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# Ergodic Control of a Singularly Perturbed Markov Process in Discrete Time with General State and Compact Action Spaces

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**Abstract.** Ergodic control of singularly perturbed Markov chains with general state and compact action spaces is considered. A new method is given for characterization of the limit of invariant measures, for perturbed chains, when the perturbation parameter goes to zero. It is also demonstrated that the limit control principle is satisfied under natural ergodicity assumptions about controlled Markov chains. These assumptions allow for the presence of transient states, a situation that has not been considered in the literature before in the context of control of singularly perturbed Markov processes with long-run-average cost functionals.

**Key Words.** Markov process, Invariant measure, Singular perturbation, Ergodic control, Limit control principle.

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## 1. Introduction

Numerous dynamic optimization problems are distinguished by the presence of so-called strong and weak interactions characterizing the dynamics of the problems (see,

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e.g., [DQ], [PG], and [G]). In discrete time many such problems are nicely modeled with the help of nearly decomposable, controlled Markov chains [BF], [PG], [KT1], [KT2]. The near decomposability of a Markov process is typically expressed in terms of an appropriately constructed infinitesimal generator of the process. The construction takes the form of a perturbation of the infinitesimal generator of some other Markov process [BS1], [BS2]. In the case of Markov processes with discrete time this translates directly into perturbation of the corresponding transition kernel.

From another perspective, a large number of dynamic optimization problems are characterized by inadequate knowledge of their parameters, most commonly the parameters of the problem's dynamics. It is often convenient to capture this inadequacy by incorporating a perturbation of the parameters into the problem description. Again, in the case of controlled, discrete time Markov processes, the perturbation takes form of perturbation of the transition kernel.

In both of the above situations the perturbation would typically be "small" in applications. "Smallness" of the perturbation is usually encoded in the form of a single parameter, which is a positive real number close to zero. It is therefore a legitimate question of practical importance to ask what is the behavior of the optimization problem as the perturbation parameter goes to zero. The study of this question has been conducted by many authors in recent years ([B], [K], [BF], [ABF], [BS1], [BS2], and [PG] among others). The problem of the asymptotic behavior of a controlled, perturbed Markov process in discrete time when the perturbation goes to zero is particularly interesting and challenging when one considers the long-run-average cost criterion and the perturbation is singular [BF], [BS1], [BS2]. Singularity of perturbation in a discrete time framework basically means that the perturbation reduces the number of ergodic classes of the unperturbed problem.

In this paper we extend the results of [BF] to the case of discrete time, singularly perturbed, controlled Markov processes with general state and compact action spaces. In particular, we do not impose any restrictions on the measurable structure of the state space. Moreover we allow for the presence of transient classes for both unperturbed and perturbed chains. This is an important step forward in the asymptotic analysis of singularly perturbed Markov processes in our opinion. The proof of our main result, Theorem 2.1, is based on the approach taken in the book by Korolyuk and Turbin (see Section 6.6 of [KT1]). Our important contribution is that we extend Korolyuk and Turbin's results in two directions: we consider a Markov chain on a general, measurable state space, and we allow for the presence of transient states.

The paper is organized as follows: In Section 2 we formulate a model for a singularly perturbed ergodic Markov chain, provide two motivating examples of applications of our model, and state Theorem 2.1 providing an asymptotic expansion of the chain's invariant measure. In Section 3 we introduce a model for a controlled singularly perturbed Markov chain and motivate it, referring to controlled versions of the two examples introduced in Section 2. Then we apply Theorem 2.1 in order to verify validity of the so-called limit control principle for the singularly perturbed, controlled Markov chain that is considered in this section. We also provide here a result on approximation of the limit control problem with an appropriately discretized one. Section 4 contains some final remarks and suggestions for future research. In the Appendix to the paper we provide a proof of Theorem 2.1.

Throughout the paper we use the following notation: $(E, \mathcal{E})$ is a measurable space. For any transition function $Q(x, A)$ on $(E, \mathcal{E})$ such that

- $\forall A \in \mathcal{E}, x \to Q(x, A)$ is $\mathcal{E}$ measurable,
- $\forall x \in E$, $Q(x, \cdot)$ is countably additive,

we let $||Q(x, \cdot)||_{\text{var}}$ denote the variation norm for $Q(x, \cdot)$ and define

$$|||Q||| = \sup_{x \in E} ||Q(x, \cdot)||_{\text{var}}.$$

For any $f \in b\mathcal{E}$ (bounded measurable, real-valued functions on $(E, \mathcal{E})$) we define

$$Qf(x) = Q(x, f) := \int_E f(y) Q(x, dy), \qquad x \in E.$$

The identity kernel $I(x, A)$ is defined by

$$I(x, A) = \begin{cases} 1 & \text{if } \quad x \in A, \\ 0 & \text{if } \quad x \notin A. \end{cases}$$

## 2. Asymptotic Expansion of Invariant Measures for a Singularly Perturbed Doeblin Process

Let $x_\bullet \equiv (x_n)_{n=0}^{\infty}$ be a Markov chain on $(E, \mathcal{E})$ with probability transition kernel $P(\,\cdot, \cdot\,)$. We impose the following assumption on $x_\bullet$:

**Assumption (D).** The transition kernel $P(\,\cdot, \cdot\,)$ satisfies the Doeblin condition $(\mathcal{D})$ (see p. 192 of [D]). That is, there is a finite measure $\varphi$, $\varepsilon > 0$ and a positive integer $n$ such that for any set $A \in \mathcal{E}$ if $\varphi(A) < \varepsilon$, then $\sup_{x \in E} P^n(x, A) \le 1 - \varepsilon$, or, equivalently, the operator $P$ considered as an operator on the space of bounded Borel functions on $E$ is quasi-compact [N, Section V-3].

The following consequences of $(\mathcal{D})$ hold:

**C1.** There exists a finite number, say $r$, of disjoint invariant sets $E_i \in \mathcal{E}$, and invariant probability measures $\pi_i$, $\pi_i(E_i) = 1$, for $i = 1, 2, \ldots, r$.

**C2.** Let $D = E \setminus \bigcup_{i=1}^r E_i$; then

$$\sup_{x \in D} E_x\{T_{\cup_{i=1}^r E_i}\} < \infty,$$

where $T_A := \inf\{n \ge 0 : x_n \in A\}$ for $A \in \mathcal{E}$.

