# VALUATION AND HEDGING OF CONTRACTS WITH FUNDING COSTS AND COLLATERALIZATION

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**Abstract**

The research presented in this work is motivated by recent papers by Brigo et al. [5, 6], Burgard and Kjaer [7, 8, 10], Crépey [14, 15], Fujii and Takahashi [21], Piterbarg [38] and Pallavicini et al. [37]. Our goal is to provide a sound theoretical underpinning for some results presented in these papers by developing a unified framework for the non-linear approach to hedging and pricing of OTC financial contracts. In particular, the impact that various funding bases and margin covenants exert on the values and hedging strategies for OTC contracts is examined.

**Keywords**: non-linear valuation and hedging, arbitrage, funding bases, collateral, BSDE

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# 1 Introduction

The research presented in this work is motivated by recent papers by Brigo et al. [5, 6], Burgard and Kjaer [7, 8, 10], Crépey [14, 15], Fujii and Takahashi [21], Piterbarg [38] and Pallavicini et al. [37]. Our goal is to provide a sound theoretical underpinning for some results presented in these papers by developing a unified framework for the non-linear approach to hedging and pricing of OTC financial contracts. Below, we briefly summarize the major aspects of the shift that contract valuation and hedging aspects financial markets underwent in the recent years. All the above works, as well as the present paper, embarked on the task to study both the theoretical and practical issues of valuation and hedging of financial contracts in the present market reality.

Let us consider a bilateral financial contract with cumulative cash flow process, seen from the perspective of one of the two parties, referred to as the hedger, denoted as $A$. The classical approach to hedging the position in such contract hinged on creation of a self-financing trading strategy, say $\varphi = (\xi^1, \dots, \xi^d, \psi^0)$, with the corresponding wealth process

$$V_t(\varphi) = \sum_{i=1}^{d} \xi^i_t S^i_t + \psi^0_t B^0_t$$

where $S^1, \dots, S^d$ are some relevant traded assets. In particular, this meant that all trades of the hedger were fully funded by the same money market account $B^0$.

Furthermore, it was assumed that both parties had access to the same traded risky assets, money market account, and market information. Since, in such a classical setup, the discounted cash flows were symmetric from the perspective of two parties in the contract, meaning that the discounted cash flows, as seen from the perspective of one party, were the negative of the discounted cash flows as seen from the perspective of the other party, the hedging and pricing exercise was symmetric in an analogous way.

Alas, things are not vanilla anymore. In particular:

- contracts now tend to be collateralized,
- parties may need to account for different funding rates,
- counterparty and systemic risks need to be accounted for,
- netting of portfolio positions becomes an important issue.

Consequently, a hedging portfolio will now refer to multiple funding accounts, denoted hereafter as $B^1, \dots, B^d$. Furthermore, the discounted cash flows (and thus also prices) will typically be asymmetric relative to the parties in the contract, since their funding sources are no longer assumed to be identical. Accordingly, the current way to hedge the position in a financial contract would be to create a trading strategy, say $\varphi = (\xi, \psi) = (\xi^1, \dots, \xi^d, \psi^0, \dots, \psi^d)$ composed of risky securities $S^i$, $i = 1, 2, \dots, d$, the cash account $B^0$ used for unsecured lending/borrowing, and funding accounts $B^i$, $i = 1, 2, \dots, d$, used for (unsecured or secured) funding of the $i$th asset, with the corresponding wealth process

$$V_t(\varphi) := \sum_{i=1}^{d} \xi^i_t S^i_t + \sum_{j=0}^{d} \psi^j_t B^j_t. \tag{1.1}$$

In fact, a trading strategy represented by (1.1) is merely a special case of more general portfolios examined in this paper. In particular, the hedger needs also to account for various netting possibilities of short/long positions in the assets comprising the hedging portfolio. Hence the classical form of the *self-financing* condition no longer holds and thus one needs to analyze a suitably modified version of this condition. Moreover, the collateral posted by a party as part of the cash flows of the contract, may depend on the hedging strategy employed by this party directly, or indirectly – through the wealth process of the hedging portfolio. This feature, in particular, makes the contract cash flows asymmetric relative to the two parties in the contract; it also makes valuation and hedging problem implicit and non-linear. For an extensive discussion and study of this aspect of valuation and hedging in the context of valuation and hedging of counterparty risk, the interested reader is referred to Bielecki et al. [3] and Crépey et al. [16].

In view of the above mentioned complexities, the problem of marking to market and hedging a financial contract in the current market environment is no longer as straightforward as it was the case in the past. Yet, as we shall argue, both the classic and novel approaches are rooted in the same principles of self-financing trading and no arbitrage, appropriately adapted to ways in which cash flows are now modified and the ways in which hedging portfolios are now formed. The aim of this work is thus to provide a framework for a systematic analysis of the presence/absence of arbitrage, as well as a systematic analysis of hedging and valuation with regard to OTC contracts. Note that that the collateral process may also be present in the security cash flows even if counterparty credit risk is not explicitly accounted for.

The goals of this paper are:

• to provide a blueprint for derivation of dynamics of the wealth process corresponding to self-financing trading strategy and to examine such dynamics under various trading covenants,
• to introduce and discuss the relevant concepts of arbitrage and no-arbitrage valuation,
• to highlight the so-called additive martingale property (see Remark 2.7) and its role in non-linear and implicit pricing via BSDEs,
• to examine how our abstract model-free framework relates to some previous works, specifically, the paper by Piterbarg [38].

For related research, the interested reader may also consult Bianchetti [2], Brigo et al. [4, 5, 6], Burgard and Kjaer [7, 8, 9, 10], Castagna [12, 13], Crépey [14, 15], Fujii and Takahashi [21], Fujii et al. [22], Henrard [23], Hull and White [24, 25], Kenyon [26, 27], Kijima et al. [28] and Mercurio [30, 31, 32] and Pallavicini et al. [37].

The paper is organized as follows. In Section 2, we start by introducing a generic market model with several risky assets and multiple funding accounts. We then derive alternative representations for the dynamics of the wealth process of a self-financing trading strategy for a given process representing all cash flows of a contract. We first solve this problem in the basic model and subsequently extend to more advanced models with various forms of netting. In Section 3, we introduce the concept of an arbitrage-free model, by proposing an essential extension of the classic definition, and we provide sufficient conditions for the no-arbitrage property of a market model under alternative assumptions about trading and netting. Surprisingly, this crucial issue was completely neglected in most existing papers that dealt with funding costs and collateralization. The authors focused instead on the 'risk-neutral valuation' under a vaguely specified martingale measure, which was assumed a priori to exist. By contrast, we propose a precise definition of the hedger's price via either replication or a suitable form of super-hedging. Moreover, we show that the problem of arbitrage under funding costs is non-trivial, but it can indeed be dealt with using a judicious definition of arbitrage opportunities and a specific form of a martingale measure.

As was already mentioned, collateralization of contracts became a widespread market practice. For this reason, we examine in Section 4 various conventions regarding margin account and we study the impact of collateralization on the dynamics of the hedger's portfolio. In our stylized approach to costs of margining, we consider both the case of segregated margin accounts and the case of rehypothecation. Moreover, we acknowledge that collateral posted or received is either in the form of cash or shares of a risky asset.

In Section 5, we deal with the fair pricing under funding costs and collateralization first in an abstract setup and then for a generic diffusion-type model. Let us stress that the pricing functional for the hedger will be typically non-linear, since hedging strategies are typically non-additive when a collection of contracts, rather than single deal, is studied. For instance, in the case of a market model with partial netting, the pricing problem can be represented in terms of a non-linear BSDE, which is shown to admit a unique solution under mild assumptions on the underlying model. For further results in this vein, the interested reader is referred to Nie and Rutkowski [33, 34, 35, 36]. To put our framework into perspective, we also analyze the valuation method for collateralized contracts proposed by Piterbarg [38] and we show that our approach covers as a special case the pricing results established in [38].

## 2 Trading under Funding Costs

Let us first introduce the notation for market models considered in this work.

**Probability space.** Throughout the paper, we fix a finite trading horizon $T$ for our model of the financial market. All processes introduced in what follows are implicitly assumed to be $\mathbb{G}$-adapted and defined on the underlying probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ models the flow of information available to all traders (in particular, any semimartingale is assumed to be càdlàg). For convenience, we assume that the initial $\sigma$-field $\mathcal{G}_0$ is trivial, although this assumption can be easily relaxed.

**Risky assets.** We denote by $S^i$ the *ex-dividend price* (or simply the *price*) of the $i$th risky asset with the *cumulative dividend stream* after time 0 represented by the process $A^i$. Note that we do not postulate that processes $S^i$, $i = 1, 2, \dots, d$ are positive. Hence by the *long cash position* (resp. *short cash position*), we mean the situation when $\xi^i_t S^i_t \le 0$ (resp. $\xi^i_t S^i_t \ge 0$) where $\xi^i_t$ is the number of hedger's positions in asset $S^i$ at time $t$.

**Funding accounts.** The *cash account* $B^0 = B$ is used for unsecured lending or borrowing of cash. In the case when the borrowing and lending cash rates are different, we use symbols $B^l$ (resp. $B^b$) to denote the process modeling the unsecured *lending* (resp. *borrowing*) cash account. Notation $B^i$ stands for the *funding account*, which may represent either unsecured or secured funding for the $i$th risky asset. A similar convention applies to this account: in case when borrowing/lending rates differ, we use symbols $B^{i,l}$ and $B^{i,b}$ to denote lending/borrowing accounts associated with the $i$th risky asset. As a general rule, we will assume the position of the hedger. Hence the superscripts $l$ (resp. $b$) will refer to rates applied to deposits (resp. loans) from the viewpoint of the hedger. Observe that funding accounts are sometimes referred to as *non-risky assets.* Unless explicitly stated otherwise, we assume that $B^l = B^b = B$ and $B^{i,l} = B^{i,b} = B^i$ for all $i$.

A more detailed mathematical and financial interpretation of funding accounts will be presented in what follows. Let us only mention here that $S^i$ is aimed to represent the price of any traded security, such as, stock, stock option, interest rates swap, currency option, cross-currency swap, CDS, CDO, etc. In essence, the rate $r^{i,l}$ (resp. $r^{i,b}$) corresponding to the *lending* (resp. *borrowing*) account $B^{i,l}$ (resp. $B^{i,b}$) represents the incremental cost of maintaining the long cash position (resp. short cash position) in asset $S^i$ (for a more precise interpretation of this statement, see Remark 2.7). Hence the actual interpretation of 'borrowing' and 'lending' accounts $B^{i,l}$ and $B^{i,b}$ will depend on a contract at hand and the relevant features of financial environment. In particular, the rates denoted here as $r^{i,l}$ and $r^{i,b}$ may in turn depend on multiple yield curves in several economies and/or other funding arrangements of a particular party (for instance, the hedger's internal funding costs).

**Assumption 2.1** It is assumed throughout that the price processes of *primary assets* satisfy:
(i) ex-dividend prices $S^i$ for $i = 1, 2, \dots, d$ are semimartingales,
(ii) cumulative dividend streams $A^i$ for $i = 1, 2, \dots, d$ are processes of finite variation with $A^i_0 = 0$,
(iii) funding accounts $B^j$ for $j = 0, 1, \dots, d$ are strictly positive and continuous processes of finite variation with $B^j_0 = 1$.

**Definition 2.1** The *cumulative dividend price* $S^{i,\text{cld}}$ is given as

$$S^{i,\text{cld}}_t := S^i_t + B^i_t \int_{(0,t]} (B^i_u)^{-1}\, dA^i_u, \quad t \in [0, T], \tag{2.1}$$

and thus the *discounted cumulative dividend price* $\widehat{S}^{i,\text{cld}} := (B^i)^{-1} S^{i,\text{cld}}$ satisfies

$$\widehat{S}^{i,\text{cld}}_t = \widehat{S}^i_t + \int_{(0,t]} (B^i_u)^{-1}\, dA^i_u, \quad t \in [0, T], \tag{2.2}$$

where we denote $\widehat{S}^i := (B^i)^{-1} S^i$.

If the $i$th traded asset does not pay any dividend up to time $T$, then the equality $S^{i,\text{cld}}_t = S^i_t$ holds for every $t \in [0, T]$. Note that the process $S^{i,\text{cld}}$ (hence also the process $\widehat{S}^{i,\text{cld}}$) is càdlàg.

**Remark 2.1** Note that formula (2.1) hinges on an implicit assumption that positive (resp. negative) dividends from the $i$th asset are invested in (resp. funded from) the $i$th funding account $B^i$. Since the main valuation and hedging results for derivative securities obtained in this section are represented in terms of primitive processes $S^i$, $B^i$ and $A^i$, rather than $S^{i,\mathrm{cld}}$, the choice of a particular convention regarding reinvestment of dividends associated with the $i$th risky asset is in fact immaterial. The implicit choice made in equation (2.1) was motivated by the mathematical convenience only.

**Remark 2.2** We adopt the following notational conventions:
(i) for any random variable $\chi$, the equality $\chi = \chi^+ - \chi^-$ is the unique decomposition of $\chi$ into its positive and negative parts,
(ii) for any stochastic process $A$ of finite variation, the equality $A = A^+ - A^-$ represents the unique decomposition of $A$ where $A^+$ and $A^-$ are increasing processes with $A_0 = A_0^+ - A_0^-$.

## 2.1 Contracts and Trading Strategies

We are in a position to introduce trading strategies based on a finite family of primary assets satisfying Assumption 2.1. In Sections 2.1 and 2.2, we consider a dynamic portfolio denoted as $\varphi = (\xi, \psi) = (\xi^1, \dots, \xi^d, \psi^0, \dots, \psi^d)$, which is composed of risky securities $S^i$, $i = 1, 2, \dots, d$, the cash account $B$ used for unsecured lending/borrowing, and funding accounts $B^i$, $i = 1, 2, \dots, d$, used for (either unsecured or secured) funding of the $i$th asset. Let us first formally define the class of contracts under our consideration.

**Definition 2.2** By a *bilateral financial contract*, or simply a *contract*, we mean an arbitrary càdlàg process $A$ of finite variation. The process $A$ is aimed to represent the *cumulative cash flows* of a given contract from time 0 till its maturity date $T$. By convention, we set $A_{0-} = 0$.

The process $A$ is assumed to model all cash flows of a given contract, which are either paid out from the wealth or added to the wealth, as seen from the perspective of the *hedger* (recall that the other party is referred to as the *counterparty*). Note that the process $A$ includes the initial cash flow $A_0$ of a contract at its inception date $t_0 = 0$. For instance, if a contract has the initial *price* $p$ and stipulates that the hedger will receive cash flows $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_k$ at future dates $t_1, t_2, \dots, t_k \in (0, T]$, then we set $A_0 = p$ so that

$$A_t = p + \sum_{l=1}^{k} \mathbb{1}_{[t_l, T]}(t) \bar{A}_l. \tag{2.3}$$

If a unique future cash flow associated with a contract is the terminal payment at time $T$, which is denoted as $X$, then the process $A$ for this security takes form $A_t = p\,\mathbb{1}_{[0,T]}(t) + X\mathbb{1}_{[T]}(t)$. For instance, if the hedger sells at time 0 a European call option on the risky asset $S^i$, then the terminal payoff equals $X = -(S_T^i - K)^+$ and thus $A_t = p\,\mathbb{1}_{[0,T]}(t) - (S_T^i - K)^+\mathbb{1}_{[T]}(t)$. The symbol $p$ is frequently used to emphasize that all future cash flows $\bar{A}_l$ for $l = 1, 2, \dots, k$ are explicitly specified by the contract's covenants, but the initial cash flow $A_0$ is yet to be formally defined and evaluated. Valuation of a contract $A$ means, in particular, searching for the range of *fair values* $p$ at time 0 from the viewpoint of either the hedger or the counterparty. Although the valuation paradigm will be the same for the two parties, due either to the asymmetry in their trading costs and opportunities, or the non-linearity of the wealth dynamics, they will typically obtain different sets of fair prices for $A$.

By a *trading strategy* associated with a contract $A$, we mean the triplet $(x, \varphi, A)$. The wealth process $V(x, \varphi, A)$ of a trading strategy depends on the *initial endowment* $x$ of the hedger, represented by an arbitrary real number $x$, his *hedging portfolio* $\varphi$ and *contractual cash flows* $A$. Note that by the hedger's initial endowment, we mean his exogenously given wealth before the initial price $p$ was received or paid by him at time 0. This means that $V_0(x, \varphi, 0) = x$, whereas for a given contract $A$, the initial wealth of the hedger's strategy at time 0 equals $V_0(x, \varphi, A) = x + A_0 = x + p$. Before stating the definition of a self-financing trading strategy, we formulate the standing assumption regarding the integrability of stochastic processes.

**Assumption 2.2** We assume that $\xi^i$ for $i = 1, 2, \dots, d$ (resp. $\psi^j$ for $j = 0, 1, \dots, d$) are arbitrary $\mathbb{G}$-predictable (resp. $\mathbb{G}$-adapted) processes such that the stochastic integrals used in what follows are well defined.

**Definition 2.3** For the hedger's initial endowment $x$, we say that a trading strategy $(x, \varphi, A)$, associated with a contract $A$, is *self-financing* whenever the wealth process $V(x, \varphi, A)$, which is given by the formula

$$V_t(x, \varphi, A) = \sum_{i=1}^{d} \xi_t^i S_t^i + \sum_{j=0}^{d} \psi_t^j B_t^j, \tag{2.4}$$

satisfies, for every $t \in [0, T]$,

$$V_t(x, \varphi, A) = x + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \, d(S_u^i + A_u^i) + \sum_{j=0}^{d} \int_0^t \psi_u^j \, dB_u^j + A_t. \tag{2.5}$$

It is important to stress that a hedging portfolio $\varphi$ and contractual cash flows $A$ cannot be dealt with separately since, in general, the wealth dynamics is obtained by a non-linear superposition of $\varphi$ and $A$. A possibility of separation of cash flows of $A$ and a hedging portfolio $\varphi$ for $A$ is, of course, a well known (and very handy) feature of the classic (that is, linear) arbitrage pricing theory, which in turn results in price additivity in a frictionless market model.

**Remark 2.3** Obviously, the wealth process always depends on the initial endowment $x$, a portfolio $\varphi$ and contractual cash flows $A$, so that the notation $V(x, \varphi, A)$ is adequate. However, for the sake of brevity, the shorthand notation $V(\varphi, A)$ (or even $V(\varphi)$) will sometimes be used in the remaining part of Section 2 if there is no danger of confusion.

**Remark 2.4** Formula (2.5) yields the following wealth decomposition

$$V_t(x, \varphi, A) = x + G_t(x, \varphi, A) + F_t(x, \varphi, A) + A_t \tag{2.6}$$

where

$$G_t(x, \varphi, A) := \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \, (dS_u^i + dA_u^i) \tag{2.7}$$

represents the gains/losses associated with holding long/short positions in risky assets $S^1, S^2, \dots, S^d$ and

$$F_t(x, \varphi, A) := \sum_{j=0}^{d} \int_0^t \psi_u^j \, dB_u^j \tag{2.8}$$

represents the portfolio's *funding costs*. This additive decomposition of the wealth process will no longer hold when more constraints will be imposed on trading.

**Remark 2.5** In some related papers (see, for instance, [38]), the process $\gamma$, which is given by, for all $t \in [0, T]$,

$$\gamma_t = x + F_t(x, \varphi, A) + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \, dA_u^i + A_t,$$

is referred to as the *cash* process financing the portfolio $\varphi$. In this context, it is important to stress that the equality

$$V_t(x, \varphi, A) = \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \, dS_u^i + \gamma_t,$$

holds but, in general, we have that $V_t(x, \varphi, A) \neq \sum_{i=1}^{d} \xi_t^i S_t^i + \gamma_t$.

## 2.2 Basic Model with Funding Costs

Let us first describe a preliminary setting, which henceforth will be referred to as the basic model with funding costs or, simply, the *basic model.*

**Definition 2.4** By the *basic model with funding costs,* we mean a market model in which the lending and borrowing accounts coincide, so that $B = B^l = B^b$, the equalities $B^i = B^{i,l} = B^{i,b}$ hold for all $i = 1, 2, \ldots, d$, and trading in funding accounts $B^i$ and risky assets $S^i$ is a priori unconstrained.

A thorough analysis of the basic model is merely a first step towards more realistic models with various trading and/or funding constraints. We will show that explicit formulae for the wealth dynamics under various constraints can be derived from results for the basic model by progressively refining the computations involving the wealth process and funding costs. For reasons that will be explained later, we are interested not only in dynamics of the wealth process, but also in dynamics of the *netted wealth*, as given by Definition 2.5. Let us only mention here that the concept of the netted wealth will be a convenient tool to examine the no-arbitrage features of a market model under funding costs and collateralization. To be a bit more specific, the concept of a martingale measure should now be applied to the discounted netted wealth, rather than to the discounted wealth of a trading strategy since the latter process includes the cash flows of $A$, whereas in the former case they are in some sense counterbalanced by the cash flows of $-A$.

**Definition 2.5** The *netted wealth* $V^{\text{net}}(x, \varphi, A)$ of a trading strategy $(x, \varphi, A)$ is given by the equality $V^{\text{net}}(x, \varphi, A) = V(x, \varphi, A) + V(0, \widetilde{\varphi}, -A)$ where $(0, \widetilde{\varphi}, -A)$ is the unique self-financing strategy such that $\xi^i_t = \psi^i_t = 0$ for every $i = 1, 2, \ldots, d$ and all $t \in [0, T]$.

We have the following lemma (for its extension to the case of different lending and borrowing accounts, see Lemma 3.1).

**Lemma 2.1** *If $B = B^l = B^b$ then the following equality holds, for all $t \in [0, T]$,*

$$V^{\text{net}}_t(x, \varphi, A) = V_t(x, \varphi, A) - B_t \int_{[0,t]} B_u^{-1} \, dA_u. \tag{2.9}$$

*Proof.* By setting $\xi^i_t = \psi^i_t = 0$ in (2.4) and (2.5) , we obtain $V_t(0, \widetilde{\varphi}, -A) = \widetilde{\psi}^0_t B_t$ and

$$V_t(0, \widetilde{\varphi}, -A) = \int_0^t \widetilde{\psi}^0_u \, dB_u - A_t.$$

Since $V_0(0, \widetilde{\varphi}, -A) = -A_0$, we obtain (2.9). □

Note that

$$V^{\text{net}}_0(x, \varphi, A) = V_0(x, \varphi, A) + V_0(0, \widetilde{\varphi}, -A) = x + A_0 - A_0 = x,$$

so that the initial netted wealth is independent of $A_0$. Nevertheless, the process $V^{\text{net}}(x, \varphi, A)$ may depend on $A_0$, in general, if the dynamics of the wealth processes $V(x, \varphi, A)$ and $V(0, \widetilde{\varphi}, -A)$ are non-linear. Intuitively, the netted wealth $V^{\text{net}}(x, \varphi, A)$ represents the wealth of the hedger, who takes the back-to-back long and short positions in $A$, uses a dynamic portfolio $\varphi$ with the initial endowment $x$ to hedge the long position, and leaves the short position *unhedged*, meaning that no investments in risky assets is undertaken to hedge the short position. In particular, the initial cash flows $A_0$ and $-A_0$ obviously cancel out, meaning that the initial price received from (or paid to) a counterparty in contract $A$ is immediately passed on to a counterparty in contract $-A$. Therefore, in the context of the computation of the netted wealth process, the value of $A_0$ should be immaterial for the hedger. This observation motivates us to make the following natural assumption, which ensures that the netted wealth is independent of $A_0$.

**Assumption 2.3** When computing the netted wealth process $V^{\text{net}}(x, \varphi, A)$, we set $A_0 = 0$.

With Assumption 2.3 in force, the representation (2.9) takes the form

$$V_t^{\mathrm{net}}(x,\varphi,A) = V_t(x,\varphi,A) - B_t \int_{(0,t]} B_u^{-1}\, dA_u. \tag{2.10}$$

In practice, the offset of cash flows at time 0 is only possible when the *market prices* of $A$ and $-A$ at time 0 satisfy $p_0^m(-A) = -p_0^m(A)$. Obviously, by the market price of $A$ (resp. $-A$), we mean here the initial price at time 0 of future cash flows of $A$ (resp. $-A$) on the time interval $(0,T]$. Then the financial interpretation of the netted wealth at time $T$ can be restated as follows: in order to assess a potential profitability of a given contract $A$ with respect to his market model, the hedger, who has the initial endowment $x$, enters at time 0 into a contract $A$ at its market price $p_0^m(A)$, and simultaneously takes a short position in the same contract at its market price $-p_0^m(A)$, so that the net cost of his two positions in the contract at time 0 is null. Subsequently, starting from his initial endowment $x$, he implements a dynamic hedging portfolio $\varphi$ for the long position and, concurrently, uses only the cash account (in general, the borrowing and lending accounts) to reinvest the incoming and outgoing cash flows associated with the short position. At terminal date $T$, the hedger aggregates the terminal wealth of a dynamically hedged long position in some contract with the outcome of the unhedged short position in the same contract. Independently of the level of the initial price of the contract, this gives him an indication whether entering into this contract could lead to an arbitrage opportunity for him. For the precise statement of this property and a detailed discussion, we refer to Section 3.1 (see, in particular, Definition 3.1).

### 2.2.1 A Preliminary Result

Let introduce the following notation, for $i = 1,2,\dots,d$,

$$K_t^i := \int_{(0,t]} B_u^i\, d\widehat{S}_u^i + A_t^i = \int_{(0,t]} B_u^i\, d\widehat{S}_u^{i,\mathrm{cld}} \tag{2.11}$$

where the second equality is an immediate consequence of (2.2), and

$$K_t^{\varphi} := \int_{(0,t]} B_u\, d\widetilde{V}_u(x,\varphi,A) - (A_t - A_0) = \int_{(0,t]} B_u\, d\widetilde{V}_u^{\mathrm{net}}(x,\varphi,A) \tag{2.12}$$

where we set $\widetilde{V}^{\mathrm{net}}(x,\varphi,A) := B^{-1}V^{\mathrm{net}}(x,\varphi,A)$ and $\widetilde{V}(x,\varphi,A) := B^{-1}V(x,\varphi,A)$, so that the second equality follows from (2.9). Obviously,

$$\widetilde{V}_t^{\mathrm{net}}(x,\varphi,A) = x + \int_{(0,t]} B_u^{-1}\, dK_u^{\varphi}. \tag{2.13}$$

The process $K^i$ is equal to the wealth, discounted by the funding account $B^i$, of a self-financing strategy that uses the risky security $S^i$ and the associated funding account $B^i$, where $B_t^i$ units of the cumulative dividend price of the $i$th asset are held at time $t$.

