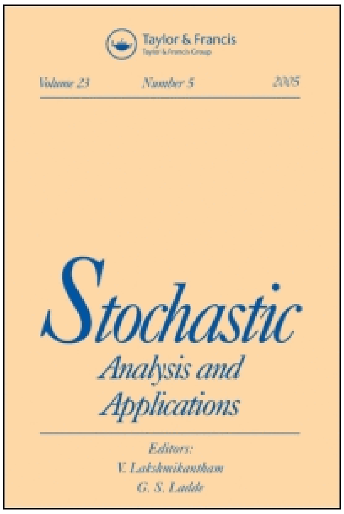

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Tomasz R. Bielecki [a]; Jacek Jakubowski [b]; Andrea Vidozzi [a]; Luca Vidozzi [a]
[a] Department of Applied Mathematics, Illinois Institute of Technology, Chicago, Illinois, USA [b] Institute of Mathematics, University of Warsaw, Warszawa, Poland

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Study of Dependence for Some Stochastic Processes

Tomasz R. Bielecki,[1] Jacek Jakubowski,[2] Andrea Vidozzi,[1] and Luca Vidozzi[1]
[1]Department of Applied Mathematics, Illinois Institute of Technology, Chicago, Illinois, USA
[2]Institute of Mathematics, University of Warsaw, Warszawa, Poland

Abstract: This article is concerned with studying the following problem: Consider a multivariate stochastic process whose law is characterized in terms of some infinitesimal characteristics, such as the infinitesimal generator in case of finite Markov chains. Under what conditions imposed on these infinitesimal characteristics of this multivariate process, the univariate components of the process agree in law with given univariate stochastic processes. Thus, in a sense, we study a stochastic processes' counterpart of the stochastic dependence problem, which in case of real valued random variables is solved in terms of Sklar's theorem.

Keywords: Compensators; Dependence; Stochastic processes.

Mathematics Subject Classification: 60G55; 60G57; 60G99.

1. INTRODUCTION

We study certain aspects of stochastic dependence between some classes of finite dimensional semimartingale processes in terms of their

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Address correspondence to Tomasz R. Bielecki, Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA; E-mail: bielecki@iit.edu

infinitesimal characteristics. This is indeed a pilot study, where we intend to present only some special cases and some special techniques. This study will be followed by a study of dependence between general classes of Markov processes and more general classes of semimartingales.

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be some underlying probability space, and let $Y = (Y_1, Y_2, \ldots, Y_n)$ be an $\mathbf{R}^n$-valued semimartingale with respect to some filtration, defined on this probability space. Let also $X_1, X_2, \ldots, X_n$ be a collection of semimartingales on $(\Omega, \mathscr{F}, \mathbf{P})$, with respect to some filtrations. We shall only consider certain classes of semimartingales that are uniquely characterized, in the sense of their probability laws, by their characteristics. For the most part, for simplicity of presentation, we shall only consider the bivariate case, that is the case of $n = 2$.

Our study is motivated by the following question: What are the sufficient conditions to be satisfied by the local characteristics of process Y so that law of Y_i is the same as the law of X_i, $i = 1, 2, \ldots, n$. This question arises naturally in various applications, such as valuation and hedging of financial derivatives written on baskets of underlying securities.

As a matter of fact, the question can be posed in a constructive way: given processes X_i, $i = 1, 2, \ldots, n$ construct process Y so that its ith univariate law, i.e., the law of the ith component Y_i, is the same as the law of X_i, $i = 1, 2, \ldots, n$. In this context, the question reminisces the concept of copula functions, and the celebrated Sklar's [4] theorem.

A progress in extending Sklar's theorem to general product spaces was made in a paper by Scarsini [3]. In this paper a family of Polish spaces L_i, $i = 1, \ldots, n$ is considered, where each L_i is endowed with a σ-field $\mathscr{L}_i$. A subclass $\mathscr{A}_i \subset \mathscr{L}_i$ is increasing if it is linearly ordered by inclusion: $\forall A, B \in \mathscr{A}_i$, either $A \subset B$, or $B \subset A$, or $A = B$. By $(\times_{i=1}^n L_i, \bigotimes_{i=1}^n \mathscr{L}_i, \mathbf{P})$ we denote a probability space such that for all $i = 1, 2, \ldots, n$

$$\mathbf{P}(L_1 \times \cdots \times L_{i-1} \times A_i \times L_{i+1} \times \cdots \times L_n) = \mathbf{P}_i(A_i), \quad \forall A_i \in \mathscr{L}_i,$$

where $\mathbf{P}_i$ is a probability measure on $(L_i, \mathscr{L}_i)$. The main result in [3] is

Theorem 1.1. *There exists a unique sub-copula function C defined on* $\times_{i=1}^n \mathrm{Range}(\mathrm{A_i} \to \mathbf{P}_\mathrm{i}(\mathrm{A_i}), \mathrm{A_i} \in \mathscr{A}_\mathrm{i})$, *such that*

$$\mathbf{P}\big(\times_{i=1}^n A_i \big) = C\big(\mathbf{P}_1(A_1), \ldots, \mathbf{P}_n(A_n)\big), \quad \forall A_i \in \mathscr{A}_i.$$

This is a nice result, but it does not suit well our needs to model dependence between stochastic processes. The reason is that this result, as stated, is limited to linearly ordered families $\mathscr{A}_i$, a condition, which, in general, is not satisfied by cylindrical sigma algebras on canonical spaces. This motivated us to study dependence between processes in terms of infinitesimal characteristics. Consequently, we somewhat abuse terminology by use of the term "copulae." We use the term for historical reasons only. Various "copulae" that we define below aren't really copula functions. Nevertheless, we find this terminology useful and convenient.

It turns out that the key role in our approach is played by canonical characteristics (canonical compensators, in particular), that is, characteristics expressed as functions of a trajectory of the process. This representation is explicit in the case of processes discussed in Sections 2, 3, and 5. In case of the point processes discussed in Section 4 this canonical representation is semi-explicit, in general.

We shall make the following standing assumption throughout the article.

Assumption. All filtrations appearing below are supposed to be appropriately augmented, so that they satisfy the usual conditions.

2. DEPENDENCE BETWEEN MARKOV DIFFUSION PROCESSES: DIFFUSION COPULAE

Let us consider two diffusion processes X_1 and X_2 with values in $\mathbf{R}^1$ and driven by the following SDEs:

$$dX_i(t) = \mu_i(X_i(t))dt + \sigma_i(X_i(t))dW_i(t), \quad X_i(0) = 1, \;\; i = 1, 2, \tag{1}$$

where W_1 and W_2 are independent SBMs. We suppose that X_i, $i = 1, 2$ are strong solutions, although this is not necessary for our needs.

Our problem is to find functions $m = [m_1, m_2]^T : \mathbf{R}^2 \to \mathbf{R}^2$ and $\Sigma = [\sigma_{ij}] : \mathbf{R}^2 \to L(\mathbf{R}^2, \mathbf{R}^2)$, so that two dimensional diffusion process Y given as a strong solution to the SDE

$$dY(t) = m(Y(t))dt + \Sigma(Y(t))dW(t), \;\; Y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{2}$$

where $W^T = (W_1, W_2)$, is such that

$$\mathscr{L}(Y_i) = \mathscr{L}(X_i), \quad i = 1, 2. \tag{3}$$

Let $\mathbf{F}$ be the filtration generated by W. Suppose that function Σ satisfies the following two conditions

$$\sigma_{11}^2(x, y) + \sigma_{12}^2(x, y) = \sigma_1^2(x), \quad \sigma_{21}^2(x, y) + \sigma_{22}^2(x, y) = \sigma_2^2(y). \tag{4}$$

In addition, suppose that function m satisfies

$$m_1(x, y) = \mu_1(x), \quad m_2(x, y) = \mu_2(y). \tag{5}$$

Assuming condition (4), and some mild technical assumptions about the coefficients σ_{ij} and σ_i, it follows from the Lévy characterization theorem that the following two process

$$B_1 = \int_0^t \frac{\sigma_{11}(Y_1(u), Y_2(u))}{\sigma_1(Y_1(u))} \, dW_1(u) + \int_0^t \frac{\sigma_{12}(Y_1(u), Y_2(u))}{\sigma_1(Y_1(u))} \, dW_2(u) \tag{6}$$

and

$$B_2 = \int_0^t \frac{\sigma_{21}(Y_1(u), Y_2(u))}{\sigma_2(Y_2(u))} \, dW_1(u) + \int_0^t \frac{\sigma_{22}(Y_1(u), Y_2(u))}{\sigma_2(Y_2(u))} \, dW_2(u) \tag{7}$$

are standard Brownian motions with respect to the filtration $\mathbf{F}$. Consequently, we obtain the following result.