**C3.** Let, for $x \in E$ and $A \in \mathcal{E}$,

$$\Pi(x, A) := \sum_{j=1}^r \rho(x, E_j) \pi_j(A),$$

where

$$\rho(x, E_j) := P_x\{\exists n : x_n \in E_j\}.$$

Then there exists $N_0 > 0$ such that for all $n \geq N_0$ the following operator is well defined:

$$\Phi_n(x,\cdot) := \left[ I(x,\cdot) + \Pi(x,\cdot) - n^{-1}\sum_{i=0}^{n-1} P^i(x,\cdot) + n^{-1}(I-P)(x,\cdot) \right]^{-1},$$

for $x \in E$. We also have $|||\Phi_n||| < \infty$ for $n \geq N_0$. Now let

$$Q_n(x, A) := I(x, A) + n^{-1}\sum_{s=1}^{n-1}\sum_{i=1}^{s-1}(P^i - \Pi)(x, A)$$

for $n \geq 1$, $x \in E$, $A \in \mathcal{E}$. Fix $n_0 \geq N_0$, and define the quasi-potential

$$R_0 = \Phi_{n_0} Q_{n_0}.$$

Then it is easy to see that

$$R_0(I - P + \Pi) = I,$$

and therefore, since $R_0\Pi = \Pi$, we have a very important identity:

$$\Pi = R_0 P - R_0 + I.$$

Note that if $P$ is aperiodic then one can choose $N_0 = n_0 = 1$ in the above, and $R_0$ coincides with the 0-potential of $P$.

We now introduce a perturbed chain $x^\varepsilon_\bullet \equiv (x^\varepsilon_n)_{n=0}^\infty$. For that, we define the probability transition kernel $P^\varepsilon(\,\cdot,\cdot\,)$ for $x^\varepsilon_\bullet$ by

$$P^\varepsilon(\,\cdot,\cdot\,) := P(\,\cdot,\cdot\,) + \varepsilon B(\,\cdot,\cdot\,),$$

where $B(\,\cdot,\cdot\,) := \overline{P}(\,\cdot,\cdot\,) - I(\,\cdot,\cdot\,)$, $\overline{P}(\,\cdot,\cdot\,)$ is a probability transition kernel on $(E, \mathcal{E})$ and $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$.

The following two examples illustrate the applicability of the above model.

**Example 2.1.** This is a discrete time version of the continuous time situation considered in Section 5.9 of [SZ], where a flexible manufacturing system is studied with machine states admitting strong and weak interactions. Capacity dynamics of a manufacturing machine are modeled by using a Markov chain with a transition operator

$$P^\varepsilon(\,\cdot,\cdot\,) := P(\,\cdot,\cdot\,) + \varepsilon B(\,\cdot,\cdot\,).$$

In the above formula $P(\,\cdot,\cdot\,)$ corresponds to $(I + Q^{(2)})$ of [SZ] and models the strong (or fast) interactions between the states of the machine. On the other hand, $B(\,\cdot,\cdot\,)$ corresponds to $Q^{(1)}$ of [SZ] and models the weak (or slow) interactions between the states of the machine. Discrete time counterpart of the ergodic assumptions about $Q^{(2)}$ made in [SZ] implies that the Doeblin condition is satisfied for $P(\,\cdot,\cdot\,)$.

**Example 2.2.** This example is motivated by Altman and Gaitsgory [AG].

In queuing networks modeling, a relationship between the number of customers $X_{n+1}$ at time $n+1$ and the number of customers $X_n$ at time $n$ can be conveniently expressed as

$$X_{n+1} = f(X_n, Y_n^\varepsilon),$$

where $(Y_n^\varepsilon)_{n\geq 0}$ is a Markov chain of random perturbations whose average time between transitions is of order $\varepsilon$ (fast time transitions), corresponding to fast changes in routing, flow control, compared with the ordinary time scale $n = 0, 1, 2, \dots$ of the network itself. Such a model gives rise to the singular perturbation form of the transition operator for the Markov chain $Z_n^\varepsilon = (X_n, Y_n^\varepsilon)$, where appropriate ergodicity assumptions hold.

To illustrate the point we consider a very simple situation in which $Y_n^\varepsilon$ lives on a finite state space. In general, the process $(Y_n^\varepsilon)_{n\geq 0}$ should be considered on a general measurable state space. Let $a, b$ be two nonnegative integers. Also, let

$$X_n \in \{a, b\}, \qquad Y_n^\varepsilon \in \{0, 1\},$$

$$f(a, 0) = f(b, 0) = b,$$

$$f(a, 1) = f(b, 1) = a,$$

$$\text{Prob}(Y_{n+1}^\varepsilon = i | Y_n^\varepsilon = j) = \frac{1}{2}, \qquad i, j \in \{0, 1\}.$$

Then the transition matrix for $(Z_n^\varepsilon)_{n\geq 0}$ is given by (recall that $(Y_n^\varepsilon)_{n\geq 0}$ is the fast chain)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \varepsilon \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},$$

where the states of $Z_n^\varepsilon$ are ordered as $(a, 0),\ (a, 1),\ (b, 0),\ (b, 1)$.

A crucial role in the asymptotic analysis of $x_0^\varepsilon$ will be played by $\widehat{P}$, where

$$\widehat{P}(\,\cdot, \cdot\,) := \Pi \overline{P} \Pi(\,\cdot, \cdot\,).$$

Clearly, $\widehat{P}$ is a probability transition kernel on $(E, \mathcal{E})$, and we have

- for $x \in E_i$,

$$\begin{aligned} \widehat{P}(x, E_j) &= \int_E \int_E \Pi(y, E_j) \overline{P}(z, dy) \Pi(x, dz) \\ &= \int_{E_i} \int_{E_j} \Pi(y, E_j) \overline{P}(z, dy) \pi_i(dz) + \int_{E_i} \int_D \Pi(y, E_j) \overline{P}(z, dy) \pi_i(dz) \end{aligned}$$

$$\begin{aligned} &= \int_{E_i} \left( \overline{P}(z, E_j) + \int_D \rho(y, E_j) \overline{P}(z, dy) \right) \pi_i(dz) \\ &=: p(i, j), \end{aligned} \tag{2.1}$$

- for $x \in D$,

$$\begin{aligned} \widehat{P}(x, E_j) &= \sum_{i=1}^{r} \int_{E_i} \int_{E} \Pi(y, E_j) \overline{P}(z, dy) \pi_i(dz) \rho(x, E_i) \\ &= \sum_{i=1}^{r} \int_{E_i} \left[ \overline{P}(z, E_i) + \int_D \rho(y, E_i) \overline{P}(z, dy) \right] \pi_i(dz) \rho(x, E_i) \\ &=: p(x, j), \end{aligned} \tag{2.2}$$

- for $x \in E$,

$$\widehat{P}(x, D) = \int_E \int_E \Pi(y, D) \overline{P}(z, dy) \Pi(x, dz) = 0 =: p(x, D), \tag{2.3}$$

with $i, j = 1, 2, \ldots, r$.

Let $\widehat{E} := \{1, 2, \ldots, r\}$ and let $\widehat{E}_e := \widehat{E} \cup D$, and let $p(\,\cdot, \cdot\,)$ be as defined in (2.1)–(2.3).

