Generalized Hawkes Processes, Generalized Multivariate Hawkes Processes and Some of Their Properties and Applications

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Abstract

This work contributes to the theory and applications of Hawkes processes. We introduce and study a new class of Hawkes processes that we call generalized Hawkes processes (GHPs), and their special subclass – the generalized multivariate Hawkes processes (GMHPs). GMHPs are multivariate marked point processes that add an important feature to the family of the (classical) multivariate Hawkes processes: they allow for explicit modelling of simultaneous occurrence of excitation events coming from different sources, i.e. caused by different coordinates of the multivariate process.

Keywords: Generalized Hawkes processes, generalized multivariate Hawkes process, Hawkes kernel, multivariate marked point process, random measure, predictable compensator, epidemiology, insurance.

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1 Introduction

A very interesting and important class of stochastic processes was introduced by Alan Hawkes in [Haw71a, Haw71b]. These processes, called now Hawkes processes, are meant to model self-exciting and mutually-exciting random phenomena that evolve in time. The self-exciting phenomena are modeled as univariate Hawkes processes, and the mutually-exciting phenomena are modeled as multivariate Hawkes processes. Hawkes processes belong to the family of marked point processes, and, of course, a univariate Hawkes process is just a special case of the multivariate one.

The present work is meant to continue our contribution to the theory and applications of multivariate Hawkes processes. Specifically, we complement here our study of generalized multivariate Hawkes processes (GMHPs), which originated in Chapter 11 of [BJN20b]. In particular, differently from what is done in Chapter 11 of [BJN20b], we define here the GMHPs as a subclass of the family generalized Hawkes processes (GHPs) that are introduced in the present paper. We provide a proof of existence of a generalized Hawkes process.In addition, we present some preliminary asymptotic in time analysis of the generalized Hawkes processes. The proofs of our results have been streamlined so to keep the presentation as clean and as transparent as possible. The theory is illustrated with a mathematical example. Some possible applications of GMHPs are indicated.

GMHPs are multivariate marked point processes of Hawkes type that add an important feature to the family of the (classical) multivariate Hawkes processes: they allow for explicit modelling of **simultaneous occurrence of excitation events** coming from different sources, i.e. **caused simultaneously** by different coordinates of the multivariate process.[1] The importance of this feature is rather intuitive, and it will be illustrated in Section 4. In this regard, GMHPs differ from the multivariate Hawkes processes that were studied, for example, in Bremaud and Massouli [BM96] and Liniger [Lin09]. We are not aware of the multivariate Hawkes processes admitting simultaneous occurrence of excitation events coming from different sources to have been introduced and studied anywhere else in the literature, except in our monograph [BJN20b]. That is why we consider this work to be an important contribution to the body of literature devoted to theory and applications of Hawkes processes.

We need to stress that we limit ourselves here to the case of linear generalized Hawkes processes and linear GMHPs, that are counterpart of the linear classical Hawkes processes. That is to say, we do not study here what would be a counterpart of the nonlinear classical Hawkes processes. We refer e.g. to Chapter 1 in [Zhu13] for comparison of linear and nonlinear Hawkes processes. We also note that the generalized Hawkes processes introduced

[1] It needs to be stressed though that in our set-up the notion of the coordinate of the multivariate process is derived from the global definition of the multivariate process, rather than the other way around, as it is classically done. We refer to Section 3.2 for details.

here should not be confused with those studied in [Vac11] or in [Che17]. In particular, we do not introduce any additional random factors, such as Brownian motions, into the compensators of the multivariate marked point process N showing in the Definition 3.1 below.

The paper is organized as follows. In Section 2 we define, prove existence of and provide some discussion of a generalized Hawkes process, including some preliminary analysis of time asymptotics of such process. Section 3 is devoted to study of the generalized multivariate Hawkes processes. We look at several aspects of these processes and provide some examples. Section 4 contains a brief description of possible applications of generalized multivariate Hawkes processes in epidemiology and insurance. We close with Section 5 that provides some concluding remarks and proposes some related future work.

In this paper we use various concepts and results from stochastic analysis. For a comprehensive study of these concepts and results we refer e.g. to [HWY92], [LB95] and [JS03].

2 Generalized Hawkes process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{X}, \boldsymbol{\mathcal{X}})$ be a Borel space. We take ∂ to be a point external to $\mathcal{X}$, and we let $\mathcal{X}^\partial := \mathcal{X} \cup \partial$. On $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a Marked Point Process (MMP) N with mark space $\mathcal{X}$, that is, a sequence of random elements

$$N = ((T_n, X_n))_{n \geq 1}, \tag{2.1}$$

where for each n:

1. T_n is a random variable with values in $(0, \infty]$,
2. X_n is a random variable with values in $\mathcal{X}^\partial$,
3. $T_n \leq T_{n+1}$, and if $T_n < +\infty$ then $T_n < T_{n+1}$,
4. $X_n = \partial$ iff $T_n = \infty$.

The explosion time of N, say T_∞, is defined as

$$T_\infty := \lim_{n \to \infty} T_n.$$

Following the typical techniques used in the theory of Marked Point Processes (MPPs), in particular following Section 1.3 in [LB95], we associate with the process N an integer-valued random measure on $(\mathbb{R}_+ \times \mathcal{X}, \mathcal{B}(\mathbb{R}_+) \otimes \boldsymbol{\mathcal{X}})$, also denoted by N and defined as

$$N(dt, dx) := \sum_{n \geq 1} \delta_{(T_n, X_n)}(dt, dx) \mathbb{1}_{\{T_n < \infty\}}, \tag{2.2}$$

so that

$$N((0, t], A) = \sum_{n \geq 1} \mathbb{1}_{\{T_n \leq t\}} \mathbb{1}_{\{X_n \in A\}},$$

where $A \in \boldsymbol{\mathcal{X}}$.

Let $\mathbb{F}^N$ be the natural filtration of N, so $\mathbb{F}^N := (\mathcal{F}_t^N,\ t \geq 0)$, where $\mathcal{F}_t^N$ is the $\mathbb{P}$–completed σ–field $\sigma(N((s,r] \times A)\ :\ 0 \leq s < r \leq t,\ A \in \mathcal{X}),\ t \geq 0$. In view of Theorem 2.2.4 in [LB95] the filtration $\mathbb{F}^N$ satisfies the usual conditions. Moreover, N is $\mathbb{F}^N$–optional, so, using Proposition 4.1.1 in [LB95] we conclude that T_n's are $\mathbb{F}^N$–stopping times and X_n are $\mathcal{F}_{T_n}$-measurable. In what follows we denote by $\mathcal{P}$ the $\mathbb{F}^N$-predictable σ-field.

We recall that for a given filtration $\mathbb{F}$ a stochastic process $X : \Omega \times [0,\infty) \to \mathbb{R}$ is said to be $\mathbb{F}$-predictable if it is measurable with respect to the predictable sigma field $\mathcal{P}^{\mathbb{F}}$ on $\Omega \times [0,\infty)$, which is generated by $\mathbb{F}$-adapted processes whose paths are continuous (equivalently left-continuous, with the left limit at $t = 0$ defined as the value of the path at $t = 0$) functions of time variable. More generally, a function $X : \Omega \times [0,\infty) \times \mathcal{X} \to \mathbb{R}$ is said to be $\mathbb{F}$-predictable function if it is measurable with respect to the sigma field $\mathcal{P}^{\mathbb{F}}(\boldsymbol{\mathcal{X}}) := \mathcal{P}^{\mathbb{F}} \otimes \boldsymbol{\mathcal{X}}$ on $\Omega \times [0,\infty) \times \mathcal{X}$. The sigma field $\mathcal{P}^{\mathbb{F}}(\boldsymbol{\mathcal{X}})$ is generated by the sets $A \times \{0\} \times \boldsymbol{\mathcal{X}}$ where $A \in \mathcal{F}_0$ and the sets of the form $B \times (s,t] \times D$ where $0 < s \leq t$, $B \in \mathcal{F}_s$ and $D \in \boldsymbol{\mathcal{X}}$.

