, the following minimization problem

$$\min_{\pi \in \Pi(\mathbb{F})} \mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi) - X)^2\big),$$

where the claim $X \in \mathcal{G}_T$ is given as

$$X = X_1 \mathbb{1}_{\{\tau > T\}} + X_2 \mathbb{1}_{\{\tau \le T\}}$$

for some $\mathcal{F}_T$-measurable and $\mathbb{P}$-square-integrable random variables X_1 and X_2. Using the same approach as in the previous section, we define the auxiliary function J_X by setting

$$J_X(y) = (y - X_1)^2(1 - F_T) + (y - X_2)^2 F_T,$$

so that its derivative equals

$$J'_X(y) = 2\big(y - X_1(1 - F_T) - X_2 F_T\big).$$

Hence

$$I_X(z) = \tfrac{1}{2} z + X_1(1 - F_T) + X_2 F_T,$$

and thus the optimal terminal wealth equals

$$V_T^{v,*} = \tfrac{1}{2}\lambda^* \zeta_T + X_1(1-F_T) + X_2 F_T,$$

where λ^* is specified through the budget constraint:

$$\mathbb{E}_{\mathbb{P}}(\zeta_T V_T^{v,*}) = \tfrac{1}{2}\lambda^* \,\mathbb{E}_{\mathbb{P}}(\zeta_T^2) + \mathbb{E}_{\mathbb{P}}(\zeta_T X_1(1-F_T)) + \mathbb{E}_{\mathbb{P}}(\zeta_T X_2 F_T) = v.$$

We deduce that

$$\begin{aligned}
&\min_{\pi} \mathbb{E}_{\mathbb{P}}((V_T^v - X)^2) \\
&\quad = \mathbb{E}_{\mathbb{P}}\Big[\big(\tfrac{1}{2}\lambda^*\zeta_T + X_1(1-F_T) + X_2F_T - X_1\big)^2 (1-F_T)\Big] \\
&\qquad + \mathbb{E}_{\mathbb{P}}\Big[\big(\tfrac{1}{2}\lambda^*\zeta_T + X_1(1-F_T) + X_2F_T) - X_2\big)^2 F_T\Big] \\
&\quad = \tfrac{1}{4}(\lambda^*)^2\,\mathbb{E}_{\mathbb{P}}(\zeta_T^2) + \mathbb{E}_{\mathbb{P}}\big((X_1 - X_2)^2 F_T(1-F_T)\big) \\
&\quad = \frac{1}{2\mathbb{E}_{\mathbb{P}}(\zeta_T^2)}\Big(v - \mathbb{E}_{\mathbb{P}}(\zeta_T(X_1 + F_T(X_2 - X_1))\Big)^2 \\
&\qquad + \mathbb{E}_{\mathbb{P}}((X_1 - X_2)^2 F_T(1-F_T)).
\end{aligned}$$

Therefore, we obtain the following result.

Proposition 3.4 *If we restrict our attention to $\mathbb{F}$-adapted strategies, the quadratic hedging price of the claim $X = X_1 1\!\!1_{\{\tau > T\}} + X_2 1\!\!1_{\{\tau \le T\}}$ equals*

$$\mathbb{E}_{\mathbb{P}}\big(\zeta_T(X_1 + F_T(X_2 - X_1))\big) = \mathbb{E}_{\mathbb{Q}}\big(X_1(1-F_T) + F_T X_2\big).$$

The optimal quadratic hedging of X is the strategy which duplicates the $\mathcal{F}_T$-measurable contingent claim $X_1(1-F_T) + F_T X_2$.

Let us now examine the case of a generic $\mathcal{G}_T$-measurable random variable X. In this case, we shall only examine the solution of the second problem introduced above, that is,

$$\min_{v,\pi} \mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi) - X)^2\big).$$

As we have explained in the previous chapter, this problem is essentially equivalent to a problem where we restrict our attention to the terminal wealth. From the properties of conditional expectations, we have

$$\min_{V \in \mathcal{F}_T} \mathbb{E}_{\mathbb{P}}\big((V - X)^2\big) = \mathbb{E}_{\mathbb{P}}\big((\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_T) - X)^2\big)$$

and the initial value of the strategy with terminal value $\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_T)$ is

$$\mathbb{E}_{\mathbb{P}}(\zeta_T \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_T)) = \mathbb{E}_{\mathbb{P}}(\zeta_T X).$$

In essence, the latter statement is a consequence of the completeness of the default-free market model. In conclusion, the quadratic hedging price equals $\mathbb{E}_{\mathbb{P}}(\zeta_T X) = \mathbb{E}_{\mathbb{Q}} X$ and the quadratic hedging strategy is the replicating strategy of the attainable claim $\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_T)$ associated with X.

3.3.2 Quadratic Hedging with $\mathbb{G}$-Adapted Strategies

Our next goal is to solve, for a given initial endowment v, the following minimization problem
$$\min_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}\big((V_T^v(\pi) - X)^2\big).$$

We have seen in Chapter 2 that one way of solving this problem is to project the random variable X on the set of stochastic integrals. Here, we present an alternative approach.