**Definition 2.1.**

(a) A Markov chain $\widehat{x}_\bullet \equiv (\widehat{x}_n)_{n=0}^{\infty}$ on $\widehat{E}$, whose transition probability matrix is $p = [p(i, j)]_{i, j \in \hat{E}}$, is called aggregated in a strict sense.

(b) A Markov chain $\widehat{x}_\bullet^e \equiv (\widehat{x}_n^e)_{n=0}^{\infty}$ on $(\widehat{E}_e, \mathcal{E}_e)$, whose probability transition kernel is $p(\,\cdot, \cdot\,)$, is called aggregated in an extended sense.

Clearly, $D$ is a transient class for $\widehat{x}_\bullet^e$, and $\widehat{x}_\bullet^e$ restricted to $\widehat{E}$ coincides with $\widehat{x}_\bullet$. In what follows we shall need the following assumption:

**Assumption (I).** There exists a unique invariant probability measure $\mu$ for $\widehat{x}_\bullet$, that is, $\mu$ is the unique row vector so that

(i) $\mu(i) \geq 0, \; i = 1, 2, \ldots, r,$

$$\sum_{i=1}^{r} \mu(i) = 1,$$

(ii) $\sum_{i=1}^{r} \mu(i) p(i, j) = \mu(j), \; j = 1, 2, \ldots, r.$

It is not difficult to see that the Doeblin condition is satisfied for any finite state space Markov chain. We may thus conclude that the quasi-potential matrix $r_0$ is well defined for $\widehat{x}_\bullet$ by (compare with the discussion in C3)

$$r_0 := \rho_{n_0} q_{n_0},$$

where $n_0 \geq 1$ is sufficiently large, $\rho_{n_0}$ and $q_{n_0}$ are matrices given by

$$\rho_{n_0}^{(-1)}(i,j) = \delta(i,j) + \mu(j) - n_0^{-1}\sum_{n=0}^{n_0-1} p^{(n)}(i,j) + n_0^{-1}(\delta(i,j) - p^{(n)}(i,j)), \qquad i,j = 1,\dots,r,$$

$$q_{n_0}(i,j) = \delta(i,j) + n_0^{-1}\sum_{s=0}^{n_0-1}\sum_{k=0}^{s-1}(p^{(k)}(i,j) - \mu(j)),$$

where $\rho_{n_0}^{(-1)}(i,j)$ is the $(i,j)$th entry of the inverse $\rho_{n_0}^{-1}$ to $\rho_{n_0}$, and where $p^{(n)}(i,j)$ is the $(i,j)$th entry in $p^n$. We have that

$$r_0(I_r - p) = I_r - \overline{\mu}, \tag{2.4}$$

where $I_r$ is the $r \times r$ identity matrix, and

$$\overline{\mu} := \begin{pmatrix} \mu(1) & \mu(2) & \cdots & \mu(r) \\ \mu(1) & \mu(2) & \cdots & \mu(r) \\ \vdots & \vdots & & \vdots \\ \mu(1) & \mu(2) & \cdots & \mu(r) \end{pmatrix}_{r\times r} = \begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix}.$$

Before we state Theorem 2.1, below, we need to introduce some more notation. For $A \in \mathcal{E}$, $i = 1,2,\dots,r$, and $m = 0,1,2,\dots,$ we let

$$\beta_m^i(A) := \int_{E_i} B(R_0 B)^m(x, A)\pi_i(dx)$$

and, additionally, for $j = 1,2,\dots,r$, we let

$$\overline{\beta}_m(i,j) := \beta_m^i(E_j) + \int_D \rho(y, E_j)\beta_m^i(dy).$$

Now we inductively define matrices

$$\beta^{(1)} := \overline{\beta}_1,$$

$$\beta^{(2)} := \overline{\beta}_2 + \beta^{(1)} r_0 \overline{\beta}_1,$$

$$\dots$$

$$\beta^{(m)} := \overline{\beta}_m + \beta^{(1)} r_0 \overline{\beta}_{m-1} + \cdots + \beta^{(m-2)} r_0 \overline{\beta}_2 + \beta^{(m-1)} r_0 \overline{\beta}_1,$$

and row vectors

$$\mu^{(0)} := \mu,$$
$$\mu^{(m)} := \mu^{(0)}\beta^{(m)} r_0, \qquad m \geq 1.$$

Finally, we let, for $A \in \mathcal{E}$ and $m \geq 1$,

$$\gamma_0(A) := \sum_{i=1}^{r} \mu^{(0)}(i)\pi_i(A),$$

$$\gamma_m(A) := \sum_{i=1}^{r} [\mu^{(0)}(i)\pi_i(BR_0)^m + \mu^{(1)}(i)\pi_i(BR_0)^{m-1} + \cdots + \mu^{(m)}(i)\pi_i](A).$$

The following theorem is inspired by Theorem 6.8 in [KT1].

**Theorem 2.1.** *Let Assumptions D and I be satisfied. Assume also that there exists a unique invariant measure* $\pi^\varepsilon$ *for* $P^\varepsilon$. *Then for* $\varepsilon < \min\{\varepsilon_0, (3a^2bc)^{-1}\}$, *where* $a = \max\{|||B|||, 1\}$, $b = \max\{|||R_0|||, 1\}$, $c = \max\{|||r_0|||, 1\}$, *we have*

$$||\gamma_m||_{\text{var}} \leq (3a^2bc)^m, \qquad m \geq 1,$$

*and*

$$\pi^\varepsilon(\cdot) = \sum_{m=0}^{\infty} \varepsilon^m \gamma_m(\cdot). \tag{2.5}$$

*Proof.* See the Appendix.

## 3. Ergodic Control of Singularly Perturbed Markov Chains

In this section we assume that the transition kernels $P$ and $\overline{P}$ depend on a control parameter $a \in U$, where $U$ is a compact, metric space. We denote this dependence by $P^a$ and $\overline{P}^a$, respectively.

Let $\mathcal{A} := B(E, U)$, the space of measurable functions from $E$ to $U$. For $u \in \mathcal{A}$, with a slight abuse of notation, we denote by $P^u$ and $\overline{P}^u$ controlled kernels such that, for each $x \in E$,

$$P^u(x, \cdot) = P^{u(x)}(x, \cdot)$$

and

$$\overline{P}^u(x, \cdot) = \overline{P}^{u(x)}(x, \cdot).$$

We suppose for the time being that the dependence of $P^a$ and $\overline{P}^a$ on $a \in U$ is such that for every $u \in \mathcal{A}$ the controlled kernels $P^u$ and $\overline{P}^u$ are in fact probability transition kernels. In particular this is true under assumption (A6) that we introduce later in the text.

The functions $u \in \mathcal{A}$ are called stationary Markovian controls. We consider such controls only. This will not reduce generality, since it is well known that, in view of the ergodic assumptions made below, optimal nonanticipating controls can be found among stationary Markovian controls (see [DY] for example).