Proposition 2.1. *Under conditions* (4) *and* (5) *the components of process* Y *satisfy*:

$$dY_i(t) = \mu_i(Y_i(t))dt + \sigma_i(Y_i(t))dB_i(t), \quad Y_i(0) = 1, \; i = 1, 2, \tag{8}$$

Thus, property (3) *is satisfied.*

Note that in view of (5) we have no flexibility in choice of the function m. However, we do have flexibility in choice of function Σ. We shall exploit this flexibility.

In view of (5) dependence between components of Y is fully described in terms of functions σ_{12} and σ_{21}. This observation and property (3) bring about the following definition,

Definition 2.1. Let functions σ_{12} and σ_{21} be such that

$$\sup_y \sigma_{12}^2(x, y) \leq \sigma_1^2(x), \quad \text{and} \quad \sup_x \sigma_{21}^2(x, y) \leq \sigma_2^2(y). \tag{9}$$

Then, these functions constitute a diffusion copula between Y_1 and Y_2 in the sense that with function m given by (5), and with function Σ given as

$$\Sigma(x, y) = \begin{pmatrix} \pm\sqrt{\sigma_1^2(x) - \sigma_{12}^2(x, y)} & \sigma_{12}(x, y) \\ \sigma_{21}(x, y) & \pm\sqrt{\sigma_2^2(x) - \sigma_{21}^2(x, y)}, \end{pmatrix} \tag{10}$$

the process Y satisfies (2) and (8). In particular, property (3) holds.

Remark 2.1. The above considerations can be generalized to the non-Markovian case, as long as functions μ_i, σ_i are functionals of trajectories of X_i, and functions m and Σ are functionals of trajectories of Y.

3. DEPENDENCE BETWEEN POISSON RANDOM MEASURES AND POISSON PROCESSES: POISSON COPULAE

Assume that E is a Polish space with Borel σ-field $\mathscr{E}$ and let $(\Omega, \mathscr{F}, \mathbf{P})$ is a probability space endowed with filtration $\mathbf{F}$. Let μ be a homogenous Poisson measure on $\mathbf{R}_+ \times E$, that is, μ is an integer-valued random measure such that

i) the intensity measure of μ, that is, the positive measure m on $(\mathbf{R}_+ \times E, \mathscr{B}(\mathbf{R}_+) \otimes \mathscr{E})$ defined by $m(A) = \mathbf{E}(\mu(A))$, is σ-finite,
ii) for every $s \in \mathbf{R}_+$ and every $A \in \mathscr{B}(\mathbf{R}_+) \otimes \mathscr{E}$ such that $A \subset (s, \infty) \times E$, $m(A) < \infty$, the variable $\mu(\cdot, A)$ is independent of σ-field $\mathscr{F}_s$,
iii) $m(\{s\} \times E) = 0$ for every $s \in \mathbf{R}_+$ and $m(dt, dx) = dt \times F(dx)$, where F is a positive σ-finite measure on $(E, \mathscr{E})$.

There exists a one-to-one correspondence between homogenous Poisson processes Y in $\mathbf{R}^n$ and homogenous Poisson measures with $E = \{0, 1\}^n \backslash (0, \dots, 0)$. The measure F is a measure on finite set, so it is uniquely determined by values on atoms. Therefore, a multivariate Poisson process Y in $\mathbf{R}^n$ is uniquely determined by function $F^{(n)} : E \to \mathbf{R}^+$, $F^{(n)}(x) = F(\{x\})$. Dependence between components of Y is fully described in terms of function $F^{(n)}$. We shal denote $\lambda_{k_1,\dots,k_n} = F^{(n)}((k_1, \dots, k_n))$ for $(k_1, \dots, k_n) \in E$.

By μ^Y we denote the jump measure associated with process Y:

$$\mu^Y(\omega, dt, dy) = \sum_s \delta_{(s, \Delta Y_s(\omega))}(dt, dy) \mathbb{1}_{\{\Delta Y_s(\omega) \neq 0\}}. \tag{11}$$

For simplicity of notation in what follows we assume that $n = 2$. We denote $E^i = \{0, 1\}$, $i = 1, 2$.

Let us consider two Poisson processes X_1 and X_2 with values in $\mathbf{R}^1$, determined by their respective intensities $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$. Our problem is to find conditions ensure that two dimensional Poisson process $Y = (Y_1, Y_2)$, determined by $F^{(2)}$, has given marginals X_1 and X_2 that is, condition analogous to (3) is satisfied.

Since, for $A_1 = \{1\}$ we have

$$\begin{aligned} \mu^{Y_1}([0, t] \times A_1) &= \sum_{0<s\le t} \mathbb{1}_{\{\Delta Y_s^1 \in A_1\}} = \sum_{0<s\le t} \mathbb{1}_{\{\Delta Y_s^1 \in A_1, \Delta Y_s^2 \in E^2\}} \\ &= \mu^Y([0, t] \times A_1 \times E^2), \end{aligned}$$

we obtain that $\mu^{Y_1}([0, t] \times A_1)$ has Poisson distribution with parameter $t\lambda_1^{(1)}$, so that $\mathscr{L}(Y_1) = \mathscr{L}(X_1)$, provided

$$\lambda_1^{(1)} = F(A_1 \times E^2) = \lambda_{1,0} + \lambda_{1,1}. \tag{12}$$

Analogous argument gives

$$\lambda_1^{(2)} = \lambda_{0,1} + \lambda_{1,1}. \tag{13}$$

Since $\lambda_{i,j} \geq 0$, then

$$\lambda_{1,1} \in \left[0, \lambda_1^{(1)} \wedge \lambda_1^{(2)}\right]. \tag{14}$$

Consequently, we obtain the following result.

Proposition 3.1. *Let $Y = (Y_1, Y_2)$ be a two dimensional Poisson process determined by $F^{(2)}$, and X_1 and X_2 be Poisson processes in $\mathbf{R}^1$. Under conditions* (12) *and* (13) *the distributions of processes Y_1 and Y_2 are equal to distributions of processes X_1 and X_2, respectively, that is,*

$$\mathscr{L}(Y_i) = \mathscr{L}(X_i), \quad i = 1, 2. \tag{15}$$

These considerations justify the following definition.

Definition 3.1. By a Poisson copula between X_1 and X_2 we understand any function $F^{(2)}$, which satisfies conditions (12) and (13).

Then, any such function constitutes a Poisson copula between X_1 and X_2 in the sense that a two dimensional Poisson process $Y = (Y_1, Y_2)$ determined by $F^{(2)}$ has marginal distributions coinciding with those of X_1 and X_2.

Example 3.1. Let $N = (N_t^1, N_t^2)$ be a two dimensional Poisson process determined by $F^{(2)}$, then

$$\begin{aligned} P(N_t^1 = k, N_t^2 = m) &= P(\mu^N([0,t] \times E) = (k, m)) \\ &= \sum_{l=0}^{k\wedge m} P\big(\mu^N([0,t] \times \{(1,1)\}\big) = l, \mu^N([0,t] \times \{(1,0)\}) \\ &= k - l, \mu^N([0,t] \times \{(0,1)\}) = m - l) \\ &= \sum_{l=0}^{k\wedge m} \frac{(\lambda_{1,1}t)^l}{l!} e^{-\lambda_{1,0}t} \frac{(\lambda_{1,0}t)^{k-l}}{(k-l)!} e^{-\lambda_{1,0}t} \frac{(\lambda_{0,1}t)^{m-l}}{(m-l)!} e^{-\lambda_{0,1}t}. \end{aligned}$$

Remark 3.1. By this method we can obtain n-dimensional Poisson process with given consistent two-dimensional marginal distributions or three-dimensional marginal distributions, etc.