We are looking for $\mathbb{G}$-adapted processes X, Θ and Ψ such that the process

$$J_t(\pi) = \big(V_t^v(\pi) - X_t\big)^2 \Theta_t + \Psi_t, \quad \forall\, t \in [0,T], \tag{3.26}$$

is a $\mathbb{G}$-submartingale for any $\mathbb{G}$-adapted trading strategy π and a $\mathbb{G}$-martingale for some strategy π^*. In addition, we require that $X_T = X$, $\Theta_T = 1$, $\Phi_T = 0$. Let us assume that the dynamics of these processes are of the form

$$\begin{aligned} dX_t &= x_t\, dt + \widehat{x}_t\, dW_t + \widetilde{x}_t\, dM_t, && (3.27)\\ d\Theta_t &= \Theta_{t-}\big(\vartheta_t\, dt + \widehat{\vartheta}_t\, dW_t + \widetilde{\vartheta}_t\, dM_t\big), && (3.28)\\ d\Psi_t &= \psi_t\, dt + \widehat{\psi}_t\, dW_t + \widetilde{\psi}_t\, dM_t, && (3.29) \end{aligned}$$

where the drifts x_t, ϑ_t and ψ_t are yet to be determined. From Itô's formula, we obtain (recall that $\xi_t = \gamma_t \mathbb{1}_{\{\tau > t\}}$)

$$\begin{aligned} d(V_t - X_t)^2 = {} & 2(V_t - X_t)(\pi_t \sigma - \widehat{x}_t)\, dW_t - 2(V_t - X_{t-})\widetilde{x}_t\, dM_t \\ & + \big[(V_t - X_{t-} - \widetilde{x}_t)^2 - (V_t - X_{t-})^2\big] dM_t \\ & + \Big(2(V_t - X_t)(\pi_t \nu - x_t) + (\pi_t \sigma - \widehat{x}_t)^2 \\ & + \xi_t \big[(V_t - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\big]\Big) dt, \end{aligned}$$

where we denote $V_t = V_t^v(\pi)$. The process $J(\pi)$ is a martingale if and only if its drift term $k(t, \pi_t, x_t, \vartheta_t, \psi_t) = 0$ for every $t \in [0,T]$.

Straightforward calculations show that

$$\begin{aligned} k(t, \pi_t, \vartheta_t, x_t, \psi_t) = {} & \psi_t + \Theta_t \Big[\vartheta_t (V_t - X_t)^2 \\ & + 2(V_t - X_t)\big[(\pi_t \nu - x_t) + \widehat{\vartheta}_t(\pi_t \sigma - \widehat{x}_t) + \xi_t \widetilde{x}_t\big] \\ & + (\pi_t \sigma - \widehat{x}_t)^2 + \xi_t(\widetilde{\vartheta}_t + 1)\big[(V_t - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\big]\Big]. \end{aligned}$$

In the first step, for any $t \in [0,T]$ we shall find π_t^* such that the minimum of $k(t, \pi_t, x_t, \vartheta_t, \psi_t)$ is attained. Subsequently, we shall choose the auxiliary processes $x = x^*, \vartheta = \vartheta^*$ and $\psi = \psi^*$ in such a way that $k(t, \pi_t^*, x_t^*, \vartheta_t^*, \psi_t^*) = 0$. This choice will imply that $k(t, \pi_t, x_t^*, \vartheta_t^*, \psi_t^*) \geq 0$ for any trading strategy π and any $t \in [0,T]$.

The strategy π^*, which minimizes $k(t,\pi_t,x_t,\vartheta_t,\psi_t)$, is the solution of the following equation:

$$(V_t^v(\pi)-X_t)(\nu+\widehat{\vartheta}_t\sigma)+\sigma(\pi_t\sigma-\widehat{x}_t)=0,\quad \forall\, t\in[0,T].$$

Hence, the strategy π^* is implicitly given by

$$\pi_t^*=\sigma^{-1}\widehat{x}_t-\sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma)(V_t^v(\pi^*)-X_t)=A_t-B_t(V_t^v(\pi^*)-X_t),$$

where we denote

$$A_t=\sigma^{-1}\widehat{x}_t,\quad B_t=\sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma).$$

After some computations, we see that the drift term of the process J admits the following representation:

$$\begin{aligned}k(t,\pi_t,\vartheta_t,x_t,\psi_t)&=\psi_t+\Theta_t(V_t-X_t)^2(\vartheta_t-\sigma^2B_t^2)\\&\quad+2\Theta_t(V_t-X_t)\big(\sigma^2A_tB_t-\widehat{\vartheta}_t\widehat{x}_t-\widetilde{\vartheta}_t\widetilde{x}_t\xi_t-x_t\big)+\Theta_t\xi_t(\widetilde{\vartheta}_t+1)\widetilde{x}_t^2.\end{aligned}$$

From now on, we shall assume that the auxiliary processes ϑ,x and ψ are chosen as follows:

$$\begin{aligned}\vartheta_t&=\vartheta_t^*=\sigma^2B_t^2,\\x_t&=x_t^*=\sigma^2A_tB_t-\widehat{\vartheta}_t\widehat{x}_t-\widetilde{\vartheta}_t\widetilde{x}_t\xi_t,\\\psi_t&=\psi_t^*=-\Theta_t\xi_t(\widetilde{\vartheta}_t+1)\widetilde{x}_t^2.\end{aligned}$$

It is rather clear that if the drift coefficients ϑ,x,ψ in (3.27)-(3.29) are chosen as above, then the drift term in dynamics of J is always non-negative, and it is equal to 0 for the strategy π^*, where $\pi_t^*=A_t-B_t(V_t^v(\pi^*)-X_t)$.