The following preliminary result is primarily tailored to cover the valuation and hedging of an *unsecured* contract. We thus mainly focus here on funding costs associated with trading in risky assets. We will argue later on that Proposition 2.1 is a convenient starting point to analyze a wide spectrum of practically appealing situations. To achieve our goals, it will be enough to impose later specific constraints on trading strategies, which will reflect particular market conditions faced by the hedger (such as: different lending, borrowing and funding rates) and/or additional covenants of an OTC contract under study (such as: a margin account, closeout payoffs, or benefits stemming from defaults). For a detailed study of trading strategies involving a *secured* (that is, *collateralized*) contract, we refer to Section 4.

**Proposition 2.1** *(i) For any self-financing strategy $\varphi$ we have that, for every $t \in [0,T]$,*

$$K_t^{\varphi} = \sum_{i=1}^d \int_{(0,t]} \xi_u^i\, dK_u^i + \sum_{i=1}^d \int_0^t (\psi_u^i B_u^i + \xi_u^i S_u^i)(\widetilde{B}_u^i)^{-1}\, d\widetilde{B}_u^i \tag{2.14}$$

*where we set* $\widetilde{B}^i := B^{-1}B^i$.
*(ii) The equality*

$$K_t^{\varphi} = \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \, dK_u^i, \quad t \in [0,T], \tag{2.15}$$

*holds if and only if*

$$\sum_{i=1}^{d} \int_0^t (\psi_u^i B_u^i + \xi_u^i S_u^i)(\widetilde{B}_u^i)^{-1} \, d\widetilde{B}_u^i = 0, \quad t \in [0,T]. \tag{2.16}$$

*(iii) In particular, if for each* $i = 1,2,\dots,d$ *we have that: either* $B_t^i = B_t$ *for all* $t \in [0,T]$ *or*

$$\zeta_t^i := \psi_t^i B_t^i + \xi_t^i S_t^i = 0, \quad t \in [0,T], \tag{2.17}$$

*then* (2.16) *is valid and thus* (2.15) *holds.*
*(iv) Assume that* $B^i = B$ *for every* $i = 1,2,\dots,d$ *and denote* $\widetilde{S}^{i,cld} = B^{-1}S^{i,cld}$. *Then*

$$d\widetilde{V}_t^{net}(x,\varphi,A) = \sum_{i=1}^{d} \xi_t^i \, d\widetilde{S}_t^{i,cld}. \tag{2.18}$$

*Proof.* Recall that (see (2.5))

$$dV_t(x,\varphi,A) = \sum_{i=1}^{d} \xi_t^i \, d(S_t^i + A_t^i) + \sum_{j=0}^{d} \psi_t^j \, dB_t^j + dA_t.$$

Using (2.4), for the discounted wealth $\widetilde{V}(\varphi,A) = B^{-1}V(\varphi,A)$ we obtain

$$\begin{aligned} d\widetilde{V}_t(x,\varphi,A) &= \sum_{i=1}^{d} \xi_t^i \, d((B_t)^{-1}S_t^i) + \sum_{i=1}^{d} \xi_t^i (B_t)^{-1} \, dA_t^i + \sum_{i=1}^{d} \psi_t^i \, d((B_t)^{-1}B_t^i) + (B_t)^{-1} \, dA_t \\ &= \sum_{i=1}^{d} \xi_t^i \, d\widetilde{S}_t^{i,\mathrm{cld}} + \sum_{i=1}^{d} \psi_t^i \, d\widetilde{B}_t^i + (B_t)^{-1} \, dA_t \end{aligned}$$

where $\widetilde{B}^i = B^{-1}B^i$ and

$$\widetilde{S}_t^{i,\mathrm{cld}} = S_t^i B_t^{-1} + \int_{(0,t]} B_u^{-1} \, dA_u^i = \widetilde{S}_t^i + \int_{(0,t]} B_u^{-1} \, dA_u^i.$$

Consequently,

$$\begin{aligned} dK_t^{\varphi} &= B_t \, d\widetilde{V}_t(x,\varphi,A) - dA_t = \sum_{i=1}^{d} B_t \xi_t^i \, d\widetilde{S}_t^{i,\mathrm{cld}} + \sum_{i=1}^{d} B_t \psi_t^i \, d\widetilde{B}_t^i \\ &= \sum_{i=1}^{d} B_t \xi_t^i \, d(\widehat{S}_t^i \widetilde{B}_t^i) + \sum_{i=1}^{d} B_t \xi_t^i \, B_t^{-1} dA_t^i + \sum_{i=1}^{d} B_t \psi_t^i \, d\widetilde{B}_t^i \\ &= \sum_{i=1}^{d} B_t \xi_t^i \widehat{S}_t^i \, d\widetilde{B}_t^i + \sum_{i=1}^{d} B_t \widetilde{B}_t^i \xi_t^i \, d\widehat{S}_t^i + \sum_{i=1}^{d} \xi_t^i \, dA_t^i + \sum_{i=1}^{d} B_t \psi_t^i \, d\widetilde{B}_t^i \\ &= \sum_{i=1}^{d} B_t \xi_t^i \widehat{S}_t^i \, d\widetilde{B}_t^i + \sum_{i=1}^{d} \xi_t^i \, (B_t^i \, d\widehat{S}_t^i + dA_t^i) + \sum_{i=1}^{d} B_t \psi_t^i \, d\widetilde{B}_t^i \\ &= \sum_{i=1}^{d} \xi_t^i \, dK_t^i + \sum_{i=1}^{d} B_t (\psi_t^i + \xi_t^i \widehat{S}_t^i) \, d\widetilde{B}_t^i. \end{aligned}$$

This completes the proof of part (i). Parts (ii) and (iii) now follow easily. By combining formulae (2.11) and (2.14), we obtain part (iv). Note that (2.18) is the classic condition for a market with a single cash account $B$. ☐

**Remark 2.6** Note that equality $B^i = B$ (resp. equality (2.17)) may correspond to unsecured (resp. secured) funding of the $i$th stock, where unsecured funding means that a risky security is not posted as collateral. In this financial interpretation, condition (2.17) would mean that at any date $t$ the value of the long or short position in the $i$th stock should be exactly offset by the value of the $i$th secured funding account. Although this condition is aimed to cover the case of the fully secured funding of the $i$th risky asset using the corresponding repo rate, it is fair to acknowledge that it is rather restrictive and thus not always practical. It would be suitable for repo contracts with the daily resettlement, but it would not cover the case of long term repo contracts.

Note also that if condition (2.17) holds for all $i = 1, 2, \dots, d$, then the wealth process satisfies $V_t(x, \varphi, A) = \psi^0_t B_t$ for every $t \in [0, T]$. This is consistent with the interpretation that all gains/losses are immediately reinvested in the cash account $B$. To make this setup more realistic, we need, in particular, to introduce different borrowing and lending rates and add more constraints on trading. More generally, the $i$th risky security can be funded in part using $B^i$ and using $B$ for another part, so that condition (2.17) may fail to hold. However, this case can also be covered by the model in which condition (2.17) is met by artificially splitting the $i$th asset into two 'sub-assets' that are subject to different funding rules. Needless to say that the valuation and hedging results for a derivative security will depend on the way in which risky assets used for hedging are funded.

### 2.2.2 Wealth Dynamics in the Basic Model

To obtain some useful representations for the wealth dynamics in the basic model, we first prove an auxiliary lemma. From equality (2.19), one can deduce that the increment $dK^i_t$ represents the change in the price of the $i$th asset net of funding cost. For the lack of the better terminology, we propose to call $K^i$ the *netted realized cash flow* of the $i$th asset.

**Lemma 2.2** *The following equalities hold, for all* $t \in [0, T]$,

$$K^i_t = S^i_t - S^i_0 + A^i_t - \int_0^t \widehat{S}^i_u \, dB^i_u \tag{2.19}$$

*and*

$$\begin{aligned} K^{\varphi}_t &= V_t(x, \varphi, A) - V_0(x, \varphi, A) - (A_t - A_0) - \int_0^t \widetilde{V}_u(x, \varphi, A) \, dB_u \\ &= F_t(x, \varphi, A) + G_t(x, \varphi, A) - \int_0^t \widetilde{V}_u(x, \varphi, A) \, dB_u. \end{aligned} \tag{2.20}$$

*Proof.* The Itô formula, (2.2) and (2.11) yield

$$\begin{aligned} \int_{(0,t]} B^i_u \, d\widehat{S}^{i,\mathrm{cld}}_u &= \int_{(0,t]} B^i_u \, d\widehat{S}^i_u + A^i_t = B^i_t \widehat{S}^i_t - B^i_0 \widehat{S}^i_0 - \int_0^t \widehat{S}^i_u \, dB^i_u + A^i_t \\ &= S^i_t - S^i_0 + A^i_t - \int_0^t \widehat{S}^i_u \, dB^i_u. \end{aligned} \tag{2.21}$$

The proof of the second equality is analogous. □

**Remark 2.7** For each $i$, the differential $K^i_t$ admits both "multiplicative" decomposition

$$K^i_t = \int_{(0,t]} B^i_u \, d\widehat{S}^{i,\mathrm{cld}}_u, \tag{2.22}$$

and "additive" decomposition

$$K^i_t = S^i_t - S^i_0 + A^i_t - \int_0^t \widehat{S}^i_u \, dB^i_u. \tag{2.23}$$

If under some probability measure, say $\widehat{\mathbb{P}}$, the process $\widehat{S}^{i,\mathrm{cld}}$ is a (local) martingale, then, in view of (2.22), process $K^i$ is also a (local) martingale under the same measure. We call this the multiplicative martingale property of $K^i$ (under $\widehat{\mathbb{P}}$). Because of (2.23), we also say that $K^i$ enjoys the additive martingale property (under $\widehat{\mathbb{P}}$). The financial meaning of this property is rather intuitive considering that that $dK^i_t = d(S^i_t + A^i_t) - \widehat{S}^i_t\, dB^i_t$ represents change in capital gains/losses of the $i$th asset, net of local holding income/holding cost of the asset. Likewise, note that if equality (2.15) is valid, then the process $K^\varphi$ is a (local) martingale. In view of (2.20), we call this additive martingale property of $K^\varphi$ (under $\widehat{\mathbb{P}}$). Again, the financial meaning of this property is fairly clear, since $dK^\varphi_t = dG_t(x,\varphi,A) + dF_t(x,\varphi,A) - \widetilde{V}_t(x,\varphi,A)\, dB_t$ represents total local change in gains/losses and funding costs, net of local wealth reinvestment income/wealth service charge.

In view of Lemma 2.2, the following corollary to Proposition 2.1 is immediate. Since the funding costs in the basic model may depend on funding accounts $B^0, B^1, \dots, B^d$, we emphasize this dependence by writing $F(\varphi) = F(\varphi; B^0, B^1, \dots, B^d)$. Recall that the processes $\zeta^i$, $i = 1, 2, \dots, d$ are given by (2.17).

**Corollary 2.1** *Formula* (2.14) *is equivalent to the following expressions*

$$d\widetilde{V}^{net}_t(x,\varphi,A) = \sum_{i=1}^d \xi^i_t \widetilde{B}^i_t\, d\widehat{S}^{i,cld}_t + \sum_{i=1}^d \zeta^i_t (B^i_t)^{-1}\, d\widetilde{B}^i_t, \tag{2.24}$$

$$d\widetilde{V}_t(x,\varphi,A) = \sum_{i=1}^d \xi^i_t \widetilde{B}^i_t\, d\widehat{S}^{i,cld}_t + \sum_{i=1}^d \zeta^i_t (B^i_t)^{-1}\, d\widetilde{B}^i_t + (B_t)^{-1}\, dA_t, \tag{2.25}$$

$$dV_t(x,\varphi,A) = \widetilde{V}_t(x,\varphi,A)\, dB_t + \sum_{i=1}^d \xi^i_t\, dK^i_t + \sum_{i=1}^d \zeta^i_t (\widetilde{B}^i_t)^{-1}\, d\widetilde{B}^i_t + dA_t. \tag{2.26}$$

*Hence the funding costs of $\varphi$ satisfy*

$$F_t(\varphi; B^0, B^1, \dots, B^d) = \int_0^t \widetilde{V}_u(x,\varphi,A)\, dB_u + \sum_{i=1}^d \int_0^t \zeta^i_u (\widetilde{B}^i_u)^{-1}\, d\widetilde{B}^i_u - \sum_{i=1}^d \int_0^t \xi^i_u \widehat{S}^i_u\, dB^i_u. \tag{2.27}$$

**Remark 2.8** Formula (2.24) may suggest that in the basic model with funding costs the dynamics of the process $V^{\mathrm{net}}(x,\varphi,A)$ do not depend on $A$. To this end, one could argue as follows: suppose that we take any processes $\xi = (\xi^1, \dots, \xi^d)$ and $\psi = (\psi^1, \dots, \psi^d)$. Then, under the present assumptions, for any two external cash flows, say $A$ and $\widehat{A}$, we may compute the unique wealth processes $V(x,\varphi,A)$ and $V(x,\widehat{\varphi},\widehat{A})$ from (2.25) and, consequently, using (2.4), also the unique processes $\psi^0$ and $\widehat{\psi}^0$ such that the full strategies $\varphi$ and $\widehat{\varphi}$ are self-financing. Then the wealth processes $V(x,\varphi,A)$ and $V(x,\widehat{\varphi},\widehat{A})$ will be manifestly different, but from (2.24) we see that the netted wealth processes $V^{\mathrm{net}}(x,\varphi,A)$ and $V^{\mathrm{net}}(x,\widehat{\varphi},\widehat{A})$ coincide and thus they do not depend on $A$. This argument is in fact flawed since, typically, the processes $\xi = (\xi^1, \dots, \xi^d)$ and $\psi = (\psi^1, \dots, \psi^d)$ may also depend on future cash flows of $A$. This feature is rather obvious when one addresses the issue of replication of a contract formally represented by the process $A$.

To illustrate this remark, let us consider a toy model with the cash account $B$ and one risky asset, namely, the unit discount bond maturing at $T$ with the price process $S^1_t = B(t,T)$. Let $0 < t_0 < T$ and let $\eta$ be a positive $\mathcal{G}_{t_0}$-measurable random variable. Recall that when dealing with the netted wealth, we may and do assume that $A_0 = 0$. We set $A_t = \eta B(t_0,T)\mathbb{1}_{[t_0,T]} - \eta \mathbb{1}_{[T]}$ and we consider the portfolio $\varphi = (\xi^1, \psi^0)$ where $\xi^1_t = \eta \mathbb{1}_{[t_0,T]}(t)$ and $\psi^0_t = 0$ for all $t$, meaning that at time $t_0$ the incoming cash flow $\eta$ is invested in the discount bond. If we assume that $x = 0$, then the wealth process $V(0,\varphi,A)$ satisfies $V_t(0,\varphi,A) = \eta B(t,T)\mathbb{1}_{[t_0,T[}(t)$ so that, in particular, $V_T(0,\varphi,A) = 0$. By contrast, equation (2.9) yields

$$V^{\mathrm{net}}_T(0,\varphi,A) = V_T(0,\varphi,A) - B_T \int_{(0,T]} B_t^{-1}\, dA_t = \eta\Big(1 - B(t_0,T)\,\frac{B_T}{B_{t_0}}\Big) \tag{2.28}$$

and thus the netted wealth manifestly depends on $\eta$, that is, on the contract $A$. Note that by modifying $\xi^1_{t_0}$ so that $\psi^0_{t_0} \neq 0$, one can easily produce an example in which $V_T(0,\varphi,A)$ is non-zero and it also depends on $A$. The advantage of the netted wealth lies in the fact that it is suitable when one wishes to identify arbitrage opportunities. For instance, from (2.28), we may deduce that an arbitrage opportunity arises for the hedger if the inequality $B(t_0,T) < \frac{B_{t_0}}{B_T}$ holds $\mathbb{P}$-a.s., that is, when the bond price is too low with respect to the cash account. Needless to say that this conclusion is trivial and it can be obtained without difficulty through other means. Our goal in this example was simply to illustrate the potential of the netted wealth as a handy tool, which can be also applied when non-linear constraints on trading are imposed, in particular, under netting or collateralization.

**Example 2.1** Suppose that the processes $B^j$, $j = 0,1,\dots,d$ are absolutely continuous, so that they can be represented as $dB^j_t = r^j_t B^j_t\,dt$ for some $\mathbb{G}$-adapted processes $r^j$, $j = 0,1,\dots,d$. Then (2.25) yields

$$dV_t(\varphi) = r_t V_t(\varphi)\,dt + \sum_{i=1}^d \zeta^i_t (r^i_t - r_t)\,dt + \sum_{i=1}^d \xi^i_t \big(dS^i_t - r^i_t S^i_t\,dt + dA^i_t\big) + dA_t \tag{2.29}$$

where, for brevity, we write $V_t(\varphi) = V_t(x,\varphi,A)$. Equation (2.29) yields

$$dV_t(\varphi) = \sum_{j=0}^d r^j_t \psi^j_t B^j_t\,dt + \sum_{i=1}^d \xi^i_t\big(dS^i_t + dA^i_t\big) + dA_t, \tag{2.30}$$

which can also be seen as an immediate consequence of (2.5). We note that the dynamics of funding costs of $\varphi$ are given by

$$dF_t(\varphi; B^0, B^1, \dots, B^d) = \sum_{j=0}^d r^j_t \psi^j_t B^j_t\,dt. \tag{2.31}$$

### 2.2.3 A Common Unsecured Account for Risky Assets

Let us analyze a special case of the basic model with a common unsecured account for risky assets. To this end, we assume that $B^i = B$ for $i = 1,2,\dots,k$ for some $k \le d$. This means that all unsecured accounts $B^1, B^2, \dots, B^k$ collapse in a single cash account, denoted as $B$, but the secured accounts $B^{k+1}, B^{k+2}, \dots, B^d$ driven by the repo rates may vary from one asset to another. Formally, it is now convenient to postulate that $\psi^i = 0$ for $i = 1,2,\dots,k$, so that a portfolio $\varphi$ may be represented as $\varphi = (\xi^1,\dots,\xi^d,\psi^0,\psi^{k+1},\dots,\psi^d)$. Hence formula (2.4) reduces to

$$V_t(\varphi) = \sum_{i=1}^d \xi^i_t S^i_t + \psi^0_t B_t + \sum_{i=k+1}^d \psi^i_t B^i_t$$

where we write $V_t(\varphi) = V_t(x,\varphi,A)$ and the self-financing condition (2.5) becomes

$$V_t(\varphi) = V_0(\varphi) + \sum_{i=1}^d \int_{(0,t]} \xi^i_u\,d(S^i_u + A^i_u) + \int_0^t \psi^0_u\,dB_u + \sum_{i=k+1}^d \int_0^t \psi^i_u\,dB^i_u + A_t.$$

Consequently, equality (2.25) takes the following form

$$dV_t(\varphi) = \widetilde{V}_t(\varphi)\,dB_t + \sum_{i=1}^k \xi^i_t B_t\,d\widetilde{S}^{i,\mathrm{cld}}_t + \sum_{i=k+1}^d \xi^i_t B^i_t\,d\widehat{S}^{i,\mathrm{cld}}_t + \sum_{i=k+1}^d \zeta^i_t (\widetilde{B}^i_t)^{-1}\,d\widetilde{B}^i_t + dA_t \tag{2.32}$$

where we denote

$$\widetilde{S}^{i,\mathrm{cld}}_t := \widetilde{S}^i_t + \int_{(0,t]} B_u^{-1}\,dA^i_u, \quad t \in [0,T],$$

where in turn $\widetilde{S}^i := B^{-1} S^i$.

**Example 2.2** If all accounts $B^j$, $j = 0, 1, \ldots, d$ are absolutely continuous so that, in particular, $r^i = r$ for $i = 1, 2, \ldots, k$, then

$$dV_t(\varphi) = \Big(r_t \psi^0_t B_t + \sum_{i=k+1}^{d} r^i_t \psi^i_t B^i_t\Big)\, dt + \sum_{i=1}^{d} \xi^i_t \big(dS^i_t + dA^i_t\big) + dA_t. \tag{2.33}$$

If, in addition, $\zeta^i_t = 0$ for $i = k+1, k+2, \ldots, d$, then $V_t(\varphi) = \sum_{i=1}^k \xi^i_t S^i_t + \psi^0_t B_t$ and (2.33) yields

$$dF_t(\varphi) = r_t \Big(V_t(\varphi) - \sum_{i=1}^{k} \xi^i_t S^i_t\Big)\, dt - \sum_{i=k+1}^{d} \xi^i_t r^i_t S^i_t \, dt.$$

## 2.3 Different Lending and Borrowing Cash Rates

In the first extension of the basic model, we assume that the unsecured borrowing and lending cash rates are different. Recall that $B^l$ and $B^b$ stand for the account processes corresponding to the lending and borrowing rates, respectively. This can be seen as a first example of a *generic market model*, in the sense explained in Section 3.1.1; further examples are given in the foregoing subsections.

It is now natural to represent a portfolio $\varphi$ as $\varphi = (\xi^1, \ldots, \xi^d, \psi^l, \psi^b, \psi^1, \ldots, \psi^d)$ where, by assumption, $\psi^l_t \geq 0$ and $\psi^b_t \leq 0$ for all $t \in [0, T]$. Since simultaneous lending and borrowing of cash is either precluded or not efficient (if $r^b \geq r^l$), we also postulate that $\psi^l_t \psi^b_t = 0$ for all $t \in [0, T]$. The wealth process of a trading strategy $(\varphi, A)$ now equals (recall that we denote $V_t(\varphi) = V_t(x, \varphi, A)$)

$$V_t(\varphi) = \sum_{i=1}^{d} \xi^i_t S^i_t + \sum_{i=1}^{d} \psi^i_t B^i_t + \psi^l_t B^l_t + \psi^b_t B^b_t, \tag{2.34}$$

and the self-financing condition reads

$$\begin{aligned} V_t(\varphi) = V_0(\varphi) &+ \sum_{i=1}^{d} \int_{(0,t]} \xi^i_u \, d(S^i_u + A^i_u) + \sum_{i=1}^{d} \int_0^t \psi^i_u \, dB^i_u \\ &+ \int_0^t \psi^l_u \, dB^l_u + \int_0^t \psi^b_u \, dB^b_u + A_t. \end{aligned} \tag{2.35}$$

It is worth noting that $\psi^l_t$ and $\psi^b_t$ satisfy

$$\psi^l_t = (B^l_t)^{-1} \Big(V_t(\varphi) - \sum_{i=1}^{d} \xi^i_t S^i_t - \sum_{i=1}^{d} \psi^i_t B^i_t\Big)^+$$

and

$$\psi^b_t = -(B^b_t)^{-1} \Big(V_t(\varphi) - \sum_{i=1}^{d} \xi^i_t S^i_t - \sum_{i=1}^{d} \psi^i_t B^i_t\Big)^-.$$

The following corollary furnishes the wealth dynamics under the present assumptions.

**Corollary 2.2** *(i) Assume that $B^l$ and $B^b$ are account processes corresponding to the lending and borrowing rates. Let $\varphi$ be any self-financing strategy such that $\psi^l_t \geq 0$, $\psi^b_t \leq 0$ and $\psi^l_t \psi^b_t = 0$ for all $t \in [0, T]$. Then the wealth process $V(\varphi)$, which is given by* (2.34)*, has the following dynamics*

$$\begin{aligned} dV_t(\varphi) = &\sum_{i=1}^{d} \xi^i_t B^i_t \, d\widehat{S}^{i,cld}_t + \sum_{i=1}^{d} \zeta^i_t (B^i_t)^{-1} \, dB^i_t + dA_t \\ &+ \Big(V_t(\varphi) - \sum_{i=1}^{d} \xi^i_t S^i_t - \sum_{i=1}^{d} \psi^i_t B^i_t\Big)^+ (B^l_t)^{-1} \, dB^l_t \\ &- \Big(V_t(\varphi) - \sum_{i=1}^{d} \xi^i_t S^i_t - \sum_{i=1}^{d} \psi^i_t B^i_t\Big)^- (B^b_t)^{-1} \, dB^b_t. \end{aligned} \tag{2.36}$$

*(ii) If, in addition, $\psi^i_t = 0$ for $i = 1, 2, \ldots, k$ and $\zeta^i_t = 0$ for $i = k+1, k+2, \ldots, d$ for all $t \in [0,T]$, then*

$$dV_t(\varphi) = \sum_{i=1}^{k} \xi^i_t \, d(S^i_t + A^i_t) + \sum_{i=k+1}^{d} \xi^i_t B^i_t \, d\widehat{S}^{i,cld}_t + dA_t \tag{2.37}$$
$$+ \Big(V_t(\varphi) - \sum_{i=1}^{k} \xi^i_t S^i_t\Big)^+ (B^l_t)^{-1} \, dB^l_t - \Big(V_t(\varphi) - \sum_{i=1}^{k} \xi^i_t S^i_t\Big)^- (B^b_t)^{-1} \, dB^b_t.$$

*Proof.* Formula (2.37) can be derived from (2.36), using also the following equality (see (2.21))

$$B^i_t \, d\widehat{S}^{i,\mathrm{cld}}_t = dS^i_t - \widehat{S}^i_t \, dB^i_t + dA^i_t.$$

The details are left to the reader. □

**Example 2.3** Under the assumptions of part (ii) in Corollary 2.2 if, in addition, the accounts $B^i$ for $i = k+1, k+2, \ldots, d$ as well as $B^l$ and $B^b$ are absolutely continuous, then (2.37) becomes

$$dV_t(\varphi) = \sum_{i=1}^{k} \xi^i_t \big(dS^i_t + dA^i_t\big) + \sum_{i=k+1}^{d} \xi^i_t \big(dS^i_t - r^i_t S^i_t \, dt + dA^i_t\big) + dA_t \tag{2.38}$$
$$+ r^l_t \Big(V_t(\varphi) - \sum_{i=1}^{k} \xi^i_t S^i_t\Big)^+ dt - r^b_t \Big(V_t(\varphi) - \sum_{i=1}^{k} \xi^i_t S^i_t\Big)^- dt$$

and thus the funding costs satisfy

$$dF_t(\varphi) = r^l_t \Big(V_t(\varphi) - \sum_{i=1}^{k} \xi^i_t S^i_t\Big)^+ dt - r^b_t \Big(V_t(\varphi) - \sum_{i=1}^{k} \xi^i_t S^i_t\Big)^- dt - \sum_{i=k+1}^{d} r^i_t \xi^i_t S^i_t \, dt.$$

In particular, by setting $k = 0$, we obtain

$$dV_t(\varphi) = \sum_{i=1}^{d} \xi^i_t \big(dS^i_t - r^i_t S^i_t \, dt + dA^i_t\big) + dA_t + r^l_t \big(V_t(\varphi)\big)^+ dt - r^b_t \big(V_t(\varphi)\big)^- dt. \tag{2.39}$$

## 2.4 Trading Strategies with Funding Costs and Netting

So far, long and short positions in funding accounts $B^j$, $j = 0, 1, \ldots, d$ were assumed to bear the same interest. This assumption will be now relaxed, so that in this section, besides postulating that $B^l \neq B^b$ we also postulate that $B^{i,l} \neq B^{i,b}$ $j = 0, 1, \ldots, d$. Accordingly, we consider trading portfolio $\varphi = (\xi^1, \ldots, \xi^d, \psi^l, \psi^b, \psi^{1,l}, \psi^{1,b}, \ldots, \psi^{d,l}, \psi^{d,b})$ whenever this is needed, and we define the corresponding wealth process as

$$V_t(x, \varphi, A) = \psi^l_t B^l_t + \psi^b_t B^b_t + \sum_{i=1}^{d} (\xi^i_t S^i_t + \psi^{i,l}_t B^{i,l}_t + \psi^{i,b}_t B^{i,b}_t). \tag{2.40}$$

Consequently, we will now deal with the extended framework in which the issue of aggregating long and short positions in risky assets becomes crucial. The concept of aggregation of long and short positions can be introduced at various levels of inclusiveness, from the total absence of offsetting and netting to the most encompassing case of netting of all positions, whenever this is possible.