In the next section we shall generalize results of this section to the case of a class of point processes.

4. DEPENDENCE BETWEEN POINT PROCESSES: PP COPULAE

Here we characterize dependence between point processes.

We start with definition of a multivariate point process.[1] As before, we consider $(\Omega, \mathcal{F}, \mathbf{P})$ – a probability space endowed with some filtration $\mathbf{F}$. We also consider the space $E = \{0, 1\}^n \backslash (0, \ldots, 0)$.

Definition 4.1. A multivariate point process, say Y, is a piece-wise constant stochastic process on $(\Omega, \mathcal{F}, \mathbf{P})$ whose associated jump measure, say μ^Y, is an integer-valued random measure on $\mathbf{R}_+ \times E$.

To simplify discussion below we make a standing assumption.

Assumption (A). We assume that $\mathbf{E}\mu^Y([0, t], E) < \infty$, for all $t \geq 0$.

From now on the filtration $\mathbf{F}$ will be the natural filtration of process Y, and will be denoted by $\mathbf{F}^Y$.

Remark 4.1. In a future work we shall consider a more general set-up where filtration $\mathbf{F}$ is given as $\mathbf{F} = \mathbf{F}^Y \vee \widetilde{\mathbf{F}}$, where $\widetilde{\mathbf{F}}$ is a filtration providing some additional information (such filtration $\widetilde{\mathbf{F}}$ is sometimes called a *reference* filtration). For example, Y may be a doubly stochastic (multivariate) Poisson process (also known as Cox process), and filtration $\widetilde{\mathbf{F}}$ may be generated by its (random) hazard process.

It is well known that a point process can be represented as a sequence of random variables (T_n, Z_n), $n = 1, 2, \ldots$, where T_n's are $\mathbf{F}^Y$ stopping times that represent jump times, and Z_n's are corresponding marks that take values in E and indicate which of the components jump at any given jump time. In particular, it is known that for each n it holds that $\mathcal{F}^Y_{T_n} = \sigma((T_i, Z_i), i = 1, 2, \ldots, n)$. It can be shown (cf., e.g., [2], Theorem 4.1.11) that the $\mathbf{F}^Y$ compensator of μ^Y, say ν^Y, can be represented as

$$\nu^Y(dt, A) = \sum_{n \geq 0} \mathbb{1}_{\{T_n < t \leq T_{n+1}\}} \frac{P\big(T_{n+1} \in dt, Z_{n+1} \in A \mid \mathcal{F}_{T_n}\big)}{P\big(T_{n+1} \geq t \mid \mathcal{F}_{T_n}\big)}, \tag{16}$$

where we use the convention that $\frac{0}{0} = 0$.

Since different processes may be defined on different probability spaces, then, in general, we are not able to compare distributional

[1]Some authors (cf. Last and Brandt [2]) refer to marked point process as to multivariate point process. This is not the definition that we have in mind, however.

properties of the processes in terms of their compensators. This is because the compensators themselves will be defined on these different probability spaces, and therefore we are not in position to assess whether they are equal to each other or not.

However, one may instead compare so called *canonical compensators* of point processes (cf. Last and Brandt [2], section 4.2).

In our set-up we have that any canonical compensator of Y, say α^Y, can be disintegrated as (cf. [2], Theorem 4.2.2)

$$\alpha^Y(y, dt, dx) = \bar{\alpha}^Y(y, dt)\kappa^Y(y, t, dx), \tag{17}$$

where y represents a path of Y, that is, $y \in E^{[0,\infty)}$. It, thus, holds that

$$\nu^Y(\omega; dt, dx) = \bar{\alpha}^Y(Y(\omega), dt)\kappa^Y(Y(\omega), t, dx). \tag{18}$$

A canonical compensator of a point process Y is the same as the compensator of this process, when process Y is considered on the canonical space. This in turn implies that canonical compensator is unique almost surely.

It follows from Theorem 4.3.9 in Last and Brandt [2] that for a point process Y its canonical compensator uniquely determines process Y (in the sense of probability law). Consequently, probabilistic properties of point processes discussed here, such as dependence between coordinate components, can be entirely characterized in terms of the respective canonical compensators, which, in view of (18), is equivalent to characterization in terms of the associated kernels: $\bar{\alpha}^Y(y, dt)$ and $\kappa^Y(y, t, dx)$.

4.1. Two-Dimensional Case

Let us consider a two-dimensional point process $Y = (Y^1, Y^2)$, so that mark space is $E = \{(1, 0), (0, 1), (1, 1)\}$. The mark space[2] of each coordinate component Y^i is $E^i = \{0, 1\}$.

In analogy to notation used in the previous section we denote by $\mu^{Y^i}(\omega, dt, dx)$ the jump measure of process Y^i, so that $\mu^{Y^i}([0, t], \{1\})$ gives the number of jumps of Y^i on the time interval $[0, t]$.

Remark 4.2. Recall that by definition of the jump measure it holds that $\mu^{Y^i}(\mathcal{T}, \{0\}) = 0$ for any Borel subset $\mathcal{T}$ in $[0, \infty)$.

We note that $\mu^{Y^1}([0, t], A_1) = \mu^Y([0, t], A_1 \times E^2)$, and $\mu^{Y^2}([0, t], A_2) = \mu^Y([0, t], E^1 \times A_2)$, for any $A_i \subset E_i, i = 1, 2$. It is clear that $\mu^{Y^i}([0, \cdot], A_i)$ is adapted to $\mathbf{F}^{Y^i}$ (of course, it is also adapted to $\mathbf{F}^Y$).

[2]We slightly abuse the term "mark space" here.

4.1.1. Auxiliary Results

We fix i. We shall use the following notation,

- ν^Y: the compensator of μ^Y with respect to $\mathbf{F}^Y$,
- $\nu_Y^{Y^i}$: the compensator of μ^{Y^i} with respect to $\mathbf{F}^Y$,
- ν^{Y^i}: the compensator of μ^{Y^i} with respect to $\mathbf{F}^{Y^i}$,
- ${}^{o_i}U$: the optional projection of a process U on $\mathbf{F}^{Y^i}$,
- ${}^{p_i}U$: the predictable projection of a process U on $\mathbf{F}^{Y^i}$,
- U^{p_i}: the dual predictable projection of a process U on $\mathbf{F}^{Y^i}$.

Since $\nu_Y^{Y^i}([0,\cdot],\{1\})$ is an increasing process, ${}^{o_i}\big(\nu_Y^{Y^i}([0,\cdot],\{1\})\big)$ is an $\mathbf{F}^{Y_i}$ (local) submartingale, thus it admits a Doob–Meyer decomposition. Hence, we have

$$ {}^{o_i}\big(\nu_Y^{Y^i}([0,\cdot],\{1\})\big)_t = M_t^i + Z_t^i, $$

where M^i is an $\mathbf{F}^{Y_i}$ (local) martingale starting from 0, and $Z^i = \big({}^{o_i}(\nu_Y^{Y^i}([0,\cdot],\{1\})\big)^{p_i}$. We now set

$$ \gamma^i(\omega, dt, dx) = dZ_t^i(\omega)\delta_1(dx). \tag{19} $$

Lemma 4.1. *Equality* (19) *defines a predictable random measure on* $\mathscr{B}(R_+) \otimes 2^{E^i}$.