We now consider a random measure ν on $(\mathbb{R}_+ \times \mathcal{X}, \mathcal{B}(\mathbb{R}_+) \otimes \boldsymbol{\mathcal{X}})$ defined as

$$\nu(\omega, dt, dy) := \mathbb{1}_{]\!]0,T_\infty(\omega)[\![}(t)\kappa(\omega,t,dy)dt, \tag{2.3}$$

where, for $A \in \boldsymbol{\mathcal{X}}$,

$$\kappa(t,A) = \eta(t,A) + \int_{(0,t)\times\mathcal{X}} f(t,s,x,A)N(ds,dx), \tag{2.4}$$

η is a finite kernel from $(\Omega \times [0,\infty), \mathcal{P})$ to $(\mathcal{X}, \boldsymbol{\mathcal{X}})$, and f is a kernel from $(\Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{X}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \boldsymbol{\mathcal{X}})$ to $(\mathcal{X}, \boldsymbol{\mathcal{X}})$.[2] We assume also that f is a kernel satisfying:

1. $f(t,s,x,A) = 0$ for $s \geq t$,
2. θ defined as

$$\theta(t,A) := \int_{(0,t)\times\boldsymbol{\mathcal{X}}} f(t,s,x,A)N(ds,dx), \quad t \geq 0,\ A \in \boldsymbol{\mathcal{X}},$$

   is a kernel from $(\Omega \times [0,\infty), \mathcal{P})$ to $(\mathcal{X}, \boldsymbol{\mathcal{X}})$, which is finite for $t < T_\infty$.

Clearly, we have

$$\theta(t,A) = \sum_{n:\ T_n<t} f(t,T_n,X_n,A). \tag{2.5}$$

Note that $\kappa(t,\mathcal{X})$ is finite for any $t < T_\infty$. We additionally assume that $\kappa(t,\mathcal{X}) > 0$ for all $t \geq 0$ and that the integral $\int_{[0,t]} \kappa(s,A)ds$ is finite for any $A \in \boldsymbol{\mathcal{X}}$ and any $t < T_\infty$. This last assumption is satisfied under mild boundedness conditions imposed on η and f.

Note that the process $\nu([0,\cdot],A) = \int_{[0,\cdot]} \mathbb{1}_{]\!]0,T_\infty(\omega)[\![}(s)\kappa(s,A)ds$ is continuous for any set $A \in \boldsymbol{\mathcal{X}}$ and thus it is $\mathbb{F}^N$–predictable. Consequently, ν is a $\mathbb{F}^N$–predictable random measure.

[2] See Appendix A.2 in Last and Brandt [LB95] for the definition of the kernel.

Before we proceed we recall that for a given filtration $\mathbb{F}$ the random measure ν is said to be $\mathbb{F}$-compensator of a random measure N if it is $\mathbb{F}$-predictable random measure such that it holds

$$\mathbb{E}\int_0^\infty \int_{E^\Delta} F(v,x)N(dv,dx) = \mathbb{E}\int_0^\infty \int_{E^\Delta} F(v,x)\nu(dv,dx)$$

for every non-negative $\mathbb{F}$-predictable function $F : \Omega \times [0,\infty) \times \mathcal{X} \to \mathbb{R}$.

We are ready to state the underlying definition in this paper.

Definition 2.1. Let N be the marked point process introduced in (2.1) with the corresponding random measure N defined in (2.2). We call N a *generalized Hawkes process* on $(\Omega, \mathcal{F}, \mathbb{P})$, if the $(\mathbb{F}^N, \mathbb{P})$–compensator of N, say ν, is of the form (2.3). The kernel κ is called the *Hawkes kernel* for N. □

Remark 2.2. We note that in our definition of the generalized Hawkes process the integral in (2.4) is taken over the interval $(0,t)$. In the definition of the classic Hawkes process, the corresponding integral is taken over the interval $(-\infty, t)$; see eg. [ELL11]. The "$(0,t)$" convention is used by several authors, though, in many applications of classical Hawkes processes (such as in Example 3.9) that do not regard stationarity and spectral properties of these processes. We use this convention here since we are not considering stationarity and spectral properties of the generalized Hawkes processes.

Remark 2.3. (i) Recall that the compensator of a random measure is unique (up to equivalence). Thus, the compensator ν of N is unique. However, the representation (2.3)-(2.4) is not unique, by any means, in general. For any given η and f in the representation (2.3)-(2.4), one can always find $\widetilde{\eta} \neq \eta$ and $\widetilde{f} \neq f$ such that

$$\kappa(t,dy) = \widetilde{\eta}(t,dy) + \int_{(0,t)\times\mathcal{X}} \widetilde{f}(t,s,x,dy)N(ds,dx). \tag{2.6}$$

(ii) With a slight abuse of terminology we refer to κ as to the Hawkes intensity kernel of N. Accordingly, we refer to the quantity $\kappa(t,A)$ as to the intensity at time t of the event regarding process N and amounting to the marks of N taking values in the set A, or, for short, as to the intensity at time t of marks of N taking values in A. □

Remark 2.4. Since $\mathbb{F}^N_0$ is a completed trivial σ-field, then it is a consequence of Theorem 3.6 in [Jac75] that the compensator ν determines the law of N under $\mathbb{P}$, and, consequently, the Hawkes kernel κ determines the law of N under $\mathbb{P}$. □

2.1 Existence of a generalized Hawkes process

We will now demonstrate that for an arbitrary measure ν of the form (2.3) there exists a Hawkes process having ν as $\mathbb{F}^N$–compensator. Towards this end we will consider the underlying canonical space. Specifically, we take $(\Omega, \mathcal{F})$ to be the canonical space of multivariate marked point processes with marks taking values in $\mathcal{X}^\partial$. That is, Ω consists of elements $\omega = ((t_n, x_n))_{n\geq1}$, satisfying $(t_n, x_n) \in (0,\infty] \times \mathcal{X}^\partial$ and

$$\begin{aligned}
&t_n \leq t_{n+1};\\
&\text{if } t_n < \infty, \text{ then } t_n < t_{n+1};\\
&t_n = \infty \text{ iff } x_n = \partial.
\end{aligned}$$

The σ–field $\mathcal{F}$ is defined to be the smallest σ–field on Ω such that the mappings $T_n : \Omega \to ([0,\infty], \mathcal{B}[0,\infty])$, $X_n : \Omega \to (\mathcal{X}^\partial, \boldsymbol{\mathcal{X}}^\partial)$ defined by

$$T_n(\omega) := t_n, \quad X_n(\omega) := x_n$$

are measurable for every n.

Note that the canonical space introduced above agrees with the definition of canonical space considered in [LB95] (see Remark 2.2.5 therein). On this space we denote by N a sequence of measurable mappings

$$N = ((T_n, X_n))_{n\geq 1}, \tag{2.7}$$

Clearly, these mappings satisfy

1. $T_n \leq T_{n+1}$, and if $T_n < +\infty$ then $T_n < T_{n+1}$,
2. $X_n = \partial$ iff $T_n = \infty$.

We call such N a canonical mapping.

The following result provides the existence of a probability measure $\mathbb{P}_\nu$ on $(\Omega, \mathcal{F})$ such that the canonical mapping N becomes a generalized Hawkes process with a given Hawkes kernel κ, which in a unique way determines the compensator ν.