Our next goal is to solve equations (3.27)-(3.29). Let us first consider equation (3.28). Since $\vartheta_t=\sigma^2B_t^2$, it suffices to find the three-dimensional process $(\Theta,\widehat{\vartheta},\widetilde{\vartheta})$ which is a solution to the following BSDE:

$$d\Theta_t=\Theta_t\big(\sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma)^2\,dt+\widehat{\vartheta}_t\,dW_t+\widetilde{\vartheta}_t\,dM_t\big),\ \Theta_T=1.$$

It is obvious that the processes $\widehat{\vartheta}=0$, $\widetilde{\vartheta}=0$ and Θ, given as

$$\Theta_t=\exp(-\theta^2(T-t)),\quad\forall\, t\in[0,T],\tag{3.30}$$

solve this equation.

In the next step, we search for a three-dimensional process $(X,\widehat{x},\widetilde{x})$, which solves equation (3.27) with $x_t=x_t^*=\sigma^2A_t(\nu/\sigma^2)=\theta\widehat{x}_t$. It is clear that $(X,\widehat{x},\widetilde{x})$ is the unique solution to the linear BSDE

$$dX_t=\theta\widehat{x}_t\,dt+\widehat{x}_t\,dW_t+\widetilde{x}_t\,dM_t,\ X_T=X.$$

The unique solution to this equation is $X_t=\mathbb{E}_{\mathbb{Q}}(X\,|\,\mathcal{G}_t)$, where $\mathbb{Q}$ is the risk-neutral probability measure, so that $d\mathbb{Q}=\eta_t\,d\mathbb{P}$, where

$$d\eta_t=-\theta\eta_t\,dW_t,\quad \eta_0=1.$$

The components $\widehat{x}$ and $\widetilde{x}$ are given by the integral representation of the $\mathbb{G}$-martingale X with respect to $W^{\mathbb{Q}}$ and M. Notice also that since $\widehat{\vartheta} = 0$, the optimal portfolio π^* is given by the feedback formula

$$\pi_t^* = \sigma^{-1}\big(\widehat{x}_t - \theta(V_t^v(\pi^*) - X_t)\big).$$

Finally, since $\widetilde{\vartheta} = 0$, we have $\psi_t = -\xi_t \widetilde{x}_t^2 \Theta_t$. Therefore, we can solve explicitly the BSDE (3.29) for the process Ψ. Indeed, we are now looking for a three-dimensional process $(\Psi, \widehat{\psi}, \widetilde{\psi})$, which is the unique solution of the BSDE

$$d\Psi_t = -\Theta_t \xi_t \widetilde{x}_t^2 \, dt + \widehat{\psi}_t \, dW_t + \widetilde{\psi}_t \, dM_t, \ \ \Psi_T = 0.$$

Noting that the process

$$\Psi_t + \int_0^t \Theta_s \xi_s \widetilde{x}_s^2 \, ds$$

is a $\mathbb{G}$-martingale under $\mathbb{P}$, we obtain the value of Ψ in a closed form:

$$\Psi_t = \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds \, \Big| \, \mathcal{G}_t \Big). \tag{3.31}$$

Substituting (3.30) and (3.31) in (3.26), we conclude that the value function for our problem is $J_t^* = J_t(\pi^*)$, where in turn

$$\begin{aligned} J_t(\pi^*) &= (V_t^v(\pi^*) - X_t)^2 e^{-\theta^2(T-t)} + \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds \, \Big| \, \mathcal{G}_t \Big) \\ &= (V_t^v(\pi^*) - X_t)^2 e^{-\theta^2(T-t)} + \int_t^T e^{-\theta^2(T-s)} \, \mathbb{E}_{\mathbb{P}}\big(\gamma_s \widetilde{x}_s^2 \mathbb{1}_{\{\tau > s\}} \, \big| \, \mathcal{G}_t \big) \, ds \\ &= (V_t^v(\pi^*) - X_t)^2 e^{-\theta^2(T-t)} \\ &\quad + \mathbb{1}_{\{\tau > t\}} \int_t^T e^{-\theta^2(T-s)} \, \mathbb{E}_{\mathbb{P}}\big(\gamma_s \widetilde{x}_s^2 e^{\Gamma_t - \Gamma_s} \, \big| \, \mathcal{F}_t \big) \, ds, \end{aligned}$$

where we have identified the process $\widetilde{x}$ with its $\mathbb{F}$-adapted version (recall that any $\mathbb{G}$-predictable process is equal, prior to default, to an $\mathbb{F}$-predictable process). In particular,

$$J_0^* = e^{-\theta^2 T} \Big((v - X_0)^2 + \mathbb{E}_{\mathbb{P}}\Big(\int_0^T e^{\theta^2 s} \gamma_s \widetilde{x}_s^2 e^{-\Gamma_s} \, ds \Big) \Big).$$

From the last formula, it is obvious that the quadratic hedging price is $X_0 = \mathbb{E}_{\mathbb{Q}} X$. We are in a position to formulate the main result of this section. A corresponding theorem for a default-free financial model was established by Kohlmann and Zhou (2000).