Proof. Let $0 \le s \le t$ and let $A \subset E^i$. We then have

$$ \gamma^i(\omega, (s,t], A) = (Z_t^i(\omega) - Z_s^i(\omega))\delta_1(A) \ge 0. \tag{20} $$

Next, let B_1 and B_2 be disjoint sets from $\mathscr{B}(R_+)$. Then, for any $A \subset E^i$ we have

$$ \begin{aligned} \gamma^i(\omega, B_1 \cup B_2, A) &= \left(\int_{B_1} dZ_t^i(\omega) + \int_{B_2} dZ_t^i(\omega)\right)\delta_1(A) \\ &= \gamma^i(\omega, B_1, A) + \gamma^i(\omega, B_2, A). \end{aligned} \tag{21} $$

Now, since process Z^i is cad-lag[3] we see that for any $s \ge 0$ and any $A \subset E^i$ we have

$$ \lim_{n\to\infty} \gamma^i\left(\omega, \left(s, s+\frac{1}{n}\right], A\right) = 0. \tag{22} $$

[3]By definition, process Z^i is right-continuous; since it is increasing—it is cad-lag.

We conclude that γ^i uniquely extends from a random measure on the algebra of sets of the form $(s, t] \times A$ to a random measure, also denoted as γ^i, on $\mathcal{B}(R_+) \otimes 2^{E^i}$.

We shall now demonstrate that γ^i is a predictable measure. Toward this end let $\widehat{W}$ be a predictable function of the form

$$\widehat{W}(\omega, t, x) = f(\omega, t)g(x),$$

where f is a predictable process. Then we have

$$(\widehat{W} * \gamma^i)_t(\omega) = g(1) \int_{(0,t]} f(\omega, s)dZ^i_s(\omega).$$

Since process Z^i is predictable, then it is clear that process $\widehat{W} * \gamma^i$ is predictable. Finally, since every predictable function $W(\omega, t, x)$ can be approximated by predictable functions of the form $f(\omega, t)g(x)$, we conclude that for any predictable function W the integral $W * \gamma^i$ is a predictable process. Consequently, measure γ^i is a predictable random measure. □

We shall see that measure γ^i is the dual predictable projection of μ^{Y^i} on $\mathbf{F}^{Y_i}$. For this, we shall use the following result.

Lemma 4.2. *For each $A \subset E^i$ we have that*

$$\mu^{Y^i}([0, t], A) - ({}^{o_i}(\nu_Y^{Y^i}([0, \cdot], A)))_t^{p_i} \tag{23}$$

is an $\mathbf{F}^{Y^i}$-local martingale.

Proof. Fix $A \subset E^i$. The process $M_t = \mu^{Y^i}([0, t], A) - \nu_Y^{Y^i}([0, t], A)$ is an $\mathbf{F}^Y$ martingale, thus its optional projection on $\mathbf{F}^{Y_i}$, denoted as ${}^{o_i}M_t$, is an $\mathbf{F}^{Y_i}$ martingale, that is, the process ${}^{o_i}M_t = \mu^{Y^i}([0, t], A) - {}^{o_i}(\nu_Y^{Y^i}([0, \cdot], A))_t$ is an $\mathbf{F}^{Y_i}$ martingale. Since $\nu_Y^{Y^i}([0, t], A)$ is an increasing process, ${}^{o_i}(\nu_Y^{Y^i}([0, \cdot], A))_t$ is an $\mathbf{F}^{Y_i}$ submartingale, thus it admits a Doob–Meyer decomposition. Hence, we have

$${}^{o_i}\big(\nu_Y^{Y^i}([0, t], A)\big) = L_t + \big({}^{o_i}\big(\nu_Y^{Y^i}([0, t], A)\big)\big)^{p_i},$$

where L_t is an $\mathbf{F}^{Y_i}$ (local) martingale starting from 0. From the above we conclude that

$$\mu^{Y^i}([0, t], A) - \big({}^{o_i}\big(\nu_Y^{Y^i}([0, \cdot], A)\big)\big)_t^{p_i} = L_t + {}^{o_i}M_t,$$

so that process $\mu^{Y^i}([0, t], A) - \big({}^{o_i}\big(\nu_Y^{Y^i}([0, \cdot], A)\big)\big)_t^{p_i}$ is an $\mathbf{F}^{Y^i}$-local martingale, as requested. □

Corollary 4.1. *The measure γ^i is the dual predictable projection of μ^{Y^i} on $\mathbf{F}^{Y_i}$, so that $\nu^{Y_i} \equiv \gamma^i$.*

Proof. It is enough to observe that for $A \subset E^i$ we have

$$\left({}^{o_i}\left(\nu_Y^{Y^i}([0,\cdot],A)\right)\right)_t^{p_i} = \gamma^i([0,t],A)$$

and then to use Theorem II.1.8 in [1]. □

In what follows we give a more direct way of computing the projections

$$\left({}^{o_i}\left(\nu_Y^{Y^i}([0,\cdot],A)\right)\right)^{p_i}$$

in a special case, namely when the compensator $\nu_Y^{Y^i}$ is absolutely continuous.

Lemma 4.3. *Assume that $\nu_Y^{Y^i}([0,t],A)$ is absolutely continuous in the sense that $\nu_Y^{Y^i}(\omega,[0,t],A) = \int_0^t K_Y^{Y^i}(\omega,s,A)ds$ for some measurable kernel $K_Y^{Y^i}$. Then, for any $t < \infty$,*

$$\left({}^{o_i}\left(\nu_Y^{Y^i}([0,\cdot],A)\right)\right)_t^{p_i} = \int_0^t {}^{p_i}(K_Y^{Y^i}(s,A))ds. \tag{24}$$

Proof. In view of our assumption we see that ${}^{o_i}(\nu_Y^{Y^i}([0,\cdot],A))_t$ is integrable. We just need to show that the process

$$M_t := {}^{o_i}\left(\nu_Y^{Y^i}([0,\cdot],A)\right)_t - \int_0^t {}^{p_i}\left(K_Y^{Y^i}(s,A)\right)ds$$

is an $\mathbf{F}^{Y^i}$ martingale as then the result will follow from uniqueness of the Doob–Meyer decomposition. Note that since ${}^{p_i}(K_Y^{Y^i}(s,A))$ is predictable it is also progressive and thus the integral $\int_0^t {}^{p_i}(K_Y^{Y^i}(s,A))ds$ is well defined in the Lebesgue sense. Integrability of the process $\int_0^t {}^{p_i}(K_Y^{Y^i}(s,A))ds$ and the martingale property of M_t are then an immediate consequence of the Fubini theorem. □

In what follows we shall denote by α^{Y_i} the canonical compensator corresponding to γ^i.

4.1.2. PP Copulae

Let us consider two point processes X_1 and X_2 with values in $\mathbf{R}^1$. We denote the corresponding compensators by ν^{X^1} and ν^{X^2}, and the corresponding canonical compensators by α^{X^1} and α^{X^2}.

Our problem is to find conditions that ensure that two-dimensional point process $Y = (Y_1, Y_2)$, with the corresponding compensators ν^Y and α^Y, has given marginals X_1 and X_2 i.e., condition analogous to (3) is satisfied.

Given all the above discussion, it turns out that the sufficient condition for this is

$$\alpha^{Y_i} = \alpha^{X_i}, \quad i = 1, 2. \tag{25}$$

Consequently, we obtain the following result, where $\tilde{\alpha}$ is a predictable kernel from the path space to $\mathscr{B}(\mathbf{R}_+) \otimes 2^E$,

Proposition 4.1. *Let $Y = (Y_1, Y_2)$ be a two dimensional point process with canonical compensator $\alpha^Y = \tilde{\alpha}$, and compensator ν^Y given through* (18). *Let X_1 and X_2 be point processes in $\mathbf{R}^1$. Under condition* (25) *the marginal distributions of process Y are equal to X_1 and X_2 respectively. Thus, property* (3) *is satisfied.*

These considerations justify the following definition.

Definition 4.2. By a PP copula between X_1 and X_2 we understand any predictable kernel $\tilde{\alpha}$, defining a two dimensional point process, which satisfies condition (25).

Of course, the above definition extends to a multivariate case in a straightforward way.