Theorem 2.5. *Consider the canonical space $(\Omega, \mathcal{F})$ and the canonical mapping N given by* (2.7)*. Let measures N and ν be associated with this canonical mapping through* (2.2) *and* (2.3)–(2.4)*, respectively. Then, there exists a unique probability measure $\mathbb{P}_\nu$ on $(\Omega, \mathcal{F})$, such that the measure ν is an $(\mathbb{F}^N, \mathbb{P}_\nu)$–compensator of N. So, N is a generalized multivariate Hawkes process on $(\Omega, \mathcal{F}, \mathbb{P}_\nu)$.*

Proof. We will use Theorem 8.2.1 in [LB95] with $\mathbf{X} = \mathcal{X}$, $\varphi = \omega$, and with

$$\bar{\alpha}(\omega, dt) := \nu(\omega, dt, \mathcal{X}) = \mathbb{1}_{]\!]0,T_\infty(\omega)[\![}(t)\kappa(\omega, t, \mathcal{X})dt, \tag{2.8}$$

from which we will conclude the assertion of theorem. Towards this end, we will verify that all assumptions of the said theorem are satisfied in the present case. As already observed, the random measure ν is $\mathbb{F}^N$–predictable. Next, let us fix $\omega \in \Omega$. Given (2.8) we see that $\bar{\alpha}$ satisfies the following equalities

$$\bar{\alpha}(\omega, \{0\}) = 0, \qquad \bar{\alpha}(\omega, \{t\}) = 0 \leq 1, \quad t \geq 0,$$

which correspond to conditions (4.2.6) and (4.2.7) in [LB95], respectively. It remains to show that condition (4.2.8) holds as well, that is

$$\bar{\alpha}(\omega, [\![\pi_\infty(\omega), \infty[\![) = 0, \tag{2.9}$$

where

$$\pi_\infty(\omega) := \inf\{t \geq 0 : \bar{\alpha}(\omega, (0,t]) = \infty\}.$$

To see this, we first note that (2.8) implies

$$\bar{\alpha}(\omega, [\![T_\infty(\omega), \infty[\![) = 0.$$

Thus it suffices to show that $\pi_\infty(\omega) \geq T_\infty(\omega)$. By definition of $\bar{\alpha}$ we can write

$$\bar{\alpha}(\omega,(0,t]) = \begin{cases} \int_0^t \kappa(\omega,s,\mathcal{X})ds, & t < T_\infty(\omega), \\ \int_0^{T_\infty(\omega)} \kappa(\omega,s,\mathcal{X})ds, & t \geq T_\infty(\omega). \end{cases}$$

If $T_\infty(\omega) = \infty$, then we clearly have $\pi_\infty(\omega) = \infty = T_\infty(\omega)$.

Next, if $T_\infty(\omega) < \infty$, then $\lim_{t\uparrow T_\infty(\omega)} \bar{\alpha}(\omega,(0,t]) = a$. We need to consider two cases now: $a = \infty$ and $a < \infty$.

If $a = \infty$, then $\bar{\alpha}(\omega,(0,t]) = \infty$ for $t \geq T_\infty(\omega)$, and, $\bar{\alpha}(\omega,(0,t]) < \infty$ for $t < T_\infty(\omega)$ in view of our assumptions imposed on κ in the beginning of this section. This implies that $\pi_\infty(\omega) = T_\infty(\omega)$.

If $a < \infty$, then $\bar{\alpha}(\omega,(0,t]) = a < \infty$ for $t \geq T_\infty(\omega)$, hence $\pi_\infty(\omega) = \infty \geq T_\infty(\omega)$. Thus, $\pi_\infty(\omega) \geq T_\infty(\omega)$, which implies that (2.9) holds.

Since ω was arbitrary, we conclude that for all $\omega \in \Omega$ conditions (4.2.6)-(4.2.8) in [LB95] are satisfied. So, applying Theorem 8.2.1 in [LB95] with $\beta = \nu$, we obtain that there exists a unique probability measure $\mathbb{P}_\nu$ such that ν is a $\mathbb{F}^N$–compensator of N under $\mathbb{P}_\nu$. $\square$

2.2 Cluster interpretation of generalized Hawkes processes

The classical Hawkes processes are conveniently interpreted, or represented, in terms of so called clusters. This kind of representation is sometimes called immigration and birth representation. We refer to [HO74] and [LTP15].

Generalized Hawkes processes also admit cluster representation. The dynamics of cluster centers, or the immigrants, is directed by η. Specifically, $\eta(t,A)$ is the time-t intensity of arrivals of immigrants with marks belonging to set A. The dynamics of the off-springs is directed by f. Specifically, $f(t,s,x,A)$ represents the time-t intensity of births of offsprings with marks in set A of either an immigrant with mark x who arrived at time s, or of an offspring with mark x who was born at time s.

Cluster interpretation is behind a very special representation, construction in fact, of some types of Hawkes processes. For example, Horst and Xu [HX19b] consider a marked Hawkes process with the Hawkes kernel with (using combination of our notation and their notation)

$$\eta(t,dx) = \mu(t,x)m(dx), \text{ and } f(t,s,x,dy) = \phi(s,x,y,t-s)m(dy),$$

for some random fields μ, ϕ and a Borel measure m. Now, starting with an exogenous Poisson measure N_0 on $[0,\infty)\times\mathcal{X}\times[0,\infty)$ with intensity $dsm(dx)dz$, they show that random measure N given as

$$N((0,t],A) = \int_0^t \int_A \int_0^\infty \mathbb{1}_{\{z\leq\lambda(s,x)\}} N_0(ds,dx,dz), \tag{2.10}$$

where

$$\lambda(t,x) = \mu(t,x) + \int_0^t \int_{\mathcal{X}} \phi(s,x,y,t-s)N(ds,dy),$$

is a marked Hawkes process N with Hawkes kernel

$$\kappa(t,dx) = \lambda(t,x)m(dx).$$

As of now we do not know whether a representation analogous to (2.10) obeys for our generalized Hawkes process.

2.3 Two results regarding Hawkes kernel

We return to our original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we give two results regarding Hawkes kernel.

First, we state sufficient conditions for $T_\infty = \infty$.

Proposition 2.6. *Assume that there exist real valued nonnegative and continuous functions α, β, γ on $[0, \infty)$ such that for $t \geq 0$*

$$0 \leq \eta(t, \mathcal{X}) \leq \alpha(t), \quad 0 \leq f(t, s, x, \mathcal{X}) \leq \beta(t)\gamma(s), \tag{2.11}$$

and, in addition, assume that $\mathbb{E}\Big(\kappa(\cdot, \mathcal{X})\Big)$ is continuous. Then $T_\infty = \infty$.

Proof. Let's first observe that for any $0 \leq s \leq t$

$$\begin{aligned}\mathbb{E}\Big(\kappa(t, \mathcal{X})\Big) &= \mathbb{E}\Big(\eta(t, \mathcal{X})\Big) + \mathbb{E}\Big(\int_0^t \int_{\mathcal{X}} f(t, s, x, \mathcal{X}) N(ds, dx)\Big) \\ &= \mathbb{E}\Big(\eta(t, \mathcal{X})\Big) + \mathbb{E}\Big(\int_0^t \int_{\mathcal{X}} f(t, s, x, \mathcal{X}) \kappa(s, dx) ds\Big).\end{aligned}$$

Then, using (2.11) we note that for each $t \geq 0$

$$\mathbb{E}\Big(\kappa(t, \mathcal{X})\Big) \leq \alpha(t) + \beta(t) \int_0^t \gamma(s) \big(\mathbb{E}\kappa(s, \mathcal{X})\big) ds$$

and applying the generalized Gronwall inequality (cf. [Pac97, Theorem 1.3.2]) we obtain that for every $T > 0$

$$\mathbb{E}\Big(\kappa(t, \mathcal{X})\Big) \leq \alpha(T) + \beta(T) \int_0^T \alpha(s)\gamma(s) \left(\exp \int_s^T \beta(u)\gamma(u) du\right) ds < \infty.$$

Hence, we conclude that for any $T > 0$ and $0 \leq t \leq T$

$$\mathbb{E} \int_0^T \kappa(t, \mathcal{X}) dt < \infty. \tag{2.12}$$

Consequently, for any $T > 0$

$$\mathbb{E}N((0, T], \mathcal{X}) < \infty$$

and thus $T_\infty = \infty$. □

Our next result is an important proposition that presents a form of compensator of a generalized Hawkes process that providers for i.i.d. marks with a given probability distribution.