We impose the following assumptions on $\{P^u, u \in \mathcal{A}\}$:

(A1) There exists a finite sequence $E_1, \dots, E_r$ of disjoint subsets of $E$, that do not depend on $u$ and are invariant for each $P^u$. That is,

$$\forall x \in E_i, \quad \forall a \in U, \qquad P^a(x, E_i) = 1$$

for $i = 1, 2, \dots, r$.

(A2) Let $D := E \setminus \bigcup_{i=1}^r E_i$. Then

$$\sup_{x \in E} \sup_{a \in U} P^a(x, D) < 1.$$

(A3) $\exists_{N<\infty}, \exists_{0<\rho<1}, \forall u \in \mathcal{A}, \forall i = 1, 2, \dots, r, \exists_{n \le N}, \exists_{\text{disjoint sets}} C_1^u(i), \dots, C_n^u(i)$ such that

(i) $\bigcup_{j=1}^n C_j^u(i) = E_i$,

(ii) $P^u(x, C_j^u(i)) = 1 \quad$ for $\quad x \in C_{j-1}^u(i) \quad$ and $\quad j = 2, 3, \dots, n$,

$P^u(x, C_1^u(i)) = 1 \quad$ for $\quad x \in C_n^u(i)$,

(iii) $$\sup_{j=1,2,\dots,n} \sup_{x,y \in C_j^u(i)} \sup_{\Gamma \in \mathcal{E}} |(P^u)^n(x, \Gamma) - (P^u)^n(y, \Gamma)| < \rho.$$

**Remark 3.1.** (a) Assumption (A2) implies that

$$\sup_{u \in \mathcal{A}} \sup_{x \in E} E_x\{T^u\} < \infty,$$

where

$$T^u = \min\left\{ j \ge 0 : x_j^u \in \bigcup_{i=1}^r E_i \right\}$$

and $(x_j^u)_{j=0}^\infty$ is a controlled Markov chain corresponding to $P^u$.

(b) Assumption (A3) implies that the process $(x_{nk}^u)_{k=0}^\infty$, with $x_0^u \in C_j^u(i)$, is uniformly ergodic on $C_j^u(i)$. That is, there exists a probability measure $\pi_j^u(i)$ such that

$$\sup_{x \in C_j^u(i)} \|(P^u)^{nk}(x, \cdot) - \pi_j^u(i)(\cdot)\|_{\text{var}} \le 2(1-\rho)^{k-1}.$$

Consequently, for the quasi-potential $R_0^u$ we have

$$|||R_0^u||| \le M(\rho) < \infty.$$

In addition, we assume what follows about $\overline{P}^u$:

(A4) $\inf_{x \in \cup_{i=1}^r E_i} \inf_{a \in U} \overline{P}^a(x, E_j) > 0, \; j = 1, 2, \dots, r.$

(A5) $\exists_{\varepsilon_0}, \forall_{\varepsilon<\varepsilon_0}, \forall_{u\in\mathcal{A}}$ there is a unique invariant measure $\pi_u^\varepsilon$ for $P^{u,\varepsilon} := P^u + \varepsilon(\overline{P}^u - I)$, and

$$\forall_{x\in E}, \qquad \lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1}(P^{u,\varepsilon})^i(x, f) = \int_E f(y)\pi_u^\varepsilon(dy)$$

for every $f \in b\mathcal{E}$.

**Remark 3.2.** (a) Assumption (A1) indicates that the ergodic structure of the underlying controlled Markov chain is invariant with respect to the control parameter. There are examples of controlled systems in which this assumption is not satisfied. Nevertheless, there are also important examples of controlled systems for which this assumption is valid (see points (b) and (c) below). The other assumptions, (A2)–(A5), are consistent with (A1).

(b) In Section 5.8 of their book Sethi and Zhang [SZ] consider machine capacities dependent on production rate. In the context of our Example 2.1 this would correspond to considering operators $P$ and $\overline{P}$ depending on control parameter $u$, as is done in this section. Note that in view of the interpretation of the perturbed chain in terms of the presence of strong and weak interactions among the capacity states, assumption (A1) about uniform (with respect to $u$) ergodic decomposition of the state space of the underlying Markov chain is quite natural.

(c) In their paper Altman and Gaitsgory [AG] consider a situation where the perturbing random process is in fact a controlled Markov chain. In the context of our Example 2.2 this would mean that the chain $Y_n^\varepsilon$ is a controlled Markov chain. Note that in this case assumption (A1) would automatically be satisfied.

(d) Time discretized versions of control problems considered in Bensoussan and Blankenship [BB] satisfy assumption (A1).

In the rest of the paper we use $(x_n^{u,\varepsilon})$ in order to denote a Markov chain corresponding to $P^{u,\varepsilon}$, for $u \in \mathcal{A}$. Moreover we let, for $x \in E$ and $u \in \mathcal{A}$,

$$J_\varepsilon(u, x) := \limsup_{n\to\infty} \frac{1}{n} E_x \left\{ \sum_{i=0}^{n-1} c(x_i^{u,\varepsilon}, u(x_i^{u,\varepsilon})) \right\},$$

where $c$: $E \times U \to R$ is a bounded, measurable function which is continuous in $a \in U$, uniformly in $x \in E$. By (A5) we have that

$$J_\varepsilon(u, x) = J_\varepsilon(u) := \int_E c(z, u(z))\pi_u^\varepsilon(dz)$$

for all $u \in \mathcal{A}$ and $x \in E$. We are interested in the following optimization problem:

$$\inf_{u\in\mathcal{A}} J_\varepsilon(u).$$

Let, for $u \in \mathcal{A}$,

$$J(u) := \sum_{i=1}^{r} \int_E c(x, u(x))\pi_{u,i}(dx)\mu_u(I),$$

where $\pi_{u,i}$, $i = 1, \ldots, r$, are the invariant measures corresponding to $P^u$, and $\mu_u$ is the unique invariant measure for the controlled, aggregated Markov chain $(\widehat{x}_n^u)$, which exists due to (A4).

The following corollary to Theorem 2.1 and Remark 3.1 is a version of what is known in the literature as the "limit control principle" (see [BF] and [BS1]) corresponding to the case considered in this paper.

**Corollary 3.1.** *Assume* (A1)–(A5). *Then for every* $\delta > 0$ *there exists* $\varepsilon_\delta > 0$ *such that, for all* $\varepsilon < \varepsilon_\delta$,

$$\sup_{u \in \mathcal{A}} |J_\varepsilon(u) - J(u)| < \delta.$$

*Proof.* By Theorem 2.1 we have, for all $u \in \mathcal{A}$,

$$J_\varepsilon(u) - J(u) = \sum_{i=1}^{\infty} \int_E c(x, u(x))\gamma_{u,i}(dx)\varepsilon^i.$$

From Remark 3.1 it follows that

$$\sup_{u \in \mathcal{A}} ||\gamma_{u,i}||_{\text{var}} \le K^i, \qquad i = 1, 2, \ldots,$$

for some constant $K > 0$. □

By Corollary 3.1 the minimization problem for $J_\varepsilon$ can be approximated by the minimization problem for $J$. In the remaining part of the paper we simplify the latter one by approximating it using appropriate discretization of the action space $\mathcal{A}$.