4.1.3. Examples

Example 4.1. Let us consider a bivariate Poisson process Y as in Section 3. Here we have

$$\alpha^Y(y, dt, \{(i, j)\}) = \lambda_{i,j} dt, \quad \text{for } (i, j) \in E \tag{26}$$

and

$$\alpha^{Y_1}(y_1, dt, \{1\}) = (\lambda_{1,0} + \lambda_{1,1})dt, \quad \alpha^{Y_2}(y_2, dt, \{1\}) = (\lambda_{0,1} + \lambda_{1,1})dt. \tag{27}$$

Given two univariate Poisson processes, X_1 and X_2, with natural intensities $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$, respectively, condition (25) takes the form

$$\lambda_{1,0} + \lambda_{1,1} = \lambda_1^{(1)}, \quad \lambda_{0,1} + \lambda_{1,1} = \lambda_1^{(2)}. \tag{28}$$

Thus, a particular PP copula between X_1 and X_2 is any triple $\{\lambda_{1,0}, \lambda_{0,1}, \lambda_{1,1}\}$ of non-negative numbers satisfying (28). This indeed is the Poisson copula between X_1 and X_2.

Of course any predictable kernel $\tilde{\alpha}$ such that

$$\alpha^{Y_1}(y_1, dt, \{1\}) = \lambda_1^{(1)}, \quad \alpha^{Y_2}(y_2, dt, \{1\}) = \lambda_1^{(2)} \tag{29}$$

also constitutes a PP copula between X_1 and X_2. Thus, a multivariate PP, which is not a multivariate Poisson process in the sense of Section 3, may have Poisson processes as marginals.

Example 4.2. Consider a stochastic basis $(\Omega, \mathcal{F}^Y, \mathbf{P})$, where $\Omega = \Omega_1 \times \Omega_2$ is the canonical space of bivariate point processes. Let $Y = (Y^1, Y^2)$ denote the canonical process on $(\Omega, \mathcal{F})$. For the illustrative purpose of this section, we make the simplifying assumption that, under $\mathbf{P}$, each component is $\mathbf{P}$-a.s bounded by the constant M, that is, it jumps at most M times. We let T_1^m and T_2^n, with $T_i^0 = 0$, denote the jump times of the coordinate processes Y^i. We endow the space with the natural filtration of the canonical process, $\mathbf{F}^Y$. We assume that, under $\mathbf{P}$, we are given the joint density of $\mathbf{P}(T_1^m \in dt, T_2^n \in dt, m, n \le M)$, say $f(s_1^1, \dots, s_1^M, s_2^1, \dots, s_2^M)$. We give here a construction of the compensator of the jump measure of Y in terms of the the joint distribution of the jump times of the components. We prove the result only for the set $\{1, 1\}$, since the proof is analogous for all other sets in the mark space. In the following we use the notation:

$$\begin{aligned} &F\big(t_1^1, \dots, t_1^{m-1}, ds, [s, \infty), \dots, [s, \infty), t_2^1, \dots, t_2^{l-1}, ds, [s, \infty) \dots, [s, \infty)\big) : \\ &\quad = \int_s^\infty \cdots \int_s^\infty f(t_1^1, \dots, t_1^{m-1}, s, s_1^{m+1}, \dots, s_1^M, t_2^1, \dots, t_2^{l-1}, \\ &\qquad\qquad s, s_2^{l=1} \dots, s_2^M) ds_1^{m+1} \dots ds_1^M ds_2^{l+1} \dots ds_2^M ds. \end{aligned} \tag{30}$$

Lemma 4.4. *Let*

$$\begin{aligned} v^Y(dt, \{1, 1\}) = \int_0^t \sum_{m,n=1}^M \mathbf{P}\big(T_1^m \in ds, T_2^n \in ds \,\big|\, \mathcal{F}_{T_1^m \vee T_2^n} \\ \vee \{T_1^m \ge s > T_1^{m-1}, T_2^n \ge s > T_2^{n-1}\}\big). \end{aligned}$$

Then

$$\begin{aligned} &v^Y(dt, \{1, 1\}) \\ &\quad = \int_0^t \sum_{m,n=1}^M \frac{\begin{array}{l} F(T_1^1, \dots, T_1^{m-1}, ds, [s, \infty), \dots, [s, \infty), T_2^1, \dots, T_2^{n-1}, ds, \\ \quad [s, \infty) \dots, [s, \infty)) \end{array}}{\begin{array}{l} F(T_1^1, \dots, T_1^{m-1}, [s, \infty), [s, \infty), \dots, [s, \infty), T_2^1, \dots, T_2^{n-1}, \\ \quad [s, \infty), \dots, [s, \infty)) \end{array}} \\ &\qquad \times \mathbb{1}_{\{T_1^{m-1} < s \le T_1^m\}} \mathbb{1}_{\{T_2^{n-1} < s \le T_2^n\}} \end{aligned} \tag{31}$$

and the process $\mu([0, t], \{1, 1\}) - \int_0^t v(ds, \{1, 1\})$ *is an* $\mathbf{F}$ *martingale.*

Proof. Equality (31) is elementary to verify.

To prove the martingale property of $\mu([0,t],\{1,1\}) - \int_0^t \nu(ds,\{1,1\})$ we first note that $\mu^Y([0,t],\{1,1\}) = \sum_{m,n} \mathbb{1}_{\{T_1^m \leq t, T_2^n \leq t, T_1^m = T_2^n\}}$, therefore it suffices to verify that process

$$M_t := \sum_{m,n} \mathbb{1}_{\{T_1^m \leq t, T_2^n \leq t, T_1^m = T_2^n\}} - \int_0^t \nu(ds,\{1,1\})$$

has the martingale property. By Lemma III.1.29 in [1], on the event $\{T_1^{m-1} < s \leq T_1^m, T_2^{n-1} < s \leq T_2^n\}$, conditioning on $\mathscr{F}^Y_{T_1^{m-1} \vee T_2^{n-1}} \vee \{T_1^m \geq s, T_2^n \geq s\}$ is equivalent to conditioning on $\mathscr{F}^Y_{s-}$, thus,

$$\begin{aligned}
\mathbf{E}(M_t \mid \mathscr{F}^Y_u) &= M_u + \mathbf{E}\Bigg(\sum_{m,n} \mathbb{1}_{\{u < T_1^m \leq t, u < T_2^n \leq t, T_1^m = T_2^n\}} \,\Big|\, \mathscr{F}^Y_u\Bigg) \\
&\quad - \mathbf{E}\int_u^t \sum_{m,n=1}^{M} \mathbf{P}\big(T_1^m \in ds, T_2^n \in ds \,\big|\, \mathscr{F}^Y_{T_1^m \vee T_2^n} \\
&\quad \vee \big\{T_1^m \geq s > T_1^{m-1}, T_2^n \geq s > T_2^{n-1}\big\} \,\big|\, \mathscr{F}^Y_u\big) \\
&= M_u + \sum_{m,n=1}^{M} \Bigg(\mathbf{P}\big(\{u < T_1^m \leq t, u < T_n^2 \leq t, T_1^m = T_2^n\} \,\big|\, \mathscr{F}^Y_u\big) \\
&\quad - \int_u^t \mathbf{P}\big(T_1^m \in ds, T_2^n \in ds \,\big|\, \mathscr{F}^Y_u\big)\Bigg) \\
&= M_u + \sum_{m,n=1}^{M} \Bigg(\mathbf{P}\big(\{u < T_1^m \leq t, u < T_n^2 \leq t, T_1^m = T_2^n\} \,\big|\, \mathscr{F}^Y_u\big) \\
&\quad - \int_u^t \mathbf{P}\big(T_1^m \in ds, T_2^n \in ds, T_1^m = T_2^n \,\big|\, \mathscr{F}^Y_u\big)\Bigg) \\
&= M_u.
\end{aligned}$$