Proposition 2.7. *Consider process N as in* (2.1) *with the compensator given by* (2.3) *with κ as in* (2.4)*. Let $\widehat{\nu}$ be a probability measure on $(\mathcal{X}, \boldsymbol{\mathcal{X}})$, and assume that $f(t,s,x,dy) = \varphi(t,s,x)\widehat{\nu}(dy)$, where φ is a deterministic nonnegative and locally integrable function, and $\eta(t,dy) = \gamma(t)\widehat{\nu}(dy)$, where γ is a deterministic, non-negative and locally integrable function, satisfying that $\int_t^\infty \gamma(s)ds = \infty$ for all $t \geq 0$. Then, $T_n < \infty$ almost surely for $n = 1, \ldots$. Moreover, the marks X_n, $n = 1, \ldots$, are i.i.d. random variables with distribution $\widehat{\nu}$, and X_{n+1} is independent of $T_1, \ldots, T_n$, for $n = 1, \ldots$.*

Proof. Let us first note that

$$\begin{aligned}
\nu(dt,dy) &= \mathbb{1}_{]\!]0,T_\infty(\omega)[\![}(t)\kappa(t,dy)dt \\
&= \mathbb{1}_{]\!]0,T_\infty(\omega)[\![}(t)\Big(\gamma(t) + \int_{(0,t)\times\mathcal{X}} \varphi(t,s,x)N(ds,dx)\Big)\widehat{\nu}(dy)dt \\
&= \widehat{\nu}(dy)\nu(dt \times \mathcal{X}).
\end{aligned}$$

The above representation is essential for using results from Last and Brandt [LB95, Section 4.3]. In particular, given the properties of function γ, it follows from (4.3.51) in [LB95] that $T_n < \infty$ almost surely for $n = 1, \ldots$.

The proof of the second assertion goes by induction. We start by noting that by Theorem 4.3.2.(ii) [LB95]

$$\begin{aligned}
\mathbb{P}(X_1 \in A_1) = \mathbb{P}(X_1 \in A_1, T_1 < \infty) &= \mathbb{E}(\mathbb{E}(\mathbb{1}_{\{X_1\in A_1\}}\mathbb{1}_{\{T_1<\infty\}}|T_1)) \\
&= \mathbb{E}(\widehat{\nu}(A_1)\mathbb{1}_{T_1<\infty}) = \widehat{\nu}(A_1).
\end{aligned}$$

Suppose now that $X_1, \ldots, X_n$ are i.i.d. random variables with distribution $\widehat{\nu}$, and consider $X_1, \ldots, X_n, X_{n+1}$. We need to show that for any sets
$A_1, \ldots, A_n, A_{n+1} \in \boldsymbol{\mathcal{X}}$ it holds

$$\mathbb{P}(X_1 \in A_1, \ldots, X_{n+1} \in A_{n+1}) = \prod_{i=1}^{n+1} \mathbb{P}(X_i \in A_i) = \prod_{i=1}^{n+1} \widehat{\nu}(A_i).$$

In view of the tower property, Theorem 4.3.2.(ii) [LB95] and induction hypothesis we obtain

$$\begin{aligned}
&\mathbb{P}(X_1 \in A_1, \ldots, X_{n+1} \in A_{n+1}) = \mathbb{E}(\mathbb{E}(\mathbb{1}_{\{X_1\in A_1\}} \ldots \mathbb{1}_{\{X_{n+1}\in A_{n+1}\}}|\mathcal{F}_{T_n}, T_{n+1})) \\
&= \mathbb{E}(\mathbb{1}_{\{X_1\in A_1\}} \ldots \mathbb{1}_{\{X_n\in A_n\}}\mathbb{1}_{\{T_{n+1}<\infty\}}\mathbb{E}(\mathbb{1}_{\{X_{n+1}\in A_{n+1}\}}|\mathcal{F}_{T_n}, T_{n+1})) \\
&= \mathbb{E}(\mathbb{1}_{\{X_1\in A_1\}} \ldots \mathbb{1}_{\{X_n\in A_n\}}\widehat{\nu}(A_{n+1})\mathbb{1}_{\{T_{n+1}<\infty\}}) \\
&= \mathbb{E}(\mathbb{1}_{\{X_1\in A_1\}} \ldots \mathbb{1}_{\{X_n\in A_n\}}\mathbb{1}_{\{T_{n+1}<\infty\}})\widehat{\nu}(A_{n+1}) = \widehat{\nu}(A_1) \ldots \widehat{\nu}(A_n)\widehat{\nu}(A_{n+1}).
\end{aligned}$$

This proves that the sequence of marks $(X_j)_{j\geq 1}$ is sequence of i.i.d. random variables with distribution $\widehat{\nu}$. The proof that X_{n+1} is independent of $T_1, \ldots, T_n$ is similar. Let us take $A_{n+1} \in \boldsymbol{\mathcal{X}}$, $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}_+)$ using again the tower property, Theorem 4.3.2.(ii) [LB95] we obtain

$$\begin{aligned}
&\mathbb{P}(X_{n+1} \in A_{n+1}, T_1 \in B_1, \ldots T_n \in B_n) \\
&= \mathbb{E}(\mathbb{E}(\mathbb{1}_{\{X_{n+1}\in A_{n+1}, T_1\in B_1, \ldots T_n\in B_n\}}|\mathcal{F}_{T_n}, T_{n+1})) \\
&= \mathbb{E}(\mathbb{1}_{\{T_1\in B_1, \ldots T_n\in B_n\}}\mathbb{E}(\mathbb{1}_{\{X_{n+1}\in A_{n+1}\}}|\mathcal{F}_{T_n}, T_{n+1})\mathbb{1}_{\{T_{n+1}<\infty\}}) \\
&= \mathbb{E}(\mathbb{1}_{\{T_1\in B_1, \ldots T_n\in B_n\}}\widehat{\nu}(A_{n+1})\mathbb{1}_{\{T_{n+1}<\infty\}}) \\
&= \mathbb{P}(T_1 \in B_1, \ldots T_n \in B_n)\widehat{\nu}(A_{n+1}) = \mathbb{P}(T_1 \in B_1, \ldots T_n \in B_n)\mathbb{P}(X_{n+1} \in A_{n+1}).
\end{aligned}$$

The proof is complete. □

2.4 Asymptotics in time of generalized Hawkes processes

Here we give a couple of preliminary results of the law of large numbers type for generalized Hawkes processes. A much more thorough and comprehensive study of asymptotics in time of generalized Hawkes processes is under preparation.

In order to consider asymptotics in time for process N we need to first provide sufficient conditions on the Hawkes kernel κ so that $T_\infty = \infty$. Towards this end we may use sufficient conditions stated in Proposition 2.6. Since in our set-up we have that $\nu(\{t\}, \mathcal{X}) = 0$, then another type of sufficient condition for $T_\infty = \infty$ comes from Theorem 4.1.7 (ii) in [LB95]: if $\inf\{t \geq 0 : \nu(\{t\}, \mathcal{X}) = \infty\} = \infty$ then $T_\infty = \infty$.

We begin with the following result, which is a weak functional law of large numbers for a generalized Hawkes process N.

Proposition 2.8. *Fix $A \in \boldsymbol{\mathcal{X}}$. Assume that $T_\infty = \infty$. Consider the compensated process $\widetilde{N}(\cdot, A) := N(\cdot, A) - \nu(\cdot, A)$ and assume that $\frac{1}{T}\widetilde{N}(Tt, A)$ converges to zero in probability, uniformly in t on any bounded interval, when $T \to \infty$. In addition, suppose that there exists a process k so that $\frac{1}{T}(\nu(Tt, A) - k(t))$ converges to zero in probability, uniformly in t on any bounded interval, when $T \to \infty$. Then $\frac{1}{T}(N(Tt, A) - k(t))$ converges to zero in probability, uniformly in t on any bounded interval, when $T \to \infty$.*

Proof. The result is an immediate consequence of the decomposition

$$N(Tt, A) - k(t) = \widetilde{N}(Tt, A) + \nu(Tt, A) - k(t)$$

and of the continuous mapping lemma (cf. e.g. [Kal02, Lemma 4.3]). □

We will illustrate the above result with an example that is based on [HX19a].