Proposition 3.5 *Let a claim X be $\mathcal{G}_T$-measurable and $\mathbb{P}$-square-integrable. The optimal trading strategy π^*, which solves the quadratic problem*

$$\min_{\pi \in \Pi(\mathbb{G})} \mathbb{E}_{\mathbb{P}}((V_T^v(\pi) - X)^2),$$

is given by the feedback formula

$$\pi_t^* = \sigma^{-1}\big(\widehat{x}_t - \theta(V_t^v(\pi^*) - X_t)\big),$$

where $X_t = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}_t)$ *for every* $t \in [0,T]$*, and the process* $\widehat{x}_t$ *is specified by*

$$dX_t = \widehat{x}_t \, dW_t^Q + \widetilde{x}_t \, dM_t.$$

The quadratic hedging price of X *is equal to* $\mathbb{E}_{\mathbb{Q}}X$*.*

Survival Claim

Let us consider a simple survival claim $X = 1\!\!1_{\{\tau > T\}}$, and let us assume that Γ is deterministic, specifically, $\Gamma(t) = \int_0^t \gamma(s)\, ds$. In that case, from the well-known representation theorem (see Bielecki and Rutkowski (2004), Page 159), we have $dX_t = \widetilde{x}_t \, dM_t$ with $\widetilde{x}_t = -e^{\Gamma(t) - \Gamma(T)}$. Hence

$$\begin{aligned}
\Psi_t &= \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds \,\Big|\, \mathcal{G}_t\Big) \\
&= \mathbb{E}_{\mathbb{P}}\Big(\int_t^T \Theta_s \gamma(s) 1\!\!1_{\{\tau > s\}} e^{2\Gamma(s) - 2\Gamma(T)} \, ds \,\Big|\, \mathcal{G}_t\Big) \\
&= 1\!\!1_{\{\tau > t\}} \, e^{\Gamma(t) - 2\Gamma(T)} \, \mathbb{E}_{\mathbb{P}}\Big(\int_t^T e^{-\theta^2(T-s)} \gamma(s) e^{\Gamma(s)} \, ds \,\Big|\, \mathcal{F}_t\Big) \\
&= 1\!\!1_{\{\tau > t\}} \, e^{\Gamma(t) - 2\Gamma(T)} \int_t^T e^{-\theta^2(T-s)} \gamma(s) e^{\Gamma(s)} \, ds.
\end{aligned}$$

One can check that, at time 0, the value function is indeed smaller that the one obtained with $\mathbb{F}$-adapted portfolios.

Case of an Attainable Claim

Assume now that a claim X is $\mathcal{F}_T$-measurable. Then $X_t = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}_t)$ is the price of X, and it satisfies $dX_t = \widehat{x}_t \, dW_t^{\mathbb{Q}}$. The optimal strategy is, in a feedback form,

$$\pi_t^* = \sigma^{-1}\big(\widehat{x}_t - \theta(V_t - X_t)\big)$$

and the associated wealth process satisfies

$$dV_t = \pi_t^*(\nu dt + \sigma dW_t) = \pi_t^* \sigma \, dW_t^{\mathbb{Q}} = \sigma^{-1}\big(\sigma \widehat{x}_t - \nu(V_t - X_t)\big)\, dW_t^{\mathbb{Q}}.$$

Therefore,

$$d(V_t - X_t) = -\theta(V_t - X_t)\, dW_t^{\mathbb{Q}}.$$

Hence, if we start with an initial wealth equal to the arbitrage price $\pi_0(X)$ of X, then we obtain that $V_t = X_t$ for every $t \in [0,T]$, as expected.

Hodges Price

Let us emphasize that the Hodges price has no real meaning here, since the problem $\min \mathbb{E}_{\mathbb{P}}((V_T^v)^2)$ has no financial interpretation. We have studied in the preceding chapter a more pertinent problem, with a constraint on the expected value of V_T^v under $\mathbb{P}$. Nevertheless, from a mathematical point of view, the Hodges price would be the value of p such that

$$(v^2 - (v-p)^2) = \int_0^T e^{\theta^2 s} \mathbb{E}_{\mathbb{P}}(\gamma_s \widetilde{x}_s^2 e^{-\Gamma_s}) \mathbb{1}_{\{\tau > t\}} ds$$

In the case of the example studied in Section 3.3.2, the Hodges price would be the non-negative value of p such that

$$2vp - p^2 = e^{-2\Gamma_T} \int_0^T e^{\theta^2 s} \gamma_s e^{\Gamma_s} \, ds.$$

Let us also mention that our results are different from results of Lim (2004). Indeed, Lim studies a model with Poisson component, and thus in his approach the intensity of this process does not vanish after the first jump.

3.4 Optimization in Incomplete Markets

In this last section, we shall briefly (and rather informally) examine a specific optimization problem associated with a defaultable claim. The interested reader is referred to Lukas (2001) for more details on the approach examined in this section. We now assume that the only risky asset available in the market is

$$dZ_t^1 = Z_t^1 \big(\nu \, dt + \sigma \, dW_t + \varphi \, dM_t \big),$$

and we assume that $r = 0$. We deal with the following problem:

$$\sup_{\pi} \mathbb{E}_{\mathbb{P}} \big(u(V_{\tau \wedge T}^v(\pi) + X) \big)$$

for the claim X of the form

$$X = \mathbb{1}_{\{\tau > T\}} g(Z_T^1) + \mathbb{1}_{\{\tau \le T\}} h(Z_\tau^1)$$

for some functions $g, h : \mathbb{R} \to \mathbb{R}$. Note that here the recovery payment is paid at hit, that is, at the time of default. In addition, we assume that the default intensity γ under $\mathbb{P}$ is constant (hence, it is constant under any equivalent martingale measure as well). After time τ, the market reduces to a standard Black-Scholes model, and thus the solution to the corresponding optimization problem is well known.