□

In the same way one can proof that $\mu^Y([0,t],\{1,0\}) - \int_0^t \nu^Y(dt,\{1,0\})$ is a local martingale with

$$\begin{aligned}
\nu(ds,\{1,0\}) &= \int_0^t \sum_{m,n=1}^{M} \mathbf{P}(T_1^m \in ds, T_2^n > s \mid \mathscr{F}^Y_{T_1^{m-1} \vee T_2^{n-1}} \\
&\quad \vee \{T_1^m \geq s > T_1^{m-1}, T_2^n \geq s > T_2^{n-1}\}) \\
&= \int_0^t \sum_{m,n=1}^{M} \frac{\begin{matrix} F(T_1^1,\ldots,T_1^{m-1}, ds, [s,\infty),\ldots,[s,\infty), \\ T_2^1,\ldots,T_2^{n-1}, (s,\infty),\ldots,[s,\infty)) \end{matrix}}{\begin{matrix} F(T_1^1,\ldots,T_1^{m-1}, [s,\infty),\ldots,[s,\infty), \\ T_2^1,\ldots,T_2^{n-1}, [s,\infty),\ldots,[s,\infty)) \end{matrix}} \\
&\quad \times \mathbb{1}_{\{T_1^{m-1} < s \leq T_1^m\}} \mathbb{1}_{\{T_2^{n-1} < s \leq T_2^n\}},
\end{aligned}$$

and similarly $\mu^Y([0,t],\{0,1\}) - \int_0^t \nu^Y(ds,\{0,1\})$ is an $\mathbf{F}^Y$ local martingale, with

$$\begin{aligned}
\nu(dt,\{0,1\}) &= \int_0^t \sum_{m,n=1}^{M} \mathbf{P}\big(T_1^m > s, T_2^n \in ds \,\big|\, \mathcal{F}_{T_1^{m-1}\vee T_2^{n-1}} \\
&\qquad \vee \{T_1^m \geq s > T_1^{m-1}, T_2^n \geq s > T_2^{n-1}\}\big) \\
&= \int_0^t \sum_{m,n=1}^{M} \frac{\begin{array}{c}F(T_1^1,\ldots,T_1^{m-1},(s,\infty),[s,\infty),\ldots,[s,\infty),\\ T_2^1,\ldots,T_2^{n-1},ds,\ldots,[s,\infty))\end{array}}{\begin{array}{c}F(T_1^1,\ldots,T_1^{m-1},[s,\infty),\ldots,[s,\infty),\\ T_2^1,\ldots,T_2^{n-1},[s,\infty),\ldots,[s,\infty))\end{array}} \\
&\qquad \times \mathbb{1}_{\{T_1^{m-1}<s\leq T_1^m\}}\mathbb{1}_{\{T_2^{n-1}<s\leq T_2^n\}}.
\end{aligned}$$

By uniqueness of the dual predictable projection it follows that ν^Y as defined above, is indeed the compensator of μ^Y. The counting measure associated to the (canonical) coordinate process Y^1 is given by: $\mu^{Y^1}((0,t],\{1\}) = \mu^Y((0,t],\{1,1\}) + \mu^Y((0,t],\{1,0\})$. It follows therefore immediately from Lemma 4.4 (and the discussion that follows Lemma 4.4), that the $\mathbf{F}^Y$ compensator of μ^{Y^1} is given by the predictable random measure $\nu_Y^{Y^1}$ defined as:

$$\begin{aligned}
\nu_Y^{Y^1}((0,t],\{1\}) &= \int_0^t \sum_{m,n=1}^{M} \mathbf{P}\big(T_1^m \in ds \,\big|\, \mathcal{F}^Y_{T_1^{m-1}\vee T_2^{n-1}} \\
&\qquad \vee \{T_1^m \geq s > T_1^{m-1}, T_2^n \geq s > T_2^{n-1}\}\big) && (32) \\
&= \int_0^t \sum_{m,n=1}^{M} \frac{\begin{array}{c}F(T_1^1,\ldots,T_1^{m-1},ds,[s,\infty),\ldots,[s,\infty),\\ T_2^1,\ldots,T_2^{n-1},[s,\infty),\ldots,[s,\infty))\end{array}}{\begin{array}{c}F(T_1^1,\ldots,T_1^{m-1},[s,\infty),\ldots,[s,\infty),\\ T_2^1,\ldots,T_2^{n-1},[s,\infty),\ldots,[s,\infty))\end{array}} \\
&\qquad \times \mathbb{1}_{\{T_1^{m-1}<s\leq T_1^m\}}\mathbb{1}_{\{T_2^{n-1}<s\leq T_2^n\}}. && (33)
\end{aligned}$$

We shall now compute the $\mathbf{F}^{Y^1}$ compensator of μ^{Y^1}.

Proposition 4.2. *The $\mathbf{F}^{Y^1}$ compensator of μ^{Y^1} is given by the process*

$$\begin{aligned}
&\nu^{Y^1}((0,t],\{1\}) \\
&\quad = \mathbf{P}\big(T_1^m \in ds \,\big|\, \mathcal{F}^{Y^1}_{T_1^{m-1}} \vee \{T_1^m \geq s > T_1^{m-1}\}\big) && (34) \\
&\quad = \int_0^t \sum_{m=1}^{M} \frac{F(T_1^1,\ldots,T_1^{m-1},ds,[s,\infty),\ldots,[s,\infty),[0,\infty),\ldots,[0,\infty))}{F(T_1^1,\ldots,T_1^{m-1},[s,\infty),\ldots,[s,\infty),[0,\infty),\ldots,[0,\infty))} \\
&\qquad \times \mathbb{1}_{\{T_1^{m-1}<s\leq T_1^m\}}. && (35)
\end{aligned}$$

Proof. In view of Corollary 4.1, it suffices to show that $\nu^{Y^1}((0,t],\{1\}) = ({}^{o_1}(\nu_Y^{Y^1}([0,t],\{1\})))^{p_1}$. In addition, in view of Lemma 4.3, $({}^{o_1}(\nu_Y^{Y^1}([0,t],\{1\})))^{p} = \int_0^t {}^{p_1}(\nu_Y^{Y^1}(ds,\{1\}))ds$. Since the process $\mathbb{1}_{\{T_1^{m-1}<s\le T_1^m\}}$ is $\mathbf{F}^{Y^1}$ predictable, and thus $\mathscr{F}_{s-}^{Y^1}$ measurable, and by definition of predictable projection:

$$
\begin{aligned}
&\nu^{Y^1}((0,t],\{1\})\\
&= \int_0^t \sum_{m,n=1}^M \mathbf{E}\Big(\mathbf{P}\big(T_1^m \in ds \,\big|\, \mathscr{F}^Y_{T_1^{m-1}\vee T_2^{n-1}} \vee \{T_1^m \ge s > T_1^{m-1},\\
&\qquad\qquad T_2^n \ge s > T_2^{n-1}\}\big) \,\big|\, \mathscr{F}^{Y^1}_{s-}\Big)\\
&= \int_0^t \sum_{m,n=1}^M \mathbf{E}\Big(\mathbf{P}(T_1^m \in ds, \{T_1^m \ge s > T_1^{m-1}, T_2^n \ge s > T_2^{n-1}\} \,\big|\, \mathscr{F}^Y_{s-}) \,\big|\, \mathscr{F}^{Y^1}_{s-}\Big)\\
&= \int_0^t \sum_{m,n=1}^M \mathbf{P}\big(T_1^m \in ds, \{T_2^n \ge s > T_2^{n-1}\} \,\big|\, \mathscr{F}^{Y^1}_{s-}\big)\mathbb{1}_{\{T_1^{m-1}<s\le T_1^m\}}\\
&= \int_0^t \sum_{m=1}^M \big(\mathbf{P}(T_1^m \in ds, \,\big|\, \mathscr{F}^1_{T^{m-1}} \vee \{T_1^{m-1} < s \le T_1^m\}\big)\mathbb{1}_{\{T_1^{m-1}<s\le T_1^m\}},
\end{aligned}
$$

yielding the result. □

Remark 4.3. The random measures ν^{Y^i} are clearly the canonical compensators of the components Y^i, therefore they uniquely characterize the respective finite dimensional distribution.

5. DEPENDENCE BETWEEN FINITE MARKOV CHAINS: MARKOV COPULAE

We shall first discuss some results regarding random measures associated with finite Markov chains.