Example 2.9. Let us consider the set-up of Horst and Xu [HX19a] with the following modifications (using their notation):

- ▷ Take the filtration $\{\mathcal{F}_t : t \geq 0\}$ considered in [HX19a] to be the natural filtration of process N;
- ▷ Take μ_0 to be a deterministic nonnegative Borel measurable function;
- ▷ Take the exogenous Poisson process N_I to be constant; that is, take the jump intensity $\lambda_I \equiv 0$;
- ▷ Set $\widehat{\nu} := \nu_H$.

Given our Proposition 2.7, their set-up, with the above modifications, can be considered as a special version of our set-up, namely (using our notation and the notation of [HX19a]):

- ▷ $\mathcal{X}$ taken to be a Lusin space;
- ▷ $f(t, s, x, dy) = \phi(t - s, x)\widehat{\nu}(dy)$ (i.e., taking $\varphi(t, s, x) = \phi(t - s, x)$ in Proposition 2.7);
- ▷ $\eta(t, dy) = \mu_0(t)\widehat{\nu}(dy)$ (i.e., taking $\gamma = \mu_0$ in Proposition 2.7);

▷ $T_\infty = \infty$.

Imposing some technical conditions, and using Burkholder-Davis-Gundy inequality, Horst and Xu [HX19a] prove that $\frac{1}{T}\widetilde{N}(Tt, A)$ converges to zero in probability, uniformly in t on any bounded interval, when $T \to \infty$, and that $\frac{1}{T}(\nu(Tt, A) - k(t))$ converges to zero in probability, uniformly in t on any bounded interval, when $T \to \infty$, where $k(t) = 0$ for $t \geq 0$. □

The next result is a version of a strong law of large numbers for a generalized Hawkes process N.

Proposition 2.10. *Fix $A \in \boldsymbol{\mathcal{X}}$. Assume that $T_\infty = \infty$. Consider the compensated process $\widetilde{N}(\cdot, A) := N(\cdot, A) - \nu(\cdot, A)$ and assume that it is a locally square integrable martingale. Next, assume that $P(\lim_{T\to\infty} \nu(T, A) = \infty) = 1$. Then*

$$P\left(\lim_{T\to\infty} \frac{N(T, A)}{\nu(T, A)} = 1\right) = 1.$$

Proof. The results follows from the strong law of large numbers for locally square integrable martingales (cf. e.g. [LS89, Corollary 1 p. 144]). □

Example 2.11. Consider the same set-up as in Example 2.9. Note that

$$\nu(t, A) = \widehat{\nu}(A) \int_0^t Z(s)ds,$$

where

$$Z(t) = \mu_0(t) + \sum_{n=1}^{N_t} \phi(t - T_k, X_k).$$

We assume that $\widetilde{N}(\cdot, A)$ is a locally square integrable martingale. Given the above, we see that a sufficient condition for $P(\lim_{T\to\infty} \nu(T, A) = \infty) = 1$ is that $P(\int_0^\infty \mu_0(s)ds = \infty) = 1$. In this case the strong law of large numbers holds for the process N. □

3 Generalized multivariate Hawkes process

We now introduce the concept of a generalized multivariate Hawkes process, which is a particular case of the concept of a generalized Hawkes process.

3.1 Definition of generalized multivariate Hawkes process

We first construct an appropriate mark space. Specifically, we fix an integer $d \geq 1$ and we let $(E_i, \mathcal{E}_i)$, $i = 1, \ldots, d$, be some non-empty Borel spaces, and Δ be a dummy mark, the meaning of which will be explained below. Very often, in practical modelling, spaces E_i are discrete. The instrumental rationale for considering a discrete mark space is that in most of the applications of the Hawkes processes that we are familiar with and/or we can imagine, a discrete mark space is sufficient to account for the intended features of the modeled phenomenon.

We set $E_i^\Delta := E_i \cup \Delta$, and we denote by $\mathcal{E}_i^\Delta$ the sigma algebra on E_i^Δ generated by $\mathcal{E}_i$. Then, we define a mark space, say E^Δ, as

$$E^\Delta := E_1^\Delta \times E_2^\Delta \times \ldots \times E_d^\Delta \setminus (\Delta, \Delta, \ldots, \Delta). \tag{3.1}$$

By $\mathcal{E}^\Delta$ we denote a trace sigma algebra of $\otimes_{i=1}^d \mathcal{E}_i^\Delta$ on E^Δ, i.e.

$$\mathcal{E}^\Delta := \Big\{ A \cap E^\Delta : A \in \otimes_{i=1}^d \mathcal{E}_i^\Delta \Big\}.$$

Moreover, denoting by ∂_i the point which is external to E_i^Δ, we define $E_i^\partial := E_i^\Delta \cup \{\partial_i\}$, and we denote by $\mathcal{E}_i^\partial$ the sigma algebra generated by $\mathcal{E}_i$ and $\{\partial_i\}$. Analogously we define

$$E^\partial := E^\Delta \cup \partial,$$

where $\partial = (\partial^1, \ldots, \partial^d)$ is a point external to $E_1^\Delta \times E_2^\Delta \times \ldots \times E_d^\Delta$ and by $\mathcal{E}^\partial$ we denote the sigma field generated by $\mathcal{E}^\Delta$ and $\{\partial\}$.

Definition 3.1. A generalized Hawkes process $N = ((T_n, X_n))_{n\geq 1}$ with the mark space $\mathcal{X} = E^\Delta$ given by (3.1), and with $\mathcal{X}^\partial = E^\partial$, is called a *generalized multivariate Hawkes process (of dimension d).* □

Remark 3.2. In Section 2.1 we provided an argument for existence of a generalized Hawkes process. This argument applies of course to the case of generalized multivariate Hawkes processes. In [BJN20a], based on the cluster interpretation of Section 2.2, we provided a constructive proof of a GMHP with deterministic kernels η and f on a finite time interval $[0, T]$.

Note that a necessary condition for generalized Hawkes processes to feature the self-excitation and mutual-excitation is that $f \neq 0$. We refer to Example 3.10 for interpretation of the components η and f of the kernel κ in case of a generalized multivariate Hawkes process.

We interpret $T_n \in (0, \infty)$ and $X_n \in E^\Delta$ as the event times of N and as the corresponding mark values, respectively. Thus, if $T_n < \infty$ we have[3]

$$X_n = (X_n^i, i = 1, 2, \ldots, d), \quad \text{where} \quad X_n^i \in E_i^\Delta.$$

Also, we interpret X^i as the marks associated with i-th coordinate of N (cf. Definition 3.4). With this interpretation, the equality $X_n^i(\omega) = \Delta$ means that there is no event taking place with regard to the i-th coordinate of N at the (general) event time $T_n(\omega)$. In other words, no event occurs with respect to the i-th coordinate of N at time $T_n(\omega)$.

Definition 3.3. We say that $T_n(\omega)$ is a common event time for a multivariate Hawkes process N if there exist i and j, $i \neq j$, such that $X_n^i(\omega) \in E_i$ and $X_n^j(\omega) \in E_j$. We say that process N admits common event times if

$$\mathbb{P}\Big(\omega \in \Omega : \exists n \text{ such that } T_n(\omega) \text{ is a common event time } \Big) > 0$$

Otherwise we say that process N admits no common event times. □

[3] Note that here d is the number of components in X_n, and n is the index of the $n-th$ element in the sequence $(X_n)_{n\geq 1}$.

Definition 3.3 generalizes that in Bremaud and Massouli [BM96] and Liniger [Lin09]. In particular, with regard to the concepts of multivariate Hawkes processes studied in Liniger [Lin09], the genuine multivariate Hawkes processes [Lin09] admits no common event times, whereas in the case of pseudo-multivariate Hawkes process [Lin09] all event times are common.

3.2 The i-th coordinate of a generalized multivariate Hawkes process N

So far in the literature multivariate Hawkes processes have been defined in terms of their coordinates. Our approach is different: note that no process coordinates are specified explicitly in Definition 3.1. In this section we define and study coordinates (components) of a generalized multivariate Hawkes process. In particular, this study is relevant in the context of the so called Hawkes structures that are introduced and investigated in [BJN20b].