Let us explain the offsetting/netting terminology adopted in this work. By *offsetting*, we mean the compensation of long and short positions either for a given risky asset or for the non-risky asset. This concept is irrelevant unless the borrowing and lending rates are different for at least one risky asset or for the cash account. By *netting*, we mean the aggregation of long or short cash positions across various risky assets, which share some funding accounts. Therefore, the possibility of netting becomes relevant when they exist some risky assets, say $S^i$ and $S^j$, for which at least one of the following equalities holds: $B^{i,b} = B^{j,b}$, $B^{i,b} = B^{j,l}$, $B^{i,l} = B^{j,b}$ or $B^{i,l} = B^{j,l}$. Needless to say that several variants of models with netting can be introduced and examined. To illustrate this concept, we will study here only one particular instance of a market model with netting (see Section 2.4.3).

For our further purposes, it will be enough to distinguish between the following cases:
(a) the complete absence of offsetting and netting of long/short positions,
(b) the offsetting of long/short positions for every risky asset, but no netting,
(c) the offsetting of long/short positions for every risky asset combined with some form of netting of long/short cash positions for all risky assets that are funded from common funding accounts.

### 2.4.1 Absence of Offsetting

To describe the case (a) of the total absence of offsetting and netting of long and short positions in all risky assets, one can postulate that for all $i = 1, 2, \dots, d$ and $t \in [0, T]$,

$$\xi_t^{i,b} S_t^i + \psi_t^{i,l} B_t^{i,l} = 0, \quad \xi_t^{i,l} S_t^i + \psi_t^{i,b} B_t^{i,b} = 0 \tag{2.41}$$

where $\xi_t^{i,b} S_t^i \leq 0$, $\xi_t^{i,l} S_t^i \geq 0$, so that $\psi_t^{i,l} \geq 0$ and $\psi_t^{i,b} \leq 0$ for all $t \in [0, T]$. In particular, even when the equality $\xi_t^{i,l} + \xi_t^{i,b} = 0$ holds for all $t$, meaning that the net position in the $i$th asset is null at any time, an incremental cost of holding open both positions may still arise, due to the spread between the rates implicit in accounts $B^{i,l}$ and $B^{i,b}$. It is clear that this case is very restrictive and not practically appealing and thus it will not be analyzed in what follows.

### 2.4.2 Offsetting of Positions in Risky Assets

Let us now examine the netting convention (b). For this purpose, we postulate that $V(\varphi) = V(x, \varphi, A)$ satisfies

$$V_t(\varphi) = \psi_t^l B_t^l + \psi_t^b B_t^b + \sum_{i=1}^d (\xi_t^i S_t^i + \psi_t^{i,l} B_t^{i,l} + \psi_t^{i,b} B_t^{i,b}) = \psi_t^l B_t^l + \psi_t^b B_t^b$$

where $\psi_t^{i,l} \geq 0$ and $\psi_t^{i,b} \leq 0$ for $t \in [0, T]$ and, for $i = 1, 2, \dots, d$ and $t \in [0, T]$,

$$\xi_t^i S_t^i + \psi_t^{i,l} B_t^{i,l} + \psi_t^{i,b} B_t^{i,b} = 0. \tag{2.42}$$

The present netting mechanism can be interpreted as follows: for the purpose of hedging, it would be pointless to hold simultaneously long and short positions in any asset $i$; it is enough to look at the net position in the $i$th asset. For example, if the hedger already holds the short position in some asset and the need to take the long position of the same size arises, it is natural to postulate that the short position is first closed.

Note also that condition (2.42) is fairly restrictive, since it prevents netting of short and long cash positions across all risky assets which share the same long and short funding accounts. By definition, the *long* (resp. *short*) *cash position* in the $i$th asset corresponds to the positive (resp. negative) sign of $\xi_t^i S_t^i$. Recall that we did not postulate that the prices processes $S^i$ of risky assets are non-negative. See also Remark 2.6 for general comments regarding condition (2.17), which also apply to condition (2.42).

Since a simultaneous lending and borrowing of cash from the funding account $i$ is precluded (or not efficient, if $r^{i,b} \geq r^{i,l}$), we also postulate that $\psi_t^{i,l} \psi_t^{i,b} = 0$ for all $t \in [0, T]$, and for $i = 0, 1, \dots, d$. This implies that

$$\psi_t^l = (B_t^l)^{-1} (V_t(\varphi))^+, \quad \psi_t^b = -(B_t^b)^{-1} (V_t(\varphi))^- \tag{2.43}$$

and, for every $i = 1, 2, \dots, d$,

$$\psi_t^{i,l} = (B_t^{i,l})^{-1} (\xi_t^i S_t^i)^-, \quad \psi_t^{i,b} = -(B_t^{i,b})^{-1} (\xi_t^i S_t^i)^+. \tag{2.44}$$

Note the essential difference between the present setup and the situation outlined in Section 2.4.1 where it was not postulated that the offsetting equality $\psi_t^{i,l}\psi_t^{i,b} = 0$ holds for all $t \in [0,T]$. The self-financing condition now reads

$$\begin{aligned} V_t(\varphi) = & V_0(\varphi) + \sum_{i=1}^d \int_{(0,t]} \xi_u^i \, d(S_u^i + A_u^i) + \int_0^t \psi_u^l \, dB_u^l + \int_0^t \psi_u^b \, dB_u^b \\ & + \sum_{i=0}^d \int_0^t \psi_u^{i,l} \, dB_u^{i,l} + \sum_{i=0}^d \int_0^t \psi_u^{i,b} \, dB_u^{i,b} + A_t \end{aligned} \tag{2.45}$$

and thus the following result is straightforward.

**Corollary 2.3** *Assume that $B^{i,l}$ and $B^{i,b}$ are account processes corresponding to the lending and borrowing rates. We postulate that $\psi_t^{i,l} \geq 0$, $\psi_t^{i,b} \leq 0$ and $\psi_t^{i,l}\psi_t^{i,b} = 0$ for all $i = 0,1,\dots,d$ and $t \in [0,T]$, and equality (2.42) holds for all $i = 1,2,\dots,d$. Then the wealth process $V(\varphi) = V(\varphi, A)$ equals $V_t(\varphi) = \psi_t^l B_t^l + \psi_t^b B_t^b$ for all $t \in [0,T]$ and the wealth dynamics are*

$$\begin{aligned} dV_t(\varphi) = & \sum_{i=1}^d \xi_t^i \, (dS_t^i + dA_t^i) + \sum_{i=1}^d (\xi_t^i S_t^i)^- (B_t^{i,l})^{-1} \, dB_t^{i,l} - \sum_{i=1}^d (\xi_t^i S_t^i)^+ (B_t^{i,b})^{-1} \, dB_t^{i,b} \\ & + (V_t(\varphi))^+ (B_t^l)^{-1} \, dB_t^l - (V_t(\varphi))^- (B_t^b)^{-1} \, dB_t^b + dA_t. \end{aligned} \tag{2.46}$$

**Remark 2.9** When the equality $B^{i,l} = B^{i,b} = B^i$ holds for all $i = 1,2,\dots,d$, then formula (2.46) can be seen as a special case of formula (2.36) with $\zeta_t^i = 0$ for all $i$ and $t \in [0,T]$ (see also dynamics (2.39)).

**Example 2.4** Under the assumptions of Corollary 2.3 if, in addition, the processes $B^{i,l}$ and $B^{i,b}$ for $i = 0,1,\dots,d$ are absolutely continuous, then (2.46) becomes (note that (2.47) extends (2.39))

$$\begin{aligned} dV_t(\varphi) = & \sum_{i=1}^k \xi_t^i \big(dS_t^i + dA_t^i\big) + \sum_{i=1}^d r_t^{i,l} (\xi_t^i S_t^i)^- \, dt - \sum_{i=1}^d r_t^{i,b} (\xi_t^i S_t^i)^+ dt \\ & + r_t^l (V_t(\varphi))^+ \, dt - r_t^b (V_t(\varphi))^- \, dt + dA_t \end{aligned} \tag{2.47}$$

and thus the funding costs satisfy

$$dF_t(\varphi) = r_t^l (V_t(\varphi))^+ \, dt - r_t^b (V_t(\varphi))^- \, dt + \sum_{i=1}^d r_t^{i,l} (\xi_t^i S_t^i)^- \, dt - \sum_{i=1}^d r_t^{i,b} (\xi_t^i S_t^i)^+ dt.$$

### 2.4.3 Model with Partial Netting

We will examine here a particular instance of netting convention (c). We now postulate that all short (but not long) cash positions in risky assets $S^1, S^2, \dots, S^d$ can be aggregated. To be more specific, all long cash positions in risky assets $S^i$ are assumed to be funded from their respective accounts $B^{i,b}$ only. We also postulate that all other incoming and outgoing cash flows, inclusive of the proceeds from short-selling of risky assets, are added to (or subtracted from) the value process of the hedger's portfolio. This convention is not necessarily satisfied in practice, since the proceeds from short-selling may be kept in the hedger's margin account with the lender of a risky asset, thus serving as a protection against non-delivery of the shares by the hedger. Hence the stylized model proposed here corresponds to the situation when the hedger is considered to be default-free with respect to his short positions in risky assets, which implies that the lender does not require a collateral. Since we deal here with the case of the netting of short positions across risky assets, the trading framework introduced in this subsection will be henceforth referred to as the *market model with partial netting.*

The present setup is formalized by postulating that the wealth process $V(\varphi) = V(x, \varphi, A)$ equals

$$V_t(\varphi) = \psi^l_t B^l_t + \psi^b_t B^b_t + \sum_{i=1}^d (\xi^i_t S^i_t + \psi^{i,b}_t B^{i,b}_t) \tag{2.48}$$

where, for every $i = 1, 2, \ldots, d$ and $t \in [0, T]$, the process $\psi^{i,b}_t$ satisfies

$$\psi^{i,b}_t = -(B^{i,b}_t)^{-1} (\xi^i_t S^i_t)^+ \leq 0. \tag{2.49}$$

Note that since in equation (2.48) we use the net position $\xi^i_t$, rather than $\xi^{i,l}_t$ and $\xi^{i,b}_t$, the offsetting of long and short positions in every risky asset $S^i$ is already implicit in this equation. From (2.48) and (2.49), we obtain

$$V_t(\varphi) = \psi^l_t B^l_t + \psi^b_t B^b_t - \sum_{i=1}^d (\xi^i_t S^i_t)^-.$$

Since, as usual, we postulate that $\psi^l_t \geq 0$ and $\psi^b_t \leq 0$, we obtain the following equalities

$$\psi^l_t = (B^l_t)^{-1} \Big( V_t(\varphi) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^+, \quad \psi^b_t = -(B^b_t)^{-1} \Big( V_t(\varphi) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^-. \tag{2.50}$$

Finally, the self-financing condition for the trading strategy $(x, \varphi, A)$ reads

$$\begin{aligned} V_t(\varphi) = V_0(\varphi) &+ \sum_{i=1}^d \int_{(0,t]} \xi^i_u \, d(S^i_u + A^i_u) + \sum_{i=1}^d \int_0^t \psi^{i,b}_u \, dB^{i,b}_u \\ &+ \int_0^t \psi^l_u \, dB^l_u + \int_0^t \psi^b_u \, dB^b_u + A_t. \end{aligned}$$

The following result gives the wealth dynamics in the present setup.

**Corollary 2.4** *Assume that $B^{i,l} = B^l$ for all $i = 1, 2, \ldots, d$ and $\psi^l_t \geq 0$ and $\psi^b_t \leq 0$ for all $t \in [0, T]$. Then, under assumptions* (2.48) *and* (2.49)*, the dynamics of $V(\varphi) = V(x, \varphi, A)$ are*

$$\begin{aligned} dV_t(\varphi) = &\sum_{i=1}^d \xi^i_t \, (dS^i_t + dA^i_t) - \sum_{i=1}^d (\xi^i_t S^i_t)^+ (B^{i,b}_t)^{-1} \, dB^{i,b}_t + dA_t \\ &+ \Big( V_t(\varphi) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^+ (B^l_t)^{-1} \, dB^l_t - \Big( V_t(\varphi) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^- (B^b_t)^{-1} \, dB^b_t. \end{aligned} \tag{2.51}$$

Note that even under an additional assumption that $B^{i,b} = B^b$ for all $i = 1, 2, \ldots, d$, expression (2.51) does not reduce to (2.36), since we work here under postulate (2.49), which explicitly states that a long cash position in the $i$th risky asset is funded exclusively from the account $B^{i,b}$.

**Example 2.5** Under the assumptions of Corollary 2.4 if, in addition, all account processes $B^{i,l}$ and $B^b$ are absolutely continuous, then (2.51) becomes

$$\begin{aligned} dV_t(\varphi) = &\sum_{i=1}^d \xi^i_t \big( dS^i_t + dA^i_t \big) - \sum_{i=1}^d r^{i,b}_t (\xi^i_t S^i_t)^+ \, dt + dA_t \\ &+ r^l_t \Big( V_t(\varphi) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^+ dt - r^b_t \Big( V_t(\varphi) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^- dt \end{aligned} \tag{2.52}$$

and thus the funding costs satisfy

$$dF_t(\varphi) = r^l_t \Big( V_t(\varphi) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^+ dt - r^b_t \Big( V_t(\varphi) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^- dt - \sum_{i=1}^d r^{i,b}_t (\xi^i_t S^i_t)^+ \, dt.$$

# 3 Pricing under Funding Costs

Our goal in the preceding section was to analyze the wealth dynamics for self-financing strategies under alternative assumptions about trading and netting. In the next step, we will provide sufficient conditions for the no-arbitrage property of a market model under various trading specifications. It is worth stressing that this issue is apparently overlooked in most papers dealing with funding costs and collateralization. Instead, most authors work under an ad hoc postulate of the existence of a very vaguely specified 'martingale measure' and they focus on the 'risk-neutral valuation' under this probability measure. Most likely, a 'martingale measure' in these papers should then be interpreted as a 'pricing' probability measure, which is obtained from the market data via a model's calibration, rather than a sound theoretical construct. The main contribution of the existing vast literature in this vein thus lies in a thorough analysis of market conventions regarding margin account and closeout payoff at default and numerical implementations of sophisticated models for risky assets and default times. By contrast, their authors show relatively little interest in searching for a sound theoretical underpinning of alternative computations of various funding and credit risk adjustments to the so-called 'clean' prices.

Obviously, this tentative approach to valuation adjustments hinges on mimicking the classic results for frictionless market models. However, due to peculiarities in the wealth dynamics under nowadays ubiquitous market frictions, the classic approach should be carefully reexamined, since its straightforward application is manifestly unjustified. To clarify this statement, we will now analyze the applicability of classic paradigms when dealing with trading under funding costs. Specifically, in Sections 3.1 and 3.2, respectively, we will address two different, albeit related, questions.

Our first question reads: given a market model and a contract $A$ with an exogenously specified market price, is it possible for the hedger to produce a risk-free profit by taking a long hedged position in $A$ and simultaneously assuming a short unhedged position in the same contract (note that the level of the market price of $A$ is not relevant for this problem)? If this is the case, the model is manifestly not viable for the hedger, since for any level of the market price for a contract $A$, he would be able to guarantee a risk-free profit for himself. Otherwise, we say that a model is *arbitrage-free for the hedger* with respect to a given contract $A$. Intuitively, the level of a model's viability rises when this desirable property holds for a sufficiently large class of contracts that encompasses $A$.

The second question is: assuming that the model is arbitrage-free for the hedger with respect to a contract $A$ (or some class of contracts that encompasses $A$), we would like to describe all possible levels of a hedger's price $p$, such that the hedger cannot make a risk-free profit by selling the contract at price $p$ and implementing a smart trading strategy $(\varphi, A)$? Any number $p$ satisfying this property is referred to as a *fair hedger's price* for a contract $A$.

We thus see that the first question deals with a possibility of making a risk-free profit by the hedger through taking back-to-back offsetting positions in a contract $A$ at an exogenously given market price $p$ for $A$ (and the market price $-p$ for $-A$), whereas the second problem addresses the situation when the hedger is an outright seller of a contract $A$ at price $p$. Let us observe that the issue how to quantify a 'risk-free profit' should be carefully analyzed as well, especially when the lending and borrowing rates differ. We will argue that thanks to a judicious specification of the netted wealth process, it is possible to give formal definitions that also enjoy plausible financial interpretations. It should be acknowledged, however, that we do not offer a satisfactory solutions to all problems arising in the context of a non-linear and asymmetric pricing, so several important issues are merely outlined.

## 3.1 Hedger's Arbitrage under Funding Costs

The arbitrage-free property of a model under funding costs is a non-trivial concept, even when no margin account (collateral) is involved. However, in some cases it can indeed be dealt with using a judicious description of an arbitrage opportunity and a suitably defined 'martingale measure'. Let us stress that the notion of a martingale measure in the present setup is far from obvious and indeed its definition will depend on adopted market conventions. Specifically, for each particular market convention, an astute choice of a definition is required in order to make this general concept useful for our purposes, namely, for verifying whether a given market model is arbitrage-free and for valuing OTC derivatives.

### 3.1.1 Generic Market Model

By a *generic market model*, we mean a general class of models encompassing, but not restricted to, all cases of trading arrangements considered in the preceding section. We only assume that the concept of the wealth process $V(x,\varphi,A)$ and the discounted wealth $\widehat{V}(x,\varphi,A)$ are well defined, where the choice of a discount factor is fairly arbitrary and thus it may depend on particular circumstances at hand. Hence all market models introduced in Section 2 should now be seen as particular instances of a generic market model.

Since, in principle, the lending and borrowing accounts, $B^l$ and $B^b$ may be different in a generic market model, the netted wealth is defined by the following natural extension of Definition 2.5. For the interpretation of the concept of the netted wealth, see Section 2.2.

**Definition 3.1** The *netted wealth* $V^{\rm net}(x,\varphi,A)$ of a trading strategy $(x,\varphi,A)$ is given by the equality $V^{\rm net}(x,\varphi,A)=V(x,\varphi,A)+V(0,\widetilde{\varphi},-A)$ where $(0,\widetilde{\varphi},-A)$ is the unique self-financing strategy satisfying the following conditions:
(i) $V_0(0,\widetilde{\varphi},-A)=-A_0$,
(ii) the equalities $\xi^i_t=\psi^i_t=0$ hold for every $i=1,2,\dots,d$ and all $t\in[0,T]$,
(iii) $\widetilde{\psi}^l_t\geq 0, \widetilde{\psi}^b_t\leq 0$ and $\widetilde{\psi}^l_t\widetilde{\psi}^b_t=0$ for all $t\in[0,T]$.

We note that

$$V^{\rm net}_0(x,\varphi,A)=V_0(x,\varphi,A)+V_0(0,\widetilde{\varphi},-A)=x+A_0-A_0=x$$

so that the initial netted wealth $V^{\rm net}(x,\varphi,A)$ is independent of $p$. In view of Assumption 2.3, we set $A_0=0$ when using the concept of the netted wealth. According to the financial interpretation, the initial cash flows $A_0$ and $-A_0$ cancel out if the market prices of $A$ and $-A$ satisfy $p^m_0(-A)=-p^m_0(A)$, so that this assumption is reasonable (albeit it reduces slightly the generality of our approach). The following result, which is an extension of Lemma 2.1, is also valid in a model in which some form of netting of positions in risky assets is postulated.

**Lemma 3.1** *The following equality holds, for all $t\in[0,T]$,*

$$V^{\rm net}_t(x,\varphi,A)=V_t(x,\varphi,A)+U_t(A) \tag{3.1}$$

*where the $\mathbb{G}$-adapted process of finite variation $U(A)$ is the unique solution to the following equation*

$$U_t(A)=\int_0^t (B^l_u)^{-1}(U_u(A))^+\,dB^l_u-\int_0^t (B^b_u)^{-1}(U_u(A))^-\,dB^b_u-A_t. \tag{3.2}$$

*Proof.* We set $\xi^i_t=\psi^i_t=0$ in (2.34) and (2.35). Then the process $V_t:=V_t(0,\widetilde{\psi}^l,\widetilde{\psi}^b,-A)$ satisfies $V_t=\widetilde{\psi}^l_tB^l_t+\widetilde{\psi}^b_tB^b_t$ and

$$V_t=\int_0^t (B^l_u)^{-1}(V_u)^+\,dB^l_u-\int_0^t (B^b_u)^{-1}(V_u)^-\,dB^b_u-A_t.$$

Hence the assertion of the lemma follows. □

The next definition is an extension of the classic definition of an arbitrage opportunity, which is suitable when dealing with the basic model with funding costs. Let us stress that we only consider here the classic concept of an *arbitrage opportunity*. For an exhaustive study of alternative versions of no-arbitrage conditions, the interested reader may consult the recent paper by Fontana [20].

Let $x$ be an arbitrary real number. We denote by $V^0(x)$ the wealth process of a self-financing strategy $(x, \varphi^0, 0)$ where $\varphi^0$ is the portfolio with all components equal to zero, except for $\psi^0$ (resp. $\psi^l$ and $\psi^b$ if the lending and borrowing rates are different). It is easy to see that the wealth process $V^0(x)$ is uniquely specified by $x$ and these conditions, specifically, it equals $xB$ (resp. $x^+B^l - x^-B^b$). For any $t \in (0,T]$, the random variable $V_t^0(x)$ represents the future value at time $t$ of the hedger's initial endowment $x$. For a given contract $A$, an arbitrage opportunity for the hedger arises if, through a clever choice of a dynamic portfolio $\varphi$, he can generate a higher netted wealth at $T$ than the future value of his initial endowment. The issue of *admissibility* of trading strategy needs to be examined for each model at hand (see, for instance, Definition 3.4).

**Definition 3.2** An admissible trading strategy $(x, \varphi, A)$ is an *arbitrage opportunity for the hedger* with respect to $A$ whenever the following conditions are satisfied: $\mathbb{P}(V_T^{\text{net}}(x, \varphi, A) \geq V_T^0(x)) = 1$ and $\mathbb{P}(V_T^{\text{net}}(x, \varphi, A) > V_T^0(x)) > 0$.

Definition 3.2 states that the hedger with the initial endowment $x$ can produce an arbitrage opportunity by entering into a contract $A$, if he can find an admissible strategy $(x, \varphi, A)$ and such that the netted wealth at the contract's maturity date $T$ is always no less than $V_T^0(x)$, and is strictly greater than $V_T^0(x)$ with a positive probability.

Let us consider the classic case when $B^l = B^b = B^{i,l} = B^{i,b} = B$ for all $i$. Then for any contract $A$, due to the additivity of self-financing strategies in the classic setting, for any self-financing strategy $(x, \varphi, A)$, we obtain

$$V^{\text{net}}(x, \varphi, A) - V^0(x) = V(x, \varphi, A) + V(0, \widetilde{\varphi}, -A) - V(x, \varphi^0, 0) = V(0, \varphi + \widetilde{\varphi} - \varphi^0, 0) = V(0, \widehat{\varphi})$$

where $V(0, \widehat{\varphi})$ is the wealth process of a trading strategy $\widehat{\varphi}$, which is self-financing in the usual sense. Also, if $\widehat{\varphi}$ is any self-financing trading strategy in the classic sense, then we may set $x = 0$ and $A = 0$, so that $V(0, \widehat{\varphi}) = V^{\text{net}}(0, \widehat{\varphi}, 0)$.

**Remark 3.1** It is fair to acknowledge that Definition 3.2 is only the first step towards a more general view of arbitrage opportunities that might arise in the context of differing funding costs and credit qualities of potential counterparties. A more sophisticated approach relies on a comparison of two opposite dynamically hedged positions, so that we would end up with the following condition: an *extended arbitrage opportunity* is a pair $(x_1, \varphi, A)$ and $(x_2, \widetilde{\varphi}, -A)$ of admissible strategies where $x_1 + x_2 = x$ and

$$\mathbb{P}(V_T(x_1, \varphi, A) + V_T(x_2, \widetilde{\varphi}, -A) \geq V_T^0(x)) = 1,$$
$$\mathbb{P}(V_T(x_1, \varphi, A) + V_T(x_2, \widetilde{\varphi}, -A) > V_T^0(x)) > 0.$$

This more general view means that an arbitrage opportunity can also be created by taking advantage of the presence of two potential counterparties with identical or different creditworthiness. The extended definition requires the possibility of taking back-to-back offsetting positions in OTC deals with identical contractual features, but initiated with different counterparties. Therefore, a minimal trading model now includes the hedger and his two counterparties. For further results in this vein, see Section 3.2 in Nie and Rutkowski [33] where the model with partial netting is examined in detail.

The arguments in favor of Definition 3.2 can be summarized as follows:
- in specific cases of market models, its implementation is relatively easy,
- it yields explicit conditions that make financial sense, and
- last but not least, it can be used to clarify and justify the use of the concept of a *martingale measure* in the general setup of a market with funding costs, collateralization and default.

To sum up, although Definition 3.2 could be further refined, it nevertheless seems to be a sufficient tool to deal with the issue of arbitrage in a non-linear trading environment. Using Definition 3.2, we may now introduce the notion of an arbitrage-free model either with respect to all contracts that can be covered by a particular model or by selecting first a particular class $\mathcal{A}$ of contracts of our interest. Note that, in principle, the arbitrage-free property may depend on the hedger's initial endowment $x$.

**Definition 3.3** We say that a generic market model is *arbitrage-free* for the hedger with respect to the class $\mathcal{A}$ of financial contracts whenever no arbitrage opportunity associated with any contract $A$ from the class $\mathcal{A}$ exists in the class of all trading strategies admissible for the hedger. In other words, a model is arbitrage-free if for any contract $A \in \mathcal{A}$ and any admissible strategy $(x, \varphi, A)$ for the hedger, we have that either $\mathbb{P}\big(V_T^{\text{net}}(x, \varphi, A) = V_T^0(x)\big) = 1$ or $\mathbb{P}\big(V_T^{\text{net}}(x, \varphi, A) < V_T^0(x)\big) > 0$.