5.1. Finite Markov Chains and Related Random Measures

As before, let $(\Omega, \mathscr{F}, \mathbf{P})$ be an underlying probability space. We consider on this space a stochastic process $X = (X_t)_{t\ge 0}$ with values in a finite set $\mathscr{X} = \{1, 2, \ldots, N\}$. By $\mathbf{F}^X$ we shall denote the natural filtration generated by X.

For any two states $i, j \in \mathscr{X}$, such that $i \ne j$, we define the following $\mathbf{F}^X$-optional random measure on $[0,\infty)$,

$$N^{ij}((0,t]) = \sum_{0<s\le t} \mathbb{1}_{\{X_{s-}=i, X_s=j\}}. \tag{36}$$

We shall simply write $N^{ij}(t)$ in place of $N^{ij}((0,t])$. Manifestly, $N^{ij}(t)$ represents the number of jumps from state i to state j that process X executes over the time interval $(0,t]$. Let us denote by ν^{ij} the dual predictable projection w.r.t. $\mathbf{F}^X$ of the random measure N^{ij}.

Next, let us define a matrix valued function A on $[0,\infty)$ by

$$A(t) = [\lambda^i_j(t)]_{i,j\in\mathscr{X}}, \tag{37}$$

where λ^i_j's are real valued, locally integrable functions on $[0,\infty)$ such that for $t \in [0,\infty)$ and $i,j \in \mathscr{X}$, $i \neq j$ we have

$$\lambda^i_j(t) \geq 0$$

and

$$\lambda^i_i(t) = -\sum_{j\neq i} \lambda^i_j(t).$$

$\lambda^i_j(t)$ is the time-t intensity of jump from state i to state j.

Then, we have the following well known result (we provide the proof for the convenience of the reader).

Lemma 5.1. *Process X is a Markov chain (with respect to $\mathbf{F}^X$) with infinitesimal generator $A(t) = [\lambda^i_j(t)]$ iff the dual predictable projections w.r.t. $\mathbf{F}^X$ of the counting measures $N^{ij}(dt)$, $i,j \in \mathscr{X}$ are of the form*:

$$\nu^{ij}(dt) = \mathbb{1}_{\{X_{t-}=i\}}\lambda^i_j(t)dt. \tag{38}$$

Proof. Let us first assume that X is a Markov chain with generator function A. It is then standard to verify that for any $i,j \in \mathscr{X}$, such that $i \neq j$, process M^{ij} defined as

$$M^{ij}_t = N^{ij}(t) - \int_0^t \nu^{ij}(ds)$$

is an $\mathbf{F}^X$ martingale. Thus, letting $n = 1,2,\ldots$ and denoting by T_n the time of the nth jump of process X, we easily deduce that process M^{ij} stopped at T_n is in fact a uniformly integrable $\mathbf{F}^X$ martingale. Since ν^{ij} given in (38) is $\mathbf{F}^X$-predictable, then, in view of Theorem 4.1.9 in Last and Brandt [2], we conclude that ν^{ij} is the compensator of N^{ij}.

Now, let us assume that ν^{ij} given in (38) is the compensator of N^{ij}, for any $i,j \in \mathscr{X}$, $i \neq j$, and define

$$N^i(t) = \sum_{k\neq i} N^{ki}(t), \quad \overline{N}^i(t) = \sum_{k\neq i} N^{ik}(t)$$

and

$$\nu^i(dt) = \sum_{k \neq i} \nu^{ki}(dt), \quad \bar{\nu}^i(dt) = \sum_{k \neq i} \nu^{ik}(dt).$$

Thus, the processes

$$M_t^i = N^i(t) - \int_0^t \nu^i(ds), \quad \text{and} \quad \overline{M}_t^i = \overline{N}^i(t) - \int_0^t \bar{\nu}^i(ds) \tag{39}$$

are $\mathbf{F}^X$ martingales, as they are each equal to a finite sum of $\mathbf{F}^X$ martingales. As usual, we denote by $\delta_i(\cdot)$ the Dirac delta function. Now, observe that the following identities are satisfied for $t \geq 0$ and $i \in \mathscr{X}$,

$$\begin{aligned} \delta_i(X_t) &= N^i(t) - \overline{N}^i(t), \quad \text{if } X_0 = j \neq i, \\ 1 - \delta_i(X_t) &= N^i(t) - \overline{N}^i(t), \quad \text{if } X_0 = i \end{aligned} \tag{40}$$

and

$$\nu^i(dt) - \bar{\nu}^i(dt) = \lambda_i^{X_{t-}}(t)dt. \tag{41}$$

Recall that, considered as a linear operator, matrix $A(t)$ acts on any function $f : \mathscr{X} \to \mathbf{R}$ in the following way

$$A(t)f(k) = \sum_{j \in \mathscr{X}} \lambda_j^k(t) f(j). \tag{42}$$

Consequently, we obtain that

$$A(t)\delta_i(k) = \lambda_i^k(t), \tag{43}$$

so that

$$A(t)\delta_i(X_{t-})\, dt = \nu^i(dt) - \bar{\nu}^i(dt). \tag{44}$$

We have then demonstrated that for any $i \in \mathscr{X}$ the process M^{δ_i} given as

$$M_t^{\delta_i} = \delta_i(X_t) - \int_0^t A(s)\delta_i(X_{s-})ds \tag{45}$$

is an $\mathbf{F}^X$ martingale. Consequently, for any function $f : \mathscr{X} \to \mathbf{R}$ the process M^f given as

$$M_t^f = f(X_t) - \int_0^t A(s)f(X_{s-})ds = f(X_t) - \int_0^t A(s)f(X_s)ds \tag{46}$$

is an $\mathbf{F}^X$ martingale. In view of the martingale characterization of Markov chains this verifies that X is a Markov chain w.r.t. $\mathbf{F}^X$. $\square$

5.2. Markov Copulae

To simplify notation we shall only consider the case of bivariate Markov chains. The general multivariate case can be treated accordingly.

We want to answer the following questions. Given a bivariate processes $Z = (X, Y)$, which is a finite Markov chain with respect to to its natural filtration $\mathbf{F}^Z = \mathbf{F}^{X,Y}$:

(Q1): What are the sufficient and necessary conditions on the infinitesimal generator of Z so that the components X and Y are Markov chains with respect to their natural filtrations?
(Q2): How do we construct a bivariate Markov chain, whose components are themselves Markov chains w.r.t. their natural filtration and have desired infinitesimal characteristics?
(Q3): What kind of dependence structure can we impose on (X, Y) and how do we do this?

In the rest of this section we denote by $\mathcal{S}$ and $\mathcal{O}$ two finite sets. Let $Z = (X, Y)$ denote a two dimensional Markov chain on $\mathcal{Z} = \mathcal{S} \times \mathcal{O}$, with generator function $A^Z(t) = \left[\lambda^{ih}_{jk}(t)\right]_{i,j\in\mathcal{S},k,h\in\mathcal{O}}$. Consider the following condition

Condition (M)

$$\sum_{k\in\mathcal{O}} \lambda^{ih}_{jk}(t) = \sum_{k\in\mathcal{O}} \lambda^{ih'}_{jk}(t), \quad \forall h, h' \in \mathcal{O}, \ \forall i, j, \in \mathcal{S}, \ i \neq j,$$

and

$$\sum_{j\in\mathcal{S}} \lambda^{ih}_{jk}(t) = \sum_{j\in\mathcal{S}} \lambda^{i'h}_{jk}(t), \quad \forall i, i' \in \mathcal{S}, \ \forall k, h \in \mathcal{O} \ \ h \neq k.$$

The following proposition addresses the sufficiency part in question (Q1).