Let $\mathcal{A}_1 = \mathcal{B}(D, U)$ and $\mathcal{A}_2 = \mathcal{B}(\bigcup_{i=1}^r E_i, U)$ be the spaces of measurable functions from $D$ to $U$ and $\bigcup_{i=1}^r E_i$ to $U$, respectively. If $u \in \mathcal{A}$ is such that

$$u(x) = \begin{cases} u_1(x), & x \in D, \\ u_2(x), & x \in \bigcup_{i=1}^{r} E_i, \end{cases}$$

for some $u_i \in \mathcal{A}_i$, $i = 1, 2$, then we write

$$u = u_1 \oplus u_2.$$

Note that

$$\inf_{u\in\mathcal{A}} J(u) = \inf_{u_1\in\mathcal{A}_1}\ \inf_{u_2\in\mathcal{A}_2} J(u_1 \oplus u_2),$$

and that ergodic assumptions about the aggregated process $(\widehat{x}_n^u)$ imply that $\int_E c(x, u(x))\pi_{u,i}(dx)$ depends on $u_2$ alone. Also, if $\overline{P}^u$ does not depend on $u$, then $\mu_u(i)$ depends on $u_2$ only through the invariant measures $\pi_{u,i}$. The above observations can be effectively used in approximations of the minimization problem for $J(u)$.

Let $\Delta$ be the metric on $U$, and let $(a_n)$ be a dense sequence in $U$. Since $U$ is compact we have that

$$\forall_{m\geq 1}, \exists_{n(m)}, \qquad U_m := \{a_1, \ldots, a_{n(m)}\} \quad \text{is } \frac{1}{m} \text{ net in } U.$$

For any $u \in \mathcal{A}$ and $x \in E$ let

$$\Delta_m^u(x) := \min\{\Delta(u(x), a_k), a_k \in U_m\},$$

$$a_m^u(x) := \min\{a_k \in U_m : \Delta_m^u(x) = \Delta(u(x), a_k)\}.$$

Define a discretization operator $S_m\colon \mathcal{A} \to \mathcal{A}$ by

$$S_m u(\cdot) = a_m^u(\cdot).$$

Clearly, we have

$$\sup_{x\in E} \Delta(u(x), S_m u(x)) \leq \frac{1}{m}$$

for all $u \in \mathcal{A}$.

Before we state our final result, Proposition 3.1, we formulate the following technical assumption:

(A6) Both $P^a(x, \cdot)$ and $\overline{P}^a(x, \cdot)$ are continuous in $a$, in variation norm, uniformly in $x$. That is to say, we have that, $\forall_{\eta>0}, \exists_{\delta>0}, \forall_{a,a'\varepsilon U}$,

$$\Delta(a, a') < \delta \quad \Rightarrow \quad \sup_{x\in E} \| Q^a(x, \cdot) - Q^{a'}(x, \cdot)\|_{\text{var}} < \eta,$$

where $Q$ stands for $P$ or $\overline{P}$.

**Proposition 3.1.** *Assume* (A1)–(A6). *Then, for each* $\delta > 0$, *there exists* $m_\delta$ *such that, for all* $m \geq m_\delta$,

$$\sup_{u\in\mathcal{A}} |J(u) - J(S_m u)| < \delta.$$

*Proof.* In the proof of Proposition 1 in [S] it has been demonstrated that, for all $u \in \mathcal{A}$,

$i = 1, \dots, r$, and $n \geq 1$,

$$||\pi_{u,i} - \pi_{S_m u,i}||_{\text{var}\, E_i} \leq \sup_{x \in E_i} \|P^u(x,\cdot) - P^{S_m u}(x,\cdot)\|_{\text{var}} K_{1,n}(i) K_{2,n}(i),$$

where

$$||\ \cdot\ ||_{\text{var}\, E_i}$$

denotes the variation norm on $E_i$, and

$$K_{1,n}(i) := \sup_{u \in \mathcal{A}} \sup_{x \in E_i} \left\| \left( I + \pi_{u,i}(\,\cdot\,) - n^{-1} \sum_{k=0}^{n-1} (P^u)^k(x,\cdot) + n^{-1}(I - P^u)(x,\cdot) \right)^{-1} \right\|_{\text{var}\, E_i},$$

$$K_{2,n}(i) := \sup_{n \in \mathcal{A}} \sup_{x \in E_i} \left\| I + n^{-1} \sum_{\ell=1}^{n-1} \sum_{k=1}^{\ell-1} (P^u)^k(x,\cdot) - \pi_{u,i}(\cdot) \right\|_{\text{var}\, E_i}.$$

By (A3), for a sufficiently large $n$ we have that

$$K_{1,n}(i) \leq K_1 < \infty,$$

$$K_{2,n}(i) \leq K_2 < \infty$$

for some $K_1$ and $K_2$. Therefore we may conclude that

$$\sup_{u \in \mathcal{A}} \sup_{i=1,2,\dots,r} \|\pi_{u,i} - \pi_{S_m u,i}\|_{\text{var}\, E_i} \leq K_1 \cdot K_2 \cdot \sup_{u \in \mathcal{A}} |||P^u - P^{S_m u}|||. \tag{3.1}$$

Similarly, for the aggregated chain $(\widehat{x}_n^u)$ we have

$$\sup_{u \in \mathcal{A}} \sup_{i=1,2,\dots,r} ||\mu_u(i) - \mu_{S_m u(i)}|| \leq \overline{K}_1 \overline{K}_2 \sup_{u \in \mathcal{A}} \sup_{i,j=1,\dots,r} |p^u(i,j) - p^{S_m u}(i,j)|, \tag{3.2}$$

for some positive and finite constants $\overline{K}_1$, $\overline{K}_2$. Next, we observe that

$$\begin{aligned} |J(u) - J(S_m u)| &= \left| \sum_{i=1}^{r} \int_E c(x, u(x)) \pi_{u,i}(dx) \mu_u(i) - \sum_{i=1}^{r} \int_E c(x, S_m u(x)) \pi_{S_m u,i}(dx) \mu_{S_m u}(i) \right| \\ &\leq \left| \sum_{i=1}^{r} \int_E c(x, u(x)) (\pi_{u,i} - \pi_{S_m u,i})(dx) \mu_u(i) \right| \end{aligned}$$

$$+\left|\sum_{i=1}^{r}\int_E c(x,u(x))\pi_{S_m u,i}(dx)(\mu_u(i)-\mu_{S_m u}(i))\right|$$
$$+\sup_{x\in E}|c(x,S_m u(x))-c(x,u(x))|. \tag{3.3}$$

Therefore, in view of (3.1)–(3.3) and assumption (A6), in order to complete the proof of Proposition 3.1 it remains to show that

$$\sup_{u\in\mathcal{A}}\sup_{i,j=1,2,\dots,r}|p^u(i,j)-p^{S_m u}(i,j)|\to 0 \tag{3.4}$$

as $m\to\infty$.