We start with

Definition 3.4. We define the $i-th$ coordinate N^i of N as

$$N^i((0,t],A) := \sum_{n\geq 1} \mathbb{1}_{\{T_n\leq t\}}\mathbb{1}_{\{X_n\in A^i\}}, \tag{3.2}$$

for $A\in\mathcal{E}_i$ and $t\geq 0$, where

$$A^i = \Big(\bigtimes_{j=1}^{i-1} E_j^\Delta\Big)\times A\times\Big(\bigtimes_{j=i+1}^{d} E_j^\Delta\Big). \tag{3.3}$$

□

Clearly, N^i is a MPP and

$$N^i((0,t],A) = N((0,t],A^i).$$

Indeed, the i-th coordinate process N^i can be represented as a sequence $N^i = (T^i_k, Y^i_k)_{k\geq 1}$, which is related to the sequence $(T_n, X^i_n)_{n\geq 1}$ as follows

$$(T^i_k, Y^i_k) = \begin{cases} (T_{m^i_k}, X^i_{m^i_k}) & \text{if } m^i_k<\infty,\\ (T_{m^i_{\widehat{k}^i}+k-\widehat{k}^i}, \Delta) & \text{if } m^i_k=\infty \text{ and } T_\infty<\infty,\\ (\infty, \partial^i) & \text{if } m^i_k=\infty \text{ and } T_\infty=\infty,\end{cases} \tag{3.4}$$

where $\widehat{k}^i = \max\{n : m^i_n<\infty\}$, with m^i defined as

$$\begin{aligned} m^i_1 &= \inf\{n\geq 1 : X^i_n\in E_i\},\\ m^i_k &= \inf\{n> m^i_{k-1} : X^i_n\in E_i\} \quad \text{for } k>1.\end{aligned}$$

We clearly have

$$N^i((0,t],A) = \sum_{k\geq 1} \mathbb{1}_{\{T^i_k \leq t\}} \mathbb{1}_{\{Y^i_k \in A\}}, \quad A \in \mathcal{E}_i. \tag{3.5}$$

In particular this means that for the i-th coordinate N^i the times $T_n(\omega)$ such that $X^i_n(\omega) = \Delta$ are disregarded as event times for this coordinate since the events occurring with regard to the entire N at these times do not affect the i-th coordinate.

We define the completed filtration $\mathbb{F}^{N^i} = (\mathcal{F}^{N^i}_t,\ t \geq 0)$ generated by N^i in analogy to $\mathbb{F}^N$; specifically $\mathcal{F}^{N^i}_t$ is the $\mathbb{P}$–completion of the σ–field $\sigma(N^i((s,r] \times A) : 0 \leq s < r \leq t,\ A \in \mathcal{E}_i),\ t \geq 0$. In view of Theorem 2.2.4 in [LB95] the filtration $\mathbb{F}^{N^i}$ satisfies the usual conditions.

We define the explosion time T^i_∞ of N^i as

$$T^i_\infty := \lim_{n\to\infty} T^i_n.$$

Clearly, $T^i_\infty \leq T_\infty$.

We conclude this section with providing some more insight into the properties of the measure N^i. Towards this end, we first observe that the measure N^i is both $\mathbb{F}^N$–optional and $\mathbb{F}^{N^i}$–optional. Subsequently, we will derive the compensator of N^i with respect to $\mathbb{F}^N$ and the compensator of N^i with respect to $\mathbb{F}^{N^i}$. The following Proposition 3.5 and Proposition 3.8 come handy in this regard.

Proposition 3.5. *Let N be a generalized multivariate Hawkes process with Hawkes kernel κ. Then the $(\mathbb{F}^N, \mathbb{P})$–compensator, say ν^i, of measure N^i defined in (3.2) is given as*

$$\nu^i(\omega, dt, dy_i) = \mathbb{1}_{]\!]0;T^i_\infty(\omega)[\![}(t)\kappa^i(\omega, t, dy_i)dt, \tag{3.6}$$

where

$$\kappa^i(t,A) := \kappa(t, A^i), \quad t \geq 0,\ A \in \mathcal{E}_i, \tag{3.7}$$

with A^i defined in (3.3).

Proof. According to Theorems 4.1.11 and 4.1.7 in [LB95] the i-th coordinate N^i admits a unique $\mathbb{F}^N$–compensator, say ν^i, with a property that $\nu^i([\![T^i_\infty;\infty[\![\times E_i) = 0$. For every n and $A \in \mathcal{E}_i$ the processes $M^{i,n,A}$ and $\widehat{M}^{i,n,A}$ given as

$$M^{i,n,A}_t = N^i((0, t\wedge T_n] \times A) - \int_0^{t\wedge T_n} \mathbb{1}_{]\!]0;T_\infty[\![}(u)\kappa^i(u,A)du,\ t \geq 0,$$

and

$$\widehat{M}^{i,n,A}_t = N^i((0,t\wedge T_n] \times A) - \nu^i((0, t\wedge T_n] \times A),\ t \geq 0,$$

are $(\mathbb{F}^N, \mathbb{P})$–martingales. Hence the process

$$\int_0^{t\wedge T_n} \Big(\mathbb{1}_{]\!]0;T_\infty[\![}(u)\kappa^i(u,A)du - \nu^i(du,A)\Big),\ t \geq 0,$$

is an $\mathbb{F}^N$-predictable martingale. Since it is of integrable variation and null at $t = 0$ it is null for all $t \geq 0$ (see e.g. Theorem VI.6.3 in [HWY92]). From the above and the fact that $T^i_\infty \leq T_\infty$ we deduce that

$$\int_0^{t\wedge T_n} \mathbb{1}_{\rrbracket 0;T^i_\infty\llbracket}(u)\kappa^i(u,A)du = \nu^i((0,t\wedge T_n]\times A), \qquad t \geq 0.$$

This proves the proposition. □

Remark 3.6. Note that for each i, the function κ^i defined in (3.7) is a measurable kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P}\otimes\mathcal{B}(\mathbb{R}_+))$ to $(E_i, \mathcal{E}_i)$. It is important to observe that, in general, there is no one-to-one correspondence between the Hawkes kernel κ and all the marginal kernels κ^i, $i = 1, \ldots, d$. We mean by this that may exist another Hawkes kernel, say $\widehat{\kappa}$, such that $\widehat{\kappa} \neq \kappa$ and

$$\kappa^i(t, A) = \widehat{\kappa}(t, A^i), \quad t \geq 0,\ A \in \mathcal{E}_i,\ i = 1, \ldots d. \tag{3.8}$$

□

Remark 3.7. As we know from Remark 2.4 the Hawkes kernel κ determines the law of N. However, in view of Remark 3.6, the kernel κ^i may not determine the law of N^i. It remains to be an open problem for now to determine sufficient conditions under which the law of N^i is determined by κ^i. This problem is a special case of a more general problem: what are general sufficient conditions under which characteristics of a semimartingale determine the law of this semimartingale. □

The following important result gives the $\mathbb{F}^{N^i}$–compensator of measure N^i.

Proposition 3.8. *Let N be a generalized multivariate Hawkes process with Hawkes kernel κ. Then the $\mathbb{F}^{N^i}$–compensator of measure N^i, say $\widetilde{\nu}^i$, is given as*

$$\widetilde{\nu}^i(\omega, dt, dy_i) = (\nu^i)^{p,\mathbb{F}^{N^i}}(\omega, dt, dy_i), \tag{3.9}$$

where $(\nu^i)^{p,\mathbb{F}^{N^i}}$ is the dual predictable projection of ν^i on $\mathbb{F}^{N^i}$ under $\mathbb{P}$.