In the particular case of the exponential utility $u(x) = 1 - \exp(-\varrho x)$, $\varrho > 0$, we are in a position to use the duality theory. This problem was studied by, among others, Rouge and El Karoui (2000), Delbaen et al. (2002) and Collin-Dufresne and Hugonnier (2002).

Let $H(\mathbb{Q}\,|\,\mathbb{P})$ stand for the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$. Recall that if a probability measure $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ then

$$H(\mathbb{Q}\,|\,\mathbb{P}) = \mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right) = \mathbb{E}_{\mathbb{Q}}\left(\ln \frac{d\mathbb{Q}}{d\mathbb{P}}\right).$$

Otherwise, the relative entropy $H(\mathbb{Q}\,|\,\mathbb{P})$ equals ∞.

It is well known that, under suitable technical assumptions (see Rouge and El Karoui (2000) or Delbaen et al. (2002) for details), we have

$$\begin{aligned}\sup_{\pi} \mathbb{E}_{\mathbb{P}}\Big(1 - e^{-\varrho(V_T^v(\pi)+X)}\Big) \\ = 1 - \exp\Big(- \inf_{\pi} \inf_{\mathbb{Q}\in\mathcal{Q}_T} \big(H(\mathbb{Q}\,|\,\mathbb{P}) + \varrho\,\mathbb{E}_{\mathbb{Q}}(V_T^v(\pi)+X)\big)\Big),\end{aligned}$$

where π runs over a suitable class of admissible portfolios, and $\mathcal{Q}_T$ stands for the set of equivalent martingale measures on the σ-field $\mathcal{G}_T$.

Since for any admissible portfolio π the expected value under any martingale measure $\mathbb{Q}\in\mathcal{Q}_T$ of the terminal wealth $V_T^v(\pi)$ equals v , we obtain

$$\sup_{\pi} \mathbb{E}_{\mathbb{P}}\Big(1 - e^{-\varrho(V_T^v(\pi)+X)}\Big) = 1 - \exp\Big(- \inf_{\mathbb{Q}\in\mathcal{Q}_T} \big(H(\mathbb{Q}\,|\,\mathbb{P}) + \varrho\,\mathbb{E}_{\mathbb{Q}}X + \varrho v\big)\Big).$$

Furthermore, since, without loss of generality, we may stop all the processes considered here at the default time τ, we end up with the following equality

$$\inf_{\pi} \mathbb{E}_{\mathbb{P}}\Big(e^{-\varrho(V_{T\wedge\tau}^v(\pi)+X)}\Big) = \exp\Big(- \inf_{\mathbb{Q}\in\mathcal{Q}_{T\wedge\tau}} \big(H(\mathbb{Q}\,|\,\mathbb{P}) + \varrho\,\mathbb{E}_{\mathbb{Q}}X + \varrho v\big)\Big),$$

where π runs over the class of all admissible trading strategies, and $\mathcal{Q}_{T\wedge\tau}$ stands the set of equivalent martingale measures on the σ-field $\mathcal{G}_{T\wedge\tau}$. The following result provides a description of the class $\mathcal{Q}_{T\wedge\tau}$.

Lemma 3.4 *The class $\mathcal{Q}_{T\wedge\tau}$ of all equivalent martingale measures on the space $(\Omega,\mathcal{G}_{T\wedge\tau})$ is the set of all probability measures $\mathbb{Q}_{k,h}$ of the form*

$$d\mathbb{Q}_{k,h}|_{\mathcal{G}_{T\wedge\tau}} = \eta_{T\wedge\tau}(k,h)\,d\mathbb{P},$$

where the Radon-Nikodym density process $\eta(k,h)$ is given by the formula

$$\eta_t(k,h) = \mathcal{E}_t(kM)\mathcal{E}_t(hW), \quad \forall\, t\in[0,T],$$

for some $\mathbb{F}$-adapted process k such that the inequality $k_t > -1$ holds for every $t\in[0,T]$, and for the associated process $h_t = -\theta - \varphi\gamma\sigma^{-1}(1+k_t)$, where $\theta = \nu/\sigma$. Under the martingale measure $\mathbb{Q} = \mathbb{Q}_{k,h}$ the process

$$W_{t\wedge\tau}^h = W_{t\wedge\tau} - \int_0^{t\wedge\tau} h_s\,ds, \quad \forall\, t\in[0,T],$$

is a stopped Brownian motion, and the process

$$M_{t\wedge\tau}^k = M_{t\wedge\tau} - \int_0^{t\wedge\tau} \gamma k_s\,ds, \quad \forall\, t\in[0,T],$$

is a martingale stopped at τ.

Straightforward calculations show that the relative entropy of a martingale measure $\mathbb{Q} = \mathbb{Q}_{k,h} \in \mathcal{Q}_{T\wedge\tau}$ with respect to $\mathbb{P}$ equals

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_0^{\tau\wedge T} h_s\, dW_s^h + \int_0^{\tau\wedge T} \big(\tfrac{1}{2}h_s^2 - \gamma k_s + \gamma(1+k_s)\ln(1+k_s)\big)\, ds\Big) \\ + \mathbb{E}_{\mathbb{Q}}\Big(\int_0^{\tau\wedge T} \ln(1+k_s)\, dM_s^k\Big).$$