By the definition of $p^u(i,j)$, taking into account (3.1) and

$$\sup_{x\in E}\sup_{u\in\mathcal{A}}\|\overline{P}^{u(x)}(x,\cdot)-\overline{P}^{S_m u(x)}(x,\cdot)\|_{\text{var}}\to 0$$

as $m\to\infty$, we only need to show that, for $i=1,2,\dots,r$,

$$\sup_{x\in E}\sup_{u\in\mathcal{A}}|\rho^u(x,E_i)-\rho^{S_m u}(x,E_i)|\to 0 \tag{3.5}$$

as $m\to\infty$. Since by Remark 3.1

$$\sup_{x\in E}\sup_{u\in\mathcal{A}}P_x^u\{T^u>N\}\le\frac{2\sup_{x\in E}\ \sup_{u\in\mathcal{A}}E_x\{T^u\}}{N}\to 0$$

as $N\to\infty$, in order to demonstrate (3.5) we need to show that, for $k=1,2,\dots$,

$$\sup_{x\in E}\sup_{i=1,2,\dots,r}\sup_{u\in\mathcal{A}}|P_x^u\{x_0^u\in D,\dots,x_{k-1}^u\in D,x_k^u\in E_i\}$$
$$-P_x^{S_m u}\{x_0^{S_m u}\in D,\dots,x_{k-1}^{S_m u}\in D,x_k^{S_m u}\in E_i\}|\to 0 \tag{3.6}$$

as $m\to\infty$.

We prove (3.6) by induction:

For $k=1$, (3.6) holds by (A5).

Assume (3.6) is true for $k\ge 1$. Then, for $k+1$ we have, for $x\in E$, $u\in\mathcal{A}$, and $j=1,\dots,r$,

$$|P_x^u\{x_0^u\in D,\dots,x_k^u\in D,x_{i+1}^u\in E_j\}-P_x^{S_m u}\{x_0^{S_m u}\in D,\dots,x_k^{S_m u},x_{k+1}^{S_m u}\in E_j\}|$$
$$=\left|\int_{E_j}\int_D\cdots\int_D\chi_D(x)P^u(x,dx_1)\cdots P^u(x_k,dx_{k+1})\right.$$
$$\left.-\int_{E_j}\int_D\cdots\int_D\chi_D(x)P^{S_m u}(x,dx_1)\cdots P^{S_m u}(x_k,dx_{k+1})\right|$$
$$\le\left|\chi_D(x)\left(\int_{E_j}\int_D\cdots\int_D P^u(x,dx_1)P^u(x_1,dx_2)\cdots P^u(x_k,dx_{k+1})\right.\right.$$
$$\left.\left.-\int_{E_j}\int_D\cdots\int_D P^u(x,dx_1)P^{S_m u}(x_1,dx_2)\cdots P^{S_m u}(x_k,dx_{k+1})\right)\right|$$

$$+\left|\chi_D(x)\int_{E_j}\int_D\cdots\int_D(P^u(x,dx_1)-P^{S_m u}(x,dx_1))\right.$$
$$\left.\cdot\, P^{S_m u}(x_1,dx_2)\cdots P^{S_m u}(x_k,dx_{k+1})\right|$$
$$\leq \chi_D(x)\{\sup_{x_1\in E}|P^u_{x_1}\{x^u_0\in D,\ldots x^u_{k-1}\in D, x^u_k\in E_j\}$$
$$-P^{S_m u}_{x_1}\{x^{S_m u}_0\in D,\ldots,x^{S_m u}_{k-1}\in D, x^{S_m u}_k\in E_j\}|+|||P^u-P^{S_m u}|||\}$$

which goes to 0 as $m\to\infty$ in view of the induction hypothesis and (A6). The proof of Proposition 3.1 is complete. □

**Remark 3.3.** In this paper we have not analyzed the structure of the optimal/suboptimal controls. That is, we have not studied the question of asymptotic expansion for these controls. It is conceivable that optimal/suboptimal controls can be split into "fast" and "slow" components, and that approximations to these components can be obtained combining Proposition 3.1 with the results of [ABF], among others. These issues are very important for applications and are under investigation.

## 4. Conclusion

In this paper we have considered a situation when the perturbed Markov chain has only one invariant measure. One step aggregation was enough to deal with asymptotic expansion of this invariant measure. Construction of an asymptotic expansion becomes much more difficult if we allow for a singular perturbation that does not reduce the number of ergodic classes of the original chain to just one. We will be considering situations of this kind in a forthcoming work.

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## Appendix. Proof of Theorem 2.1

*Step* 1. We have

$$||\beta^i_m||_{\mathrm{var}}\leq |||R_0|||^m|||B|||^{m+1},$$

$$|||\overline{\beta}_m|||\leq 2|||R_0|||^m|||B|||^{m+1}$$

for $i=1,2,\ldots,r$ and $m\geq 0$. Therefore

$$|||\beta^{(1)}|||\leq 2|||R_0|||\,|||B|||^2\leq 2ba^2$$

and

$$|||\beta^{(2)}||| \le 2|||R_0||| \cdot |||B|||^2 + |||\beta^{(1)}|||\ |||r_0|||\ |||\overline{\beta}_1|||$$
$$\le 2ba^2 + |||\beta^{(1)}|||c \cdot 2ba^2 \le 6b^2a^4c.$$

We shall show by induction that

$$||\beta^{(m)}||| \le 2a^2b(3a^2bc)^{m-1} \tag{A.1}$$

for $m \ge 1$. For $m = 1, 2$, (A.1) has already been verified. Suppose (A.1) holds for $k \le m$. Then, for $m + 1$, we have

$$\begin{aligned}|||\beta^{(m+1)}||| &\le 2b^{m+1}a^{m+2} + 2ba^2 \cdot c \cdot 2b^m a^{m+1} \\ &\quad + \cdots + 2a^2b(3a^2b \cdot c)^{m-1} \cdot c \cdot 2b \cdot a^2 \\ &= a^{2(m+1)}b^{m+1}c^m \left[2 + 4\sum_{i=1}^{m} 3^{i-1}\right] \\ &= 2a^2b(a^2 \cdot b \cdot c)^m \left(1 + 2\frac{1-3^m}{1-3}\right) = 2a^2b(3a^2bc)^m.\end{aligned}$$