Let us stress that if a model is arbitrage-free for the hedger, it is not necessarily true that it is arbitrage-free for the counterparty as well. Observe also that in the classic case when $B^l = B^b = B^{i,l} = B^{i,b} = B$ for all $i$, Definition 3.3 reduces to the classic definition of an arbitrage-free market model. Hence, as expected, the methodology developed here agrees with the standard arbitrage pricing theory if there are no frictions in trading strategies or, at least, when they do not affect the class of contracts at hand, so that they can be safely ignored.

### 3.1.2 Basic Model with Funding Costs

Let us now specify the concepts introduced in the preceding subsection to the basic model with funding costs of Section 2.2 with the cash account $B^0 = B$. We now have that $V_T^0(x) = xB_T$ and thus conditions of Definition 3.2 become

$$\mathbb{P}(V_T^{\text{net}}(x, \varphi, A) \geq xB_T) = 1, \quad \mathbb{P}(V_T^{\text{net}}(x, \varphi, A) > xB_T) > 0 \tag{3.3}$$

or, equivalently,

$$\mathbb{P}(\widetilde{V}_T^{\text{net}}(x, \varphi, A) \geq x) = 1, \quad \mathbb{P}(\widetilde{V}_T^{\text{net}}(x, \varphi, A) > x) > 0$$

where the netted wealth $V^{\text{net}}(x, \varphi, A)$ is given by Definition 2.5 or, equivalently, Lemma 2.1. Recall also that for an arbitrary self-financing trading strategy $(x, \varphi, A)$, equation (2.24) yields

$$\widetilde{V}_t^{\text{net}}(x, \varphi, A) = x + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \widetilde{B}_u^i \, d\widehat{S}_u^{i,\text{cld}} + \sum_{i=1}^{d} \int_0^t (\psi_u^i + \xi_u^i \widehat{S}_u^i) \, d\widetilde{B}_u^i. \tag{3.4}$$

We thus observe that to examine the arbitrage-free property of the basic model, it suffices to consider trading strategies with null initial value. In other words, the no-arbitrage property of the basic model does not depend on the hedger's initial endowment. Note that according to (3.4), the hedger's trading in risky assets is unrestricted, meaning that each risky asset can be funded from arbitrarily chosen funding accounts. In addition, we make the usual postulate that a strategy $(x, \varphi, A)$ need to satisfy some form of admissibility. In the framework of the basic model with funding costs, we adopt the following definition of the class of admissible strategies; they are usually referred to as *tame strategies.*

**Definition 3.4** A self-financing trading strategy $(x, \varphi, A)$ is *admissible for the hedger* whenever the discounted netted wealth process $\widetilde{V}^{\text{net}}(x, \varphi, A)$ is bounded from below by a constant.

The condition that the discounted netted wealth process $\widetilde{V}^{\text{net}}(x, \varphi, A)$ is bounded from below by a constant is a commonly used requirement of *admissibility*, which ensures that, if the process $\widetilde{V}^{\text{net}}(x, \varphi, A)$ a local martingale under some equivalent probability measure, then it is also a supermartingale. It is well known that some technical assumption of this kind cannot be avoided even in the classic case of the Black and Scholes model. Let us stress that the choice of a discount factor was left unspecified in Definition 3.2. If a constant mentioned in Definition 3.4 equals zero, so that the netted wealth of an admissible strategy is bound to stay non-negative, then it suffices to consider the netted wealth without any discounting and thus the choice of a discount factor in Definition 3.2 is manifestly irrelevant. Otherwise, this choice will depend on the problem and model under study (see, for instance, Proposition 3.3).

**Lemma 3.2** *Assume that for any admissible trading strategy $(\varphi, A)$ there exists a probability measure $\widetilde{\mathbb{P}}^{\varphi,A}$ on $(\Omega, \mathcal{G}_T)$ such that $\widetilde{\mathbb{P}}^{\varphi,A}$ is equivalent to $\mathbb{P}$ and the process $\widetilde{V}^{net}(x, \varphi, A)$ is a $(\widetilde{\mathbb{P}}^{\varphi,A}, \mathbb{G})$-local martingale. Then the basic market model with funding costs is arbitrage-free for the hedger.*

A probability measure $\widetilde{\mathbb{P}}^{\varphi,A}$ is then called a *equivalent local martingale measure* (ELMM) for the process $\widetilde{V}^{\mathrm{net}}(x,\varphi,A)$. Of course, the sufficient condition of Lemma 3.2 is very cumbersome to check, in general, and thus it does not seem to be of practical interest. For this reason, we will search for more explicit conditions that will be relatively easy to verify. They will refer to the existence of some universal equivalent local martingale measure for a given trading framework and for a sufficiently large class of contracts under study.

To this end, we will first re-examine the concepts of an arbitrage opportunity and arbitrage price since, as we will argue in what follows, the classic definitions do not reflect adequately the present general framework. In particular, we show that the study of the arbitrage-free property of a market model cannot be separated from an analysis of hedging strategies for a given class of contracts. This is due to the fact that the presence of either incoming or outgoing cash flows associated with a contract (that is, external cash flows $A$) may exert a non-additive impact on the dynamics of the wealth process, and thus also on the total gains and/or losses from hedger's trading activities.

Obviously, if there exists $B^k \neq B$, then an arbitrage opportunity arises. Indeed, it is easy to produce it by taking $\xi^1 = \ldots = \xi^d = 0$ and $\psi^j = 0$ for every $j$, except for $j = k$. Then we obtain

$$\widetilde{V}_t^{\mathrm{net}}(x,\varphi,A) = x + \int_0^t \psi_u^k \, d\widetilde{B}_u^k$$

and thus we see that the existence of an equivalent local martingale measure $\widetilde{\mathbb{P}}^{\varphi,A}$ for the process $\widetilde{V}^{\mathrm{net}}(x,\varphi,A)$ is by no means ensured, in general. Therefore, additional conditions need to be imposed on the class of trading strategies and/or funding rates to guarantee that the basic model with funding costs is arbitrage-free. In the next result, we thus preclude the occurrence of a mixed funding for any risky asset. Recall that condition (2.16) holds if, for instance, the equality $\psi_t^i B_t^i + \xi_t^i S_t^i = 0$ is satisfied for all $t \in [0,T]$. We write $\mathbb{Q} \sim \mathbb{P}$ to denote that the probability measures $\mathbb{Q}$ and $\mathbb{P}$ are equivalent on $(\Omega, \mathcal{G}_T)$.

**Proposition 3.1** *Assume that all strategies available to the hedger are admissible and satisfy condition* (2.16)*. If there exists a probability measure $\widetilde{\mathbb{P}}$ on $(\Omega,\mathcal{G}_T)$ such that $\widetilde{\mathbb{P}} \sim \mathbb{P}$ and the processes $\widehat{S}^{i,cld}$, $i = 1,2,\ldots,d$ are $(\widetilde{\mathbb{P}},\mathbb{G})$-local martingales then the basic model with funding costs is arbitrage-free for the hedger.*

*Proof.* It suffices to observe that, under the present assumptions, equation (3.4) reduces to

$$\widetilde{V}_t^{\mathrm{net}}(x,\varphi,A) = x + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \widetilde{B}_u^i \, d\widehat{S}_u^{i,\mathrm{cld}} \tag{3.5}$$

and to apply the usual argument that any local martingale (or even a sigma-martingale, which may arise as a stochastic integral in (3.5) when the semimartingales $S^1, S^2, \ldots, S^d$ are not continuous) that is bounded from below by a constant, is necessarily a supermartingale. □

## 3.2 Hedger's Fair Valuation under Funding Costs

In the next step, we focus on fair pricing of contracts. The fair pricing of contracts in a market model that allows for arbitrage opportunities is obviously not viable, so we henceforth work under the standing assumption that a model under study is arbitrage-free for the hedger with a given initial endowment $x$ and for a sufficiently large class $\mathcal{A}$ of contracts, which encompasses a given contract $A$ (see Definition 3.3).

Our goal is to propose a realistic definition of a *hedger's fair price* and to show how to apply it to some models with funding costs. Let us observe that the definition of an arbitrage-free model is not symmetric, that is, a model in which no arbitrage opportunities for the hedger exist may still allow for arbitrage opportunities for the counterparty. Moreover, even when the market conditions are identical for both parties, they have the same initial endowment and a given model is arbitrage-free for both parties, the cash flows of a contract are obviously asymmetric and thus the range of fair prices computed by the two counterparties may be different. By the usual convention adopted throughout this paper, we will focus the following discussion on one party, which is called the hedger.

### 3.2.1 Generic Market Model

Our next goal is to describe the range of hedger's arbitrage prices of a contract with cash flows $A$. Let $x$ be the hedger's initial endowment and let $p$ stand for a generic price of a contract at time 0 from the perspective of the hedger. A positive value of $p$ means that the hedger receives at time 0 the cash amount $p$ from the counterparty, whereas a negative value of $p$ means that he makes the payment $-p$ to the counterparty at time 0. It is clear from the next definition that a *hedger's price* may depend on the hedger's initial endowment $x$ and it may fail to be unique, in general.

It is important to stress that the *admissibility* of a trading strategy is now defined using the discounted wealth, as opposed to the discounted netted wealth, as was the case in Section 3.1. For this reason, to avoid a possibility of confusion with Definition 3.4, we decided to state explicitly the admissibility condition each time when it is relevant. Moreover, the choice of a discount factor depends on a model under consideration, but it is otherwise fairly arbitrary, so that the *discounted wealth process*, formally represented by the generic symbol $\widehat{V}(x, \varphi, A)$, is not necessarily given as $B^{-1}V(x, \varphi, A)$. For instance, in Section 3.4, the discounted wealth will be given by $\widehat{V}(x, \varphi, A) := (B^l)^{-1}V(x, \varphi, A)$. As a rule of thumb, we suggest that the choice of discounting should be the same when we address either the first or the second question stated at the beginning of this section.

**Definition 3.5** We say that a real number $\bar{p}_0 = A_0$ is a *hedger's fair price* for $A$ at time 0 whenever for any self-financing trading strategy $(x, \varphi, A)$ such that the discounted wealth process $\widehat{V}(x, \varphi, A)$ is bounded from below by a constant, we have that either

$$\mathbb{P}\big(V_T(x, \varphi, A) = V_T^0(x)\big) = 1 \tag{3.6}$$

or

$$\mathbb{P}\big(V_T(x, \varphi, A) < V_T^0(x)\big) > 0. \tag{3.7}$$

One may observe that the two conditions in Definition 3.5 are analogous to conditions of Definition 3.3, although they are not identical and, indeed, they have quite different financial interpretations. Recall that Definition 3.3 deals with a possibility of offsetting a dynamically hedged contract $A$ by an unhedged contract $-A$, whereas Definition 3.5 is concerned with finding a fair price for $A$ from the viewpoint of the hedger as a contract's seller. In the latter case, it is natural to say that a price level $\bar{p}_0$ is too high for the hedger, if he can produce an arbitrage opportunity (in the sense that is implicit in Definition 3.5) by selling $A$ at price $\bar{p}_0$ and devising a suitable hedging strategy for his short position. Once again, the hedger's profits are measured with respect to his idiosyncratic cost of raising cash or, more precisely, with respect to the future value of his current endowment $x$, as represented by the random variable $V_T^0(x)$. This leads to the following natural definition of a hedger's arbitrage opportunity for $A$ at price $p$. As usual in the arbitrage pricing theory, we need to postulate that trading strategies are admissible.

**Definition 3.6** We say that a quadruplet $(p, x, \varphi, A)$, where $p = A_0$ is a real number and $(x, \varphi, A)$ is an admissible trading strategy such that the discounted wealth process $\widehat{V}(x, \varphi, A)$ is bounded from below by a constant, is a *hedger's arbitrage opportunity for $A$ at price $p$* if

$$\mathbb{P}\big(V_T(x, \varphi, A) \geq V_T^0(x)\big) = 1$$

and

$$\mathbb{P}\big(V_T(x, \varphi, A) > V_T^0(x)\big) > 0.$$

Assume that the hedger has the initial endowment $x$ and he sells the contract $A$ at price $\bar{p}_0$. Then $\bar{p}_0$ is a hedger's fair price for $A$, in the sense of Definition 3.5, whenever he is not able to find an arbitrage opportunity for $A$ at price $p = \bar{p}_0$, in the sense of Definition 3.6.

In practice, the hedger's initial endowment $x < 0$ can be interpreted as the amount of cash borrowed by the trading desk from the bank's internal funding unit, which should be repaid with interest $B^b_T$ at a given horizon date $T$. Therefore, an arbitrage opportunity at price $\bar{p}_0$ for $A$ means that the price $\bar{p}_0$ is high enough to allow the hedger to make a risk-free profit, where the 'profits' are assessed in relation to the hedger's idiosyncratic cost of capital, as formally represented by the account $B^b$.

### 3.2.2 Basic Model with Funding Costs

In the basic model with funding costs, we obtain the following result, which bears a close resemblance to its classic counterpart, which deals with a market model with a single cash account. Note that we work here under the assumptions of Proposition 3.1 so that the basic model with funding costs is arbitrage-free for the hedger with respect to any contract $A$. Recall that here $B^l = B^b = B$, so that the discounted wealth is defined as $\widetilde{V}(x,\varphi,A) = B^{-1}V(x,\varphi,A)$, the admissibility is specified by Definition 3.4, and trading strategies are assumed to satisfy condition (2.16).

**Proposition 3.2** *Under the assumptions of Proposition 3.1, a real number $\bar{p}_0$ is a hedger's fair price whenever, for any admissible trading strategy $(x,\varphi,A)$ satisfying condition* (2.16)*, we have that either*

$$\mathbb{P}\bigg(\bar{p}_0 + \sum_{i=1}^d \int_{(0,T]} \xi^i_u \widetilde{B}^i_u \, d\widehat{S}^{i,cld}_u + \int_{(0,T]} B_u^{-1}\, dA_u = 0\bigg) = 1$$

*or*

$$\mathbb{P}\bigg(\bar{p}_0 + \sum_{i=1}^d \int_{(0,T]} \xi^i_u \widetilde{B}^i_u \, d\widehat{S}^{i,cld}_u + \int_{(0,T]} B_u^{-1}\, dA_u < 0\bigg) > 0.$$

*Proof.* It suffices to combine Definition 3.5 with equation (2.25). □

Note that, in this basic framework where $B^l = B^b = B$, the set of all hedger's fair prices does not depend on the hedger's initial endowment $x$, although it manifestly depends on funding accounts $B^i$, $i = 1,2,\dots,d$. Also, the real-world probability measure $\mathbb{P}$ can be replaced by an equivalent local martingale measure $\widehat{\mathbb{P}}$ for processes $\widehat{S}^{i,\mathrm{cld}}$, $i = 1,2,\dots,d$.

As an example, let us take $A_t = -X\mathbb{1}_{\{t=T\}}$ and let us assume $B^i = B$ for every $i = 1,2,\dots,d$. Then we obtain the following characterization of a hedger's price $\bar{p}_0$: for any admissible trading strategy $(\varphi, A)$, either

$$\mathbb{P}\bigg(\bar{p}_0 + \sum_{i=1}^d \int_{(0,T]} \xi^i_u \, d\widetilde{S}^{i,\mathrm{cld}}_u = B_T^{-1}X\bigg) = 1$$

or

$$\mathbb{P}\bigg(\bar{p}_0 + \sum_{i=1}^d \int_{(0,T]} \xi^i_u \, d\widetilde{S}^{i,\mathrm{cld}}_u < B_T^{-1}X\bigg) > 0.$$

We recognize here the classic case, namely, the notion of the hedger's fair price as an arbitrary level of $\bar{p}_0$ that does not allow for creation of a hedger's super-hedging strategy for a European claim $X$.

## 3.3 Example: Valuation of FRA

As a simple illustration of pricing problems studied in this section, we propose to consider the extension of the Black-Scholes model to the case of different lending and borrowing rates, which satisfy $r^b \geq r^l \geq 0$. It is known that this model is arbitrage-free in the classic sense when only self-financing trading strategies with a non-negative wealth are allowed (see, for instance, Bergman [1] and Example 1.1 in El Karoui et al. [18]). We assume that the hedger's initial endowment equals $x > 0$ and we complement the model by the contract $A$ with only two cash flows after time 0, namely, the outgoing cash flow of $\bar{A}_1 := \alpha$ units of cash at time $0 < t_0 < T$ for some $\alpha > 0$ and the incoming cash flow of $\bar{A}_2 := \alpha e^{\widehat{r}(T-t_0)}$ units of cash at time $T$. The contract $A$ initiated at time 0 is thus the forward rate agreement (FRA) over $[t_0, T]$ with the continuously compounded forward rate $\widehat{r}$ (note that the existence of the account associated with the rate $\widehat{r}$ is not postulated). The process $A$ is thus given by $A_t = p - \bar{A}_1 \mathbb{1}_{[t_0,T]}(t) + \bar{A}_2 \mathbb{1}_{[T]}(t)$ for every $t \in [0,T]$ where $p$ is undetermined (see equation (2.2)).

### 3.3.1 Hedger's Arbitrage

We first focus on the hedger's arbitrage, in the sense of Definition 3.2, when the FRA is available to the hedger. In this subsection, the initial value $p$ of this contract at time 0 is left unspecified since, without loss of generality, we may assume that its market value equals zero when analyzing the hedger's arbitrage using the concept of the netted wealth. Let us consider a self-financing strategy $(x, \widehat{\varphi}, A)$ in which the initial endowment $x$ is invested in lending and borrowing accounts $B^l$ and $B^b$ only and a part of this investment is used by the hedger at time $t_0$ to deliver $\alpha$ units of cash to the counterparty. It is postulated that no investment in the risky asset is ever made by the hedger. Hence, from (2.4) and (2.5), we obtain $V_t(x, \widehat{\varphi}, A) = \psi^l_t B^l_t + \psi^b_t B^b_t$ where, by assumption, $\psi^l_t \geq 0$, $\psi^b_t \leq 0$ and $\psi^{i,l}_t \psi^{i,b}_t = 0$ and

$$V_t(x, \widehat{\varphi}, A) = x + \int_0^t \psi^l_u \, dB^l_u + \int_0^t \psi^b_u \, dB^b_u + A_t. \tag{3.8}$$

If we assume that $xe^{r^l t_0} \geq \alpha$, then the unique strategy $\widehat{\varphi} = (\psi^l, \psi^b)$ satisfying these assumptions is given as: $\psi^b_t = 0$ for all $t \in [0,T]$, and

$$\psi^l_t = x \mathbb{1}_{[0,t_0)} + \frac{\widetilde{x}}{B^l_{t_0}} \mathbb{1}_{[t_0,T)} + \frac{\widehat{x}}{B^l_T} \mathbb{1}_{[T,T]}, \quad t \in [0,T],$$

where

$$\widetilde{x} := xe^{r^l t_0} - \alpha, \quad \widehat{x} := xe^{r^l T} - \alpha e^{r^l (T-t_0)} + \alpha e^{\widehat{r}(T-t_0)}.$$

Hence, in this case, the wealth of the hedger's strategy $(\widehat{\varphi}, A)$ at time $T$ equals

$$V_T(x, \widehat{\varphi}, A) = \big(xe^{r^l t_0} - \alpha\big) e^{r^l (T-t_0)} + \alpha e^{\widehat{r}(T-t_0)} = xe^{r^l T} - \alpha e^{r^l (T-t_0)} + \alpha e^{\widehat{r}(T-t_0)}. \tag{3.9}$$

If, on the contrary, the inequality $xe^{r^l t_0} < \alpha$ is valid, then the hedger's wealth at $T$ necessarily satisfies

$$V_T(x, \widehat{\varphi}, A) = \big(xe^{r^l t_0} - \alpha\big) e^{r^b (T-t_0)} + \alpha e^{\widehat{r}(T-t_0)} = xe^{r^l t_0} e^{r^b (T-t_0)} - \alpha e^{r^b (T-t_0)} + \alpha e^{\widehat{r}(T-t_0)}$$

since now the unique portfolio $\widehat{\varphi} = (\psi^l, \psi^b)$ available to the hedger involves borrowing of $\alpha - xe^{r^l t_0}$ units of cash at time $t_0$ (this is needed to pay $\alpha$ units of cash to the counterparty). Similar arguments show that if we set $x = 0$ and consider the contract $-A$, then the wealth at $T$ of the unique portfolio $\widetilde{\varphi} = (\widetilde{\psi}^l, \widetilde{\psi}^b)$ available to the hedger equals

$$V_T(0, \widetilde{\varphi}, -A) = \alpha e^{r^l (T-t_0)} - \alpha e^{\widehat{r}(T-t_0)}.$$

We thus see that, if $xe^{r^l t_0} \geq \alpha$, then the netted wealth equals

$$\begin{aligned} V^{\text{net}}_T(x, \widehat{\varphi}, A) &= V_T(x, \widehat{\varphi}, A) + V_T(0, \widetilde{\varphi}, -A) \\ &= xe^{r^l T} - \alpha e^{r^l (T-t_0)} + \alpha e^{\widehat{r}(T-t_0)} + \alpha e^{r^l (T-t_0)} - \alpha e^{\widehat{r}(T-t_0)} \\ &= xe^{r^l T} = V^0_T(x) \end{aligned}$$

and for $xe^{r^l t_0} < \alpha$ it satisfies

$$\begin{aligned} V^{\text{net}}_T(x, \widehat{\varphi}, A) &= V_T(x, \widehat{\varphi}, A) + V_T(0, \widetilde{\varphi}, -A) \\ &= xe^{r^l t_0} e^{r^b (T-t_0)} - \alpha e^{r^b (T-t_0)} + \alpha e^{\widehat{r}(T-t_0)} + \alpha e^{r^l (T-t_0)} - \alpha e^{\widehat{r}(T-t_0)} \\ &= xe^{r^l T} + \big(\alpha - xe^{r^l t_0}\big)\big(e^{r^l (T-t_0)} - e^{r^b (T-t_0)}\big) \leq xe^{r^l T} = V^0_T(x) \end{aligned}$$

where the last inequality is strict whenever $r^b > r^l$. This means that the strategy $(\widehat{\varphi}, A)$ is not an arbitrage opportunity for the hedger, in the sense of Definition 3.2. Of course, more sophisticated hedger's strategies $(\varphi, A)$ should be examined as well, but it is unlikely that an arbitrage opportunity for the hedger may arise when a possibility of investing in the risky asset is also taken into account if the hedger's model without the contract $A$ is arbitrage-free.

### 3.3.2 Hedger's Fair Valuation

We will now focus on fair valuation of the forward rate agreement from the hedger's perspective. If we assume, in addition, that $\widehat{r} > r^l$ and $xe^{r^l t_0} \geq \alpha$, then equation (3.9) with $A_0 = 0$ yields $V_T(x, \widehat{\varphi}, A) > xe^{r^l T}$, and thus it is obvious that $p = 0$ is not the fair hedger's price for the contract, in the sense of Definition 3.5. It is thus natural to expect that any hedger's fair price for $A$ is a strictly negative number. We maintain the assumption that the hedger does not invest in the risky asset and we postulate that

$$x + p \geq 0, \quad (x + p)e^{r^l t_0} > \alpha. \tag{3.10}$$

Then for $A_0 = p$ the unique hedger's strategy satisfies

$$V_T(x, \widehat{\varphi}, A) = \big((x+p)e^{r^l t_0} - \alpha\big)e^{r^l (T-t_0)} + \alpha e^{\widehat{r}(T-t_0)} = (x+p)e^{r^l T} - \alpha e^{r^l (T-t_0)} + \alpha e^{\widehat{r}(T-t_0)}$$

and thus the equality $V_T(x, \widehat{\varphi}, A) = xe^{r^l T}$ holds whenever

$$p = \alpha e^{-r^l T}\big(e^{r^l (T-t_0)} - e^{\widehat{r}(T-t_0)}\big). \tag{3.11}$$

Observe that $p$ given by (3.11) is strictly negative, since we assumed that $\widehat{r} > r^l$. It is thus natural to conjecture that the value given by (3.11) is the upper bound for hedger's fair prices for $A$, in the sense of Definition 3.5, provided that $p$ given by (3.11) is such that conditions (3.10) are also met, that is, the absolute value of $p$ is indeed sufficiently small with respect to $x$. Otherwise, the computations leading to the fair value of $p$ should be modified accordingly and a different result is expected. This example, albeit not completely solved, makes it clear that the classic arbitrage pricing paradigm relying on "taking the risk-neutral expected value of cash flows discounted using the risk-free rate" is not applicable when dealing with more realistic models of trading by financial institutions. This statement will be further illustrated and examined in the foregoing sections (for a more detailed study of non-linear pricing rules, we refer to related works [33]–[34]).

## 3.4 Model with Funding Costs and Partial Netting

To provide a non-trivial illustration of the novel concepts introduced in this section, we will now consider the market model with partial netting of short cash positions, which was introduced in Section 2.4.3. Recall that, in principle, the choice of a discount factor is unrestricted, so any particular choice is motivated by its convenience in handling a particular problem.

Let the hedger's initial endowment be $x \geq 0$. We will first show that, under mild assumptions, the model is arbitrage-free for the hedger with respect to a contract $A$. To this end, we define the discounted wealth and the discounted wealth of $(\varphi, A)$ by setting $\widetilde{V}^l_t(x, \varphi, A) := (B^l_t)^{-1} V_t(x, \varphi, A)$ and $\widetilde{V}^{l,\text{net}}_t(x, \varphi, A) := (B^l_t)^{-1} V^{\text{net}}_t(x, \varphi, A)$, respectively. The choice of $B^l$ for discounting is related here to the assumption that $x \geq 0$; when $x < 0$ it is more natural to take $B^b$ instead, since in that case the hedger has a debt at time 0 that has to be repaid with interest determined by $B^b$.

### 3.4.1 Hedger's Arbitrage

The following result hinges on a plausible assumption that all borrowing rates $r^{i,b}$ are higher than the common lending rate $r^l$. Note that we assume here that all cash accounts are absolutely continuous.