Consequently, the optimization problem

$$\inf_{\mathbb{Q}\in\mathcal{Q}_{T\wedge\tau}} \big(H(\mathbb{Q}\,|\,\mathbb{P}) + \varrho\, \mathbb{E}_{\mathbb{Q}} X\big)$$

can be reduced to the following problem

$$\inf_{k,h} \mathbb{E}_{\mathbb{Q}}\Big(\int_0^{\tau\wedge T} \big(\tfrac{1}{2}h_s^2 - \gamma k_s + \gamma(1+k_s)\ln(1+k_s)\big)\, ds + \varrho X\Big), \tag{3.32}$$

where the processes k and h are as specified in the statement of Lemma 3.4. Let us set

$$\ell(k_s) = \tfrac{1}{2}h_s^2 - \gamma k_s + \gamma(1+k_s)\ln(1+k_s)$$

so that

$$\ell(k) = \tfrac{1}{2}\big(\theta + \varphi\gamma(1+k)\big)^2 - \gamma k + \gamma(1+k)\ln(1+k). \tag{3.33}$$

Consider a dynamic version of the minimization problem (3.32)

$$\inf_{k,h} \mathbb{E}_{\mathbb{Q}}\Big(\int_t^{\tau\wedge T} \ell(k_s)\, ds + \varrho \mathbb{1}_{\{\tau\le T\}} h(Z_\tau^1) + \varrho \mathbb{1}_{\{\tau> T\}} g(Z_T^1)\,\Big|\, \mathcal{G}_t\Big).$$

Let us denote $K_s^t = e^{-\int_t^s \gamma(1+k_u)\, du}$ for $t \le s$. Then, on the pre-default event $\{\tau > t\}$, we obtain the following problem:

$$\inf_{k,h} \mathbb{E}_{\mathbb{Q}}\Big(\int_t^{T} K_s^t\big(\ell(k_s) + \varrho\gamma(1+k_s)h(Z_s^1(1+\varphi))\big)\, ds + \varrho K_T^t g(Z_T^1)\,\Big|\, \mathcal{F}_t\Big).$$

The value function $J(t,x)$ of the latter problem satisfies the HJB equation

$$\partial_t J(t,x) + \tfrac{1}{2}\sigma^2 x^2 \partial_{xx} J(t,x) \\ + \inf_{k>-1} \big(-\varphi\gamma(1+k)x\partial_x J(t,x) - \gamma(1+k)J(t,x) + \psi(k,x)\big) = 0$$

with the terminal condition $J(T,x) = \varrho g(x)$, where we denote

$$\psi(k,x) = \ell(k) + \varrho\gamma(1+k)h(x(1+\varphi))$$

and where the function ℓ is given by (3.33). The minimizer is given by $k = k^*(t,x)$, which is the unique root of the following equation:

$$\frac{\varphi}{\sigma^2}\big(\nu + \varphi\gamma(1+k)\big) + \ln(1+k) = J(t,x) + \varphi x \partial_x J(t,x) - \varrho h(x(1+\varphi)),$$

and the optimal portfolio π^* is given by the formula

$$\pi_t^* = (\varrho\sigma^2)^{-1}\big(\nu + \varphi\gamma(1 + k^*(t, Z_t^1)) - \sigma^2 Z_{t-}^1 \partial_x J(t, Z_{t-}^1)\big).$$

Remark. Note that in the case $\varphi = 0$ this result is consistent with our result established in Section 3.2.1. When $\varphi = 0$, the process Z^1 is continuous, and thus we obtain

$$\pi_t^* = (\varrho\sigma)^{-1}\big(\theta - \sigma Z_t^1 \partial_x J(t, Z_t^1)\big),$$

where the value function $J(t,x)$ satisfies the simplified HJB equation

$$\begin{aligned}&\partial_t J(t,x) + \tfrac{1}{2}\sigma^2 x^2 \partial_{xx} J(t,x)\\ &\qquad + \inf_{k>-1} \big(\ell(k) - \gamma(1+k)J(t,x) + \varrho\gamma(1+k)h(x)\big) = 0,\end{aligned}$$

where in turn

$$\ell(k) = \tfrac{1}{2}\theta^2 - \gamma k + \gamma(1+k)\ln(1+k).$$

References

Arvanitis A. and Gregory, J. (2001) *Credit: The Complete Guide to Pricing, Hedging and Risk Management.* Risk Publications.

Arvanitis, A. and Laurent, J.-P. (1999) On the edge of completeness. *Risk*, October, 61–65.

Barles, G., Buckdahn, R. and Pardoux, E. (1997) Backward stochastic differential equations and integral-partial differential equations. *Stochastics and Stochastics Reports* **60**, 57–83.

Bélanger, A., Shreve, S. and Wong, D. (2001) A unified model for credit derivatives. To appear in *Mathematical Finance.*

Bernis, G. and Jeanblanc, M. (2003) Hedging defaultable derivatives *via* utility theory. Preprint, Evry University.

Bielecki, T.R. and Jeanblanc, M. (2003) Genuine mean-variance hedging of credit risk: A case study. Working paper.

Bielecki, T.R., Jeanblanc, M. and Rutkowski, M. (2004a) Modeling and valuation of credit risk. In: *CIME-EMS Summer School on Stochastic Methods in Finance, Bressanone, July 6-12, 2003*, Springer-Verlag, Berlin Heidelberg New York.

Bielecki, T.R. and Rutkowski, M. (2003) Dependent defaults and credit migrations. *Applicationes Mathematicae* **30**, 121–145.

Bielecki, T.R. and Rutkowski, M. (2004) *Credit Risk: Modeling, Valuation and Hedging.* Corrected 2nd printing. Springer-Verlag, Berlin Heidelberg New York.