*Step* 2. From Step 1 it follows that

$$||\mu^{(m)}||_{\text{var}} \le 2 \cdot 3^{m-1}(a^2bc)^m$$

for $m = 1, 2, \ldots$. Therefore, since $||\gamma_0||_{\text{var}} = 1$, we have, for $m = 1, 2, \ldots$,

$$\begin{aligned}||\gamma_m||_{\text{var}} &\le ||\mu^{(0)}||_{\text{var}}|||B|||^m|||R_0|||^m \\ &\quad + ||\mu^{(1)}||_{\text{var}}|||B|||^{m-1}|||R_0|||^{m-1} + \cdots + ||\mu^{(m)}||_{\text{var}} \\ &\le a^{2m}b^m c^m \left[1 + 2\sum_{i=1}^{m} 3^{i-1}\right] = (3a^2bc)^m.\end{aligned}$$

*Step* 3. By Step 2 we have that for all $\varepsilon < \min(\varepsilon_0, (3a^2bc)^{-1})$ the series

$$\sum_{i=0}^{m} \varepsilon^i \gamma_i$$

is convergent in variation norm as $n \to \infty$. Therefore we can define a countably additive function

$$\eta_\varepsilon(\,\cdot\,) := \sum_{i=0}^{\infty} \gamma_i(\,\cdot\,)\varepsilon^i.$$

Note that $B(E) = 0$ implies $\gamma_m(E) = 0$ for $m \ge 1$. Therefore, since $\gamma_0(E) = 1$, we have that $\eta_\varepsilon(E) = 1$. We demonstrate in Step 4 below that if we suppose that

$$\eta_\varepsilon P^\varepsilon = \eta_\varepsilon, \tag{A.2}$$

then, since $\eta_\varepsilon(E) = 1$, we have

$$\eta_\varepsilon = \pi^\varepsilon. \tag{A.3}$$

*Step* 4. Assume (A.2) is satisfied. By Jordan decomposition we have

$$\eta_\varepsilon = \eta_\varepsilon^+ - \eta_\varepsilon^-$$

and, for $A \in \mathcal{E}$, $* = +, -$,

$$\eta_\varepsilon^* P^\varepsilon(A) - \eta_\varepsilon^*(A) = \eta_\varepsilon^*(E \backslash A) - \eta_\varepsilon^* P^\varepsilon(E \backslash A). \tag{A.4}$$

Since $\eta_\varepsilon^+ P^\varepsilon(E) = \eta_\varepsilon^+(E)$ and $\eta_\varepsilon^- P^\varepsilon(E) = \eta_\varepsilon^-(E)$, then (A.4) implies

$$\eta_\varepsilon^+ P^\varepsilon = \eta_\varepsilon^+, \qquad \eta_\varepsilon^- P^\varepsilon = \eta_\varepsilon^-. \tag{A.5}$$

Therefore $\eta_\varepsilon^+$, $\eta_\varepsilon^-$ are invariant measures for $P^\varepsilon$. By the uniqueness of invariant probability measure for $P^\varepsilon$ we conclude that

$$\frac{\eta_\varepsilon^+}{\eta_\varepsilon^+(E)} = \pi^\varepsilon$$

and, if $\eta_\varepsilon^-(E) > 0$,

$$\frac{\eta_\varepsilon^-}{\eta_\varepsilon^-(E)} = \pi^\varepsilon.$$

Thus

$$\begin{aligned}\eta_\varepsilon &= (\eta_\varepsilon^+(E) - \eta_\varepsilon^-(E))\pi^\varepsilon \\ &= \eta_\varepsilon(E)\pi^\varepsilon = \pi^\varepsilon.\end{aligned}$$

*Step* 5. It remains to demonstrate that (A.2) is satisfied. This will take some time. We start by observing that (A.2) is equivalent to

$$\sum_{i=0}^{\infty} \varepsilon^i \gamma_i (P + \varepsilon B) = \sum_{i=0}^{\infty} \varepsilon^i \gamma_i, \tag{A.6}$$

and thus to

$$\sum_{i=1}^{\infty} (\gamma_i P + \gamma_{i-1} B - \gamma_i)\varepsilon^i + \gamma_0 P - \gamma_0 = 0. \tag{A.7}$$

Now, since

$$\gamma_0 P = \gamma_0$$

then (A.7) will be verified once we show that

$$\gamma_i P + \gamma_{i-1} B - \gamma_i = 0, \qquad i = 1, 2, \ldots . \tag{A.8}$$

Applying formulas for $\gamma_{i-1}, \gamma_i$ we get

$$\begin{aligned}\gamma_i P + \gamma_{i-1} B - \gamma_i &= \sum_{j=1}^{r} [\mu^{(0)}(j)\pi_j (BR_0)^i P + \mu^{(1)}(j)\pi_j (BR_0)^{i-1} P \\ &\qquad + \cdots + \mu^{(i)}(j)\pi_j P + \mu^{(0)}(j)\pi_j (BR_0)^{i-1} B \\ &\qquad + \mu^{(1)}(j)\pi_j (BR_0)^{i-2} B \\ &\qquad + \cdots + \mu^{(i-1)}(j)\pi_j B - \mu^{(0)}(j)\pi_j (BR_0)^i \\ &\qquad - \mu^{(1)}(j)\pi_j (BR_0)^{i-1} - \cdots - \mu^{(i)}(j)\pi_j] \\ &= \sum_{j=1}^{r} \{[\mu^{(0)}(j)\pi_j (BR_0)^{i-1} + \mu^{(1)}(j)\pi_j (BR_0)^{i-2} \\ &\qquad + \cdots + \mu^{(i-1)}(j)\pi_j] \\ &\qquad \times B(R_0 P + I - R_0) + \mu^{(i)}(j)\pi_j (P - I)\}. \qquad \text{(A.9)}\end{aligned}$$

Note that

$$\pi_j (P - I) = 0, \qquad j = 1, \ldots, r, \qquad \text{(A.10)}$$

and, for $f \in b\mathcal{E}$, by C3 of Section 2 we get

$$R_0 P + I - R_0 = \Pi. \qquad \text{(A.11)}$$

From (A.9)–(A.11) it follows that we will have demonstrated (A.8) once we have verified that

$$\sum_{j=1}^{r} [\mu^{(0)}(j)\pi_j (BR_0)^{i-1} + \mu^{(1)}(j)\pi_j (BR_0)^{i-2} + \cdots + \mu^{(i-1)}(j)\pi_j] B\Pi = 0 \qquad \text{(A.12)}$$

for $i = 1, 2, \ldots$.