**Proposition 3.3** *Assume that $x \geq 0$, $r^l_t \leq r^b_t$ and $r^l_t \leq r^{i,b}_t$ for $i = 1, 2, \dots, d$. Let us denote*

$$\widetilde{S}^{i,l,cld}_t = (B^l_t)^{-1} S^i_t + \int_{(0,t]} (B^l_u)^{-1} \, dA^i_u. \tag{3.12}$$

*If there exists a probability measure $\widetilde{\mathbb{P}}^l \sim \mathbb{P}$ such that the processes $\widetilde{S}^{i,l,cld}$, $i = 1, 2, \dots, d$ are $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-local martingales, then the market model of Section 2.4.3 is arbitrage-free for the hedger with respect to any contract $A$.*

*Proof.* From Corollary 2.4, we know that the wealth process $V(x, \varphi, A)$ of a self-financing strategy $(x, \varphi, A)$ satisfies (see equation (2.52))

$$\begin{aligned} dV_t(x,\varphi,A) = &\sum_{i=1}^d \xi^i_t \big(dS^i_t + dA^i_t\big) - \sum_{i=1}^d r^{i,b}_t (\xi^i_t S^i_t)^+ \, dt + dA_t \\ &+ r^l_t \Big(V_t(x,\varphi,A) + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^+ dt - r^b_t \Big(V_t(x,\varphi,A) + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^- dt. \end{aligned}$$

Since we assumed that $r^l_t \leq r^b_t$, we obtain

$$\begin{aligned} dV_t(x,\varphi,A) \leq &\sum_{i=1}^d \xi^i_t \big(dS^i_t + dA^i_t\big) - \sum_{i=1}^d r^{i,b}_t (\xi^i_t S^i_t)^+ \, dt + dA_t \\ &+ r^l_t \Big(V_t(x,\varphi,A) + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^+ dt - r^l_t \Big(V_t(x,\varphi,A) + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^- dt \\ = &\, r^l_t V_t(x,\varphi,A)\, dt + \sum_{i=1}^d \xi^i_t \big(dS^i_t + dA^i_t\big) + dA_t - \sum_{i=1}^d r^{i,b}_t (\xi^i_t S^i_t)^+ \, dt + r^l_t \sum_{i=1}^d (\xi^i_t S^i_t)^- \, dt \\ \leq &\, r^l_t V_t(x,\varphi,A)\, dt + \sum_{i=1}^d \xi^i_t \big(dS^i_t - r^l_t S^i_t \, dt + dA^i_t\big) + dA_t \end{aligned}$$

where the last inequality holds, since it is also postulated that $r^l_t \leq r^{i,b}_t$. Consequently, the discounted wealth $\widetilde{V}^l_t(x,\varphi,A) = (B^l_t)^{-1} V_t(x,\varphi,A)$ satisfies

$$d\widetilde{V}^l_t(x,\varphi,A) \leq \sum_{i=1}^d \xi^i_t (B^l_t)^{-1} \big(dS^i_t - r^l_t S^i_t \, dt + dA^i_t\big) + (B^l_t)^{-1} \, dA_t$$

and thus, in view of (3.12), we obtain

$$d\widetilde{V}^l_t(x,\varphi,A) \leq \sum_{i=1}^d \xi^i_t \, d\widetilde{S}^{i,l,\mathrm{cld}}_t + (B^l_t)^{-1} \, dA_t.$$

Furthermore, the netted wealth equals $V^{\mathrm{net}}_t(x,\varphi,A) = V_t(x,\varphi,A) + U_t(A)$ (see Lemma 3.1) where the $\mathbb{G}$-adapted process of finite variation $U(A)$ is the unique solution to the following equation

$$U_t(A) = \int_0^t (B^l_u)^{-1} (U_u(A))^+ \, dB^l_u - \int_0^t (B^b_u)^{-1} (U_u(A))^- \, dB^b_u - A_t \tag{3.13}$$

where $U(A) = (U(A))^+ - (U(A))^-$ is the decomposition of the process $U(A)$ into its increasing and decreasing components. Hence the netted discounted wealth $\widetilde{V}^{l,\mathrm{net}}_t(x,\varphi,A) := (B^l_t)^{-1} V^{\mathrm{net}}_t(x,\varphi,A)$

satisfies

$$
\begin{aligned}
d\widetilde{V}^{l,\mathrm{net}}_t(x,\varphi,A) &= d\widetilde{V}^l_t(x,\varphi,A) + d((B^l_t)^{-1}U_t(A)) \\
&\le \sum_{i=1}^d \xi^i_t\, d\widetilde{S}^{i,l,\mathrm{cld}}_t + (B^l_t)^{-2}(U_t(A))^+\, dB^l_t - (B^l_t)^{-1}(B^b_t)^{-1}(U_t(A))^-\, dB^b_t + U_t(A)\, d(B^l_t)^{-1} \\
&= \sum_{i=1}^d \xi^i_t\, d\widetilde{S}^{i,l,\mathrm{cld}}_t + r^l_t(B^l_t)^{-1}(U_t(A))^+\, dt - r^b_t(B^l_t)^{-1}(U_t(A))^-\, dt - r^l_t(B^l_t)^{-1}U_t(A)\, dt \\
&= \sum_{i=1}^d \xi^i_t\, d\widetilde{S}^{i,l,\mathrm{cld}}_t + (r^l_t - r^b_t)(B^l_t)^{-1}(U_t(A))^-\, dt
\end{aligned}
$$

and thus

$$
\widetilde{V}^{l,\mathrm{net}}_t(x,\varphi,A) - \widetilde{V}^{l,\mathrm{net}}_0(x,\varphi,A) \le \sum_{i=1}^d \int_{(0,t]} \xi^i_u\, d\widetilde{S}^{i,l,\mathrm{cld}}_u. \tag{3.14}
$$

The arbitrage-free property of the model for the hedger can now be established using the standard arguments. First, from (3.14) and the assumption that the process $\widetilde{V}^{l,\mathrm{net}}(x,\varphi,A)$ is bounded from below by a constant, we deduce that the right-hand side in (3.14) is a $(\widetilde{\mathbb{P}}^l,\mathbb{F})$-supermartingale, which is null at $t=0$. Next, since the initial endowment $x$ is non-negative, we have that $V^0_T(x) = B^l_T x$. From inequality (3.14), we obtain

$$
(B^l_T)^{-1}\big(V^{\mathrm{net}}_T(x,\varphi,A) - V^0_T(x)\big) \le \sum_{i=1}^d \int_0^T \xi^i_t\, d\widetilde{S}^{i,l,\mathrm{cld}}_t.
$$

Since $\widetilde{\mathbb{P}}^l$ is equivalent to $\mathbb{P}$, we conclude that either the equality $V^{\mathrm{net}}_T(x,\varphi,A) = V^0_T(x)$ holds or the inequality $\mathbb{P}(V^{\mathrm{net}}_T(x,\varphi,A) < V^0_T(x)) > 0$ is satisfied. This means that arbitrage opportunities are indeed precluded and thus the market model with partial netting is arbitrage-free for the hedger in respect of any contract $A$. □

**Remark 3.2** We claim that assertion of Proposition 3.3 is also true for $x \le 0$ under the stronger assumption that $r^b \le r^{i,b}$ for all $i$ provided that the processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i=1,2,\dots,d$ are replaced by $\widetilde{S}^{i,b,\mathrm{cld}}$, $i=1,2,\dots,d$, where the process $\widetilde{S}^{i,b,\mathrm{cld}}$ is obtained by replacing $B^l$ by $B^b$ in the right-hand side of (3.12). Since $r^l \le r^b \le r^{i,b}$, we now obtain

$$
\begin{aligned}
dV_t(x,\varphi,A) \le{} & \sum_{i=1}^d \xi^i_t\big(dS^i_t + dA^i_t\big) - \sum_{i=1}^d r^{i,b}_t(\xi^i_tS^i_t)^+\, dt + dA_t \\
& + r^b_t\Big(V_t(x,\varphi,A) + \sum_{i=1}^d(\xi^i_tS^i_t)^-\Big)^+ dt - r^b_t\Big(V_t(x,\varphi,A) + \sum_{i=1}^d(\xi^i_tS^i_t)^-\Big)^- dt \\
={} & r^b_tV_t(x,\varphi,A)\, dt + \sum_{i=1}^d \xi^i_t\big(dS^i_t + dA^i_t\big) + dA_t - \sum_{i=1}^d r^{i,b}_t(\xi^i_tS^i_t)^+\, dt + r^b_t\sum_{i=1}^d(\xi^i_tS^i_t)^-\, dt \\
\le{} & r^b_tV_t(x,\varphi,A)\, dt + \sum_{i=1}^d \xi^i_t\big(dS^i_t - r^b_tS^i_t\, dt + dA^i_t\big) + dA_t.
\end{aligned}
$$

Therefore, the discounted wealth $\widetilde{V}^b_t(x,\varphi,A) = (B^b_t)^{-1}V_t(x,\varphi,A)$ satisfies

$$
d\widetilde{V}^b_t(x,\varphi,A) \le \sum_{i=1}^d \xi^i_t\, d\widetilde{S}^{i,b,\mathrm{cld}}_t + (B^b_t)^{-1}\, dA_t
$$

and for the netted discounted wealth $\widetilde{V}^{b,\mathrm{net}}_t(x,\varphi,A) := (B^b_t)^{-1}V^{\mathrm{net}}_t(x,\varphi,A)$, we obtain from (3.13)

$$\begin{aligned}
d\widetilde{V}^{b,\mathrm{net}}_t(x,\varphi,A) &= d\widetilde{V}^b_t(x,\varphi,A) + d((B^b_t)^{-1}U_t(A)) \\
&\leq \sum_{i=1}^d \xi^i_t\, d\widetilde{S}^{i,b,\mathrm{cld}}_t + (B^b_t)^{-1}(B^l_t)^{-1}(U_t(A))^+\, dB^l_t - (B^b_t)^{-2}(U_t(A))^-\, dB^b_t + U_t(A)\, d(B^b_t)^{-1} \\
&= \sum_{i=1}^d \xi^i_t\, d\widetilde{S}^{i,b,\mathrm{cld}}_t + r^l_t(B^b_t)^{-1}(U_t(A))^+\, dt - r^b_t(B^b_t)^{-1}(U_t(A))^-\, dt - r^b_t(B^b_t)^{-1}U_t(A)\, dt.
\end{aligned}$$

Since $r^l \leq r^b$, this yields

$$(B^b_T)^{-1}\big(V^{\mathrm{net}}_T(x,\varphi,A) - V^0_T(x)\big) \leq \sum_{i=1}^d \int_0^T \xi^i_t\, d\widetilde{S}^{i,b,\mathrm{cld}}_t$$

where $V^0_T(x) = B^b_T x$ since $x \leq 0$. Hence the conclusion follows if there exists a probability measure $\widetilde{\mathbb{P}}^b \sim \mathbb{P}$ such that the processes $\widetilde{S}^{i,b,\mathrm{cld}}$, $i = 1,2,\dots,d$ are $(\widetilde{\mathbb{P}}^b,\mathbb{G})$-local martingales.

### 3.4.2 Hedger's Fair Valuation

We now address the issue of the hedger's fair valuation of a contract $A$. In the present setup, Definition 3.5 is applied to the discounted wealth $\widehat{V}(x,\varphi,A)$ with $A_0 = \bar{p}_0$, which is given by the following equation

$$\widehat{V}_t(x,\varphi,A) = \widetilde{V}^l_t(x,\varphi,A) = (B^l_t)^{-1}V_t(x,\varphi,A),$$

that is, the admissibility of a trading strategy $(x,\varphi,A)$ is defined using the discounted wealth $\widetilde{V}^l(x,\varphi,A)$. In view of Corollary 2.4 (see also equation (2.52)), the set of hedger's fair prices $\bar{p}_0$ in the model with partial netting can be characterized as follows: for any admissible strategy $(x,\varphi,A)$, we have that either

$$\begin{aligned}
\mathbb{P}\bigg( x + \bar{p}_0 &+ \sum_{i=1}^d \int_{(0,T]} \xi^i_t\big(dS^i_t + dA^i_t\big) - \sum_{i=1}^d \int_0^T r^{i,b}_t(\xi^i_t S^i_t)^+\, dt + \int_{(0,T]} dA_t \\
&+ \int_0^T r^l_t\Big(V_t(x,\varphi,A) + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^+ dt - \int_0^T r^b_t\Big(V_t(x,\varphi,A) + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^- dt < V^0_T(x)\bigg) > 0
\end{aligned}$$

or

$$\begin{aligned}
\mathbb{P}\bigg( x + \bar{p}_0 &+ \sum_{i=1}^d \int_{(0,T]} \xi^i_t\big(dS^i_t + dA^i_t\big) - \sum_{i=1}^d \int_0^T r^{i,b}_t(\xi^i_t S^i_t)^+\, dt + \int_{(0,T]} dA_t \\
&+ \int_0^T r^l_t\Big(V_t(x,\varphi,A) + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^+ dt - \int_0^T r^b_t\Big(V_t(x,\varphi,A) + \sum_{i=1}^d (\xi^i_t S^i_t)^-\Big)^- dt = V^0_T(x)\bigg) = 1.
\end{aligned}$$

It is clear that that $\int_{(0,T]} dA_u = A_T - A_0$. However, the terms $\bar{p}_0$ and $-A_0$ do not cancel out in the formulae above, since $\bar{p}_0$ is the yet unknown initial fair price of the contract, whereas the random variable $A_T - A_0$ represents all contract's cash flows on $(0,T]$, and thus it is explicitly specified through the contract's covenants. Of course, this formal characterization of a fair price $\bar{p}_0$ does not offer any specific computational algorithm. Hence one needs to develop more explicit methods for finding fair prices (for instance, via a suitable extension of the BSDE approach, which is examined in Section 5.2).

# 4 Trading under Funding Costs and Collateralization

In this section, we will examine the situation when the hedger enters a contract with cash flows $A$ and either receives or posts collateral with the value formally represented by a stochastic process $C$.

The process $C$ is called the *margin account* or the *collateral amount* and the mechanism of either posting or receiving a collateral is referred to as *margining*. Let

$$C_t = C_t \mathbb{1}_{\{C_t \geq 0\}} + C_t \mathbb{1}_{\{C_t < 0\}} = C_t^+ - C_t^- \tag{4.1}$$

be the usual decomposition of the random variable $C_t$ into the positive and negative components. By convention, $C_t^+$ is the cash value of collateral received by the hedger, whereas $C_t^-$ represents the cash value of collateral posted by him.

For simplicity of presentation, it is postulated throughout that only shares of particular *collateral assets*, henceforth denoted by $S^{d+1}$ (resp. $S^{d+2}$) may be delivered (resp. received) by the hedger as a collateral. In principle, this assumption can be relaxed to cover the case where a collateral asset is not predetermined, but it may be chosen from a larger class of assets. However, the notation and computations would become heavier, so we decided to consider a simple case only. Unless explicitly stated otherwise, we work under the following standing assumptions:
(a) lending and borrowing cash rates $B^l$ and $B^b$ are equal, so that $B^l = B^b = B$,
(b) long and short funding rates for each risky asset $S^i$ are identical, that is, $B^{i,l} = B^{i,b} = B^i$ for $i = 1, 2, \dots, d$.

We make the following standing assumption regarding the behavior of the margin account at the contract's maturity date.

**Assumption 4.1** We postulate that the $\mathbb{G}$-adapted collateral amount process satisfies $C_T = 0$. Hence any particular specification of the collateral amount $C_t$ discussed in that follows will only be valid for $0 \leq t < T$ and, invariably, we set $C_T = 0$.

The postulated equality $C_T = 0$ is a convenient way of ensuring that any collateral amount posted is returned in full to its owner when a contract matures, provided that the default event does not occur at $T$. Of course, if the default event is also modeled, then one needs to specify the closeout payoff. Let us mention that the case of an exogenously given collateral was studied in [33, 34], whereas the case of an endogenous collateral (as given, for instance by equation (4.10) below) was examined in [35].

## 4.1 Collateral Conventions

In the market practice, the complexity of the issue of collateralization is enormous and obviously beyond the scope of this work, in which we will only focus on the impact of collateralization on the dynamics of the hedger's portfolio and thus on the valuation from the perspective of the hedger. Let us first make some comments regarding the crucial features of the margin accounts that underpin our stylized approach to the costs of margining. As usual, we take the perspective of the hedger.

- The current financial practice typically requires the collateral amounts to be held in *segregated* margin accounts, so that the hedger, when he is a collateral taker, cannot make use of the collateral amount for trading. Therefore, under segregation the hedger's wealth dynamics do not depend on whether the collateral amount was posted by the counterparty in cash or shares of a risky asset $S^{d+2}$. By contrast, the character of delivered assets always matters to him when the hedger is a collateral giver.

- Another collateral convention encountered in practice is *rehypothecation*, which refers to the situation where a bank is allowed to reuse the collateral pledged by its counterparties as collateral for its own borrowing. In our approach to rehypothecation, we will distinguish between the case when the collateral amount was delivered to the hedger in the form of shares of a risky asset (and thus it can only be reused as a collateral) and the case of cash collateral where it can be used for an outright trading.

- If the hedger is a collateral giver, then a particular convention regarding segregation or rehypothecation is immaterial for the wealth dynamics of his portfolio. Of course, the distinction between segregation and rehypothecation becomes important when the closeout payoff at default of either the hedger or the counterparty is evaluated. However, the latter issue, as well as a rather complex mechanism of updating the margin account, are left aside, since they were already thoroughly studied in the literature.

We first introduce the general notation, which will be used when analyzing various conventions regarding collateralization. Let us make clear that we set here to introduce an abstract setup, which is flexible enough to cover various collateral convention. By contrast, we do not pretend that any particular convention should be seen as either a prevailing or a desirable one.

**Definition 4.1** A *collateralized hedger's trading strategy* is a quadruplet $(x, \varphi, A, C)$ where a portfolio $\varphi$, given by

$$\varphi = \left(\xi^1, \dots, \xi^{d+1}, \psi^0, \dots, \psi^{d+1}, \eta^b, \eta^l, \eta^{d+2}\right) \tag{4.2}$$

is composed of the *risky assets* $S^i$, $i = 1, 2, \dots, d+1$, the *unsecured cash account* $B^0 = B$, the *funding accounts* $B^i$, $i = 1, 2, \dots, d+1$, the *borrowing account* $B^{d+1}$ for the posted cash collateral, the *collateral accounts* $B^{c,b}$ and $B^{c,l}$, and the *lending account* $B^{d+2}$ associated with the received collateral asset $S^{d+2}$.

**Remark 4.1** The collateral account $B^{c,b}$ (resp. $B^{c,l}$) plays the following role: if the hedger receives (resp. posts) cash or risky collateral with the equivalent cash value $C^+$ (resp. $C^-$), then he pays (resp. receives) interest on this nominal amount, as specified by the process $B^{c,b}$ (resp. $B^{c,l}$).

All funding and collateral accounts are assumed to be continuous processes of finite variation, whereas the price of the collateral asset $S^{d+1}$ is assumed to be a càdlàg semimartingale. Let us formulate two definitions, which clarify the distinction between the conventions of the *risky asset collateral* and the *cash collateral* and introduce the notation, which will be used in what follows.

**Definition 4.2** The *risky collateral* is described by the following postulates:

- If the hedger receives at time $t$ the number $\xi_t^{d+2} > 0$ of shares of the risky asset $S^{d+2}$ as collateral, then he pays to the counterparty interest determined by the amount $C_t^+ = \xi_t^{d+2} S_t^{d+2}$ and the collateral account $B^{c,b}$. Formally, there is no reason to postulate that the process $\xi^{d+2}$ is a component of the hedger's trading strategy. However, in cases where the collateral amount is related to the hedging strategy, this process is explicitly given in terms of the wealth of the hedger's portfolio. Under *segregation*, the hedger also receives (possibly null) interest determined by the amount $C_t^+$ and the account $B^{d+2,s}$, whereas under *rehypothecation*, he also receives interest determined by the amount $C_t^+$ and the secured funding account $B^{d+2,h}$.
- If the hedger posts a collateral at time $t$, then he delivers $\xi_t^{d+1} > 0$ of shares of the risky asset $S^{d+1}$ funded from the (unsecured) funding account $B^{d+1}$ and he receives interest determined by the amount $C_t^- = \xi_t^{d+1} S_t^{d+1}$ and the collateral account $B^{c,l}$. Formally, we thus postulate that
$$\xi_t^{d+1} S_t^{d+1} = C_t^-, \quad \xi_t^{d+1} S_t^{d+1} + \psi_t^{d+1} B_t^{d+1} = 0. \tag{4.3}$$
This implies, in particular, that the equality $\psi_t^{d+1} B_t^{d+1} = -C_t^-$ holds for all $t$.

Note that the lending account $B^{d+2}$ is equal to $B^{d+2,s}$ or $B^{d+2,h}$, depending on the adopted collateral convention. In practice, deliverable collateral assets should have low credit risk and should be uncorrelated with the underlying trading portfolio. For this reason, it is assumed in Definition 4.2 that, even under rehypothecation, the received risky asset $S^{d+2}$ cannot be used for hedging purposes. We assume instead that it yields interest, denoted by $B^{d+2,h}$, by being pledged as collateral in a repo contract and thus raising an equivalent amount $C^+$ of cash. Note that the hedger's relative advantages when $S^{d+2}$ can be pledged as collateral in another contract are not examined here. For this purpose, one would need to consider a portfolio of hedger's contracts, rather than to study a single contract. Under segregation, the account $B^{d+2,s}$ reflects a (perhaps unlikely in practice) possibility that the hedger receives interest from the collateral custodian, whenever he maintains a positive equivalent cash amount $C^+$ in the segregated account. We now proceed to the case when all collateral amounts are delivered in cash.

**Definition 4.3** The *cash collateral* is described by the following postulates:

- If the hedger receives at time $t$ the amount $C_t^+$ as collateral in cash, then he pays to the counterparty interest determined by the amount $C_t^+$ and the account $B^{c,b}$. Under segregation, he receives interest determined by the amount $C_t^+$ and the account $B^{d+2,s}$. When rehypothecation is considered, the hedger may temporarily (that is, before the contract's maturity date or the default time, whichever comes first) utilize the cash amount $C_t^+$ for his trading purposes.
- If the hedger posts cash collateral at time $t$, then the collateral amount is borrowed from the dedicated *collateral borrowing account* $B^{d+1}$ (which, of course, may coincide with $B$). He receives interest determined by the amount $C_t^-$ and the collateral account $B^{c,l}$. Instead of (4.3), we now postulate that

$$\xi_t^{d+1} = 0, \quad \psi_t^{d+1} B_t^{d+1} = -C_t^- . \tag{4.4}$$

In the context of a collateralized contract, we find it convenient to introduce the following three processes:
• the process $V_t(x, \varphi, A, C)$ representing the hedger's wealth at time $t$,
• the process $V^p(x, \varphi, A, C)$ representing the value of hedger's portfolio at time $t$,
• the adjustment process $V_t^c(x, \varphi, A, C) := V_t(x, \varphi, A, C) - V_t^p(x, \varphi, A, C)$, which measures the impact of the margin account.

An explicit specification of the process $V^c(x, \varphi, A, C)$ depends on the adopted collateral convention, however, we always have that $V^c(x, \varphi, A, C) = 0$ when $C$ vanishes, so that the adjustment is not needed. Let us consider, for example, the case where $V_t^c(x, \varphi, A, C) = -C_t$. Then the portfolio's value satisfies $V_t^p(x, \varphi, A, C) = V_t(x, \varphi, A, C) + C_t$, meaning that the hedger also invests in his portfolio of traded assets the collateral amount $C_t^+$ received at time $t$. By contrast, when he posts collateral at time $t$, then the portfolio's value is computed by subtracting the posted amount $C_t^-$ from his wealth. At time 0, for the hedger with the initial endowment $x$, we have $V_0(x, \varphi, 0, 0) = x$, $V_0(x, \varphi, A, C) = x + A_0$ and $V_0^p(x, \varphi, A, C) = x + A_0 + C_0$, where the first two equalities are always true and the last one is a special case of the general relationship: $V_0^p(x, \varphi, A, C) = x + A_0 - V_0^c(x, \varphi, A, C)$.

We are now in a position to formally define the processes $V(x, \varphi, A, C)$, $V_t^p(x, \varphi, A, C)$ and $V_t^c(x, \varphi, A, C)$ in our framework. For alternative explicit specifications of the process $\eta^{d+2}$, we refer to Propositions 4.1, 4.2 and 4.3. Similarly, in the next definition, we formally identify $B^{d+2,h}$ with $B^{d+2,s}$ and we denote them generically as $B^{d+2}$. This is possible, since these two accounts will play a similar role in our further computations, although their financial interpretation is different and thus in practice they are not necessarily equal.

**Definition 4.4** The hedger's *portfolio's value* $V^p(x, \varphi, A, C)$ is given by

$$V_t^p(x, \varphi, A, C) = \sum_{i=1}^{d+1} \xi_t^i S_t^i + \sum_{j=0}^{d+1} \psi_t^j B_t^j . \tag{4.5}$$

The hedger's *wealth* $V(x, \varphi, A, C)$ equals

$$V_t(x, \varphi, A, C) = \sum_{i=1}^{d+1} \xi_t^i S_t^i + \sum_{j=0}^{d+1} \psi_t^j B_t^j + \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l} + \eta_t^{d+2} B_t^{d+2}. \tag{4.6}$$

The *adjustment process* $V^c(x, \varphi, A, C)$ satisfies

$$V_t^c(x, \varphi, A, C) = \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l} + \eta_t^{d+2} B_t^{d+2} = -C_t + \eta_t^{d+2} B_t^{d+2} \tag{4.7}$$

where $\eta_t^b = -(B_t^{c,b})^{-1} C_t^+$ and $\eta_t^l = (B_t^{c,l})^{-1} C_t^-$ (see Remark 4.1).

Various specifications of the adjustment process $V^c(\varphi)$ are now encoded in the process denoted generically as $\eta^{d+2}$, which will be sometimes complemented by superscripts $s$ or $h$, so that it can also be denoted as $\eta^{d+2,s}$ or $\eta^{d+2,h}$. For explicit specifications of these processes, we refer to Propositions 4.1, 4.2 and 4.3.

The self-financing property of the hedger's strategy is defined in terms of the dynamics of the value process of his portfolio. This definition is a natural extension of Definition 2.3 the case of collateralized contracts. Note that we use here the process $V^p(x,\varphi,A,C)$, and not $V(x,\varphi,A,C)$ as was done in Definition 2.3, to emphasize the role of $V^p(x,\varphi,A,C)$ as the value of the hedger's portfolio of traded assets (recall also that $V^p(x,\varphi,A,C) = V(x,\varphi,A,C)$ when the process $C$ vanishes).

**Definition 4.5** A collateralized hedger's trading strategy $(x,\varphi,A,C)$ with $\varphi$ given by (4.2) is *self-financing* whenever the *portfolio's value* $V^p(x,\varphi,A,C)$, which is given by (4.5), satisfies, for every $t \in [0,T]$,

$$V^p_t(x,\varphi,A,C) = x + \sum_{i=1}^{d+1} \int_{(0,t]} \xi^i_u \, d(S^i_u + A^i_u) + \sum_{j=0}^{d+1} \int_0^t \psi^j_u \, dB^j_u + A_t \tag{4.8}$$
$$+ \int_0^t \eta^b_u \, dB^{c,b}_u + \int_0^t \eta^l_u \, dB^{c,l}_u + \int_0^t \eta^{d+2}_u \, dB^{d+2}_u - V^c_t(x,\varphi,A,C).$$

It is clear that the terms $\int_0^t \eta^b_u \, dB^{c,b}_u$, $\int_0^t \eta^l_u \, dB^{c,l}_u$ and $\int_0^t \eta^{d+2}_u \, dB^{d+2}_u$ represent the cumulative interest due to the presence of the margin account. The first two processes are given explicitly in terms of $C$ since $\eta^b_t = -(B^{c,b}_t)^{-1}C^+_t$ and $\eta^l_t = (B^{c,l}_t)^{-1}C^-_t$, whereas the last one depends on the collateral convention.