Proof. Using Theorems 4.1.9 and 3.4.6 in [LB95], as well as the uniqueness of the compensator, it is enough to show that for any $A \in \mathcal{E}^i$ and any $n \geq 1$ the process $(\nu^i)^{p,\mathbb{F}^{N^i}}((0, t \wedge T^i_n], A)$, where $(\nu^i)^{p,\mathbb{F}^{N^i}}$ is the dual predictable projection of ν^i on $\mathbb{F}^{N^i}$ under $\mathbb{P}$, is the $\mathbb{F}^{N^i}$–compensator of the increasing process $N^i((0, t \wedge T^i_n], A),\ t \geq 0$. This however follows from Theorem 3.3 in [BJJN20]. □

3.3 Examples

We will provide now some examples of generalized multivariate Hawkes processes.

For $\omega = (t_n, x_n)_{n\geq 1}$, $t \geq 0$ and $A \in \mathcal{E}^\Delta$ we set

$$N(\omega, (0, t], A) := \sum_{n\geq 1} \mathbb{1}_{\{t_n\leq t, x_n\in A\}}. \tag{3.10}$$

In all examples below we define the kernel κ of the form (2.4) with η and f properly chosen, so that we may apply Theorem 2.5 to the effect that there exists a probability measure $\mathbb{P}_\nu$ on $(\Omega, \mathcal{F})$ such that process N given by (3.10) is a Hawkes process with the Hawkes kernel equal to κ. In other words, there exists a probability measure $\mathbb{P}_\nu$ on $(\Omega, \mathcal{F})$ such that ν given in (2.3)–(2.4) is the $\mathbb{F}^N$–compensator of N under $\mathbb{P}_\nu$.

For a Hawkes process N with a mark space E^Δ we introduce the following notation

$$N_t = N((0,t], E^\Delta), \quad t \geq 0.$$

Likewise, we denote for $i = 1, \ldots, d$,

$$N^i_t = N^i((0,t], E_i), \quad t \geq 0.$$

Example 3.9. (Classical univariate Hawkes process)

We take $d = 1$ and $E_1 = \{1\}$, so that $E^\Delta = E_1 = \{1\}$. As usual, and in accordance with (2.2), we identify N with a point process $(N_t)_{t\geq 0}$. Now we take

$$\eta(t, \{1\}) = \lambda(t),$$

where λ is positive, locally integrable function, and, for $0 \leq s \leq t$, we take

$$f(t, s, 1, \{1\}) = w(t - s)$$

for some non-negative function w defined on $\mathbb{R}_+$ (recall that $f(t, s, 1, \{1\}) = 0$ for $s \geq t$). Using these objects we define κ by

$$\kappa(t, dy) = \bar{\kappa}(t)\delta_{\{1\}}(dy),$$

where

$$\bar{\kappa}(t) = \lambda(t) + \int_{(0,t)} w(t - s) dN_s.$$

In case of the classical univariate Hawkes process sufficient conditions under which the explosion time is almost surely infinite, that is

$$T_\infty = \infty \quad \mathbb{P}_\nu - \text{a.s.}$$

are available in terms of the Hawkes kernel. Specifically, sufficient conditions for no-explosion are given in [BDHM13]:

$$\lambda \text{ is locally bounded,} \quad \text{and} \quad \int_0^\infty w(u) du < \infty. \qquad \square$$

Example 3.10. (Generalized bivariate Hawkes process with common event times)

In the case of a generalized bivariate Hawkes process N we have $d = 2$ and the mark space is given as

$$E^\Delta = E^\Delta_1 \times E^\Delta_2 \setminus \{(\Delta, \Delta)\} = \{(\Delta, y_2), (y_1, \Delta), (y_1, y_2) : y_1 \in E_1, y_2 \in E_2\}.$$

Here, in order to define kernel κ, we take kernel η in the form

$$\eta(t, dy) = \eta_1(t, dy_1) \otimes \delta_\Delta(dy_2) + \delta_\Delta(dy_1) \otimes \eta_2(t, dy_2) + \eta_c(t, dy_1, dy_2),$$

where δ_Δ is a Dirac measure, η_i for $i = 1, 2$ are probability kernels, from $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ to $(E_i, \mathcal{E}_i)$ and η_c is a probability kernel from $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ to $(E^\Delta, \mathcal{E}^\Delta)$, satisfying

$$\eta_c(t, E_1 \times \Delta) = \eta_c(t, \Delta \times E_2) = 0.$$

Kernel f is given, for $0 \le s \le t$ and $x = (x_1, x_2)$, by

$$\begin{aligned} f(t,s,x,dy) = \Big(& w_{1,1}(t,s) g_{1,1}(x_1) \mathbb{1}_{E_1 \times \Delta}(x) + w_{1,2}(t,s) g_{1,2}(x_2) \mathbb{1}_{\Delta \times E_2}(x) \\ &+ w_{1,c}(t,s) g_{1,c}(x) \mathbb{1}_{E_1 \times E_2}(x) \Big) \phi_1(x, dy_1) \otimes \delta_\Delta(dy_2) \\ &+ \Big(w_{2,1}(t,s) g_{2,1}(x_1) \mathbb{1}_{E_1 \times \Delta}(x) + w_{2,2}(t,s) g_{2,2}(x_2) \mathbb{1}_{\Delta \times E_2}(x) \\ &+ w_{2,c}(t,s) g_{2,c}(x) \mathbb{1}_{E_1 \times E_2}(x) \Big) \delta_\Delta(dy_1) \otimes \phi_2(x, dy_2) \\ &+ \Big(w_{c,1}(t,s) g_{c,1}(x_1) \mathbb{1}_{E_1 \times \Delta}(x) + w_{c,2}(t,s) g_{c,2}(x_2) \mathbb{1}_{\Delta \times E_2}(x) \\ &+ w_{c,c}(t,s) g_{c,c}(x_1, x_2) \mathbb{1}_{E_1 \times E_2}(x) \Big) \phi_c(x, dy_1, dy_2), \end{aligned} \tag{3.11}$$

where ϕ_i is a probability kernel from $(E^\Delta, \mathcal{E}^\Delta)$ to $(E_i, \mathcal{E}_i)$ for $i = 1, 2$ and ϕ_c is a probability kernel from $(E^\Delta, \mathcal{E}^\Delta)$ to $(E^\Delta, \mathcal{E}^\Delta)$ satisfying

$$\phi_c(x, E_1 \times \Delta) = \phi_c(x, \Delta \times E_2) = 0.$$

The *decay functions* $w_{i,j}$ and the *impact functions* $g_{i,j}$, $i, j = 1, 2, c$, are appropriately regular and deterministic. Moreover, the decay functions are positive and the impact functions are non-negative. In particular, this implies that the kernel f is deterministic and non-negative.

In what follows we will need the concept of idiosyncratic group of I coordinates of a generalized bivariate Hawkes process N. For $I = \{1\}$ we define

$$N^{idio,\{1\}}((0,t], A) := N((0,t], A \times \Delta), \quad t \ge 0,\ A \in \mathcal{E}_1$$

and, likewise, for $I = \{2\}$ we define

$$N^{idio,\{2\}}((0,t], A) := N((0,t], \Delta \times A), \quad t \ge 0,\ A \in \mathcal{E}_2.$$

Finally, for $I = \{1, 2\}$ we define

$$N^{idio,\{1,2\}}((0,t], A) := N((0,t], A), \quad t \ge 0,\ A \in \mathcal{E}_1 \otimes \mathcal{E}_2.$$

Clearly, $N^{idio,\mathcal{I}}$ is a MPP. For example, $N^{idio,i}$ is a MPP which records idiosyncratic events occurring with regard to X^i; that is, events that only regard to X^i, so that $X^j_n = \Delta$ for $j \neq i$ at times T_n at which these events take place. Likewise, $N^{idio,\{1,2\}}$ is a MPP which records idiosyncratic events occurring with regard to X^1 and X^2 simultaneously. Let us note that

$$N^1 = N^{idio,\{1\}} + N^{idio,\{1,2\}}, \qquad N^2 = N^{idio,\{2\}} + N^{idio,\{1,2\}}.$$

We will now interpret various terms that appear in the expressions for η and f above:

- ▷ $\eta_1(t, dy_1) \otimes \delta_\Delta(dy_2)$ represents autonomous portion of the intensity, at time t, of marks of the coordinate N^1 taking values in the set $dy_1 \subset E_1$ and no marks occurring for N^2;
- ▷ $\eta_c(t, dy_1, dy_2)$ represents autonomous portion of the intensity, at time t, of an event amounting to the marks of both coordinates N^1 and N^2 taking values in the set $dy_1 dy_2 \subset E_1 \times E_2$;
- ▷

$$\begin{aligned}&\int_{(0,t)\times E^\Delta} w_{1,1}(t,s) g_{1,1}(x_1) \mathbb{1}_{E_1\times\Delta}(x)\phi_1(x, dy_1) \otimes \delta_\Delta(dy_2) N(ds, dx)\\ &= \int_{(0,t)\times E_1} w_{1,1}(t,s) g_{1,1}(x_1)\phi_1((x_1,\Delta), dy_1) \otimes \delta_\Delta(dy_2) N^{idio,1}(ds, dx_1)\end{aligned}$$

  represents idiosyncratic impact of the coordinate N^1 alone on the intensity, at time t, of marks of the coordinate N^1 taking values in the set $dy_1 \subset E_1$ and no marks occurring for N^2;
- ▷

$$\int_{(0,t)\times E_2} w_{1,2}(t,s) g_{1,2}(x_2)\phi_1((\Delta, x_2), dy_1) \otimes \delta_\Delta(dy_2) N^{idio,2}(ds, dx_2)$$

  represents idiosyncratic impact of the coordinate N^2 alone on the intensity, at time t, of an event amounting to the marks of coordinate N^1 taking value in the set $dy_1 \subset E_1$ and no marks occurring for N^2;
- ▷

$$\int_{(0,t)\times E^\Delta} w_{1,c}(t,s) g_{1,c}(x) \mathbb{1}_{E_1\times E_2}(x)\phi_1(x, dy_1) \otimes \delta_\Delta(dy_2) N(ds, dx)$$

  represents joint impact of the coordinates N^1 and N^2 on the intensity, at time t, of an event amounting to the marks of coordinate N^1 taking value in the set $dy_1 \subset E_1$ and no marks occurring for N^2;
- ▷

$$\int_{(0,t)\times E_1} w_{c,1}(t,s) g_{c,1}(x_1)\phi_c((x_1,\Delta), dy_1, dy_2) N^{idio,1}(ds, dx_1)$$

  represents idiosyncratic impact of the coordinate N^1 alone on the intensity, at time t, of an event amounting to the marks of both coordinates N^1 and N^2 taking values in the set $dy_1 dy_2 \subset E_1 \times E_2$;
- ▷

$$\int_{(0,t)\times E^\Delta} w_{c,c}(t,s) g_{c,c}(x_1) \mathbb{1}_{E_1\times\Delta}(x)\phi_c(x, dy_1, dy_2) N(ds, dx)$$

  represents joint impact of the coordinates N^1 and N^2 on the intensity, at time t, of an event amounting to the marks of both coordinates N^1 and N^2 taking values in the set $dy_1 dy_2 \subset E_1 \times E_2$.

In particular, the terms contributing to occurrence of common events are $\eta_c(t, dy_1, dy_2)$ and

$$\Big(g_{c,1}(x_1)\mathbb{1}_{E_1\times\Delta}(x) + g_{c,2}(x_2)\mathbb{1}_{\Delta\times E_2}(x) + g_{c,c}(x_1,x_2)\mathbb{1}_{E_1\times E_2}(x)\Big)\phi_c(x, dy_1, dy_2).$$

Upon integrating $\kappa(t, dy)$ over $A_1 \times \{\Delta, E_2\}$ we get

$$\begin{aligned}
\kappa^1(t, A_1) &= \kappa(t, A_1 \times \{\Delta, E_2\}) \\
&= \eta_1(t, A_1) + \eta_c(t, A_1 \times E_2) \\
&+ \int_{(0,t)\times E_1} \Big(w_{1,1}(t,s)g_{1,1}(x_1)\phi_1((x_1,\Delta), A_1) \\
&\qquad + w_{c,1}(t,s)g_{c,1}(x_1)\phi_c((x_1,\Delta), A_1\times E_2)\Big)N^{idio,1}(ds, dx_1) \\
&+ \int_{(0,t)\times E_2} \Big(w_{1,2}(t,s)g_{1,2}(x_2)\phi_1((\Delta, x_2), A_1) \\
&\qquad + w_{c,2}(t,s)g_{c,2}(x_2)\phi_c((\Delta, x_2), A_1\times E_2)\Big)N^{idio,2}(ds, dx_2) \\
&+ \int_{(0,t)\times E} \Big(w_{1,c}(t,s)g_{1,c}(x)\phi_1((x_1,x_2), A_1) \\
&\qquad + w_{c,c}(t,s)g_{c,c}(x_1,x_2)\phi_c((x_1,x_2), A_1\times E_2)\Big)N^{idio,\{1,2\}}(ds, dx).
\end{aligned}$$

To complete this example we note that upon setting $\eta_c = 0$ and $\phi_c = 0$ we produce a generalized bivariate Hawkes process with no common event times. □

4 Applications

The are numerous potential applications of the generalized multivariate Hawkes processes. Here we present a brief description of possible applications in epidemiology and insurance. For other applications we refer to [BJN20b].

4.1 Insurance

As it has been observed, clustering and self-exciting arrivals of claims is an inherent feature of claims arrival processes that is faced by the insurance companies. See e.g. [DZ12] or [Mag15]. Our theory of GMHPs can potentially be applied to enrich that analysis of insurance risk management allowing for accounting for mutual and simultaneous excitation between claims streams corresponding to different lines of insurance, as well as allowing for accounting for various types/severities of claims arriving within the insurance lines.

Consider an insurance company with d major lines of insurance. Let N be a generalized multivariate Hawkes process that models the claims arrivals. Specifically, we interpret the coordinate N^i as a model for claims arrival process corresponding the the i-th line of insurance. We take discrete mark space, say $E^i = \{e^i_k,\ k = 1, 2, \ldots, K^i\}$, where e^i_ks represent type/severity of claims in the i-th line.

Now, let $Z_n^{i,k}$, $n = 1, 2, \ldots$ be i.i.d. random variables, independent of N, representing the claim sizes triggered by the claim of type/severity e_k^i.

The surplus process for the i-th line, say R^i, is then given as the following extension of the classical Cramer-Lundberg model (termed a compound, generalized, multivariate Hawkes process)

$$R_t^i = r^i + c^i t - \sum_{k=1}^{K^i} \sum_{n=1}^{N^i((0,t],\{e_k^i\})} Z_n^{i,k}, \tag{4.1}$$

where r^i is the initial capital for the i-th line, and c^i is the insurance premium rate for the i-th line. This model generalizes the compound Hawkes surplus model originally introduced in [ST10].

4.2 Epidemiology

It was already observed by Hawkes in [Haw71b] that Hawkes processes may find applications in epidemiology for modeling spread of epidemic diseases accounting for various types of cases, such as children or adults, that can be taken as marks. This insight has been validated over the years in numerous studies. We refer for example to [SHH19, RMK$^+$18, KPHea19] and the references therein.

It is important to account for the temporal and spatial aspects in the modeling of spread and intensity of epidemic and pandemic diseases, such as COVID-19. We believe that the variant of the generalized multivariate Hawkes process that we described at the end of Section 12.4.1 in [BJN20b] may offer a valuable tool in this regard. This will be investigated in a follow-up work.

5 Conclusion and future work

In this paper we have defined and studied generalized Hawkes processes, as well as their subclass - the multivariate generalized Hawkes processes. We also indicated some possible applications of the latter class of processes.

In particular, we provided relevant existence results, and some preliminary results regarding time-asymptotics of GHPs, that are applicable to GMHPs of course. In a future work a more in-depth analysis of asymptotics in time of GHPs and GMHPs will be carried. This, in particular, will cover the relevant central limit theorems, as well as relevant large-deviations and moderate-deviations results.

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