Bielecki, T.R., Jin, H., Pliska, S.R. and Zhou, X.Y. (2004b) Dynamic mean-variance portfolio selection with bankruptcy prohibition. Forthcoming in *Mathematical Finance.*

Black, F. and Cox, J.C. (1976) Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance* **31**, 351–367.

Blanchet-Scalliet, C. and Jeanblanc, M. (2004) Hazard rate for credit risk and hedging defaultable contingent claims. *Finance and Stochastics* **8**, 145–159.

Bobrovnytska, O. and Schweizer, M. (2004) Mean-variance hedging and stochastic control: Beyond the Brownian setting. *IEEE Transactions on Automatic Control* **49**, 396–408.

Buckdahn, R. (2001) Backward stochastic differential equations and viscosity solutions of semilinear parabolic deterministic and stochastic PDE of second order. In: *Stochastic Processes and Related Topics*, R. Buckdahn, H.-J. Engelbert and M. Yor, editors, Taylor and Francis, pp. 1–54.

Collin-Dufresne, P. and Hugonnier, J.-N. (2001) Event risk, contingent claims and the temporal resolution of uncertainty. Working paper, Carnegie Mellon University.

Collin-Dufresne, P. and Hugonnier, J.-N. (2002) On the pricing and hedging of contingent claims in the presence of extraneous risks. Preprint, Carnegie Mellon University.

Collin-Dufresne, P., Goldstein, R.S. and Hugonnier, J.-N. (2003) A general formula for valuing defaultable securities. Preprint.

Cossin, D. and Pirotte, H. (2000) *Advanced Credit Risk Analysis.* J. Wiley, Chichester.

Davis, M. (1997) Option pricing in incomplete markets. In: *Mathematics of Derivative Securities*, M.A.H. Dempster and S.R. Pliska, editors, Cambridge University Press, Cambridge, pp. 216–227.

Delbaen, F., Grandits, P., Rheinländer, Th., Samperi, D., Schweizer, M. and Stricker, Ch. (2002) Exponential hedging and entropic penalties. *Mathematical Finance* **12**, 99–124.

Duffie, D. (2003) *Dynamic Asset Pricing Theory.* 3rd ed. Princeton University Press, Princeton.

Duffie, D. and Singleton, K. (2003) *Credit Risk: Pricing, Measurement and Management.* Princeton University Press, Princeton.

El Karoui, N. and Mazliak, L. (1997) Backward stochastic differential equations. *Pitman Research Notes in Mathematics*, Longman.

El Karoui, N., Peng, S. and Quenez, M.-C. (1997) Backward stochastic differential equations in finance. *Mathematical Finance* **7**, 1–71.

El Karoui, N. and Quenez, M.-C. (1995) Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM Journal on Control and Optimization* **33**, 29–66.

El Karoui, N. and Quenez, M.-C. (1997) Imperfect markets and backward stochastic differential equations. In: *Numerical Methods in Finance*, L.C.G. Rogers, D. Talay, editors, Cambridge University Press, Cambridge, pp. 181–214.

Elliott, R.J., Jeanblanc, M. and Yor, M. (2000) On models of default risk. *Mathematical Finance* **10**, 179–195.

Gouriéroux, C., Laurent, J.-P. and Pham, N. (1998) Mean-variance hedging and numéraire. *Mathematical Finance* **8**, 179–200.

Greenfield, Y. (2000) Hedging of credit risk embedded in derivative transactions. Thesis, Carnegie Mellon University.

Gregory, J. and Laurent, J.-P. (2003) I will survive. *Risk*, June, 103–107.

Hodges, S.D. and Neuberger, A. (1989) Optimal replication of contingent claim under transaction costs. *Review of Futures Markets* **8**, 222–239.

Hu, Y. and Zhou, X.Y. (2004) Constrained stochastic LQ control with random coefficients and application to mean-variance portfolio selection. Preprint.

Hugonnier, J.-N., Kramkov, D. and Schachermayer, W. (2002) On the utility based pricing of contingent claims in incomplete markets. Preprint.

Jamshidian, F. (2002) Valuation of credit default swap and swaptions. To appear in *Finance and Stochastics.*

Jankunas, A. (2001) Optimal contingent claims. *Annals of Applied Probability* **11**, 735–749

Jarrow, R.A. and Yu, F. (2001) Counterparty risk and the pricing of defaultable securities. *Journal of Finance* **56**, 1756–1799.

Jeanblanc, M. and Rutkowski, M. (2000) Modelling of default risk: Mathematical tools. Working paper, Université d'Evry and Politechnika Warszawska.

Jeanblanc, M. and Rutkowski, M. (2002) Default risk and hazard processes. In: *Mathematical Finance – Bachelier Congress 2000*, H. Geman, D. Madan, S.R. Pliska and T. Vorst, editors, Springer-Verlag, Berlin Heidelberg New York, pp. 281–312.

Jeanblanc, M. and Rutkowski, M. (2003) Modelling and hedging of default risk. In: *Credit Derivatives: The Definitive Guide*, J. Gregory, editor, Risk Books, London, pp. 385–416.

Jeanblanc, M., Yor, M. and Chesney, M. (2004) *Mathematical Methods for Financial Markets.* Springer-Verlag. Forthcoming.

Karatzas, I. and Shreve, S.E. (1998) *Brownian Motion and Stochastic Calculus.* 2nd ed. Springer-Verlag, Berlin Heidelberg New York.