**Remark 4.2** As was already mentioned, the process $V^c(x,\varphi,A,C)$ is aimed to measure the impact of the margin account on the part of hedger's wealth that can be used for trading in primary traded assets. Typically, it is given as $V^c_t(\varphi) = g(C_t(\varphi))$ for some real function $g$ (typically, $g(x) = -x$ or $g(x) = x^-$). Hence, in view of Assumption 4.1, the equality $V_T(x,\varphi,A,C) = V^p_T(x,\varphi,A,C)$ is always satisfied provided that $g(0) = 0$. In the remainder of this work (with one exception, namely, Section 5.3.4), we have that either $V^c(x,\varphi,A,C) = -C$ or $V^c(x,\varphi,A,C) = C^-$ for an exogenously given process $C$. Then equations (4.5) and (4.8) are autonomous, so that they uniquely specify the portfolio's value $V^p(\varphi)$, meaning that we do not use (4.6) for this purpose. One can observe that we formally deal here with an example of a self-financing strategy with the wealth $V^p_t(x,\varphi,A,C)$, in the sense of Definition 2.3, but where the process $A$ is substituted with $A - V^c(x,\varphi,A,C)$.

Although these are obviously very important practical issues, neither an explicit specification of the process $C$, nor the rules governing the way in which the margin account is adjusted, are studied in detail here. Let us only remark that the collateral amount is typically tied to the regularly updated *marked-to-market value* of a contract, whose level at time $t$ is henceforth denoted as $M_t$. In that case, the process $C$ can be specified as follows

$$C_t = (1+\delta^1_t)M_t \mathbb{1}_{\{M_t>0\}} + (1+\delta^2_t)M_t \mathbb{1}_{\{M_t<0\}} = (1+\delta^1_t)M^+_t - (1+\delta^2_t)M^-_t \tag{4.9}$$

for some non-negative *haircut processes* $\delta^1$ and $\delta^2$. In our theoretical framework, the goal is to develop valuation of a contract based on hedging, so that it is natural to relate the marked-to-market value to the (so far unspecified) hedger's value of a contract. To be more specific, since the wealth process $V(\varphi)$ of the hedger is aimed to cover his future liabilities, it is natural to postulate that the stylized 'market value' of a contract, as seen by the hedger, coincides with the negative of his wealth. Consequently, we formally identify the marked-to-market value $M$ with the negative of the wealth of the hedger's portfolio. More precisely, one may set $M_t = V^0_t(x) - V(x,\varphi,A,C)$ (see also Definition 5.3 of the ex-dividend price of a contract $(A,C)$ for a justification of this postulate). Then formula (4.9) becomes

$$C_t = C_t(\varphi) := (1+\delta^1_t)(V^0_t(x) - V_t(x,\varphi,A,C))^+ - (1+\delta^2_t)(V^0_t(x) - V_t(x,\varphi,A,C))^-. \tag{4.10}$$

The case of a *fully collateralized contract* is obtained by setting $\delta_t^1 = \delta_t^2 = 0$ for all $t$ in (4.10), which implies that the process $C(\varphi)$ is implicitly given by the equation $C(\varphi) = V_t^0(x) - V(x,\varphi,A,C)$. Of course, an analogous analysis can be done for the counterparty. However, since the market conditions will typically be different for the two parties, it is unlikely that their computations of the contract's value (hence the collateral amount) will yield the same value. Obviously, specification (4.10) of collateral amount with the wealth $V(\varphi)$ computed by the hedger makes practical sense only when it is bilaterally accepted in the contract's CSA (*Credit Support Annex*).

The remaining part of this section is organized as follows. First, in Proposition 4.1, which covers both segregation and rehypothecation, we derive more explicit representation for the dynamics of the hedger's wealth in the case of a risky collateral. Subsequently, in Propositions 4.2 and 4.3, we examine the case of cash collateral under segregation and rehypothecation, respectively.

## 4.2 Risky Collateral

In this subsection, we work under the assumption that the collateral amount is delivered by the hedger in the form of shares of the risky asset $S^{d+1}$ and we follow the conventions described in Definition 4.2. In particular, from (4.3), we see that the net wealth invested in risky asset $S^{d+1}$ and the account $B^{d+1}$ is null. We denote by $F^h$ the process given by

$$F_t^h := F_t^c + \int_0^t C_u^+ (B_u^{d+2,h})^{-1}\, dB_u^{d+2,h} \tag{4.11}$$

where $F^c$ is the *cumulative interest of margin account*

$$F_t^c := \int_0^t C_u^- (B_u^{c,l})^{-1}\, dB_u^{c,l} - \int_0^t C_u^+ (B_u^{c,b})^{-1}\, dB_u^{c,b}. \tag{4.12}$$

We will show that the process $F^h$ represents all positive and negative cash flows from the margin account under rehypothecation, as specified by Definition 4.2. Note that if segregation of the delivered asset $S^{d+2}$ is postulated, then all statements in Proposition 4.1 remain valid if $B^{d+2,h}$ is substituted with $B^{d+2,s}$ and thus this result also covers the case of a segregated risky collateral. In the latter case, the symbol $F^h$ will be replaced by $F^s$.

**Proposition 4.1** *Consider the case of the segregated margin account when the collateral is posted in shares of a risky asset $S^{d+1}$ and received in any form. Assume that a trading strategy $(x,\varphi,A,C)$ is self-financing and the following equalities hold, for all $t \in [0,T]$,*

$$\xi_t^{d+1} = (S_t^{d+1})^{-1} C_t^-, \quad \psi_t^{d+1} = -(B_t^{d+1})^{-1} C_t^-, \quad \eta_t^{d+2} = (B_t^{d+2,h})^{-1} C_t^+. \tag{4.13}$$

*Then the hedger's wealth $V(\varphi) = V(x,\varphi,A,C)$ equals, for every $t \in [0,T]$,*

$$V_t(\varphi) = V_t^p(\varphi) + C_t^- = \sum_{i=1}^d \xi_t^i S_t^i + \sum_{j=0}^d \psi_t^j B_t^j + C_t^- \tag{4.14}$$

*and the dynamics of the portfolio's value $V^p(\varphi) = V^p(x,\varphi,A,C)$ are*

$$dV_t^p(\varphi) = \widetilde{V}_t^p(\varphi)\, dB_t + \sum_{i=1}^d \xi_t^i\, dK_t^i + (S_t^{d+1})^{-1} C_t^-\, dK_t^{d+1} + \sum_{i=1}^d \zeta_t^i (\widetilde{B}_t^i)^{-1}\, d\widetilde{B}_t^i + dA_t^h - dC_t^- \tag{4.15}$$

*where $\widetilde{V}_t^p(\varphi) := (B_t)^{-1} V_t^p(\varphi)$ and $A^h := A + F^h$. In particular, under assumption (2.16), we obtain*

$$dV_t(\varphi) = \widetilde{V}_t(\varphi)\, dB_t + \sum_{i=1}^d \xi_t^i B_t^i\, d\widehat{S}_t^{i,cld} + (S_t^{d+1})^{-1} C_t^- B_t^{d+1}\, d\widehat{S}_t^{d+1,cld} + d\bar{F}_t^h + dA_t \tag{4.16}$$

*where*

$$\bar{F}^h_t := F^c_t + \int_0^t C^+_u (B^{d+2,h}_u)^{-1}\, dB^{d+2,h}_u - \int_0^t C^-_u (B_u)^{-1}\, dB_u. \tag{4.17}$$

*The hedger's wealth admits the following decomposition*

$$V_t(\varphi) = x + G_t(\varphi) + F_t(\varphi) + F^h_t + A_t \tag{4.18}$$

*where $G_t(\varphi)$ is given by* (2.7) *with $d$ replaced by $d+1$ and $F_t(\varphi)$ satisfies* (2.8) *with $d$ replaced by $d+1$.*

*Proof.* Equality (4.14) is an immediate consequence of the specification of $\varphi$ and assumptions (4.13). We now focus on dynamics of the process $V^p(\varphi)$. First, we observe that, in view of (4.13), we have $\zeta^{d+1}_t := \xi^{d+1}_t S^{d+1}_t + \psi^{d+1}_t B^{d+1}_t = 0$. Second, the term $F^h$, which is deduced from (4.8) and (4.13), may be combined with $A$ to yield $A^h = A + F^h$. We are now in a position to apply Corollary 2.1 to the process $V^p(\varphi)$ satisfying (4.5)–(4.8). This yields equality (4.15), which in turn after simple computations becomes (4.16) when $\zeta^i = 0$ for all $i = 1, 2, \dots, d$. Finally, decomposition is immediate from (4.14) and (4.8). □

**Remark 4.3** If the assumption that $B^l = B^b = B$ is relaxed, then the dynamics of the portfolio's value (hence also the hedger's wealth) should be adjusted along the same lines as in Section 2.3. Specifically, if $\zeta^i = 0$ for all $i$, then we obtain the following equality, which combines formulae (2.35) and (4.15),

$$dV^p_t(\varphi) = \psi^l_t\, dB^l_t + \psi^b_t\, dB^b_t + \sum_{i=1}^d \xi^i_t\, dK^i_t + (S^{d+1}_t)^{-1} C^-_t\, dK^{d+1}_t + dA^h_t - dC^-_t \tag{4.19}$$

where the processes $\psi^l_t$ and $\psi^b_t$ satisfy

$$\psi^l_t = (B^l_t)^{-1}\Big(V^p_t(\varphi) - \sum_{i=1}^d \xi^i_t S^i_t - \sum_{i=1}^d \psi^i_t B^i_t - C^-_t\Big)^+$$

and

$$\psi^b_t = -(B^b_t)^{-1}\Big(V^p_t(\varphi) - \sum_{i=1}^d \xi^i_t S^i_t - \sum_{i=1}^d \psi^i_t B^i_t - C^-_t\Big)^-.$$

Formula (4.19) leads to a suitable extension of Proposition 4.1. Similar extensions can be derived for the case of cash collateral; since they are rather straightforward, they are rather to the reader.

## 4.3 Cash Collateral

In this section, we work under the conventions of cash collateral, as specified in Definition 4.3. Of course, the risky assets $S^{d+1}$ plays no role in this subsection and thus its existence can be safely ignored. Formally, we will postulate that $\xi^{d+1}_t = 0$ for all $t$.

### 4.3.1 Margin Account under Segregation

Assume first that the cash amount received by the hedger as collateral cannot be used for trading. Then only the interest on $C^+$, denoted as $B^{d+2,s}$, matters and the fact that the collateral is received in cash is immaterial here. The cash amount $C^-$ posted by the hedger is borrowed from the account $B^{d+1}$ and it yields interest paid by the counterparty, as determined by the process $B^{c,l}$. These features of the margin account are reflected through equalities (4.20) in the statement of the next result. Recall that the process $F^s$ is given by (4.11) with the superscript $h$ substituted with $s$.

**Proposition 4.2** *Consider the case of the segregated margin account when the collateral is posted by the hedger in cash borrowed from the account $B^{d+1}$ and it is received in any form. Assume that a trading strategy $(\varphi, A, C)$ is self-financing and the following equalities hold, for all $t \in [0,T]$,*

$$\xi_t^{d+1} = 0, \quad \psi_t^{d+1} = -(B_t^{d+1})^{-1} C_t^-, \quad \eta_t^{d+2,s} = (B_t^{d+2,s})^{-1} C_t^+. \tag{4.20}$$

*Then the hedger's wealth $V(\varphi) = V(x, \varphi, A, C)$ equals, for every $t \in [0,T]$,*

$$V_t(\varphi) = V_t^p(\varphi) + C_t^- = \sum_{i=1}^d \xi_t^i S_t^i + \sum_{j=0}^{d+1} \psi_t^j B_t^j + C_t^- \tag{4.21}$$

*and the dynamics of the portfolio's value $V^p(\varphi) = V^p(x, \varphi, A, C)$ are*

$$dV_t^p(\varphi) = \widetilde{V}_t^p(\varphi)\, dB_t + \sum_{i=1}^d \xi_t^i\, dK_t^i + \sum_{i=1}^d \zeta_t^i (\widetilde{B}_t^i)^{-1}\, d\widetilde{B}_t^i - C_t^- (B_t^{d+1})^{-1}\, dB_t^{d+1} + dA_t^s - dC_t^- \tag{4.22}$$

*where $\widetilde{V}_t^p(\varphi) := (B_t)^{-1} V_t^p(\varphi)$ and $A^s := A + F^s$. In particular, under assumption* (2.16) *we obtain*

$$dV_t(\varphi) = \widetilde{V}_t(\varphi)\, dB_t + \sum_{i=1}^d \xi_t^i B_t^i\, d\widehat{S}_t^{i,cld} - C_t^- (B_t^{d+1})^{-1}\, dB_t^{d+1} + dF_t^s + dA_t \tag{4.23}$$

*or, equivalently,*

$$dV_t(\varphi) = \widetilde{V}_t(\varphi)\, dB_t + \sum_{i=1}^d \xi_t^i B_t^i\, d\widehat{S}_t^{i,cld} + d\widehat{F}_t^s + dA_t \tag{4.24}$$

*where*

$$\widehat{F}_t^s := F_t^c + \int_0^t C_u^+ (B_u^{d+2,s})^{-1}\, dB_u^{d+2,s} - \int_0^t C_u^- (B_u^{d+1})^{-1}\, dB_u^{d+1}.$$

*The hedger's wealth admits the following decomposition*

$$V_t(\varphi) = x + G_t(\varphi) + F_t(\varphi) + \widehat{F}_t^s + A_t \tag{4.25}$$

*where $G_t(\varphi)$ is given by* (2.7) *and $F_t(\varphi)$ satisfies* (2.8).

*Proof.* We use the arguments similar to those in the proof of Proposition 4.1. We start by noting that (4.21) yields

$$V_t^p(\varphi) = \sum_{i=1}^d \xi_t^i S_t^i + \psi_t^0 B_t + \sum_{i=1}^{d+1} \psi_t^i B_t^i.$$

Hence, in view of (4.8), we may observe that we deal here with a self-financing strategy $(\varphi, A)$ introduced in Definition 4.5 with $d$ replaced by $d+1$ and $A^s = A + F^s$ such that $\zeta_t^{d+1} = \psi_t^{d+1} B_t^{d+1} = -C_t^-$. An application of Corollary 2.1 gives (4.22). The equivalence of (4.23) and (4.24) follows by direct computations using the equality $V(\varphi) = V^p(\varphi) + C^-$ and the assumptions that $B$ and $B^{d+1}$ are continuous processes of finite variation. □

### 4.3.2 Margin Account under Rehypothecation

In the case of cash collateral under rehypothecation, we assume that the hedger, when he is a collateral taker, is granted an unrestricted use of the full collateral amount $C^+$. As usual, we postulate that the hedger then pays interest to the counterparty determined by the collateral amount $C^+$ and $B^{c,b}$. Furthermore, we assume that when the hedger is a collateral giver, then collateral is delivered in cash and he receives interest specified by $C^-$ and $B^{c,l}$. We maintain the assumption that the hedger borrows cash for collateral delivered to the counterparty from the dedicated account $B^{d+1}$. Of course, the case when $B^{d+1} = B$ is not excluded, but we decided to use a different symbol for the dedicated account to facilitate identification of each cash flow. Recall that the process $F^h$, which is now given by expression (4.11) with $B^{d+2,h} = B$, is aimed to represent the cash flows from the margin account under rehypothecation, as specified by Definition 4.3. The proof of the next result is also based on Corollary 2.1 and thus it is omitted.

**Proposition 4.3** *Consider the case of a rehypothecated margin account when the cash collateral is posted and received by the hedger. We assume that a trading strategy $(\varphi, A, C)$ is self-financing and the following equalities hold, for all $t \in [0,T]$,*

$$\xi_t^{d+1} = 0, \quad \psi_t^{d+1} = -(B_t^{d+1})^{-1} C_t^-, \quad \eta_t^{d+2,h} = 0. \tag{4.26}$$

*Then the hedger's wealth $V(\varphi) = V(x, \varphi, A, C)$ equals, for every $t \in [0,T]$,*

$$V_t(\varphi) = V_t^p(\varphi) - C_t = \sum_{i=1}^d \xi_t^i S_t^i + \sum_{j=0}^{d+1} \psi_t^j B_t^j - C_t \tag{4.27}$$

*and the dynamics of the portfolio's value $V^p(\varphi) = V^p(x, \varphi, A, C)$ are*

$$dV_t^p(\varphi) = \widetilde{V}_t^p(\varphi)\, dB_t + \sum_{i=1}^d \xi_t^i \, dK_t^i + \sum_{i=1}^d \zeta_t^i (\widetilde{B}_t^i)^{-1} \, d\widetilde{B}_t^i - C_t^- (B_t^{d+1})^{-1} \, dB_t^{d+1} + dA_t^h + dC_t \tag{4.28}$$

*where $\widetilde{V}_t^p(\varphi) := (B_t)^{-1} V_t^p(\varphi)$ and $A^h := A + F^h$. In particular, under assumption* (2.16) *we obtain*

$$dV_t(\varphi) = \widetilde{V}_t(\varphi)\, dB_t + \sum_{i=1}^d \xi_t^i B_t^i \, d\widehat{S}_t^{i,cld} - C_t^- (B_t^{d+1})^{-1} \, dB_t^{d+1} + dF_t^h + dA_t \tag{4.29}$$

*or, equivalently,*

$$dV_t(\varphi) = \widetilde{V}_t(\varphi)\, dB_t + \sum_{i=1}^d \xi_t^i B_t^i \, d\widehat{S}_t^{i,cld} + d\widehat{F}_t^h + dA_t \tag{4.30}$$

*where*

$$\widehat{F}_t^h := F_t^c + \int_0^t C_u^+ (B_u)^{-1} \, dB_u - \int_0^t C_u^- (B_u^{d+1})^{-1} \, dB_u^{d+1}. \tag{4.31}$$

*The hedger's wealth admits the following decomposition*

$$V_t(\varphi) = x + G_t(\varphi) + F_t(\varphi) + \widehat{F}_t^h + A_t \tag{4.32}$$

*where $G_t(\varphi)$ is given by* (2.7) *and $F_t(\varphi)$ satisfies* (2.8).

# 5 Pricing under Funding Costs and Collateralization

We will now focus on valuation of a collateralized contract that can be replicated by the hedger with the initial endowment $x$ at time 0. We consider throughout the hedger's self-financing trading strategies $(x, \varphi, A, C)$, as specified by Definition 4.5 and, unless explicitly stated otherwise, we postulate that condition (2.16) is met. It will be implicitly assumed that all trading strategies considered in what follows are *admissible*, in a suitable sense.

As usual, the price of a contract will be defined from the perspective of a hedger. We assume that $p_0 = A_0$ is an unknown real number, which should be found through contract's replication, whereas the *cumulative dividend stream* $A - A_0$ of a contract $A$ is predetermined. Therefore, by pricing of $A$, we mean in fact valuation of the cumulative dividend stream $A - A_0$ (or $A - A_t$ if we search for the price of $A$ at time $t$), which is supplemented by the collateral process $C$.

**Definition 5.1** For a fixed $t \in [0,T]$, let $p_t$ be a $\mathcal{G}_t$-measurable random variable. We say that a self-financing trading strategy $(V_t^0(x)+p_t,\varphi,A-A_t,C)$ *replicates the collateralized contract* $(A,C)$ on $[t,T]$ whenever $V_T(V_t^0(x)+p_t,\varphi,A-A_t,C)=V_T^0(x)$.

In the next definition, we consider the situation when the hedger with the initial wealth $x$ at time 0 enters the contract $A$ at time $t$.

**Definition 5.2** Any $\mathcal{G}_t$-measurable random variable $p_t$ for which a replicating strategy for $(A,C)$ over $[t,T]$ exists is called the *ex-dividend price at time t of the contract A associated with* $\varphi$ and it is denoted by $S_t(x,\varphi,A,C)$.

It is worth noting that for $t=0$ we always have that $p_0=A_0$ and thus, for any portfolio $\varphi$, the strategies $(x+p_0,\varphi,A-A_0,C)$ and $(x,\varphi,A,C)$ are in fact identical. Therefore, we may simply say that a self-financing trading strategy $(x,\varphi,A,C)$ *replicates* $(A,C)$ on $[0,T]$ whenever the equality $V_T(x,\varphi,A,C)=V_T^0(x)$ holds. This equality is in fact consistent with equation (3.6) in Definition 3.5 of a hedger's fair price, so we conclude that any ex-dividend price $p_0$ of $A$ at time 0 is also a hedger's fair price $\bar{p}_0$ for $A$ at time 0 (though the converse does not hold in general).

**Remark 5.1** In general, the ex-dividend price $S_t(x,\varphi,A,C)$ depends on $x$, so that the knowledge of the hedger's initial endowment is essential for our (non-linear) pricing rule. In the special case when $x=0$, the price $p_t$ at time $t$ corresponds to the existence of a trading strategy $\varphi$ such that $V_T(p_t,\varphi,A-A_t,C)=0$. In particular, when $x=0$, $C=0$ and the process $A-A_t$ is given as a single cash flow $X$ at time $T$, then $p_t$ is the initial wealth of a self-financing strategy $\varphi$ with the wealth equal to $-X$ just prior to $T$, more precisely, the wealth satisfying the equality $V_{T-}(\varphi)=-X$ (since here $\Delta A_T=A_T-A_{T-}=X$). In a frictionless market model, we thus obtain the classic definition of the replicating price of a European claim $X$.

It is not difficult to check that necessarily $S_T(x,\varphi,A,C)=0$ for any contract $A$. By contrast, it is not clear a priori whether $S_t(x,\varphi,A,C)$ for some $t<T$ depends on the initial endowment $x$ and a portfolio $\varphi$ (recall also that $C=C(\varphi)$, in general). If model's where the uniqueness of $S_t(x,\varphi,A,C)$ fails to hold, it would be natural to search for the least expensive way of replication for a given initial endowment $x$. One could also address the issue of finding the least expensive way of super-hedging a contract $A$ by imposing the weaker condition that $V_T(x,\varphi,A,C)\geq V_T^0(x)$ instead of insisting on the equality $V_T(x,\varphi,A,C)=V_T^0(x)$.

If we assume that the hedger can replicate the contract $A$ on $[0,T]$ using a trading strategy initiated at time 0, then it is not necessarily true that, starting with the initial endowment $V_t^0(x)$ at some date $0<t<T$, he can also replicate the cumulative dividend stream $A-A_t$ representing the contract $A$ restricted to the interval $[t,T]$. Let us thus consider the situation when a contract $(A,C)$ can be replicated on $[0,T]$. Then we may propose an alternative definition of an ex-dividend price at time $t$. In fact, Definition 5.3 mimics the classic definition of arbitrage price obtained through replication of a contingent claim when $x=0$. Of course, in the classic case, we may assume, without loss of generality that $x=0$, since arbitrage prices obtained through replication is independent of the hedger's initial endowment.

**Definition 5.3** Assume that a self-financing trading strategy $(x,\varphi,A,C)$ replicates $(A,C)$ on $[0,T]$. Then the process $\widehat{p}_t := V_t(x,\varphi,A,C)-V_t^0(x)$ is called the *valuation ex-dividend price of A associated with* $\varphi$ and it is denoted by $\widehat{S}_t(x,\varphi,A,C)$.

We note that the equality $\widehat{S}_T(x,\varphi,A,C)=0$ is always satisfied. Furthermore, when $x=0$, Definition 5.3 states that the reduced ex-dividend price of $A$ associated with $\varphi$ is simply the wealth $V(0,\varphi,A,C)$ of a replicating strategy. Observe also that a replicating strategy for the hedger with null initial endowment starts from the initial wealth $p_0$ at time 0 and terminates with null wealth at time $T$. We will argue that Definitions 5.1 and 5.3 of ex-dividend prices are equivalent in the basic model with funding costs (where indeed the ex-dividend prices will be shown to be independent of $x$ and $\varphi$, provided that the collateral process $C$ is exogenously given), but the two prices do not necessarily coincide in a generic market model with different borrowing and lending rates and/or other restrictions on trading. The latter observation and the aim to cover all sorts of market restrictions, not necessarily exemplified in what follows, motivated us to introduce a more general Definition 5.1, which is subsequently used in Definition 5.3, which is sufficient in simpler models.

## 5.1 Basic Model with Funding Costs and Collateralization

We consider the basic model with funding costs introduced in Section 3.2.2 and we postulate that:
(i) the assumptions of Proposition 3.1 are met, so that the model is arbitrage-free for the hedger,
(ii) the collateral process $C$ is exogenously given, that is, it is independent of a hedger's portfolio $\varphi$.

We assume that the random variables whose conditional expectations are evaluated are integrable and we write $\widetilde{\mathbb{E}}_t(\cdot) := \mathbb{E}_{\widetilde{\mathbb{P}}}(\cdot \,|\, \mathcal{G}_t)$ where $\widetilde{\mathbb{P}}$ is any martingale measure for the processes $\widehat{S}^{i,\mathrm{cld}}$, $i = 1,2,\dots,d$ (for the existence of $\widetilde{\mathbb{P}}$, see Proposition 3.1). We use a generic symbol $\widehat{A}^c$ to denote either of the processes $A + \bar{F}^h$, $A + \widehat{F}^s$ or $A + \widehat{F}^h$, depending on the adopted convention for the margin account, and we assume that the process $\widehat{A}^c$ is bounded. Also, we postulate that the cash account process $B$ is increasing.

We first show that, under mild technical assumptions, the price can be computed using the conditional expectation under $\widetilde{\mathbb{P}}$. It is worth noting that the impact of collateralization is relatively easy to handle in the present setting by quantifying additional gains or losses generated by the margin account, as explicitly given by either of processes $\bar{F}^h$, $\widehat{F}^s$ or $\widehat{F}^h$, and aggregating them with the cumulative cash flows $A$. We write here $S(A,C)$, rather than $S(x,\varphi,A,C)$, in order to emphasize that, under the present assumptions, the price does not depend on $(x,\varphi)$.