However, (A.12) means that, for all $A \in \mathcal{E}$ and for $i = 1, 2, \ldots$, we need to have

$$\begin{aligned}0 &= \sum_{j=1}^{r} \int_E \Pi(y, A)[\mu^{(0)}(j)\pi_j B(R_0 B)^{i-1}(dy) + \mu^{(1)}(j)\pi_j B(R_0 B)^{i-2}(dy) \\ &\qquad + \cdots + \mu^{(i-1)}(j)\pi_j B(dy)] \\ &= \sum_{j=1}^{r} \sum_{k=1}^{r} \int_E \rho(y, E_k)\pi_k(A)[\mu^{(0)}(j)\pi_j B(R_0 B)^{i-1}(dy) \\ &\qquad + \mu^{(1)}(j)\pi_j B(R_0 B)^{i-2}(dy) \\ &\qquad + \cdots + \mu^{(i-1)}(j)\pi_j B(dy)] \\ &= \sum_{j=1}^{r} \sum_{k=1}^{r} \Big\{ \pi_k(A)[\mu^{(0)}(j)\pi_j B(R_0 B)^{i-1}(E_k) + \mu^{(1)}(j)\pi_j B(R_0 B)^{i-2}(E_k) \\ &\qquad + \cdots + \mu^{(i-1)}(j)\pi_j B(E_k)]\end{aligned}$$

$$+ \int_D \rho(y, E_k)\pi_k(A)[\mu^{(0)}(j)\pi_j B(R_0 B)^{i-1}(dy) + \cdots + \mu^{(i-1)}(j)\pi_j B(dy)]\Big\}$$

$$= \sum_{k=1}^{r} \pi_k(A) \sum_{j=1}^{r} \Big[\mu^{(0)}(j)\Big(\pi_j B(R_0 B)^{i-1}(E_k) + \int_{\mathcal{D}} \rho(y, E_k)\pi_j B(R_0 B)^{i-1}(dy)\Big)$$

$$+ \mu^{(1)}(j)\Big(\pi_j B(R_0 B)^{i-2}(E_k) + \int_D \rho(y, E_k)\pi_j B(R_0 B)^{i-2}(dy)\Big)$$

$$+ \cdots + \mu^{(i-1)}(j)\Big(\pi_j B(E_k) + \int_D \rho(y, E_k)\pi_j B(dy)\Big)\Big]$$

$$= \sum_{k=1}^{\infty} \pi_k(A) \sum_{j=1}^{r} \Big[\mu^{(0)}(j)\Big(\beta_{i-1}^{j}(E_k) + \int_D \rho(y, E_k)\beta_{i-1}^{j}(dy)\Big)$$

$$+ \mu^{(1)}(j)\Big(\beta_{i-2}^{j}(E_k) + \int_D \rho(y, E_k)\beta_{i-2}^{j}(dy)\Big)$$

$$+ \cdots + \mu^{(i-1)}(j)\Big(\beta_0^{j}(E_k) + \int_D \rho(y, E_k)\beta_0^{j}(dy)\Big)\Big]$$

$$= \sum_{k=1}^{r} \pi_k(A) \sum_{j=1}^{r} [\mu^{(0)}(j)\overline{\beta}_{i-1}(j,k) + \mu^{(1)}(j)\overline{\beta}_{i-2}(j,k) + \cdots + \mu^{(i-1)}(j)\overline{\beta}_0(j,k)]. \tag{A.13}$$

Now, using the definitions of $\mu^{(m)}$, we have that, for $i = 1, 2, \ldots$, and $k = 1, \ldots, r$,

$$\sum_{j=1}^{r} [\mu^{(0)}(j)\overline{\beta}_{i-1}(j,k) + \mu^{(1)}(j)\overline{\beta}_{i-2}(j,k) + \cdots + \mu^{(i-1)}(j)\overline{\beta}_0(j,k)]$$

$$= \sum_{j=1}^{r} [\mu^{(0)}(j)\overline{\beta}_{i-1}(j,k) + \mu^{(0)}\beta^{(1)}r_0(j)\overline{\beta}_{i-2}(j,k) + \cdots + \mu^{(0)}\beta^{(i-1)}r_0(j)\overline{\beta}_0(j,k)]. \tag{A.14}$$

Observe that since

$$r_0(I - p) = I - \overline{\mu}$$

and

$$\overline{\beta}_0 = p - I$$

we have

$$I = -r_0\overline{\beta}_0 + \overline{\mu}.$$

Therefore we may continue (A.14) as

$$\sum_{j=1}^{r}[\mu^{(0)}(j)\overline{\beta}_{i-1}(j,k)+\mu^{(0)}\beta^{(1)}r_0(j)\overline{\beta}_{i-2}(j,k)+\cdots+\mu^{(0)}\beta^{(i-1)}r_0(j)\overline{\beta}_0(j,k)]$$

$$\begin{aligned}&=\sum_{j=1}^{r}\sum_{\ell=1}^{r}[\mu^{(0)}(j)\overline{\beta}_{i-1}(j,\ell)(-r_0\overline{\beta}_0(\ell,k)+\mu(k))\\ &\qquad+\mu^{(0)}\beta^{(1)}r_0(j)\overline{\beta}_{i-2}(j,\ell)(-r_0\overline{\beta}_0(\ell,k)+\mu(k))\\ &\qquad+\cdots+\mu^{(0)}\beta^{(i-2)}r_0(j)\overline{\beta}_1(j,\ell)(-r_0\overline{\beta}_0(\ell,k)+\mu(k))]\\ &\quad+\sum_{j=1}^{r}\mu^{(0)}\beta^{(i-1)}r_0(j)\overline{\beta}_0(j,k).\end{aligned}\tag{A.15}$$

Now notice that, for $m \geq 0$ and $j = 1, \ldots, r$,

$$\begin{aligned}\sum_{\ell=1}^{r}\overline{\beta}_m(j,\ell)&=\int_{E_j}B(R_0B)^m\left(x,\bigcup_{s=1}^{r}E_s\right)\pi_j(dx)+\int_D\rho\left(y,\bigcup_{s=1}^{r}E_s\right)\beta_m^i(dy)\\ &=\int_{E_j}B(R_0B)^m\left(x,\bigcup_{s=1}^{r}E_s\right)\pi_j(dx)+\beta_m^i(D)\\ &=\int_{E_j}B(R_0B)^m(x,E)\pi_j(dx)=0.\end{aligned}\tag{A.16}$$

Finally, (A.15) and (A.16) imply that, for $k = 1, \ldots, r$, we have

$$\begin{aligned}&\sum_{j=1}^{r}[\mu^{(0)}(j)\overline{\beta}_{i-1}(j,k)+\mu^{(0)}\beta^{(1)}r_0(j)\overline{\beta}_{i-2}(j,k)+\cdots+\mu^{(0)}\beta^{(i-1)}r_0(j)\overline{\beta}_0(j,k)]\\ &\quad=\mu^{(0)}[-\overline{\beta}_{i-1}-\beta^{(1)}r_0\overline{\beta}_{i-2}-\cdots-\beta^{(i-2)}r_0\overline{\beta}_1+\beta^{(i-1)}]r_0\overline{\beta}_0(k)=0,\end{aligned}\tag{A.17}$$

where the least equality holds by the definition of $\beta^{(I-1)}$. We see now that (A.17) implies (A.13).

The proof of the theorem is complete. □

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