Karatzas, I. and Shreve, S.E. (1998) *Methods of Mathematical Finance.* Springer-Verlag, Berlin Heidelberg New York.

Kobylanski, M. (2000) Backward stochastic differential equations and partial differential equations with quadratic growth. *Annals of Probability* **28**, 558–602.

Kohlmann, M. and Zhou, X.Y. (2000) Relationship between backward stochastic differential equations and stochastic controls: A linear-quadratic approach. *SIAM Journal on Control and Optimization* **38**, 1392–1407.

Kramkov, D. (1996) Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probability Theory and Related Fields* **105**, 459–479.

Kusuoka, S. (1999) A remark on default risk models. *Advances in Mathematical Economics* **1**, 69–82.

Lando, D. (1998) On Cox processes and credit-risky securities. *Review of Derivatives Research* **2**, 99–120.

Last, G. and Brandt, A. (1995) *Marked Point Processes on the Real Line: The Dynamic Approach.* Springer-Verlag, Berlin Heidelberg New York.

Laurent, J.-P. and Gregory, J. (2002) Basket defaults swaps, CDOs and factor copulas. Working paper.

Lepeltier, J.-P. and San-Martin, J. (1997) Existence for BSDE with superlinear-quadratic coefficient. *Stochastics and Stochastics Reports* **63**, 227–240.

Li, X., Zhou, X.Y. and Lim, A.E.B. (2001) Dynamic mean-variance portfolio selection with no-shorting constraints. *SIAM Journal on Control and Optimization* **40**, 1540–1555.

Lim, A.E.B. (2004) Hedging default risk. Preprint.

Lim, A.E.B. and Zhou, X.Y. (2002) Mean-variance portfolio selection with random parameters. *Mathematics of Operations Research* **27**, 101–120.

Lotz, C. (1998) Locally minimizing the credit risk. Working paper, University of Bonn.

Lukas, S. (2001) On pricing and hedging defaultable contingent claims. Thesis, Humboldt University.

Ma, J. and Yong, Y. (1999) *Forward-Backward Stochastic Differential Equations and Their Applications.* Springer-Verlag, Berlin Heidelberg New York.

Mania, M. (2000) A general problem of an optimal equivalent change of measure and contingent claim pricing in an incomplete market. *Stochastic Processes and their Applications* **90**, 19–42.

Mania, M. and Tevzadze, R. (2003) Backward stochastic PDE and imperfect hedging. *International Journal of Theoretical and Applied Finance* **6**, 663–692.

Merton, R. (1974) On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance* **29**, 449–470.

Musiela, M. and Rutkowski, M. (1997) *Martingale Methods in Financial Modelling.* Springer-Verlag, Berlin Heidelberg New York.

Musiela, M. and Zariphopoulou, T. (2004) An example of indifference prices under exponential preferences. *Finance and Stochastics* **8**, 229–239.

Pham, N., Rheinländer, Th. and Schweizer, M. (1998) Mean-variance hedging for continuous processes: New results and examples. *Finance and Stochastics* **2**, 173–98.

Pliska, S.R. (2001) Dynamic Markowitz problems with no bankruptcy. Presentation at JAFEE Conference, 2001.

Protter, P. (2003) *Stochastic Integration and Differential Equations.* 2nd ed. Springer-Verlag, Berlin Heidelberg New York.

Rheinländer, Th. (1999) Optimal martingale measures and their applications in mathematical finance. Ph.D. Thesis, TU-Berlin.

Rheinländer, Th. and Schweizer, M. (1997) On L^2-projections on a space of stochastic integrals. *Annals of Probability* **25**, 1810–1831.

Rong, S. (1997) On solutions of backward stochastic differential equations with jumps and applications. *Stochastic Processes and their Applications* **66**, 209–236.

Rouge, R. and El Karoui, N. (2000) Pricing via utility maximization and entropy. *Mathematical Finance* **10**, 259–276.

Royer, M. (2003) Equations différentielles stochastiques rétrogrades et martingales non-linéaires. Doctoral thesis.

Schönbucher, P.J. (1998) Pricing credit risk derivatives. Working paper, University of Bonn.

Schönbucher, P.J. (2003) *Credit Derivatives Pricing Models.* J.Wiley, Chichester.

Schweizer, M. (2001) A guided tour through quadratic hedging approaches. In: *Option Pricing, Interest Rates and Risk Management*, E. Jouini, J. Cvitanić and M. Musiela, editors, Cambridge University Press, pp. 538–574.

Vaillant, N. (2001) A beginner's guide to credit derivatives. Working paper, Nomura International.

Wong, D. (1998) A unifying credit model. Working paper, Research Advisory Services, Capital Markets Group, Scotia Capital Markets.

Yong, J. and Zhou, X.Y. (1999) *Stochastic Controls: Hamiltonian Systems and HJB Equations.* Springer-Verlag, Berlin Heidelberg New York.

Zhou, X.Y. (2003) Markowitz's world in continuous time, and beyond. In: *Stochastic Modeling and Optimization*, D.D. Yao et al., editors, Springer, New York, pp. 279–310.

Zhou, X.Y. and Li, D. (2000) Continuous time mean-variance portfolio selection: A stochastic LQ framework. *Applied Mathematics and Optimization* **42**, 19–33.

Zhou, X.Y. and Yin, G. (2004) Dynamic mean-variance portfolio selection with regime switching: A continuous-time model. *IEEE Transactions on Automatic Control* **49**, 349–360.