**Proposition 5.1** *Under assumptions (i)–(ii), if the collateralized contract $(A,C)$ can be replicated by an admissible trading strategy $(x,\varphi,A,C)$ on $[0,T]$ and the stochastic integrals with respect to $\widehat{S}^{i,cld}$, $i = 1,2,\dots,d$ in (5.4) (or with respect to $\widehat{S}^{i,cld}$, $i = 1,2,\dots,d+1$ in (5.5)) are $\widetilde{\mathbb{P}}$-martingales, then its ex-dividend price process $S(x,\varphi,A,C)$ is independent of $(x,\varphi)$ and equals, for all $t \in [0,T]$,*

$$S_t(A,C) = -B_t \, \widetilde{\mathbb{E}}_t \bigg( \int_{(t,T]} B_u^{-1} \, d\widehat{A}^c_u \bigg). \tag{5.1}$$

*Proof.* Assume that a strategy $(x,\varphi,A,C)$ replicates the collateralized contract $(A,C)$ on $[t,T]$. By applying (4.16), we obtain

$$d\widetilde{V}_t(x,\varphi,A,C) = \sum_{i=1}^d \xi^i_t \widetilde{B}^i_t \, d\widehat{S}^{i,\mathrm{cld}}_t + B_t^{-1} (S^{d+1}_t)^{-1} C^-_t B^{d+1}_t \, d\widehat{S}^{d+1,\mathrm{cld}}_t + B_t^{-1} d\widehat{A}^c_t \tag{5.2}$$

whereas (4.24) and (4.30) yield

$$d\widetilde{V}_t(x,\varphi,A,C) = \sum_{i=1}^d \xi^i_t \widetilde{B}^i_t \, d\widehat{S}^{i,\mathrm{cld}}_t + B_t^{-1} d\widehat{A}^c_t \tag{5.3}$$

where the specification of the process $\widehat{A}^c$ depends on the convention regarding the margin account. Using equation (2.25) in Corollary 2.1 with $\zeta^i = 0$ for all $i$ and a suitable choice of $A$, we deduce that the trading strategy given by (5.3) is self-financing, in the sense of Definition 2.3 and, obviously, it satisfies condition (2.16).

For a fixed $0 \le t < T$, equality $V_T(V^0_t(x) + p_t, \varphi, A - A_t, C) = V^0_T(x)$ where $V^0_t(x) = xB_t$, combined with equation (5.3), yields

$$-B_t^{-1} p_t = \sum_{i=1}^d \int_{(t,T]} \xi^i_u \widetilde{B}^i_u \, d\widehat{S}^{i,\mathrm{cld}}_u + \int_{(t,T]} B_u^{-1} \, d\widehat{A}^c_u. \tag{5.4}$$

By the definition of $\widetilde{\mathbb{P}}$, the processes $\widehat{S}^{i,\mathrm{cld}}$, $i = 1,2,\dots,d$ are $\widetilde{\mathbb{P}}$-local martingales. Consequently, equality (5.1) follows provided that the integrals with respect to $\widehat{S}^{i,\mathrm{cld}}$, $i = 1,2,\dots,d$ are martingales under $\widetilde{\mathbb{P}}$, rather than merely local (or sigma) martingales. The arguments used in the case of a risky collateral, as described by (5.2), are analogous. We now obtain

$$-B_t^{-1}p_t = \sum_{i=1}^{d}\int_{(t,T]} \xi_u^i \widetilde{B}_u^i \, d\widehat{S}_u^{i,\mathrm{cld}} + \int_{(t,T]} B_u^{-1}(S_u^{d+1})^{-1}C_u^{-}B_u^{d+1}\, d\widehat{S}_u^{d+1,\mathrm{cld}} + \int_{(t,T]} B_u^{-1}\, d\widehat{A}_u^{c} \tag{5.5}$$

and we postulate that all integrals with respect to $\widehat{S}^{i,\mathrm{cld}}$, $i = 1,2,\dots,d+1$ in equation (5.5) are martingales under $\widetilde{\mathbb{P}}$. The last postulate is indeed justified, since we assumed, in particular, that a replicating strategy is admissible. □

Note that the minus sign in equation (5.1) is due to the fact that all cash flows and prices are considered from the viewpoint of the hedger. For instance, a negative payoff $X$ at $T$, which represents the hedger's liability at time $T$ to his counterparty, is compensated by a positive price collected by the hedger at time 0.

The existence of a replicating strategy can be ensured by postulating that the local martingales $\widehat{S}^{i,\mathrm{cld}}$, $i = 1,2,\dots,d$ have the predictable representation property with respect to $\mathbb{G}$ under $\widetilde{\mathbb{P}}$. Moreover, since the ex-dividend price is independent of $x$ and $\varphi$, it is easy to verify that the equality $S_t(A,C) = \widehat{S}_t(A,C)$ holds for every $t \in [0,T]$. In essence, we deal here with only a minor modification of the standard linear pricing rule, which is very well understood in a market model with a single cash account. A more complex situation where the pricing mechanism is non-linear is a subject of the next subsection.

**Remark 5.2** Proposition 5.1 sheds some light on the connection between arbitrage-free property of the model, in the sense of Definition 3.3, and existence and representation of the hedger's fair price, in the sense of Definition 3.5.

## 5.2 Model with Partial Netting and Collateralization

We now consider the market model from Section 3.4, and we work under the assumptions of Proposition 3.3. Specifically, we assume that $x \geq 0$, $0 \leq r^l \leq r^b$ and $r^l \leq r^{i,b}$ for $i = 1,2,\dots,d$, and we postulate the existence of a probability measure $\widetilde{\mathbb{P}}^l$ equivalent to $\mathbb{P}$ and such that the processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i = 1,2,\dots,d$ are $(\widetilde{\mathbb{P}}^l,\mathbb{G})$-local martingales, where

$$\widetilde{S}_t^{i,l,\mathrm{cld}} = (B_t^l)^{-1}S_t^i + \int_{(0,t]} (B_u^l)^{-1}\, dA_u^i.$$

For a collateralized contract $(A,C)$, we search for the value process of a replicating strategy (of course, we will also need to show that such a strategy exists). We consider here the special case of an exogenous margin account with rehypothecated cash collateral $C$.

### 5.2.1 Dynamics of Discounted Portfolio's Wealth

By applying a slight extension of Definition 4.5 (see also Proposition 4.3) to the case of different lending and borrowing rates, one notes that a hedger's trading strategy $(x,\varphi,A,C)$ is self-financing whenever the hedger's wealth, which is given by the equality

$$V(x,\varphi,A,C) = \sum_{i=1}^{d}\xi_t^i S_t^i + \psi_t^l B_t^l + \psi_t^b B_t^b + \sum_{j=1}^{d}\psi_t^j B_t^j - C_t = V_t^p(x,\varphi,A^c) - C_t$$

where $\psi_t^l \geq 0$, $\psi_t^b \leq 0$ and $\psi_t^l\psi_t^b = 0$ for all $t \in [0,T]$, is such that the portfolio's value satisfies

$$V_t^p(x,\varphi,A^c) = x + \sum_{i=1}^{d}\int_{(0,t]} \xi_u^i \, d(S_u^i + A_u^i) + \int_0^t \psi_t^l \, dB_t^l + \int_0^t \psi_t^b \, dB_t^b + \sum_{j=1}^{d}\int_0^t \psi_u^j \, dB_u^j + A_t^c$$

where we set $A^c = A + C + F^c$ and the process $F^c$ is given by equation (4.12).

Observe that here $V_t^p(x,\varphi,A^c) = V_t(x,\varphi,A,C) + C_t$ for every $t \in [0,T)$ and $V_T^p(x,\varphi,A^c) = V_T(x,\varphi,A,C)$ since, by Assumption 4.1, the equality $C_T = 0$ holds. The following lemma shows, in particular, that one could also write $\widetilde{V}_t^{p,l}(x,\varphi,A^c) = \widetilde{V}_t^{p,l}(x,\xi,A^c)$ in order to emphasize that within the present framework the process $\xi$ uniquely determines the trading strategy $\varphi$, as can be seen from Corollary 2.4 and equations (2.49)–(2.50) in Section 2.4.3.

**Lemma 5.1** *The discounted wealth $Y_t := \widetilde{V}_t^{p,l}(\varphi,A^c) = (B_t^l)^{-1}V_t^p(x,\varphi,A^c)$ satisfies*

$$dY_t = \sum_{i=1}^d \xi_t^i \, d\widetilde{S}_t^{i,l,cld} + \widetilde{f}_l(t,Y_t,\xi_t)\,dt + (B_t^l)^{-1}\,dA_t^c \tag{5.6}$$

*where*

$$\widetilde{f}_l(t,Y_t,\xi_t) := (B_t^l)^{-1} f_l(t,B_t^lY_t,\xi_t) - r_t^lY_t \tag{5.7}$$

*where in turn for any process $X$ (note that $f_l$ depends on $(t,\omega)$ through $r_t^l, r_t^b, r_t^{i,b}$ and $S_t^i$)*

$$f_l(t,X_t,\xi_t) := \sum_{i=1}^d r_t^l\xi_t^iS_t^i - \sum_{i=1}^d r_t^{i,b}(\xi_t^iS_t^i)^+ + r_t^l\Big(X_t + \sum_{i=1}^d(\xi_t^iS_t^i)^-\Big)^+ - r_t^b\Big(X_t + \sum_{i=1}^d(\xi_t^iS_t^i)^-\Big)^-.$$

*Proof.* We note that the processes $C$ and $F^c$, which represent additional cash flows due to the presence of the margin account, do not depend on $\varphi$. It thus follows from (2.52) that the portfolio's value $V^p(\varphi,A^c)$ satisfies

$$\begin{aligned} dV_t^p(\varphi,A^c) = {} & \sum_{i=1}^d \xi_t^i\big(dS_t^i + dA_t^i\big) - \sum_{i=1}^d r_t^{i,b}(\xi_t^iS_t^i)^+\,dt + dA_t^c \\ & + r_t^l\Big(V_t^p(\varphi,A^c) + \sum_{i=1}^d(\xi_t^iS_t^i)^-\Big)^+dt - r_t^b\Big(V_t^p(\varphi,A^c) + \sum_{i=1}^d(\xi_t^iS_t^i)^-\Big)^-dt \end{aligned}$$

so that we may represent the dynamics of $V^p(\varphi,A^c)$ as follows

$$dV_t^p(\varphi,A^c) = \sum_{i=1}^d \xi_t^i\big(dS_t^i - r_t^lS_t^i\,dt + dA_t^i\big) + f_l\big(t,\xi_t,V_t^p(\varphi,A^c)\big)\,dt + dA_t^c.$$

Consequently, the discounted wealth $\widetilde{V}_t^{p,l}(\varphi,A^c) = (B_t^l)^{-1}V_t^p(\varphi,A^c)$ is governed by

$$d\widetilde{V}_t^{p,l}(\varphi,A^c) = \sum_{i=1}^d \xi_t^i\,d\widetilde{S}_t^{i,l,\mathrm{cld}} - r_t^l\widetilde{V}_t^{p,l}(\varphi,A^c)\,dt + (B_t^l)^{-1}f_l\big(t,B_t^l\widetilde{V}_t^{p,l}(\varphi,A^c),\xi_t\big)\,dt + (B_t^l)^{-1}\,dA_t^c,$$

which means that

$$d\widetilde{V}_t^{p,l}(\varphi,A^c) = \sum_{i=1}^d \xi_t^i\,d\widetilde{S}_t^{i,l,\mathrm{cld}} + \widetilde{f}_l\big(t,\widetilde{V}_t^{p,l}(\varphi,A^c),\xi_t\big)\,dt + (B_t^l)^{-1}\,dA_t^c$$

where the mapping $\widetilde{f}_l$ is given by (5.7). □

**Remark 5.3** Using analogous arguments, it is possible to show that the discounted wealth process $\widehat{Y}_t := \widetilde{V}_t^{p,b}(\varphi,A^c) = (B_t^b)^{-1}V_t^p(x,\varphi,A^c)$ satisfies

$$d\widehat{Y}_t = \sum_{i=1}^d \xi_t^i\,d\widetilde{S}_t^{i,b,\mathrm{cld}} + \widetilde{f}_b(t,\widehat{Y}_t,\xi_t)\,dt + (B_t^b)^{-1}\,dA_t^c \tag{5.8}$$

where the mapping $\widetilde{f}_b$ is given by

$$\widetilde{f}_b(t, \widehat{Y}_t, \xi_t) := (B^b_t)^{-1} f_b(t, B^b_t \widehat{Y}_t, \xi_t) - r^b_t \widehat{Y}_t \tag{5.9}$$

where in turn $f_b$ is given by, for any process $X$,

$$f_b(t, X_t, \xi_t) := \sum_{i=1}^d r^b_t \xi^i_t S^i_t - \sum_{i=1}^d r^{i,b}_t (\xi^i_t S^i_t)^+ + r^l_t \Big( X_t + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^+ - r^b_t \Big( X_t + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^-.$$

### 5.2.2 An Auxiliary BSDE

We focus here on the case where $x \geq 0$; an analogous analysis can be done for the case where $x < 0$ examined in Remark 5.3. Assume that the processes $M^i$, $i = 1, 2, \ldots, d$ are continuous local martingales on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \widetilde{\mathbb{P}}^l)$. To proceed further, we need to address the problem of existence and uniqueness of a solution $(Y, Z)$ to the following BSDE

$$dY_t = \sum_{i=1}^d Z^i_t \, dM^i_t + \widetilde{f}_l(t, Y_t, Z_t) \, dt + dU_t$$

with a terminal value $Y_T = \eta$ and a given process $U$. Equivalently,

$$Y_t = \eta - \int_t^T \sum_{i=1}^d Z^i_u \, dM^i_u - \int_t^T \widetilde{f}_l(u, Y_u, Z_u) \, du - (U_T - U_t). \tag{5.10}$$

If we set $\widehat{Y}_t = Y_t - U_t$, then equation (5.10) can be written as

$$\widehat{Y}_t = \widehat{Y}_T - \int_t^T \sum_{i=1}^d Z^i_u \, dM^i_u - \int_t^T \widehat{f}_l(t, \widehat{Y}_u, Z_u) \, du \tag{5.11}$$

where the terminal value satisfies $\widehat{Y}_T = \eta - U_T$ and where the driver $\widehat{f}_l$ satisfies

$$\widehat{f}_l(t, \widehat{Y}_t, Z_t) := \widetilde{f}_l(t, \widehat{Y}_t + U_t, Z_t). \tag{5.12}$$

Equation (5.11) is a special case of general BSDE studied in El Karoui and Huang [19] (see also Carbone et al. [11]). Note, that if a pair $(\widehat{Y}, Z)$ is a solution to (5.11) with terminal condition $\widehat{Y}_T = \eta - U_T$, then the pair $(Y, Z)$ with $Y := \widehat{Y} + U$ is a solution to (5.10) with terminal condition $Y_T = \eta$.

Under the assumption that the processes $r^l, r^b$ and $r^{i,b}$, $i = 1, 2, \ldots, d$ are non-negative and bounded, and the prices of risky assets are bounded, it is easy to check that the mapping $\widetilde{f}_l : [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ given by (5.7) is a standard *driver* (in the terminology of El Karoui and Huang [19]). Consequently, under mild integrability assumptions imposed on the process $U$, the mapping $\widehat{f}_l : [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ given by (5.12) is a standard driver as well. Therefore, the existence and uniqueness of a solution $(\widehat{Y}, Z)$ to BSDE (5.11) in a suitable space of stochastic processes holds, provided that the $\mathbb{R}^k$-valued local martingale $M = (M^1, \ldots, M^d)$ is continuous and has the predictable representation property with respect to the filtration $\mathbb{G}$ under $\widetilde{\mathbb{P}}^l$ (see, for instance, El Karoui and Huang [19]) and the terminal condition $\eta - Y_T$ satisfies a suitable integrability condition. We conclude that the existence and uniqueness of a solution $(Y, Z)$ to BSDE (5.10) holds under mild technical assumptions. For technical details, the reader is referred to Nie and Rutkowski [33, 34]. We also observe that if BSDE (5.10) has a solution $(Y, Z)$, then the process

$$\bar{M}_t := \sum_{i=1}^d \int_0^t Z^i_u \, dM^i_u$$

is a $\widetilde{\mathbb{P}}^l$-martingale, since the property that $\bar{M}$ is a square-integrable martingale is a part of the definition of a solution to BSDE (5.10) . Consequently, the process $Y$ admits the following recursive representation

$$Y_t = -\,\widetilde{\mathbb{E}}^l_t\bigg(\int_t^T \widetilde{f}_l(u, Y_u, Z_u)\,du + U_T - U_t\bigg)$$

where we denote $\widetilde{\mathbb{E}}^l_t(\cdot) := \mathbb{E}_{\widetilde{\mathbb{P}}^l}(\,\cdot\,|\,\mathcal{G}_t)$.

### 5.2.3 Pricing and Hedging Result

Assume that the processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i = 1, 2, \dots, d$ are continuous. In the next result, we assume that the $d$-dimensional continuous local martingale $\widetilde{S}^{l,\mathrm{cld}}$ has the predictable representation property with respect to the filtration $\mathbb{G}$ under $\widetilde{\mathbb{P}}^l$, meaning that any square-integrable $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-martingale $N$ admits the following integral representation for some process $(\eta^1, \dots, \eta^d)$

$$N_t = N_0 + \sum_{i=1}^d \int_0^t \eta^i_u\, d\widetilde{S}^{i,l,\mathrm{cld}}_u.$$

For the sake of concreteness, one may assume, for instance, that under $\widetilde{\mathbb{P}}^l$ the processes $\widetilde{S}^{i,l,\mathrm{cld}}$ satisfy, for every $i = 1, 2, \dots, d$ and $t \in [0, T]$,

$$d\widetilde{S}^{i,l,\mathrm{cld}}_t = \sum_{j=1}^d \widetilde{S}^{i,l,\mathrm{cld}}_t \sigma^{ij}_t\, d\widetilde{W}^j_t$$

where $(\widetilde{W}^1, \dots, \widetilde{W}^d)$ is the $d$-dimensional standard Brownian motion generating the filtration $\mathbb{G}$ and the matrix-valued process $\sigma = [\sigma^{ij}]$ is non-singular.

We are in a position to establish the following pricing result in which we write $S_t(x, A, C)$ instead of $S_t(x, \varphi, A, C)$, in order to emphasize that, for any fixed $x \geq 0$, the replicating strategy $\xi^x$ for the collateralized contract $(A, C)$ is unique. Note also that the price $S_t(x, A, C)$ manifestly depends on the hedger's initial endowment $x$ through the terminal condition in BSDE (5.13). For the existence and uniqueness of a solution to (5.13) and further properties of the price $S_t(x, A, C)$, the reader is referred to the follow-up papers by Nie and Rutkowski [33, 36].

**Proposition 5.2** *Let the random variables*

$$U_T := \int_{(0,T]} (B^l_t)^{-1} dA^c_t \quad \text{and} \quad \int_0^T (U_t)^2\, dt$$

*be square-integrable under* $\widetilde{\mathbb{P}}^l$*. Then, for any fixed real number* $x \geq 0$*, the unique replicating strategy* $\xi^x$ *equals* $Z^x$ *and the ex-dividend price satisfies, for every* $t \in [0, T)$*,*

$$S_t(x, A, C) = B^l_t(Y^x_t - x) - C_t$$

*where the pair* $(Y^x, Z^x)$ *is the unique solution to the BSDE*

$$Y^x_t = x - \int_t^T \sum_{i=1}^d Z^{x,i}_u\, d\widetilde{S}^{i,l,cld}_u - \int_t^T \widetilde{f}_l\big(u, Y^x_u, Z^x_u\big)\,du - \int_{(t,T]} (B^l_u)^{-1}\, dA^c_u. \tag{5.13}$$

*Consequently, the following representation is valid*

$$S_t(x, A, C) = -B^l_t\, \widetilde{\mathbb{E}}^l_t\bigg(\int_t^T \widetilde{f}_l(u, Y^x_u, \xi^x_u)\,du + \int_{(t,T]} (B^l_u)^{-1} dA^c_u\bigg) - C_t. \tag{5.14}$$

*Proof.* For a fixed $0 \le t < T$, we consider replication on the interval $[t,T]$ and valuation at time $t$. Recall that $V^p_t(x,\varphi,A^c) = V_t(x,\varphi,A,C) + C_t$ for every $t \in [0,T)$ and $V^p_T(x,\varphi,A^c) = V_T(x,\varphi,A,C)$. On the one hand, the definition of replication on the interval $[t,T]$ requires that

$$V_T(V^0_t(x) + p_t, \varphi, A - A_t, C) = V^0_T(x)$$

where $V^0(x) = xB^l$ (recall that we work here under the assumption that $x \ge 0$), so that

$$\widetilde{V}^l_T(V^0_t(x) + p_t, \varphi, A - A_t, C) - \widetilde{V}^l_t(V^0_t(x) + p_t, \varphi, A - A_t, C) = x - (B^l_t)^{-1}(p_t + xB^l_t) = -(B^l_t)^{-1}p_t.$$

On the other hand, $V(x,\varphi,A,C) = V^p(x,\varphi,A^c) - C$ and thus, since $C_T = 0$,

$$\widetilde{V}^l_T(V^0_t(x) + p_t, \varphi, A - A_t, C) - \widetilde{V}^l_t(V^0_t(x) + p_t, \varphi, A - A_t, C) = \widetilde{V}^{p,l,x}_T - \widetilde{V}^{p,l,x}_t + (B^l_t)^{-1}C_t$$

where the dynamics of the process $\widetilde{V}^{p,l,x}$ are given by (5.6) with the terminal condition

$$\widetilde{V}^{p,l,x}_T = (B^l_T)^{-1}(V_T(V^0_t(x) + p_t, \varphi, A - A_t, C) + C_T) = (B^l_T)^{-1}V^0_T(x) = x$$

where the last equality is obvious, since $V^0_T(x) = B^l_T x$ for every $x \ge 0$. Therefore, the ex-dividend price $p_t = S_t(x,\varphi,A,C)$ satisfies

$$-(B^l_t)^{-1}S_t(x,\varphi,A,C) = \widetilde{V}^{p,l,x}_T - \widetilde{V}^{p,l,x}_t + (B^l_t)^{-1}C_t.$$

This in turn implies that $S_t(x,\varphi,A,C)$ equals

$$S_t(x,\varphi,A,C) = B^l_t\widetilde{V}^{p,l,x}_t - C_t - xB^l_t = B^l_t(Y^x_t - x) - C_t$$

where the pair $(Y^x, Z^x)$ solves the BSDE (5.13) with the terminal condition $Y^x_T = x$. This in turn yields equality (5.14). □

For any fixed $t \in [0,T)$, equation (5.14) can also be rewritten as follows

$$\begin{aligned} S_t(x,A,C) = {} & -B^l_t\,\widetilde{\mathbb{E}}^l_t\Big(\int_t^T \widetilde{f}_l\big(u, Y^x_u, \xi_u\big)\,du + \int_{(t,T]} (B^l_u)^{-1}(dA_u + dF^c_u)\Big) \\ & - B^l_t\,\widetilde{\mathbb{E}}^l_t\Big(\int_{[t,T]} (B^l_u)^{-1} dC^{[t]}_u\Big) \end{aligned} \tag{5.15}$$

where the process $C^{[t]}$ is given by: $C^{[t]}_u := C_u$ for $u \in [t,T]$ and $C^{[t]}_u := 0$ for $u \in [0,t)$. Equation (5.15) follows easily from (5.14) and the fact that, for any fixed $t$, the process $C^{[t]}$ in equation (5.15) has the jump at time $t$ equal to $\Delta C^{[t]}_t = C^{[t]}_t - C^{[t]}_{t-} = C^{[t]}_t = C_t$. Note also that the last integral in this equation is taken over $[t,T]$, whereas the penultimate one over $(t,T]$. This discrepancy is due to markedly different financial interpretations of the cumulative cash flows process $A$ and the collateral process $C$. Alternative collateral conventions can also be covered through a suitable modification of BSDE (5.14). Although we do not offer here any general result in this vein, some special cases are presented in Section 5.3.

**Remark 5.4** In contrast to the linear case studied in Section 5.1, we no longer claim here that the ex-dividend price $S(x,\varphi,A,C)$ and the valuation ex-dividend price $\widehat{S}(x,\varphi,A,C)$ necessarily coincide in the present non-linear setting.

**Remark 5.5** In view of Remarks 3.2 and 5.3, it is easy to check that if $x \le 0$, then the ex-dividend price $S_t(x,A,C)$ satisfies, for every $t \in [0,T)$,

$$S_t(x,A,C) = B^b_t(Y^x_t - x) - C_t$$

where $(Y^x, Z^x)$ is the unique solution to the following BSDE under $\widetilde{\mathbb{P}}^b$

$$Y^x_t = x - \int_t^T \sum_{i=1}^d Z^{x,i}_u\, d\widetilde{S}^{i,b,\mathrm{cld}}_u - \int_t^T \widetilde{f}_b\big(u, Y^x_u, Z^x_u\big)\,du - \int_{(t,T]} (B^b_u)^{-1}\, dA^c_u$$

where the mapping $\widetilde{f}_b$ is given by equation (5.9). It can be checked that for $x = 0$ the pricing algorithm of Proposition 5.2 and the one outlined in this remark coincide, as was expected.

### 5.2.4 Illustrative Example

As a sanity check for pricing equation (5.14), let us consider a toy model where $S^i = 0$ for all $i$. so that $A^c = A$. We assume that the interest rates $r^l$ and $r^b$ are constant and, for simplicity, we set $r^{c,l} = r^l$ and $r^{c,b} = r^b$. We fix $0 \leq t_0 < T$, and we first consider the contract $(A, C)$ where

$$A_t = \mathbb{1}_{[t_0,T]}(t) - e^{r^l(T-t_0)}\mathbb{1}_{[T]}(t)$$

and for some constant $0 \leq \alpha < 1$

$$C_t = -\alpha e^{r^l(t-t_0)}\mathbb{1}_{[t_0,T)}(t).$$

Let us assume that $x = 0$. We claim that the contract is fair, in the sense that the hedger's price at time $t_0$ is null. To this end, we observe that the hedger may easily replicate his net liability at time $T$ by investing $1 - \alpha$ units of cash received from the counterparty at time $t_0$ in the lending account $B^l$. When the collateral amount $\alpha e^{r^l(T-t_0)}$ is returned to him at time $T$, then the hedger will have the right amount $e^{r^l(T-t_0)}$ units of cash to deliver to the counterparty.