Proposition 5.1. *Suppose that condition (M) holds, and define*

$$f^i_j(t) := \sum_{k\in\mathcal{O}} \lambda^{ih}_{jk}(t), \quad i, j \in \mathcal{S}, \ i \neq j, \quad f^i_i(t) = -\sum_{j\in\mathcal{S}, j\neq i} f^i_j(t), \quad \forall i \in \mathcal{S},$$

and

$$g^h_k(t) := \sum_{j\in\mathcal{S}} \lambda^{ih}_{jk}(t), \quad k, h \in \mathcal{O}, \ h \neq k, \quad g^h_h(t) = -\sum_{k\in\mathcal{O}, k\neq h} g^h_k(t), \quad \forall h \in \mathcal{O}.$$

Then the components X and Y of the Markov chain Z are Markov chains with respect to their natural filtrations with generator functions $A^X(t) = [f^i_j(t)]_{i,j\in\mathcal{S}}$ and $A^Y(t) = [g^h_k(t)]_{k,h\in\mathcal{O}}$, respectively.

Proof. Let $N_Z^{ih,jk}$ be the counting measure associated to the process Z counting jumps from state ih to state jk (cf. (36)). It follows from Lemma 5.1 that the compensator of $N_Z^{ih,jk}$ is given by the random measure

$$\nu_Z^{ih,jk}(dt) = \mathbb{1}_{\{X_{t-}=i, Y_{t-}=h\}} \lambda_{jk}^{ih}(t)dt.$$

The random measure counting the number of jumps of the component X from state i to state j is given as

$$N_X^{ij}(t) = \sum_{k\in\mathcal{O}} \sum_{h\in\mathcal{O}} N_Z^{ih,jk}(t).$$

The $\mathbf{F}^Z$ compensator of N_X^{ij} is given by

$$\begin{aligned} \nu_X^{ij}(dt) &= \sum_{k\in\mathcal{O}} \sum_{h\in\mathcal{O}} \nu_{jk}^{ih}(dt) \\ &= \sum_{k\in\mathcal{O}} \sum_{h\in\mathcal{O}} \mathbb{1}_{\{X_{t-}=i, Y_{t-}=h\}} \lambda_{jk}^{ih}(t)dt \\ &= \sum_{h\in\mathcal{O}} \mathbb{1}_{\{X_{t-}=i, Y_{t-}=h\}} f_j^i(t)dt \\ &= \mathbb{1}_{\{X_{t-}=i\}} f_j^i(t)dt \end{aligned}$$

Note that this is also the $\mathbf{F}^X$ compensator of $N_Z^{ih,jk}$. Thus, invoking Lemma 5.1 again, we see that X is a Markov chain with respect to $\mathbf{F}^X$ and that its generator function is $A^X(t)$. Analogous argument verifies that the Y component of Z is a Markov chain w.r.t. $\mathbf{F}^Y$ with generator function $A^Y(t)$. □

For the necessity part of question (Q1) we have

Proposition 5.2. *For the components X and Y of the Markov chain Z to be Markov chains with respect to their natural filtrations, with generator functions $A^X(t) = [f_j^i(t)]_{i,j\in\mathcal{S}}$ and $A^Y(t) = [g_k^h(t)]_{k,h\in\mathcal{O}}$, respectively, it is necessary that the following conditions hold for almost all $t \geq 0$, $\mathbf{P}$-a.s.:*

$$\left({}^{o_x}\Lambda^{ij}\right)_t^{p_x} = \int_0^t \mathbb{1}_{\{X_{u-}=i\}} f_j^i(u)du, \tag{47}$$

$$\left({}^{o_y}\Gamma^{hk}\right)_t^{p_y} = \int_0^t \mathbb{1}_{\{Y_{u-}=i\}} g_k^h(u)du, \tag{48}$$

where ${}^{o_x}(\cdot)$ $[(\cdot)^{p_x}]$ and ${}^{o_y}(\cdot)$ $[(\cdot)^{p_y}]$ denote the optional [predictable] projection on $\mathbf{F}^X$ and $\mathbf{F}^Y$ respectively, and where

$$\Lambda_t^{ij} = \int_0^t \sum_{k\in\mathcal{O}} \sum_{h\in\mathcal{O}} \mathbb{1}_{\{X_{u-}=i, Y_{u-}=h\}} \lambda_{jk}^{ih}(u)du$$

and

$$\Gamma_t^{hk} = \int_0^t \sum_{j\in\mathcal{S}} \sum_{i\in\mathcal{S}} \mathbb{1}_{\{X_{u-}=i, Y_{u-}=h\}} \lambda_{jk}^{ih}(u) du.$$

Proof. Similarly as in Corollary 4.1 one can show that $({}^{o_x}\Lambda^{ij})^{p_x}$ determines the $\mathbf{F}^X$ compensator of N_X^{ij}, and that $({}^{o_y}\Gamma^{hk})^{p_y}$ determines $\mathbf{F}^Y$ compensator of N_Y^{ij}. Thus, the result follows from Lemma 5.1. □

The following corollary addresses question (Q2).

Corollary 5.1. *Consider two Markov chains X and Y, with respect to their own filtrations, and with values in $\mathcal{S}$ and $\mathcal{O}$, respectively. Suppose that their respective generators are $A^X(t) = [\alpha_j^i(t)]_{i,j\in\mathcal{S}}$ and $A^Y(t) = [\beta_k^h(t)]_{h,k\in\mathcal{O}}$. Next, consider the system of equations in the unknowns $\lambda_{jk}^{ih}(t)$, where $i, j \in \mathcal{S}$, $h, k \in \mathcal{O}$ and $(i, h) \neq (j, k)$:*

$$\sum_{k\in\mathcal{O}} \lambda_{jk}^{ih}(t) = \alpha_j^i(t), \quad \forall h \in \mathcal{O}, \ \forall i, j \in \mathcal{S}, \ i \neq j \tag{49}$$

$$\sum_{j\in\mathcal{S}} \lambda_{jk}^{ih}(t) = \beta_k^h(t), \quad \forall i \in \mathcal{S}, \ \forall h, k \in \mathcal{O}, \ h \neq k. \tag{50}$$

Suppose that the above system admits solution such that the matrix function $A(t) = [\lambda_{jk}^{ih}(t)]_{i,j\in\mathcal{S},k,h\in\mathcal{O}}$, with

$$\lambda_{ih}^{ih}(t) = -\sum_{(j,k)\in\mathcal{S}\times\mathcal{O},(j,k)\neq(i,h)} \lambda_{jk}^{ih}(t), \tag{51}$$

properly defines an infinitesimal generator function of a Markov chain with values in $\mathcal{S} \times \mathcal{O}$. Consider, a bivariate Markov chain $Z := (Z_1, Z_2)$ on $\mathcal{S} \times \mathcal{O}$ with generator function $A^Z(t) = A(t)$. Then, the components Z_1 and Z_2 are Markov chains with respect to to their own filtrations, and their generators are $A^{Z_1}(t) = A^X(t)$ and $A^{Z_2}(t) = A^Y(t)$.

Note that, typically, system (49)–(50) contains many more unknowns than equations. In fact, given that cardinalities of $\mathcal{S}$ and $\mathcal{O}$ are $K_{\mathcal{S}}$ and $K_{\mathcal{O}}$, respectively, the system consists of $K_{\mathcal{S}}(K_{\mathcal{S}} - 1) + K_{\mathcal{O}}(K_{\mathcal{O}} - 1)$ equations in $K_{\mathcal{S}}K_{\mathcal{O}}(K_{\mathcal{S}}K_{\mathcal{O}} - 1)$ unknowns.

Thus, in principle, one can create several bivariate Markov chains Z with the given margins X and Y. Thus, indeed, the system (49)–(50) essentially serves as a "copula" between the Markovian margins X, Y and the bivariate Markov chain Z. This observation leads to the following definition,

Definition 5.1. A Markov copula between the Markov chains X and Y is any solution to system (49)–(50) such that the matrix function

$A(t) = [\lambda_{jk}^{ih}(t)]_{i,j\in\mathcal{S},k,h\in\mathcal{O}}$, with $\lambda_{ih}^{ih}(t)$ given in (51), properly defines an infinitesimal generator function of a Markov chain with values in $\mathcal{S} \times \mathcal{O}$.

Different Markov copulae will entail different dependence structure between the margins X and Y. This relates to our third question, that is (Q3) above, which we shall discuss elsewhere.

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