We thus expect that the price $S_t(0, A, C)$ equals zero for every $t < t_0$, Under the present assumptions, equation (5.6) reduces to

$$d\widetilde{V}^{p,l}_t(\varphi, A^c) = (r^l - r^b)\big(\widetilde{V}^{p,l}_t(\varphi, A^c)\big)^- dt + (B^l_t)^{-1}\, dA^c_t \tag{5.16}$$

where $A^c = A + C + F^c$ where $F^c_t = -\int_0^t r^l C_u\, du$ (note that $C = -C^-$). For $x = 0$, the portfolio's wealth $V^p(\varphi, A^c)$ is always non-negative, so that $d\widetilde{V}^{p,l}_t(\varphi, A^c) = (B^l_t)^{-1}\, dA^c_t$. Using (5.15) with $\widetilde{f} = 0$, we obtain, for every $t < t_0$,

$$\begin{aligned}
(B^l_t)^{-1} S_t(0, A, C) &= -\int_{(t,T]} (B^l_u)^{-1} d(A_u + F^c_u) - \int_{[t,T]} (B^l_u)^{-1} dC_u \\
&= -(B^l_{t_0})^{-1} + (B^l_T)^{-1} e^{r^l(T-t_0)} - \alpha \int_{t_0}^T (B^l_u)^{-1} r^l e^{r^l(u-t_0)}\, du + \alpha (B^l_{t_0})^{-1} \\
&\quad + \alpha \int_{t_0}^T (B^l_u)^{-1} d\big(e^{r^l(u-t_0)}\big) - \alpha (B^l_T)^{-1} e^{r^l(T-t_0)} = 0.
\end{aligned}$$

If we take instead the process

$$A_t = -\mathbb{1}_{[t_0,T]}(t) + e^{r^b(T-t_0)}\mathbb{1}_{[T]}(t),$$

then the hedger pays one unit of cash at time $t_0$ and thus if $C = 0$ then his wealth will be negative, specifically, $V^p_t(\varphi, A, 0) = -e^{r^b(t-t_0)}$ for $t \in [t_0, T)$. Hence (5.16) and (5.15) with $\widetilde{f}(t, Y_t) = (r^l - r^b)(Y_t)^-$ now yield, for $t < t_0$,

$$\begin{aligned}
(B^l_t)^{-1} S_t(0, A, 0) &= -\int_{t_0}^T (B^l_u)^{-1} (r^l - r^b) e^{r^b(u-t_0)}\, du - \int_{[t_0,T]} (B^l_u)^{-1} dA_u \\
&= e^{-r^b t_0}\big(e^{(r^b - r^l)T} - e^{(r^b - r^l)t_0}\big) + e^{-r^l t_0} - e^{-r^l T} e^{r^b(T-t_0)} = 0.
\end{aligned}$$

Once again, this was expected since if the hedger borrows one unit of cash at time $t_0$ then his debt at time $T$ will match the cash amount, which he receives from the counterparty at this date.

## 5.3 Diffusion-Type Market Models

To give an illustration of the general hedging and pricing methodology developed in preceding sections, we will now present a detailed study of the valuation problem under various conventions regarding collateralization. A special case of this model was previously examined by Piterbarg [38]. We assume that the processes $B^j$, $j = 0, 1, \ldots, d$ are absolutely continuous, so that they can be represented as $dB^j_t = r^j_t B^j_t \, dt$ for some $\mathbb{G}$-adapted processes $r^j$, $j = 0, 1, \dots, d+1$ (see Example 2.1). It is also postulated in this section that the lending and borrowing rates are identical, that is, $r^l = r^b = r$ for some non-negative $\mathbb{G}$-adapted process $r$. For this reason, we can mimic (but also slightly extend) here the approach developed in Section 5.1.

We postulate the existence of $d+2$ traded risky assets $S^i$, $i = 1, 2, \dots, d+2$, where the asset $S^{d+1}$ (resp. $S^{d+2}$) can be posted by the hedger (resp. the counterparty) as collateral. Of course, the situation where $S^{d+1} = S^{d+2}$ is not excluded. However, if the risky assets $S^{d+1}$ and $S^{d+2}$ are distinct, then we do not need to model the dynamics of $S^{d+2}$; it suffices to know the identity of this asset or, more precisely, the corresponding repo rate $r^{d+2,h}$. By contrast, an explicit specification of the dynamics of $S^{d+2}$ (but not of $S^{d+1}$) would be needed if the valuation problem were solved from the perspective of the counterparty. Unless explicitly stated otherwise, we postulate in this section that condition (2.16) is satisfied for $i = 1, 2, \dots, d+1$.

### 5.3.1 Martingale Measure

We assume that each risky asset $S^i$, $i = 1, 2, \dots, d+1$ pays continuously dividends at stochastic rate $\kappa^i$ and has the (ex-dividend) price dynamics under the real-world probability $\mathbb{P}$

$$dS^i_t = S^i_t \big( \mu^i_t \, dt + \sigma^i_t \, dW^i_t \big), \quad S^i_0 > 0,$$

where $W^1, W^2, \dots, W^d$ are correlated Brownian motions and the volatility processes $\sigma^1, \sigma^2, \dots, \sigma^d$ are positive and bounded away from zero. The corresponding dividend processes are given by

$$A^i_t = \int_0^t \kappa^i_u S^i_u \, du.$$

As usual, we write $\widehat{S}^i_t = (B^i_t)^{-1} S^i_t$ and $\widehat{S}^{i,\mathrm{cld}}_t = (B^i_t)^{-1} S^{i,\mathrm{cld}}_t$. Recall that we denote by $\widetilde{\mathbb{P}}$ a martingale measure for the basic model with funding costs (see Proposition 3.1).

**Lemma 5.2** *The price process $S^i$ satisfies under $\widetilde{\mathbb{P}}$*

$$dS^i_t = S^i_t \big( (r^i_t - \kappa^i_t) \, dt + \sigma^i_t \, d\widetilde{W}^i_t \big)$$

*where $\widetilde{W}^i$ is a Brownian motion under $\widetilde{\mathbb{P}}$. Equivalently, the process $\widehat{S}^{i,cld}$ satisfies*

$$d\widehat{S}^{i,cld}_t = \widehat{S}^{i,cld}_t \sigma^i_t \, d\widetilde{W}^i_t. \tag{5.17}$$

*The process $K^i$ given by* (2.11) *satisfies*

$$dK^i_t = dS^i_t - r^i_t S^i_t \, dt + \kappa^i_t S^i_t \, dt = S^i_t \sigma^i_t \, d\widetilde{W}^i_t \tag{5.18}$$

*and thus it is a (local) martingale under $\widetilde{\mathbb{P}}$.*

*Proof.* By the definition of a martingale measure $\widetilde{\mathbb{P}}$, the discounted cumulative-dividend price $\widehat{S}^{i,\mathrm{cld}}$ is a (local) martingale under $\widetilde{\mathbb{P}}$. Recall that the process $\widehat{S}^{i,\mathrm{cld}}$ is given by

$$\widehat{S}^{i,\mathrm{cld}}_t = \widehat{S}^i_t + \int_{(0,t]} (B^i_u)^{-1} \, dA^i_u, \quad t \in [0,T].$$

Consequently,

$$\widehat{S}^{i,\mathrm{cld}}_t = \widehat{S}^i_t + \int_0^t \kappa^i_u (B^i_u)^{-1} S^i_u \, du = \widehat{S}^i_t + \int_0^t \kappa^i_u \widehat{S}^i_u \, du.$$

Since

$$d\widehat{S}^i_t = \widehat{S}^i_t \big( (\mu^i_t - r^i_t) \, dt + \sigma^i_t \, dW^i_t \big), \tag{5.19}$$

we obtain

$$d\widehat{S}_t^{i,\mathrm{cld}} = d\widehat{S}_t^i + \kappa_t^i \widehat{S}_t^i \, dt = \widehat{S}_t^i \big( (\mu_t^i + \kappa_t^i - r_t^i) \, dt + \sigma_t^i \, dW_t^i \big).$$

Hence $\widehat{S}^{i,\mathrm{cld}}$ is a (local) martingale under $\widetilde{\mathbb{P}}$ provided that the process

$$d\widetilde{W}_t^i = dW_t^i + (\sigma_t^i)^{-1} (\mu_t^i + \kappa_t^i - r_t^i) \, dt \tag{5.20}$$

is a Brownian motion under $\widetilde{\mathbb{P}}$. By combining (5.19) with (5.20) we obtain expression (5.17). Other asserted formulae now follow easily. □

### 5.3.2 Wealth Dynamics for Collateralized Contracts

We postulate, in addition, that the processes $B^{c,b}, B^{c,l}, B^{d+2,s}$ and $B^{d+2,s}$ are absolutely continuous as well, so that

$$\begin{aligned} dB_t^{c,b} &= r_t^{c,b} B_t^{c,b} \, dt, \quad dB_t^{c,l} = r_t^{c,l} B_t^{c,l} \, dt, \\ dB_t^{d+2,s} &= r_t^{d+2,s} B_t^{d+2,s} \, dt, \quad dB_t^{d+2,h} = r_t^{d+2,h} B_t^{d+2,h} \, dt, \end{aligned}$$

for some processes $r^{c,b}, r^{c,l}, r^{d+2,s}$ and $r^{d+2,h}$, which are assumed to be non-negative.

- **Risky collateral.**

We first consider the case of risky collateral under the assumptions of Proposition 4.1. Formally, the cases of rehypothecation and segregation differ only in the choice of either $r^{d+2,s}$ or $r^{d+2,h}$ as the hedger's interest on the collateral amount posted by the counterparty. In practice, it is clear that the repo rate $r^{d+2,h}$ is positive, whereas the conventional rate $r^{d+2,s}$ is likely to be zero. In the case of rehypothecation, $\bar{F}_t^h$ is given here by the following expression (see (4.17))

$$\bar{F}_t^h = \int_0^t \big( r_u^{d+2,h} - r_u^{c,b} \big) C_u^+ \, du - \int_0^t \big( r_u^{d+1} - r_u \big) C_u^- \, du$$

and thus, as expected, the term $\bar{F}_t^h$ vanishes when the equalities $r^{d+2,h} = r^{c,b}$ and $r^{d+1} = r$ hold, since then the negative and positive cash flows related to the margin account cancel out. From equation (5.2), we obtain the dynamics of the hedger's wealth $V(\varphi) = V(x, \varphi, A, C)$

$$\begin{aligned} dV_t(\varphi) &= r_t V_t(\varphi) \, dt + \sum_{i=1}^d \xi_t^i \left( dS_t^i - r_t^i S_t^i \, dt + dA_t^i \right) \\ &\quad + (S_t^{d+1})^{-1} C_t^- \left( dS_t^{d+1} - r_t^{d+1} S_t^{d+1} \, dt + dA_t^{d+1} \right) + d\bar{F}_t^h + dA_t. \end{aligned}$$

If the collateral $C$ is predetermined, then the sum of the last three terms in the formula above defines a single process $\bar{A}^{c,h}$, which represents all cash flows associated with a collateralized contract except for the gains or losses from trading in risky assets $S^1, S^2, \ldots, S^d$. Then we may rewrite the last equation as follows

$$dV_t(\varphi) = r_t V_t(\varphi) \, dt + \sum_{i=1}^d \xi_t^i S_t^i \sigma_t^i \, d\widetilde{W}_t^i + d\bar{A}_t^{c,h}. \tag{5.21}$$

We note that the process $\bar{A}^{c,h}$ depends also on the dynamics of the risky asset $S^{d+1}$. As was already mentioned, the dynamics of the asset $S^{d+2}$ are irrelevant, so they are left unspecified.

- **Cash collateral under segregation.**

We now consider the case of cash collateral under segregation and we place ourselves within the setup of Proposition 4.2. Under the present assumptions, the expression for $\widehat{F}^s$ reduces to

$$\widehat{F}_t^s = \int_0^t \big( r_u^{d+2,s} - r_u^{c,b} \big) C_u^+ \, du - \int_0^t \big( r_u^{d+1} - r_u^{c,l} \big) C_u^- \, du.$$

Formula (4.24) yields

$$dV_t(\varphi) = r_t V_t(\varphi)\,dt + \sum_{i=1}^{d} \xi_t^i \left(dS_t^i - r_t^i S_t^i\,dt + dA_t^i\right) + d\widehat{F}_t^s + dA_t,$$

so, if we denote the sum of the last three terms by $\widehat{A}^{c,s}$, then we obtain

$$dV_t(\varphi) = r_t V_t(\varphi)\,dt + \sum_{i=1}^{d} \xi_t^i S_t^i \sigma_t^i\,d\widetilde{W}_t^i + d\widehat{A}_t^{c,s} \tag{5.22}$$

where the process $\widehat{A}^{c,s}$ does not depend on the dynamics of the risky asset $S^{d+1}$.

• **Cash collateral under rehypothecation.**

Recall that the case of cash collateral under rehypothecation was examined in Proposition 4.3. Under the present assumptions, we deduce from (4.31) that

$$\widehat{F}_t^h = \int_0^t C_u^+ \left(r_u - r_u^{c,b}\right)du - \int_0^t C_u^- \left(r_u^{d+1} - r_u^{c,l}\right)du \tag{5.23}$$

and thus (4.30) becomes

$$dV_t(\varphi) = r_t V_t(\varphi)\,dt + \sum_{i=1}^{d} \xi_t^i \left(dS_t^i - r_t^i S_t^i\,dt + dA_t^i\right) + d\widehat{F}_t^h + dA_t. \tag{5.24}$$

If we denote the sum of the last three terms by $\widehat{A}^{c,h}$, then, using also (5.18), we obtain

$$dV_t(\varphi) = r_t V_t(\varphi)\,dt + \sum_{i=1}^{d} \xi_t^i S_t^i \sigma_t^i\,d\widetilde{W}_t^i + d\widehat{A}_t^{c,h} \tag{5.25}$$

where, once again, the process $\widehat{A}^{c,h}$ does not depend on the dynamics of $S^{d+1}$.

### 5.3.3 Pricing with an Exogenous Collateral

Our goal is to value and hedge a collateralized contract within the framework of a diffusion-type model. We postulate that the process $A$ is adapted to the filtration $\mathbb{F}^S$ generated by risky assets $S^1, S^2, \dots, S^d$. We first assume that a collateral process $C$ is predetermined, so it does not depend on the hedger's trading strategy. We use the generic symbol $A^c$ to denote either of the processes $\bar{A}^{c,h}, \widehat{A}^{c,h}, \widehat{A}^{c,s}$ introduced in the preceding subsection. Assume that all short-term rates and the processes $A$ and $C$ are bounded, so that the process $A^c$ is bounded as well. In fact, it would be enough to postulate that the conditional expectation in (5.26) is well defined for all $t \in [0,T]$. The following result can be seen as a corollary to Proposition 5.1.

**Proposition 5.3** *A collateralized contract $(A,C)$ with the predetermined collateral process $C$ can be replicated by an admissible trading strategy. The ex-dividend price $S(A,C)$ satisfies, for every $t \in [0,T)$,*

$$S_t(A,C) = -B_t\,\widetilde{\mathbb{E}}_t\bigg(\int_{(t,T]} B_u^{-1}\,dA_u^c\bigg). \tag{5.26}$$

*Proof.* We formally consider $A^c$ as the total cash flow process associated with the contract $A$. Hence it suffices to check that the assumptions of Proposition 5.1 are met. For the existence of an admissible replicating strategy under the cash collateral convention, we note that the processes $C$

and $A$ are adapted with respect to filtration generated by risky assets $S^1, S^2, \ldots, S^d$ and thus the predictable representation property of the Brownian filtration entails that

$$\int_{(0,T]} B_u^{-1}\, dA^c_u = S_0 + \sum_{i=1}^{d} \int_0^T \xi^i_u S^i_u \sigma^i_u \, d\widetilde{W}^i_u.$$

In the case of the risky collateral, the trading strategy is complemented by $\xi^{d+1} = (S^{d+1}_t)^{-1} C^-_t$, so that we now use the following representation

$$\int_{(0,T]} B_u^{-1}\,(d\bar{F}^h_t + dA_u) = \widetilde{S}_0 + \sum_{i=1}^{d+1} \int_0^T \xi^i_u S^i_u \sigma^i_u \, d\widetilde{W}^i_u \tag{5.27}$$

where, by assumption, the process $A$ is adapted to the filtration $\mathbb{F}^S$ generated by the risky assets $S^1, S^2, \ldots, S^d$, and thus the right-hand side in (5.27) defines a bounded $\mathcal{F}^S_T$-measurable random variable. Hence the existence of an admissible replicating strategy satisfying condition (2.16) follows. □

For the sake of concreteness, let us consider a particular instance of a collateralized contract, specifically, the valuation of a single cash flow $X$ at maturity date $T$ under the convention of cash collateral with rehypothecation. We assume that $X$ is a bounded random variable, which is measurable with respect to the $\sigma$-field $\mathcal{F}^S_T$. It is natural to assume that $r^{d+1} = r$, meaning that the cash for collateral posted is borrowed from the risk-free account. We first obtain the non-linear pricing formula (5.28). Under an additional assumption of symmetry, $r^{c,b} = r^{c,l} = r^c$, we denote by $B^c$ the process satisfying $dB^c_t = r^c_t B^c_t\, dt$, and we obtain the linear pricing formula (5.29).

**Corollary 5.1** *A collateralized contract with the cumulative dividend* $A_t = p\,\mathbb{1}_{[0,T]}(t) + X\mathbb{1}_{[T]}(t)$ *and the predetermined collateral process* $C$ *can be replicated by an admissible trading strategy. The ex-dividend price* $S(A,C)$ *satisfies, for every* $t \in [0,T)$,

$$S_t(A,C) = -B_t\, \widetilde{\mathbb{E}}_t\bigg(B_T^{-1}X + \int_t^T B_u^{-1} C^+_u \left(r_u - r^{c,b}_u\right) du - \int_t^T B_u^{-1} C^-_u \left(r^{d+1}_u - r^{c,l}_u\right) du\bigg). \tag{5.28}$$

*In particular, if* $r^{d+1} = r$ *and* $r^{c,b} = r^{c,l} = r^c$, *then*

$$S_t(A,C) = -B_t\, \widetilde{\mathbb{E}}_t\bigg(B_T^{-1}X + \int_t^T B_u^{-1}(r_u - r^c_u) C_u \, du\bigg). \tag{5.29}$$

*Proof.* Equality (5.28) is an immediate consequence of (5.23) and (5.26). To obtain (5.29), it suffices to observe that equalities $r^{d+1} = r$ and $r^{c,b} = r^{c,l} = r^c$ imply that

$$\widehat{F}^h_t = \int_0^t C_u \left(r_u - r^c_u\right) du$$

and thus (5.29) is an immediate consequence of (5.28). □

**Remark 5.6** Piterbarg [38] examined a diffusion-type market model with three cash accounts

$$B_t = e^{\int_0^t r_u\, du}, \quad B^1_t = e^{\int_0^t r^1_u\, du}, \quad B^c_t = e^{\int_0^t r^c_u\, du},$$

where the spreads $r^1 - r^c$, $r^1 - r$, $r^c - r$ represent the *bases* between the funding rates, that is, the *funding bases.* No distinction between the borrowing and lending rates is made in [38], in particular, $r^2 = r$ and $r^{c,b} = r^{c,l} = r^c$. Our formulae agree with those derived by Piterbarg [38], although our convention for the collateral amount is slightly different than the one adopted in [38], specifically, our collateral process $C$ corresponds to the process $-C$ in [38].

**Remark 5.7** Observe that the equivalence of formulae (5.29) and (5.37) shows that the choice of a particular discount factor can be rather arbitrary, as long as the (cumulative) cash flow process of a security under valuation is appropriately adjusted. In the case of formula (5.29), the discount factor is chosen as the price process $B$ representing a traded asset, whereas in the case of formula (5.37), we deal with the process $B^c$, which does not even represent the price process of a traded asset in the present setup.

Suppose, for instance, that $d = 1$ and the dividend rate $\kappa^1 = 0$. Then none of the two above mentioned choices of the discount factor correspond to the usual martingale measure for the stock price which corresponds to the choice of $B^1$ as the discount factor.

### 5.3.4 Pricing with Hedger's Collateral

As already mentioned in Section 4, the collateral amount $C$ can be specified in terms of the marked-to-market value of a contract and thus, at least in theory, it can be given in terms of the wealth process $V(\varphi)$ of the hedger's strategy. To this end, we introduce the process $\widehat{V}(\varphi) := V(\varphi) - xB$; for the interpretation of the process $\widehat{V}(\varphi)$, see Definition 5.3. Then, for instance, the process $C$ may be given as follows (see (4.10))

$$C_t(\varphi) = (1+\delta^1_t)\widehat{V}^-_t(\varphi) - (1+\delta^2_t)\widehat{V}^+_t(\varphi) = \bar{\delta}^1_t\widehat{V}^-_t(\varphi) - \bar{\delta}^2_t\widehat{V}^+_t(\varphi) \tag{5.30}$$

for some bounded, $\mathbb{F}^S$-adapted processes $\delta^1$ and $\delta^2$, where for brevity we set $\bar{\delta}^i_t = 1 + \delta^i_t$. Hence the generic process $A^c$, which, as before, is aimed to represent either of the processes $\bar{A}^{c,h}, \widehat{A}^{c,h}, \widehat{A}^{c,s}$, depends here in a non-linear manner on the hedger's wealth when he implements a replicating portfolio. Consequently, the conditional expectation in equation (5.26) can now be informally interpreted as a BSDE with the shorthand notation

$$V_t(\varphi) = -B_t\,\widetilde{\mathbb{E}}_t\bigg(\int_{(t,T]} B^{-1}_u\,dA^c_u(\widehat{V}(\varphi))\bigg) \tag{5.31}$$

where the notation $A^c(\widehat{V}(\varphi))$ is used to emphasize that the process $A^c$ depends on $\widehat{V}(\varphi)$. A more explicit form of BSDE (5.31) can be derived as soon as a particular convention for the margin account is adopted. Let us consider, for instance, the special case of cash collateral with rehypothecation (recall that this was also our choice in Section 5.2). To simplify expressions, we also assume that $r^{d+1} = r$ and $r^{c,b} = r^{c,l} = r^c$, so that the process $F^c$ satisfies $F^c_t = \int_0^t r^c_u C_u\,du$ for all $t \in [0,T]$. Then, from equations (5.23) and (5.24), the wealth process of a self-financing strategy $\varphi$ satisfies

$$dV_t(\varphi) = r_tV_t(\varphi)\,dt + \sum_{i=1}^d \xi^i_tS^i_t\sigma^i_t\,d\widetilde{W}^i_t + (r_t - r^c_t)(\bar{\delta}^1_t\widehat{V}^-_t(\varphi) - \bar{\delta}^2_t\widehat{V}^+_t(\varphi))\,dt + dA_t. \tag{5.32}$$

In the next pricing result, we once again focus on a collateralized contract $(A, C)$ where $A_t = p\,\mathbb{1}_{[0,T]}(t) + X\mathbb{1}_{[T]}(t)$. For any fixed $t \in [0,T]$, we search for a $\mathcal{G}_t$-measurable random variable $p_t$ such that

$$V_T(V^0_t(x) + p_t, \varphi, A - A_t, C) = V^0_T(x)$$

for some admissible trading strategy $\varphi$. Obviously, for any fixed $t \in [0,T)$, we have $A_u - A_t = X\mathbb{1}_{[T]}(u)$ for all $u \in [t,T]$.

It is worth noting that in Proposition 5.4 we obtain a non-linear pricing rule, although we work there under the assumption that the lending and borrowing rates are identical. Due to this postulate, the price process $S(x,A,C)$ is in fact independent of the hedger's initial endowment – this property can be easily deduced from equation (5.33). The non-linearity of the pricing rule is now due to specification (5.30) of the collateral amount $C$, which implies that the non-linear BSDEs (5.13) and (5.33) have different shapes. For a detailed study of pricing BSDEs and fair prices for both parties when $C$ is given by an extension of (5.30), the interested reader is referred to Nie and Rutkowski [35].

**Proposition 5.4** *Let $X$ be an $\mathcal{F}_T^S$-measurable, bounded random variable. The BSDE*

$$dY_t^x = rY_t^x\,dt + \sum_{i=1}^d Z_t^{x,i} S_t^i \sigma_t^i\,d\widetilde{W}_t^i + (r_t - r_t^c)\big(\bar{\delta}_t^1 (Y_t^x - xB_t)^- - \bar{\delta}_t^2 (Y_t^x - xB_t)^+\big)\,dt \tag{5.33}$$

*with the terminal condition $xB_T - X$ has a unique solution $(Y^x, Z^x)$. For any fixed $x$ and $t \in [0,T)$, the contract $A_t = p\,\mathbb{1}_{[0,T]}(t) + X\mathbb{1}_{[T]}(t)$ with the collateral process $C$ given by (5.30) can be replicated on $[t,T]$ by an admissible trading strategy $\xi^x = Z^x$ and the ex-dividend price satisfies $S_t(x,A,C) = Y_t^x - xB_t$. Furthermore, the price $S_t(x,A,C)$ admits the following representation, for every $t \in [0,T)$,*

$$S_t(x,A,C) = -B_t\,\widetilde{\mathbb{E}}_t\bigg(B_T^{-1}X + \int_t^T B_u^{-1}(r_u - r_u^c)\big(\bar{\delta}_u^1(Y_u^x - xB_u)^- - \bar{\delta}_u^2(Y_u^x - xB_u)^+\big)\,du\bigg). \tag{5.34}$$

*Equivalently, the price $S_t(x,A,C) = Y_t$ for every $t \in [0,T)$, where the process $Y$ solves the following BSDE*

$$dY_t = rY_t\,dt + \sum_{i=1}^d Z_t^i S_t^i \sigma_t^i\,d\widetilde{W}_t^i + (r_t - r_t^c)\big(\bar{\delta}_t^1 Y_t^- - \bar{\delta}_t^2 Y_t^+\big)dt \tag{5.35}$$

*with the terminal condition $Y_T = -X$. Consequently, the price $S(x,A,C) = S(A,C)$ is independent of the hedger's initial endowment $x$.*

*Proof.* The proof is similar to the proof of Proposition 5.3 and thus we omit the details. Let us only observe that the uniqueness of a solution to BSDE (5.35) follows from the general theory of BSDEs with Lipschitz continuous coefficients (see, e.g., [19]). □

In view of (5.30), equation (5.32) may also be represented as follows

$$dV_t(\varphi) = r_t^c V_t(\varphi)\,dt + \sum_{i=1}^d \xi_t^i S_t^i \sigma_t^i\,d\widetilde{W}_t^i + (r_t - r_t^c)(C_t + V_t(\varphi) - xB_t)\,dt + dA_t. \tag{5.36}$$

This yield the following representation (note that we may write here $V(\varphi) = V(\xi^x)$)

$$S_t(A,C) = -B_t^c\,\widetilde{\mathbb{E}}_t\bigg((B_T^c)^{-1}X + \int_t^T (B_u^c)^{-1}(r_u - r_u^c)(C_u + V_u(\xi^x) - xB_u)\,du\bigg). \tag{5.37}$$

Furthermore, in the case of the fully collateralized contract, we postulate that $\delta_t^1 = \delta_t^2 = 0$ so that the equalities

$$C_t = -\widehat{V}_t(\xi^x) = xB_t - V_t(\xi^x)$$

are satisfied for all $t \in [0,T]$. Hence (5.36) reduces to

$$dV_t(\varphi) = r_t^c V_t(\varphi)\,dt + \sum_{i=1}^d \xi_t^i S_t^i \sigma_t^i\,d\widetilde{W}_t^i + dA_t$$

and this in turn yields the following explicit representation for the ex-dividend price of the fully collateralized contract, for every $t \in [0,T)$,

$$S_t(A,C) = -B_t^c\,\widetilde{\mathbb{E}}_t\big((B_T^c)^{-1}X\big). \tag{5.38}$$

Note that the price given by equation (5.38) not only does not depend on the initial endowment $x$, but it is also linear as a mapping from the space of contingent claims to the space of processes representing their prices.

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